RINGS
GENERATED BY THEIR UNITS

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ABSTRACT

RINGS GENERATED BY THEIR UNITS

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This thesis is a study of the article [5] written by R. Raphael. The work contains a systematic theory of rings generated by their invertable elements. Such rings are called S-rings. Special attention is paid to those S-rings which are also regular.

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CONTENTS

INTRODUCTION

In 1953, K. Wolfson [11] proved that the ring of all linear transformations of a vector space of dimension at least two, is generated by its idempotents. The following year, D. Zelinsky showed that every element of this ring is a sum of two nonsingular ones. Motivated by these results, in 1958, Skórnyakov posed the question [6, p.167]: "Can every element of a regular ring with unit element be represented as a sum of elements having inverses?"

The answer to the Skornyakov question is negative in general, for a Boolean ring with more than two elements is never generated by its invertable elements (units). Thus the question became, which regular rings are generated by their units? For some time, it was suspected that every element of a regular ring, in which two is a unit, can be written as a sum of units. This conjecture was settled in the negative by G. Bergman. Finally, in [5], R. Raphael developed a general theory of rings generated by their invertable elements. Such rings he calls S-rings, after Skornyakov.

In the first part of his article, R. Raphael discusses S-rings in general. He then answers when Artinian, perfect and semiperfect rings are generated by their units, arguing directly from the Wedderburn theorem. After this, he demonstrates that the familiar examples of regular rings (matrix, commutative, and self-injective ones) are S-rings if they satisfy a generalization of the condition that two is a unit. Furthermore, using the work of Utumi, he shows that any regular ring satisfying this condition can be embedded into a regular S-ring. The article is closed with questions and comments.

This work is a study of Raphael's article [5]. Some results of this article are generalized and the question [5, question 1., p.602] is answered. For a better understanding, other results of general ring theory are proved. Lambek's "Lectures on Rings and Modules" [4] serves here as a general reference.

CHAPTER I

THE S-RING

1.1 DEFINITION OF THE S-RING

We begin with the assumption that by ring is meant ring with identity such that $0 \neq 1$.

<u>DEFINITION 1.1</u>: An element r of a ring R is called a <u>unit</u> if $rs = 1 = sr$ for some element s of R.

If a and b are units, we have $a^{-1}a = aa^{-1} = 1$ and $(b^{-1}a^{-1})ab = ab(b^{-1}a^{-1}) = 1$ and this shows that a^{-1} and ab are units. It follows that in a ring the units form a group with respect to multiplication.

<u>DEFINITION 1.2</u>: Let R be a ring, define U(R) as the set of elements of R which can be written as the sum of a finite number of units of R.

<u>LEMMA 1.1</u>: U(R) is a subring of R and it is the smallest subring of R that contains the group of units of R.

<u>PROOF</u>: $1 \in U(R)$ so U(R) is a non-empty subset of R.

If u is a unit of R then there exists s in R such that $1 = us = su = (-u)(-s) = (-s)(-u)$ and it follows that $-u$ is also a unit of R. Therefore, if $a = \Sigma u_i \in U(R)$ where u_i are units then $-a = \Sigma -u_i \in U(R)$. Also, if $a, b \in U(R)$ then a and b are sums of units and therefore $a + b$ is a sum of units, so $a + b \in U(R)$.

Moreover, if $a = \Sigma u_i$ and $b = \Sigma v_j$ are in U(R) where u_i and v_i are units then $ab = \Sigma u_i \Sigma v_j = \Sigma u_i v_j \in U(R)$, where $u_i v_j$ are units. This shows that U(R) is a subring of R.

Now, let U be the group of units in R and let S be a subring of R such that $U \subset S$. If $a = \Sigma u_i \in U(R)$ where $u_i \in U \subset S$ then $a \in S$, and therefore $U(R) \subset S$. Thus $U(R)$ is the smallest subring of R that contains U.

DEFINITION 1.3: We call a ring R an S-ring if $U(R) = R$ and say that R is generated by its units.

1.2 EXAMPLES OF S-RINGS

EXAMPLE 1. The ring of integers Z is an S-ring, since any integer can be written as the sum of 1's and -1's.

Before the next example of an S-ring we will introduce some definitions and results.

DEFINITION 1.4: A right R-module A_R is called irreducible if it has exactly two submodules. These submodules must be A and 0, and the definition is meant to imply that $A \neq 0$.

THEOREM 1.1: The following conditions concerning the ring R are equivalent:

(1) 0 is a maximal right ideal.

(2) R is irreducible as a right R-module.

(3) Every nonzero element is right invertable.

(4) Every nonzero element is a unit.

DEFINITION 1.5: Under the conditions of theorem 1.1, R is called a division ring.

PROOF: (1) $\Rightarrow$ (2): 0 is the maximal right ideal

$\Rightarrow rR = R$ for all nonzero r in R

$\Rightarrow R_R$ has exactly two submodules, O and R.

(2) ⇒ (3): R_R has exactly two submodules, 0 and R

⇒ $rR = R_R = R$ for all nonzero $r \in R$

⇒ For every nonzero $r \in R$ there exists $s \in R$ such that $rs = 1$.

(3) ⇒ (4): Assume (3) and let $0 \neq r \in R$, then $rs = 1$ for some $s \in R$. Now $0 \neq s$, hence $st = 1$ for some $t \in R$. But

$$t = 1\cdot t = (rs)t = r(st) = r1 = r,$$

hence $sr = 1$, and so r is also left invertable. Therefore every nonzero r in R is a unit.

(4) ⇒ (1): Assume (4), then $rR = R$ for every $0 \neq r \in R$ and hence 0 is a maximal ideal.

DEFINITION 1.5: An ordered set (sometimes called "partially" ordered) is a system $(S, \leq)$ where S is a set and $\leq$ is a binary relation on S satisfying the reflexive, transitive, and antisymmetric laws: $a \leq a$, $(a \leq b \text{ and } b \leq c) \Rightarrow a \leq c$, $(a \leq b \text{ and } b \leq a) \Rightarrow a = b$. (Universal quantifiers are assumed.)

DEFINITION 1.6: An ordered set is called simply ordered (also called "totally" ordered) if for any two elements $a \leq b$ or $b \leq a$.

Let us now state an axiom, so called Zorn's Lemma, which is often used in ring theory.

ZORN'S LEMMA: If every simply ordered subset of a nonempty ordered set $(S, \leq)$ has an upper bound in S, then S has at least one maximal element m, maximal in the sense that $m \leq s$ implies $m = s$, for all $s \in S$.

LEMMA 1.2: Every proper (right) ideal in a ring is contained in a maximal proper (right) ideal.

PROOF: Let I be any proper (right) ideal of a ring R. Consider the set S of all proper (right) ideals of R which contain I. S is non-empty since $I \in S$, and it is evident that $(S, \subset)$ is a partially ordered set. Moreover, if $\{I_i | i \in A\}$ is any simply ordered subset of S then its upper bound $\bigcup_{i\in A} I_i$ is also in S, since $\bigcup_{i\in A} I_i$ is an ideal and $I \subset \bigcup_{i\in A} I_i \neq R$. The conditions of Zorn's Lemma are satisfied, thus S has a maximal element M. Therefore $I \subset M$ where M is maximal (right) ideal of R.

DEFINITION 1.7: The intersection of all maximal right ideals of the ring R is called The Jacobson radical of R and is denoted by Rad R.

LEMMA 1.3: The Jacobson radical of R is the set of all $r \in R$ such that $1 - rs$ is right invertable for all $s \in R$.

PROOF: $r \in \text{Rad } R \Rightarrow r \in M$ for all maximal right ideals M of R

$\Rightarrow 1 \notin M = M + rR$ for all maximal right ideals M of R

$\Rightarrow 1 - rs \notin M$ for all $s \in R$ and all maximal right ideals M

$\Rightarrow (1 - rs)R$ is not a proper right ideal, by Lemma 1.2,

$\Rightarrow (1 - rs)R = R$ for all $s \in R$

$\Rightarrow 1 - rs$ is right invertable for all $s \in R$.

Conversely, assume that M is a maximal right ideal and $r \notin M$ where $1 - rs$ is right invertable for all $s \in R$.

Then $M + rR = R$ and hence $m + rs = 1$ for some $m \in M$ and $s \in R$.

But $m = 1 - rs$ is right invertable, and this contradicts the fact that M is a proper right ideal.

THEOREM 1.2: The following conditions concerning the ring R are equivalent:

(1) R/Rad R is a division ring.

(2) R has exactly one maximal right ideal.

(3) All nonunits of R are contained in a proper ideal.

(4) The nonunits of R form a proper ideal.

(5) For every element r of R, either r or $1 - r$ is a unit.

(6) For every element r of R, either r or $1 - r$ is right invertable.

DEFINITION 1.8: A ring R is called a local ring if it satisfies one of these equivalent conditions.

PROOF: (1) ⇒ (2): R/Rad R is a division ring

⇒ $\bar{r}$ is a unit for all $\bar{0} \neq \bar{r} = r + \text{Rad } R \in R/\text{Rad } R$.

⇒ For all nonzero $\bar{r} = r + \text{Rad } R \in R/\text{Rad } R$ there exists $\bar{s} = s + \text{Rad } R \in R/\text{Rad } R$ such that $\bar{r}\,\bar{s} = \bar{1}$.

⇒ For all $r \notin \text{Rad } R$ there exists $s \in R$ such that $1 - rs \in \text{Rad } R$.

⇒ For every $r \notin \text{Rad } R$ there exists $s \in R$ such that $1 - (1 - rs) = rs$ is right invertable (by Lemma 1.3).

⇒ For every $r \notin \text{Rad } R$, r is right invertable.

⇒ Rad R is a maximal right ideal.

⇒ R has exactly one maximal right ideal.

(2) ⇒ (3): Let M be the unique maximal right ideal of R. Assume that $x \notin M$.

Since M is unique and every proper ideal is contained in some maximal ideal, then $xR = R$. This implies that $xy = 1$ for some $y \in R$. If $y \in M$ then $yx \in M$, and $(yx)(yx) = y(xy)x = y1x = yx$, so yx is an idempotent, say $yx = e$. But $e + (1 - e) = 1 \notin M$, hence $1 - e \notin M$. Thus there exists $s \in R$ such that $(1 - e)s = 1$ and consequently $e = e(1 - e)s = (e - e^2)s = 0$. But $0 = e = yx$ implies that $x = 1x = xyx = x0 = 0$, and this contradicts the fact that $x \notin M$. Therefore $y \notin M$. This again implies that there exists $z \in R$ such that $yz = 1$, hence we have that $xy = 1 = yz$, which implies that $xy = 1 = yx$.

It follows that for every $x \notin M$, x is a unit, and thus all nonunits are in M. Moreover, since M is a proper ideal, M is the set of all nonunits of R. This shows that M is also a left ideal. Therefore all nonunits are contained in a proper ideal.

(3) ⇒ (4): Assume that all nonunits of R are contained in a proper ideal I. Since I is a proper ideal therefore all elements of I are nonunits. It follows that I is the set of all nonunits of R.

(4) ⇒ (5): Let I be the proper ideal of all nonunits of R. For every $r \in R$, if $r \in I$ then $1 - r \notin I$, since $r + (1 - r) = 1 \notin I$. Therefore, if r is a nonunit then $1 - r$ is a unit, for all $r \in R$. Thus for every element r of R, either r or $1 - r$ is a unit.

(5) ⇒ (6): This implication follows from the fact that a unit is a right invertable element.

(6) ⇒ (1): Assume that for every $r \in R$, either r or $1 - r$ is right invertable. Let $\bar{r} = r + \text{Rad } R$ be an element of R/Rad R. Then,

$\bar{r} \neq \bar{o}$

⇒ $r \notin \text{Rad } R$

⇒ there exists $s \in R$ such that $1 - rs$ is not right invertable (by Lemma 1.3)

⇒ $1 - (1 - rs) = rs$ is right invertable (by assumption)

⇒ r is right invertable

⇒ $\bar{r}$ is right invertable

So we have that every nonzero element of R/Rad R is right invertable. It follows (by Theorem 1.1, (3)) that R/Rad R is a division ring.

EXAMPLE 2: Any local ring is an S-ring.

PROOF: Let R be a local ring.

If r is a nonunit of R then 1 − r is a unit (by Theorem 1.2, (5)), and hence r = 1 − (1 − r) is the sum of two units. Thus R is an S-ring. Note that in particular any division ring is an S-ring.

EXAMPLE 3: If X is a topological space then C(X), the ring of real valued continuous functions on X, is an S-ring.

PROOF: It is easy to see that if $f \in C(X)$, then the function $|f|$ (defined as $|f|(x) = |f(x)|$) is also in C(X).

This implies that

$u_1 = 2^{-1}(f + |f|) + 1 \in C(X)$ and $u_2 = 2^{-1}(f - |f|) - 1 \in C(X)$, where

$f = u_1 + u_2$. Moreover, since for all $x \in X$

$$u_1(x) \geq 1 \quad \text{and} \quad u_2(x) \leq -1$$

hence $u_1^{-1}(x) = \dfrac{1}{u_1(x)}$ and $u_2^{-1}(x) = \dfrac{1}{u_2(x)}$ exist, and are in C(X). Thus u_1 and u_2 are units of C(X), and so C(X) is an S-ring.

The same argument shows the following.

EXAMPLE 4: If X is a topological space then C*(X) the ring of bounded functions in C(X), and Q(X) their common full ring of quotients are S-rings.

Note that in the above examples, X can be considered as a completely regular Hasdorff space which is an important topological space. For more details about the subject see [2].

One can see that there are many examples of $\S$-rings in ring theory, therefore it is proper to study their abstract structure.

CHAPTER II

THE GROUP RING GENERATED BY ITS UNITS

2.1 DEFINITION OF THE GROUP RING AND ITS SIMPLEST PROPERTIES

DEFINITION 2.1: Given a group G and a ring A, the group ring R = AG consists of all functions r: G → A with finite support. The support of r is $\{g \in G \mid r(g) \neq 0\}$. R is endowed with ring operations by defining:

$$0(g) = 0 \quad ,$$

$$1(g) = \begin{cases} 1 \text{ if } g = 1 \\ 0 \text{ if } g \neq 1 \end{cases} \quad ,$$

$$(-r)(g) = -r(g) \quad ,$$

$$(r + r')(g) = r(g) + r'(g) \quad ,$$

$$(rr')(g) = \sum_{g=hh'} r(h)r'(h') \quad .$$

Let us verify that R(0, 1, −, +, ·) is in fact a ring.

PROOF: It is obvious that the addition and the multiplication defined above are binary operations.

$$(0 + r)(g) = 0(g) + r(g) = 0 + r(g) = r(g), \text{ therefore}$$

$0 \in R$.

Also, $(1 \cdot r)(g) = \sum_{g=hh'} 1(h)r(h') = 1(1)r(g) + \sum_{\substack{g=hh' \\ h\neq 1}} 1(h)r(h') =$

$1r(g) + \sum_{\substack{g=hh' \\ h\neq 1}} 0\cdot r(h') = r(g) + 0 = r(g)$, and similarly $r \cdot 1 = r$.

Therefore, $1 \in R$.

Addition in R is associative and commutative since an addition in a ring A is.

Now,

$$(r_1(r_2r_3))(g) = \sum_{g=hh'} r_1(h)\cdot(r_2r_3)(h') = \sum_{g=hh'} r_1(h)(\sum_{h'=tt'} r_2(t)r_3(t')) =$$

$$\sum_{\substack{g=hh'\\h'=tt'}} r_1(h)r_2(t)r_3(t') = \sum_{g=htt'} r_1(h)r_2(t)r_3(t') = \sum_{\substack{g=kt'\\k=ht}} r_1(h)r_2(t)r_3(t') =$$

$$\sum_{g=kt'}(\sum_{k=ht} r_1(h)r_2(t))r_3(t') = \sum_{g=kt'}(r_1r_2)(k)r_3(t') = ((r_1r_2)r_3)(g).$$

So the multiplication in R is associative.

Thus $(R,0,-,+)$ is an abelian group and $(R,1,\cdot)$ is a semigroup.

Moreover,

$$(r_1(r_2 + r_3))(g) = \sum_{g=hh'} r_1(h)(r_2 + r_3)(h') = \sum_{g=hh'} r_1(h)(r_2(h') +$$

$$r_3(h')) = \sum_{g=hh'}(r_1(h)r_2(h') + r_1(h)r_3(h')) = \sum_{g=hh'} r_1(h)r_2(h') +$$

$$\sum_{g=hh'} r_1(h)r_3(h') = (r_1r_2)(g) + (r_1r_3)(g).$$

Similarly, $(r_1 + r_2)r_3 = r_1r_3 + r_2r_3$

Therefore R is a ring.

With any $a \in A$ and $g \in G$ we associate elements a^* and g^+ of $R = AG$ as follows. For any $h \in G$, put

$$a^*(h) = \begin{cases} a \text{ if } h = 1 \\ 0 \text{ if } h \neq 1 \end{cases},$$

$$g^+(h) = \begin{cases} 1 \text{ if } h = g \\ 0 \text{ if } h \neq g \end{cases}.$$

<u>LEMMA 2.1</u>: If $\phi: A \to R$ such that $\phi(a) = a^*$, then ϕ is a ring monomorphism of A into R.

<u>PROOF</u>: It is obvious that ϕ is well defined.

Since $\phi(a) = \phi(b)$

$\Rightarrow a^* = b^*$

$\Rightarrow a^*(1) = b^*(1)$

$\Rightarrow a = b$, therefore ϕ is $1 - 1$.

Also, because $(ab)^*(1) = ab = a^*(1)b^*(1)$ and for any $h \neq 1$, $(ab)^*(h) = 0 = 0.0 = a^*(h)b^*(h)$, thus $\phi(ab) = \phi(a)\phi(b)$. Moreover, since $(a + b)^*(1) = a + b = a^*(1) + b^*(1)$ and for every $h \neq 1$, $(a + b)(h) = 0 = 0 + 0 = a^*(h) + b^*(h)$, hence $\phi(a + b) = \phi(a) + \phi(b)$. It follows that ϕ is a ring monomorphism of A into R.

LEMMA 2.2: If $\psi: G \to R$ such that $\psi(g) = g^+$, then ψ is a semigroup monomorphism.

PROOF: Notice that ψ is well defined.

Moreover, since for $g_1, g_2 \in G$, $\psi(g_1) = \psi(g_2)$

$$\Rightarrow g_1^+ = g_2^+$$

$$\Rightarrow g_1^+(h) = g_2^+(h), \text{ for all } h \in G$$

$$\Rightarrow g_1^+(g_1) = g_2^+(g_2) = 1$$

$\Rightarrow g_1 = g_2$, thus ψ is $1 - 1$. Also, since

$$(g_1^+g_2^+)(h) = \sum_{h=tt'} g_1^+(t)g_2^+(t') = \begin{cases} 1 \text{ if } h = g_1g_2 \\ 0 \text{ if } h \neq g_1g_2 \end{cases} = (g_1g_2)^+(h),$$

hence $\psi(g_1)\psi(g_2) = \psi(g_1g_2)$.

Therefore it follows that ψ is a semigroup monomorphism of G into R.

LEMMA 2.3: For every element $r \in R = AG$,

$$r = \sum_{g\in G} r(g)^*g^+ = \sum_{g\in G} g^+r(g)^*.$$

PROOF: Notice that the above sums are finite since r has a finite support. For every $h \in G$, $(\sum_{g\in G} r(g)^*g^+)(h) = \sum_{g\in G} (r(g)^*g^+)(h) =$

$\sum_{g\in G} (\sum_{h=tt'} r(g)^*(t)g^+(t')) = S$ (call it S).

Observe that, since $r(g)^*(t)g^+(t') \neq 0$

$\Rightarrow t = 1$ and $t' = g$

$\Rightarrow h = tt' = 1g = g$, therefore $s = r(h)^*(1)g^+(g) = r(h)1 =$

$r(h)$.

Thus $r = \sum_{g \in G} r(g)^* g^+$, and similarly $r = \sum_{g \in G} g^+ r(g)^*$.

DEFINITION 2.2: A module M_R is called free if it has a basis $\{m_i | i \in I\}$, $m_i \in M$, such that every element $m \in M$ can be written uniquely in the form

$$m = \sum_{i \in I} m_i r_i$$

where $r_i \in R$ and all but a finite number of the r_i are 0.

LEMMA 2.4: If we write $ra = ra^*$ for any $r \in R$ and $a \in A$, then R becomes a free A-module R_A with basis $\{g^+ | g \in G\}$.

PROOF: R is an additive Abelian group and A is a ring.

Also, the mapping $R \times A \to A$ defined by $(r, a) \to ra = ra^*$ is such that:

$$(r + s)a = (r + s)a^* = ra^* + sa^* = ra + sa,$$

$$r(a + b) = r(a + b)^* = r(a^* + b^*) = ra^* + rb^* = ra + rb$$

$$r(ab) = r(ab)^* = r(a^*b^*) = (ra^*)b^* = (ra^*)b = (ra)b$$

$$r1_A = r(1_A)^* = r1_R = r ,$$

for all $r, s \in R$ and $a, b \in A$.

Therefore R is an A-module R_A.

Now, by Lemma 2.3, for all $r \in R_A$

$$r = \sum_{g \in G} g^+ r(g)^* = \sum_{g \in G} g^+ r(g) .$$

This implies that $\{g^+ | g \in G\}$ spans R_A. Moreover, if $\sum_{g \in G} g^+ a_g = 0$ for some $a_g \in A$, then for all $h \in G$, $0 = (\sum_{g \in G} g^+ a_g)(h) = (\sum_{g \in G} g^+ a_g^*)(h) =$

$\sum_{g \in G} (g^+ a_g^*)(h) = \sum_{g \in G} (\sum_{h = tt'} g^+(t) a_g^*(t')) = \sum_{g \in G} g^+(h) a_g^*(1) = \sum_{g \in G} g^+(h) a_g =$

$h^{+}(h)a_h = a_h'$.

It follows that $\{g^{+} | g \in G\}$ is a linearly independent set, and so it is a basis of R_A. Hence R is a free A-module.

2.2 FUNDAMENTAL THEOREM

In this section we introduce the theorem, which tells us a necessary and sufficient condition for the group ring to be an S-ring.

LEMMA 2.5: Let A be a ring, let G be a group and let R be the group ring defined by A and G. Then A is a homomorphic image of R.

PROOF: Define $\phi: R \to A$ such that for all $r = \sum_{g \in G} g^{+}r(g)^{*} \in R$,

$$\phi(r) = \sum_{g \in G} r(g)$$

It is clear that ϕ is well defined.

ϕ is onto, since for all $a \in A$ there exists $a^{*} \in R$ such that

$$\phi(a^{*}) = \sum_{g \in G} a^{*}(g) = a^{*}(1) = a.$$

Moreover, for every $r_1, r_2 \in R$,

$\phi(r_1 + r_2) = \sum_{g \in G}(r_1 + r_2)(g) = \sum_{g \in G}(r_1(g) + r_2(g)) = \sum_{g \in G} r_1(g) + \sum_{g \in G} r_2(g) =$

$\phi(r_1) + \phi(r_2)$ and, $\phi(r_1 r_2) = \phi(\sum_{g \in G} g^{+}r_1(g)^{*} \sum_{h \in G} h^{+}r_2(h)^{*}) =$

$\phi(\sum_{g,h \in G} g^{+}r_1(g)^{*}h^{+}r_2(h)^{*}) = \phi(\sum_{g,h \in G} g^{+}h^{+}r_1(g)^{*}r_2(h)^{*}) =$

$\phi(\sum_{g,h \in G} (gh)^{+}(r_1(g)r_2(g))^{*}) = \sum_{g,h \in G} r_1(g)r_2(h) = \sum_{g \in G} r_1(g) \sum_{h \in G} r_2(g) =$

$\phi(r_1)\phi(r_2)$.

So ϕ is a ring homomorphism of R onto A. Thus A is a homomorphic image of R.

THEOREM 2.1: Let A be a ring, let G be a group and let R be a group ring defined by A and G. Then R is an S-ring if and only if A is an S-ring.

PROOF: ($\Rightarrow$): Assume that the group ring $R = AG$ is an S-ring. By Lemma 2.5, A is a homomorphic image of R. Moreover, it is obvious that a homomorphic image of an S-ring is an S-ring. Thus A is an S-ring.

($\Leftarrow$): Let A be an S-ring.

If u is a unit in A then $uv = vu = 1$ for some $v \in A$. This implies that $u^*v^* = (uv)^* = 1_A^* = 1_R$ and similarly $v^*u^* = 1_R$, which means that u^* is a unit in R. Also, since $g^+(g^{-1})^+ = (gg^{-1})^+ = 1_G^+ = 1_R$ and similarly $(g^{-1})^+g^+ = 1_R$, then g^+ is a unit in R for all $g \in G$. Now, for every $r \in R$, $r = \sum_{g \in G} g^+ r(g)^*$

and, $r(g) \in A \Rightarrow r(g) = \Sigma u_i$ for some units u_i in A

$\Rightarrow$ $r(g)^* = (\Sigma u_i)^* = \Sigma u_i^*$ where u_i^* are units in R.

Hence it follows that r is a sum of units of R and therefore R is an S-ring.

From the above results, it is clear that any S-ring can be imbedded in the group ring which is also an S-ring.

CHAPTER III

UNIT GENERATION AND RADICALS

3.1 JACOBSON RADICAL AND UNIT GENERATION

LEMMA 3.1: Let R be a ring. Then every element of Rad R is the sum of two units.

PROOF: $r \in \text{Rad } R \Rightarrow 1 - r$ is right invertable

$\Rightarrow (1 - r)u = 1$ for some $u \in R$

$\Rightarrow 1 - u = -r\,u \in \text{Rad } R$

$\Rightarrow 1 - (1 - u) = u$ is right invertable

$\Rightarrow uv = 1$ for some $v \in R$

$\Rightarrow u(1 - r) = u(1 - r)uv = u\,1\,v = uv = 1$

$\Rightarrow 1 - r$ is a unit.

Therefore for any $r \in \text{Rad } R$, $r = 1 - (1 - r)$ is the sum of two units.

LEMMA 3.2: Let R be a ring and let I be an ideal of R contained in Rad R. Then units can be lifted modulo I, in the sense that, if $\bar{x} = x + I$ is a unit of R/I then x is a unit of R.

PROOF: $\bar{x} = x + I$ is a unit of R/I

$\Rightarrow \bar{x}\,\bar{y} = \bar{1}$ and $\bar{y}\,\bar{x} = \bar{1}$ for some $\bar{y} \in R/I$

$\Rightarrow 1 - xy \in I \subset \text{Rad } R$ and $1 - yx \in I \subset \text{Rad } R$

$\Rightarrow 1 - (1 - xy) = xy$ is a unit and $1 - (1 - yx) = yx$ is a unit, by Lemma 3.1

$\Rightarrow xy$ is right invertable and yx is left invertable

$\Rightarrow x$ is right invertable and x is left invertable

$\Rightarrow x$ is a unit of R.

THEOREM 3.1: Let R be a ring and let I be an ideal contained in Rad R. Then R is an S-ring if R/I is.

PROOF: Assume that R/I is an S-ring.
Therefore for all $\bar{x} = x + I \in R/I$, $\bar{x} = \Sigma \bar{u}_i$ where u_i are units of R/I, and hence u_i are units of R, by Lemma 3.2. This implies that $x - \Sigma u_i \in I \subset \text{Rad } R$. Thus $x - \Sigma u_i$ is the sum of two units, by Lemma 3.1. Say $x - \Sigma u_i = u + u'$ where u and u' are units of R. So we have that $x = u + u' + \Sigma u_i$ is the sum of units. Hence R is an S-ring.

3.2 RADICALS OF R RELATIVE TO RADICALS OF U(R)

THEOREM 3.2: Let R be a ring. Then Rad R $\subseteq$ Rad U(R).

PROOF: We know that Rad R $\subset$ U(R), by Lemma 3.1.
Thus, $j \in \text{Rad } R \subset U(R) \Rightarrow 1 - jr$ is right invertable for all $r \in R$

$\Rightarrow j \in U(R)$ and $1 - jr$ is right invertable for all $r \in U(R)$

$\Rightarrow j \in \text{Rad } U(R)$.

Hence, Rad R $\subset$ Rad U(R).

DEFINITION 3.1: An ideal P of a ring R is prime if it is proper (that is $P \neq R$) and $AB \subset P \Rightarrow A \subset P$ or $B \subset P$ for any ideals A and B of R.

DEFINITION 3.2: The prime radical of a ring R is the intersection of all prime ideals of R and is denoted by rad R.

DEFINITION 3.3: An element a of a ring R is called nilpotent if $a^n = 0$ for some natural number n.

DEFINITION 3.4: An element a of a ring R is called strongly nilpotent provided every sequence $a_0, a_1, a_2, \cdots$, such that

$$a_0 = a, \quad a_{n+1} \in a_n R a_n$$

is ultimately zero.

Note that every strongly nilpotent element is nilpotent, and if R is commutative every nilpotent element is strongly nilpotent.

LEMMA 3.3: The prime radical of R is the set of all strongly nilpotent elements.

PROOF: Assume that $a \notin \text{rad } R$, then there exists a prime ideal P of R such that $a_0 = a \notin P$. Therefore $a_0 R a_0 \not\subset P$, and so there is $a_1 \in a_0 R a_0$ such that $a_1 \notin P$. Continuing in this manner, we find $a_{n+1} \in a_n R a_n$ such that $a_{n+1} \notin P$. Thus, for all natural numbers n, $a_n \notin P$, hence $a_n \neq 0$, and so a is not strongly nilpotent.

Conversely, assume that a is not strongly nilpotent. Then there exists a sequence $a_0, a_1, a_2, \cdots$, such that $a_0 = a$, $a_{n+1} \in a_n R a_n$ and all $a_n \neq 0$. Let T be a set of all a_n, then $0 \notin T$. Let $K = \{I \mid I$ is an ideal of R and $T \cap I = \emptyset\}$, then $K \neq \emptyset$ since $\{0\} \in K$, and so by Zorn's Lemma K has a maximal element P. If we can show that P is a prime ideal, it will follow from $a \notin P$ that $a \notin \text{rad } R$.

Now suppose A and B are ideals of R such that $A \not\subset P$ and $B \not\subset P$. Thus, by maximality of P, $(A + P) \cap T \neq \emptyset$ and $(B + P) \cap T \neq \emptyset$, hence $a_i \in A + P$ and $a_j \in B + P$ for some $a_i, a_j \in T$. Note that for any ideal I of R, $a_n \in I \Rightarrow a_{n+1} \in a_n R a_n \subset I R I = I^2 \subset I \Rightarrow a_{n+1} \in I$. This implies that $a_m \in A + P$ and $a_m \in B + P$ where $m = \max(i,j)$. Consequently,

$a_{m+1} \in a_m R a_m \subset (A + P)R(B + P) \subseteq (A + P)(B + P) \subset (A + P)B =$ $AB + PB \subset AB + P$. Therefore $a_{m+1} \in AB + P$ and $a_{m+1} \notin P$, and so $AB \not\subset P$. Moreover, P is proper since $a \notin P$. Hence P is a prime ideal.

<u>REMARK:</u> If R is a commutative ring, then rad R is the set of all nilpotent elements of R.

<u>DEFINITION 3.5</u>: Let S be a subring of a ring R. Then R is an <u>integral extension</u> of S if for all $x \in R$ there exists $a_0, a_1, \cdots, a_{n-1} \in S$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o = 0$.

<u>LEMMA 3.5</u>: If M is a maximal ideal of a commutative ring R and R is an integral extension of S, then $M \cap S$ is a maximal ideal of S.

<u>PROOF:</u> It is obvious that $M \cap S$ is a proper ideal of S. Moreover, it is clear that the proper ideal M of a commutative ring R is maximal if and only if for all $r \notin M$ there exists $x \in R$ such that $1 - rx \in M$.

Assume that $M \cap S$ is not maximal in S. Therefore there exists $s_o \notin M \cap S$, $(s_o \in S)$, such that for all $s \in S$,

(*) $$1 - s_o s \notin M \cap S$$

But, since M is maximal in R, for s_o there exists $x \in R$ such that $1 - s_o x \in M$. It follows that $s_o x \equiv 1 \bmod M$, and $x^n + a_{n-1}x^{n-1} + \cdots + a_o = 0$ for some $a_o, \cdots, a_{n-1} \in S$, since R is an integral extension of S.

Thus we have $s_o^n(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o) = 0$

$$\Rightarrow (s_o x)^n + a_{n-1} s_o (s_o x)^{n-1} + \cdots + a_1 s_o^{n-1}(s_o x) + a_o s_o^n = 0$$

$$\Rightarrow 1 + a_{n-1} s_o + \cdots + a_1 s_o^{n-1} + a_o s_o^n \equiv 0 \bmod M$$

$$\Rightarrow 1 - (-a_{n-1} - \dots - a_1 s_o^{n-2} - a_o s^{n-1}) s_o \equiv 0 \bmod M$$

$$\Rightarrow 1 - s_o s_1 \equiv 0 \bmod M \text{ where } s_1 = -a_{n-1} - \dots - a_o^{n-1} \in S$$

$$\Rightarrow 1 - s_o s_1 \in M \text{ and } 1 - s_o s_1 \in S$$

$$\Rightarrow 1 - s_o s_1 \in M \cap S$$

Hence for s_o there exists $s_1 \in S$ such that $1 - s_o s_1 \in M \cap S$, which is the contradiction to (*).

<u>COROLLARY</u>: Let R be a commutative ring. Then

$$\text{rad } U(R) \subset \text{rad } R \subset \text{Rad } R \subset \text{Rad } U(R).$$

In particular, if R is an integral extension of U(R), then

$$\text{Rad } R = \text{Rad } U(R).$$

<u>PROOF</u>: Since a nilpotent of any subring of R is the nilpotent of R, and U(R) is the subring of R, therefore rad U(R) $\subset$ rad R, by the above remark. Also, we know that every proper ideal (hence prime ideal) is contained in a maximal ideal, thus

$$\text{rad } R = \cap\{\text{prime ideals}\} \subset \cap\{\text{maximal ideals}\} = \text{Rad } R,$$

and by Theorem 3.2 Rad R $\subset$ Rad U(R). So the first statement of the corollary is proved. For the second statement it suffices to show that Rad U(R) $\subset$ Rad R. Let $x \in$ Rad U(R) and let M be any maximal ideal of R. By Lemma 3.5, $M \cap U(R)$ is a maximal ideal of U(R), so $x \in M$. Thus, $x \in$ Rad R and so Rad U(R) $\subset$ Rad R.

We conclude this section with the example which illustrates that R can be integral over U(R).

<u>DEFINITION 3.6</u>: A ring R is called <u>Boolean</u> if $x = x^2$ for each $x \in R$.

LEMMA 3. : If R is a Boolean ring then $U(R) = \{0,1\}$.

PROOF: Let u be a unit of R.

Then, $u^2 = u \Rightarrow u^{-1}u^2 = u^{-1}u$

$\Rightarrow u = 1$

Thus 1 is the only unit of R. Moreover, $1 = (-1)(-1) = -1$

$\Rightarrow 1 + 1 = 0$

It follows that if $a \in U(R)$ then a is either the sum of an even number of 1, or the sum of an odd number of 1, and so a is either 0 or 1.

If R is a Boolean ring with more than two elements then for all $x \in R$, $x^2 + x + 0 = 0$ and hence R is integral over U(R).

CHAPTER IV

EVEN S-RING

4.1 DEFINITION AND SIMPLEST PROPERTY

DEFINITION 4.1: A ring R is called an <u>even S-ring</u> if each element of R can be written as the sum of an even number of units.

Notice that the two-element field is an S-ring that is not even. For if 1 can be written as the sum of an even number of units then 1 = 0, since 1 + 1 = 0. Thus, in the two-element field 1 can not be written as the sum of an even number of units.

<u>LEMMA 4.1</u>: In an even S-ring, 0 can be written as the sum of an odd number of units.

<u>PROOF</u>: 0 = 1 + (-1) and by definition of an even S-ring -1 can be written as the sum of an even number of units. Thus 0 can be written as the sum of an odd number of units.

We can generalize the above and state the following immediate result.

<u>LEMMA 4.2</u>: Let R be an S-ring. The following conditions are equivalent:

(1) R is an even S-ring

(2) 0 can be written as the sum of an odd number of units.

(3) Every element can be written as the sum of an odd number of units.

<u>PROOF</u>: <u>(1) ⇒ (2)</u>: By Lemma 4.1

<u>(2) ⇒ (3)</u>: If x is the sum of an even number of units then

write 0 as the sum of an odd number of units, and so $x = x + 0$ is the sum of an odd number of units.

(3) $\Rightarrow$ (1): Write 0 as the sum of an odd number of units. If x is the sum of an odd number of units, then $x + 0$ is the sum of an even number of units.

4.2 RESULTS ON EVEN S-RINGS

LEMMA 4.3: If R is an S-ring that contains a unit u such that $u + 1$ is a unit, then R is an even S-ring.

PROOF: Let u be a unit such that $u + 1$ is a unit.
Then $0 = (u + 1) - (u + 1) = (u + 1) - u - 1$ is the sum of three units, and so 0 can be written as the sum of an odd number of units. Thus R is an even S-ring by Lemma 4.2.

We can generalize the above as follows:

LEMMA 4.4: If R is an S-ring that contains a unit which can be written as the sum of an even number of units, then R is an even S-ring.

PROOF: Let u be a unit such that $u = u_1 +\cdots+ u_n$ for some units u_i and some even natural number n. Then, $0 = u_1 +\cdots+ u_n - u$ is the sum of an odd number of units. Hence R is an even S-ring by Lemma 4.2.

LEMMA 4.5: A finite product of even S-rings is an even S-ring.

PROOF: It suffices to show the result for the product $R_1 \oplus R_2$. For any $r \in R_1$ write $r = u_1 +\ldots+ u_m$ where m is even and u_i are units of R_1.

Then, $(r,0) = (u_1,1) + (u_2,-1) + \cdots + (u_{m-1},1) + (u_m,-1)$ is the sum of an even number of units. Similarly, $(0,s)$ can be written as the sum of an even number of units, for every $s \in R_2$. Thus for every $(r,s) \in R_1 \oplus R_2$, $(r,s) = (0,s) + (r,0)$ can be written as the sum of an even number of units.

REMARK 4.1: A finite product of S-rings is not necessarily an S-ring.

PROOF: Let R be the two-element field. Observe that $(1,0) \in U(R \oplus R)$

$$\Rightarrow (1,0) = (1,1) + \cdots + (1,1), \text{ (n-times)}.$$

$$\Rightarrow (1,0) = (0,0) \text{ if n is even and } (1,0) = (1,1) \text{ if n is odd.}$$

Therefore, $(1,0) \notin U(R \oplus R)$ and so $R \oplus R$ is not an S-ring.

LEMMA 4.6: If R is any ring, then R_n (the ring of n x n matrices over R) is an even S-ring, for all $n > 1$.

PROOF: Let $r \in R$. Because the elementary matrices are units, it suffices to show that the n x n matrix with entry r in position one-two and zeros elsewhere can be written as the sum of an even number of units.

Let $A = \begin{pmatrix} 1 & r & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & \dots & & 0 & 1 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & & \dots & 0 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 1 & -r & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & \dots & & 0 & 1 \end{pmatrix}$

and so $\begin{pmatrix} 0 & r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & \dots & & & 0 \end{pmatrix} = A - I$ is the sum of two units.

LEMMA 4.7 If R is an even S-ring, and S is an S-ring, then $R \oplus S$ is an S-ring.

PROOF: Let $(r,s) \in R \oplus S$.

$(r,0)$ can be written as the sum of units, by the same argument as in the proof of Lemma 4.5.

Now, consider $(0,s)$ where $s = u_1 + \ldots + u_n$ and u_i are units of S. If n is even then $(0,s) = (1,u_1) + (-1,u_2) + \ldots + (-1,u_n)$ is the sum of units. If n is odd then write $0 \in R$ as the sum of an odd number of units, it is possible by Lemma 4.1). Say, $0 = v_1 + \ldots + v_m$ where v_i are units and m is odd. Therefore $(0,s) = (v_1 + \ldots + v_m, u_1 + \ldots + u_n)$. If $m = n$ then $(0,s) = (v_1,u_1) + \ldots + (v_n,u_n)$ is the sum of units. If $m < n$, then $n - m$ is even and $(0,s) = (v_1,u_1) + \ldots + (v_m,u_m) + (1,u_{m+1}) + (-1,u_{m+2}) + \ldots + (-1,u_n)$ is the sum of units. Similarly, $(0,s)$ can be written as the sum of units if $n < m$. Thus, it follows that $(r,s) = (r,0) + (s,0)$ can be written as the sum of units. Hence $R \oplus S$ is an S-ring.

It is clear that S is a homomorphic image of $R \oplus S$ and that a homomorphic image of an even S-ring is an even S-ring. Therefore, as for $R \oplus S$ to be an even S-ring it is necessary that S be an even S-ring, one cannot hope to strengthen the result of Lemma 4.7.

DEFINITION 4.2: The centre of a ring R, denoted by cent R, is the set

$$\text{cent } R = \{a \in R \mid ar = ra \text{ for all } r \in R\}$$

It is easy to see that the cent R is the subring of R.

REMARK 4.2: The centre of an S-ring need not be an S-ring.

PROOF: Let R be the two element field.

R_2 is an even S-ring by Lemma 4.6. This implies that $R_2 \oplus R_2$ is an

S-ring by Lemma 4.7. But, since

$$\text{cent } R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \bar{0}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \bar{1} \right\}$$

hence $\text{cent}(R_2 \oplus R_2) = \text{cent } R_2 \times \text{cent } R_2 = \{(\bar{0},\bar{0}),(\bar{1},\bar{1}),(\bar{0},\bar{1}),(\bar{1},\bar{0})\} \cong R \oplus R$, and thus $\text{cent}(R_2 \oplus R_2)$ is not an S-ring by Remark 4.1.

CHAPTER V

INFORMATION ON S-RINGS INSPIRED BY THE WEDDERBURN THEOREM

5.1 THE WEDDERBURN THEOREM

DEFINITION 5.1 The socle Soc A of a module A_R is the sum of all irreducible submodules of A. If there are no such submodules, Soc A = A.

DEFINITION 5.2 A module A_R is called completely reducible if A = Soc A.

DEFINITION 5.3 A ring R is called completely reducible if a module R_R is completely reducible.

DEFINITION 5.4 A vector space is a module over a division ring.

DEFINITION 5.5 It is known that a vector space V_D is a direct sum of copies of D_D. The number of these is called the dimension of the vector space.

DEFINITION 5.6 The ring $E = \mathrm{Hom}_D(V,V)$ of endomorphisms of a vector space V_D is called the ring of linear transformations of V_D.

DEFINITION 5.7 A ring R is called simple if it has exactly two ideals, that is, if 0 is a maximal ideal.

THEOREM 5.1 (Wedderburn-Artin).

(a) A ring R is completely reducible if and only if it is isomorphic to a finite product of completely reducible simple rings.

(b) A ring R is completely reducible and simple if and only if it is the ring of all linear transformations of a finite dimensional vector space.

PROOF: See [4 ,p. 65] .

If V_D is a finite dimensional vector space over the division ring D, the ring $Hom_D(V,V)$ of endomorphisms of V_D is well known to be isomorphic to the ring of n x n matrices over D, where n is the dimension of V_D. Thus we have the following immediate result:

COROLLARY 5.1 A ring R is completely reducible if and only if it is isomorphic to a finite direct product of $D^i_{n_i}$, where $D^i_{n_i}$ are rings of $n_i \times n_i$ matrices over division rings D^i.

5.2 RESULTS ON S-RINGS

In this section we will introduce the results on S-rings, which follow directly from the above Wedderburn theorem.

THEOREM 5.2: Let R be completely reducible. Then R is an S-ring if and only if the two element field occurs at most once in the decomposition of R into completely reducible simple rings. R is an even S-ring if and only if this field does not occur at all.

PROOF: Let R be a completely reducible ring. Then by the above results, the decomposition of R into completely simple rings is as follows:

$$(*) \qquad R \cong D^1_{n_1} \oplus D^2_{n_2} \oplus \ldots \oplus D^m_{n_m}$$

where $D^i_{n_i}$ is a ring of $n_i \times n_i$ matrices over a division ring D^i.

If $n_i > 1$ then $D^i_{n_i}$ is an even S-ring by Lemma 4.6, and if $n_i = 1$ then

$D^i_{n_i} \cong D^i$ is a division ring. Notice that any division ring with more than two elements is an even S-ring, and the two-element division ring is the two-element field. Hence for every i, $D^i_{n_i}$ is either an even S-ring or the two-element field which is the S-ring that is not even. Now, let F be the two-element field.

F occurs more than once in the decomposition (*)
$\Rightarrow$ F occurs in (*) at least twice,
$\Rightarrow$ $F \oplus F$ is an epimorphic image of R, and $F \oplus F$ is not an S-ring (Remark 4.1),
$\Rightarrow$ R has an epimorphic image that is not an S-ring,
$\Rightarrow$ R is not an S-ring.

Therefore, if R is an S-ring then F occurs at most once in the decomposition of R.

Conversely F does not occur at all in (*),
$\Rightarrow$ every $D^i_{n_i}$ is an even S-ring,
$\Rightarrow$ R is an even S-ring by Lemma 4.5

and F occurs exactly once in (*),
$\Rightarrow$ $R = R_1 \oplus F$, where R_1 is an even S-ring by Lemma 4.5
$\Rightarrow$ R is an S-ring by Lemma 4.7

Thus, if F occurs at most once in the decomposition of R then R is an S-ring. Hence the first statement of the theorem is proved. For the second it suffices to observe that if R is an even S-ring, then F does not occur at all in the decomposition of R.

Let I be an ideal of R contained in Rad R. It is obvious that if R is an (even) S-ring then R/I is an (even) S-ring. Also, if R/I is an S-ring then R is an S-ring by Theorem 3.1. Moreover, observing the proof of Theorem 3.1, it is easy to see that if R/I is an even S-ring then R is an even S-ring. Thus we have the following remark:

REMARK 5.1: Let I be an ideal of R contained in Rad R. Then R is an (even) S-ring if and only if R/I is an (even) S-ring.

Now combining the preceding remark with the Theorem 5.2, we have the following immediate result:

COROLLARY 5.2: Let I be an ideal of a ring R such that $I \subset$ Rad R and R/I is completely reducible. Then R is an (even) S-ring if and only if the two element field occurs (never) at most once in the Wedderburn representation of R/I.

DEFINITION 5.8: Let I be an ideal of a ring R. We say that idempotents modulo I can be lifted provided for every element v of I such that $v^2 - v \in I$ there exists an element $e^2 = e \in R$ such that $e - v \in I$.

DEFINITION 5.9: We call the ring R semiperfect if idempotents modulo Rad R can be lifted and if R/Rad R is completely reducible.

In the case where I = Rad R, the Corollary 5.2 tells us under what conditions semiperfect rings are (even) S-rings. Thus we have the following:

COROLLARY 5.3: Let R be a semiperfect ring. Then R is an (even) S-ring, if and only if the two-element field occurs (never) at most once in the Wedderburn representation of R/Rad R.

Notice that left or right Artinian rings are semiperfect [4, p.74]. Moreover, perfect rings are also semiperfect [4, p.170]. Therefore the Corollary 5.3 describes, in particular, when Artinian or perfect rings are (even) S-rings. One can see that these results produce a large class of rings generated by units.

CHAPTER VI

REGULAR S-RINGS

6.1 GENERAL INFORMATION ON REGULAR RINGS

DEFINITION 6.1: A ring R is called <u>regular</u> if for every $a \in R$ there exists $x \in R$ such that $axa = a$.

DEFINITION 6.2: A ring R is called <u>strongly regular</u> if for every $a \in R$ there exists $x \in R$ such that $a^2x = a$.

<u>LEMMA 6.1</u>: Let R be a strongly regular ring. Then R has no non-zero nilpotent elements.

<u>PROOF</u>: Let a be a nilpotent element of R. Then $a^n = 0$ for some natural number n, and $a = a^2x$ for some $x \in R$. This implies that

$$a = a^2x = a^3x^2 = \ldots = a^nx^{n-1} = 0\, x^{n-1} = 0$$

Thus 0 is the only nilpotent of R.

<u>LEMMA 6.2</u>: Let e be an idempotent element of a ring R, which commutes with all nilpotent elements of R, then e is central.

<u>PROOF</u>: For any $a \in R$, $(ea(1-e))^2 = ea(1-e)ea(1-e) = ea0a(1-e) = 0$ and similarly $((1-e)ae)^2 = 0$. Thus for any $a \in R$, $ea(1-e)$ and $(1-e)ae$ are nilpotent elements. Therefore, $ea(1-e) = e\, ea(1-e) = ea(1-e)e = 0$ and similarly $(1-e)ae = 0$. Hence $ea = ea\, e = ae$ and so e is central.

<u>COROLLARY 6.1</u>: Every idempotent element of a strongly regular ring R is central.

PROOF: By Lemma 6.1, 0 is the only nilpotent of R and 0 commutes with every element of R. Now the result follows by Lemma 6.2.

LEMMA 6.3: Let a be an element of a strongly regular ring R and $x \in R$ be such that $a^2x = a$. Then $a^2x = xa^2 = axa = a$ and $ax = xa$ is an idempotent of R.

PROOF: Assume that $a^2x = a$. Then

$$(a - axa)^2 = a^2 - a^2xa - axa^2 + axa^2xa = a^2 - a^2 - axa^2 + axa^2 = 0$$

Hence $a - axa$ is a nilpotent and so $a = axa$ by Lemma 6.1.

Also, $(a - xa^2)^2 = a^2 - axa^2 - xa^3 + x(a^2x)a^2 = a^2 - (axa)a - xa^3 + xa^3 =$ $a^2 - a^2 = 0$. Thus $a - xa^2$ is a nilpotent and so $a = xa^2$.

This shows that $a^2x = xa^2 = axa = a$.

Moreover $(ax)^2 = (axa)x = ax$ and $ax = (xa^2)x = x(a^2x) = xa$, thus $ax = xa$ is an idempotent.

From the above Lemma we have the following immediate corollary.

COROLLARY 6.2: A ring R is strongly regular if and only if for every element $a \in R$ there exists $x \in R$ such that $a = axa$ and $ax = xa$.

LEMMA 6.4: Let R be a regular ring. Then for every $a \in R$ there exists $a_1 \in R$ such that $a\, a_1 a = a$ and $a_1\, a\, a_1 = a_1$. If R is strongly regular then a_1 is uniquely determined by a.

PROOF: Since R is regular, for each $a \in R$ there exists $x \in R$ such that $a = axa$. Let $a_1 = xax$, then $aa_1a = a(xax)a = (axa)xa = axa = a$, and $a_1aa_1 = (xax)aa_1 = x(axa)a_1 = xaa_1 = xa(xax) = x(axa)x = xax = a_1$.

Now assume that R is strongly regular.

If for $a \in R$ there exists a_1 and a_2 in R such that $a = aa_1a = aa_2a$ and

$a_1 = a_1aa_1$, $a_2 = a_2aa_2$, then by Lemma 6.3 and Corollary 6.1, $a_2a = aa_2$ is a central idempotent, and thus

$$a_1 = a_1aa_1 = a_1(aa_2a)a_1 = a_1(aa_2)aa_1 = a_1aa_1(aa_2) = a_1aa_2 = a_1(aa_2a)a_2 =$$

$$a_1a(a_2a)a_2 = (a_2a)a_1aa_2 = a_2(aa_1a)a_2 = a_2aa_2 = a_2, \text{ hence } a_1 = a_2.$$

LEMMA 6.5: Let R be a strongly regular ring. Then for every $a \in R$ there exists a unit $u \in R$ such that $a = aua$.

PROOF: By Lemma 6.4 for every $a \in R$ there exists $a_1 \in R$ such that

$$a = aa_1a \quad \text{and} \quad a_1 = a_1aa_1.$$

We know that $aa_1 = a_1a$ is a central idempotent, say $aa_1 = e$.

Let $u = a_1 + 1 - e$ and $v = a + 1 - e$, then

$$uv = (a_1 + 1 - e)(a + 1 - e) = a_1a + a_1 - a_1e + a + 1 - e - ea - e(1 - e)$$

where $a_1e = a_1$ and $ea = a$, thus $uv = e + a_1 - a_1 + a + 1 - e - a = 1$.

Similarly, $vu = (a + 1 - e)(a_1 + 1 - e) = aa_1 + a - ae + a_1 + 1 - e -$

$ea_1 - e(1 - e) = e + a - a + a_1 + 1 - e - a_1 = 1$.

Therefore u is a unit of R, and

$$a\,u\,a = a(a_1 + 1 - e)a = (e + a - a)a = ea = a.$$

LEMMA 6.6: Let R be a regular ring. Then the centre of R is a commutative regular ring.

PROOF: Let cent R be the center of R. It is known that cent R is a commutative subring of R. Since R is regular, then by Lemma 6.4 for every $a \in$ cent R there exists $x \in R$ such that

$$a = axa = a^2x \quad \text{and} \quad x = xax = x^2a$$

We shall show that $x \in$ cent R. For every element $r \in R$,

$xr = x^2ar = x^2ra = x^2ra^2x = a^2x^2rx = axrx = axrx^2a = a^2xrx^2 = arx^2 = rx^2a = rx$. Thus $xr = rx$ and so $x \in$ cent R.

<u>LEMMA 6.7</u>: If e is an idempotent of a regular ring R, then e R e is a regular ring.

<u>PROOF</u>: It is easy to see that e R e is a ring with e as its identity. Since R is regular, for every $a \in e R e$ there exists $x \in R$ such that $axa = a$. Moreover $ae = ea = a$, since $a \in e R e$. Define $y = e x e$. Then $y \in e R e$ and $aya = a\ exe\ a = axa = a$. Thus e R e is regular.

<u>LEMMA 6.8</u>: A ring R is regular if and only if every principal right ideal of R is generated by an idempotent.

<u>PROOF</u>: Let R be a regular ring and aR be a principal right ideal of R. Then there exists $x \in R$ such that $a = axa$, where $ax = axax$ is an idempotent. Since $a = axa \in (ax)R$, we have $a R \subset (ax)R$, and the inverse inclusion is obvious. Thus $aR = (ax)R$.

Conversely, assume that every principal right ideal of R is generated by an idempotent. Then for every $a \in R$ there exists an idempotent $e \in R$ such that $aR = eR$. This implies that there exists $x \in R$ and $y \in R$ such that $a = ex$ and $e = ay$. Hence we have that

$$aya = ea = eex = ex = a.$$

Thus for every $a \in R$ there exists $y \in R$ such that $a = aya$, and so R is a regular ring.

<u>LEMMA 6.9</u>: In a regular ring every finitely generated ideal is principal.

PROOF: Let R be a regular ring. It suffices to consider a right ideal $aR + bR$. Now by Lemma 6.8, $aR = eR$ for some $e = e^2 \in R$, and $bR \subset ebR + (1 - e)bR$ since $br = [e + (1 - e)]br = ebr + (1 - e)br$. Therefore $aR + bR \subset eR + ebR + (1 - e)bR$ where $ebR \subset eR$ and $(1 - e)bR = fR$ for some $f = f^2$. Thus $aR + bR \subset eR + fR$ where $fr = (1 - e)br = br - ebr \in aR + bR$ and so $aR + bR = eR + fR$ where $ef = e(1 - e)br = 0$. Put $g = f(1 - e)$, then

$$gf = f(1 - e)f = f(f - ef) = f(f - 0) = f^2 = f,$$

$$g^2 = gf(1 - e) = f(1 - e) = g,$$

and $$eg = 0 = ge.$$

Since $g = f(1 - e) \in fR$ and $f = gf \in gR$, hence $fR = gR$. Thus we have that $aR + bR = eR + fR = eR + gR$. Moreover for any $r, s \in R$, since $(e + g)(er + gs) = e^2r + egs + ger + g^2s = er + 0s + 0r + gs = er + gs$, hence $eR + gR \subset (e + g)R$, and the inverse inclusion is obvious. So we have that $aR + bR = eR + gR = (e + g)R$ is a principal ideal.

DEFINITION 6.3: A ring R is called <u>π-regular</u> if for each $a \in R$, there exists an $x \in R$ and a positive integer n such that $a^n = a^n x a^n$.

<u>LEMMA 6.10</u>: If R is a commutative ring then R/Rad has no non-zero nilpotent element.

PROOF: Let $\bar{a} = a + \text{Rad } R$ be in R/Rad R.
Assume that $\bar{a} \neq \bar{0}$. Then $a \notin \text{Rad } R$ and so $1 - as$ is not invertable for some $s \in R$. This implies that $1 - a^2s^2 = (1 - as)(1 + as)$ is not invertable. Continuing in this manner we find that $1 - a^{2n}s^{2n}$ is not invertable for every positive integer n. Thus $1 - a^n(a^ns^{2n})$ is not invertable for every positive integer n. Therefore $a^n \notin \text{Rad } R$ and

hence $(\bar{a})^n \neq \bar{0}$ for every positive integer n. So we have that $\bar{a}$ is not a nilpotent element of R/Rad R. This shows that the only nilpotent element of R/Rad is $\bar{0}$.

<u>LEMMA 6.11</u>: If R is a commutative π-regular ring then R/Rad R is strongly regular.

<u>PROOF</u>: For every $a \in R$ there exists $x \in R$ such that $a^n = a^n x a^n$ for some positive integer n. Let $y = a^{n-1}x$, then

$$a^{n+1}y = a^{n+1}a^{n-1}x = a^n a^n x = a^n x a^n = a^n$$

Thus, for every $a \in R$ there exists $y \in R$ such that,

$$a^n = a^{n+1}y$$

for some positive integer n.

This implies that for every $\bar{a} = a + \text{RadR} \in R/\text{Rad } R$,

$$\bar{a}^n = \bar{a}^{n+1}\bar{y}.$$

Now observe that

$$(\bar{a}^n\bar{y} - \bar{a}^{n-1})^2 = \bar{a}^n\bar{y}\bar{a}^n\bar{y} - \bar{a}^n\bar{y}\bar{a}^{n-1} - \bar{a}^{n-1}\bar{a}^n\bar{y} + \bar{a}^{n-1}\bar{a}^{n-1} =$$

$$\bar{a}^n\bar{y}^n\bar{a}^n\bar{y} - \bar{a}^{n+1}\bar{y}\bar{y}\bar{a}^{n-1} - \bar{a}^{n-1}\bar{a}^n\bar{y} + \bar{a}^n\bar{a}^{n-2} =$$

$$\bar{a}^{2n}\bar{y}^2 - \bar{a}^{2n}\bar{y}^2 - \bar{a}^{2n-1}\bar{y} + \bar{a}^{n+1}\bar{y}\bar{a}^{n-2} = \bar{a}^{2n-1}\bar{y} - \bar{a}^{2n-1}\bar{y} = \bar{0}$$

Therefore $\bar{a}^n\bar{y} - \bar{a}^{n-1}$ is a nilpotent element of R/Rad R, and hence $\bar{a}^n\bar{y} - \bar{a}^{n-1} = \bar{0}$ by Lemma 6.10. Thus we have that,

$$\bar{a}^{n-1} = \bar{a}^n\bar{y}$$

Similarly, we get $\bar{a}^{n-2} = \bar{a}^{n-1}\bar{y}$, and continuing in this manner we get $\bar{a} = \bar{a}^2\bar{y}$. This proves that R/Rad R is strongly regular.

6.2 RESULTS ON REGULAR S-RINGS

Recall that $U(R) = \{0,1\}$ for any Boolean ring R, and so a Boolean ring with more than two elements is never generated by its units.

Moreover, it is obvious that every Boolean ring is strongly regular. Therefore, there are regular rings which are not S-rings.

<u>LEMMA 6.12</u>: If 2 is a unit of a ring R then any idempotent element of R can be written as the sum of two units.

<u>PROOF</u>: Let e be any idempotent element of R. It is clear that 2 and 2^{-1} are central elements. Observe that
$(e + 1)(1 - e2^{-1}) = e - e2^{-1} + 1 - e2^{-1} = e + 1 - e(1 + 1)2^{-1} = e + 1 - e22^{-1} = e + 1 - e = 1$, and
$(1 - e2^{-1})(e + 1) = e + 1 - e2^{-1}e - e2^{-1} = e + 1 - ee2^{-1} - e2^{-1} = e + 1 - e2^{-1} - e2^{-1} = e + 1 - e(1 + 1)2^{-1} = e + 1 - e22^{-1} = e + 1 - e = 1$
Therefore $e + 1$ is a unit, and hence $e = (e + 1) - 1$ is the sum of two units.

One can generalize the above result as follows:

<u>LEMMA 6.13</u>: If a ring R contains a unit u with the property that u commutes with all idempotents and $u + 1$ is also a unit, then every idempotent element of R can be written as the sum of two units.

<u>PROOF</u>: Let e be any idempotent and let u be a unit such that u commutes with e and $u + 1$ is a unit. Since e commutes with u then e commutes with $u + 1$, and so e commutes with u^{-1} and $(u + 1)^{-1}$. Observe that

$$\begin{aligned} &(e + u)(u^{-1} - eu^{-1}(u + 1)^{-1}) \\ &= eu^{-1} - eu^{-1}(u + 1)^{-1} + 1 - ueu^{-1}(u + 1)^{-1} \\ &= 1 + eu^{-1} - eu^{-1}(u + 1)^{-1} - eu^{-1}u(u + 1)^{-1} \\ &= 1 + eu^{-1} - eu^{-1}(1 + u)(u + 1)^{-1} \\ &= 1 + eu^{-1} - eu^{-1} = 1, \end{aligned}$$

and $$(u^{-1} - eu^{-1}(u + 1)^{-1})(e + u)$$
$$= u^{-1}e + 1 - eu^{-1}(u + 1)^{-1}e - eu^{-1}(u + 1)^{-1}u$$
$$= u^{-1}e + 1 - eeu^{-1}(u + 1)^{-1} - eu^{-1}(u + 1)^{-1}$$
$$= u^{-1}e + 1 - eu^{-1}(u + 1)^{-1}(1 + u)$$
$$= 1 + eu^{-1} - eu^{-1} = 1.$$

Therefore $e + u$ is a unit of R, and so $e = (e + u) - u$ is the sum of two units.

It is clear that the property of R, required in the preceding lemma, is inherited under ring homomorphism and that any S-ring with this property is an even S-ring (Lemma 4.3). If a ring R is regular then R_n is well known to be regular and by Lemma 4.6 it is an even S-ring. One can ask the question, which regular rings are generated by their units? We will consider regular rings with the following assumption:

ASSUMPTION: Throughout this section we assume that the regular rings discussed have the property that any idempotent element can be written as the sum of two units.

THEOREM 6.1: If a ring R is strongly regular, in particular, if R is commutative regular, then every element of R can be written as the sum of two units.

PROOF: Let R be a strongly regular ring. Then by Lemma 6.5, for every element $a \in R$ there exists a unit $u \in R$ such that

$$a = aua$$

Notice that $au = auau$ is an idempotent element, say $au = e$. This implies that $a = eu^{-1}$. Moreover, by Corollary 6.2 R is a regular ring, thus e can be written as the sum of two units by Assumption. Say, $e = u_1 + u_2$ where u_1 and u_2 are units of R. Therefore $a = eu^{-1} = (u_1 + u_2)u^{-1} = u_1u^{-1} + u_2u^{-1}$ is the sum of two units.

DEFINITION 6.3: A ring R is called unit regular if for every $a \in R$ there exists a unit $u \in R$ such that $a = aua$.

COROLLARY 6.3: If a ring R is unit regular, then every element of R can be written as the sum of two units.

LEMMA 6.14: Every nilpotent element of a ring R can be written as the sum of two units.

PROOF: Let a be a nilpotent element of R. Then there exists a positive integer n such that $a^n = 0$. It follows that

$(1 - a)(1 + a + \cdots + a^{n-1}) = 1 + a + \cdots + a^{n-1} - (a + a^2 + \cdots + a^{n-1} + a^n)$

$= 1 - a^n = 1 - 0 = 1$ and similarly

$(1 + a + \cdots + a^{n-1})(1 - a) = 1 - a^n = 1.$

Therefore $1 - a$ is a unit and hence $a = 1 - (1 - a)$ is the sum of two units.

THEOREM 6.2: If for each element $a \in R$ there exists $x \in R$ such that $a = axa$ and $a^2x = xa^2$, then every element of R can be written as the sum of four units.

PROOF: Let a be any element of R. Then $a = axa$ and $a^2x = xa^2$ for some $x \in R$. Let $a_1 = a^2x$, therefore

$a_1xa_1 = (a^2x)x(a^2x) = (xa^2)x(a^2x) = xa(axa)ax = xa^3x = a^2xax = a(axa)x = a^2x = a_1$, and

$a_1x = (a^2x)x = (xa^2)x = x(a^2x) = xa_1$

Thus we have that $a_1 x a_1 = a_1$ and $a_1 x = x a_1$. Hence, observing the proof of Theorem 6.1, one can see that a_1 can be written as the sum of two units. Say $a_1 = u_1 + u_2$. Moreover, let $a_2 = a - a_1$, then

$$\begin{aligned} a_2^2 &= (a - a_1)(a - a_1) = a^2 - aa_1 - a_1 a + a_1^2 \\ &= a^2 - a(a^2x) - (a^2x)a + (a^2x)(a^2x) \\ &= a^2 - a^3x - a(axa) + a(axa)ax \\ &= a^2 - a^3x - a^2 + a^3x = 0. \end{aligned}$$

Therefore a_2 is a nilpotent element, and so it can be written as the sum of two units by Lemma 6.14. Say $a_2 = u_3 + u_4$. Thus $a = a_1 + a_2 = u_1 + u_2 + u_3 + u_4$ is the sum of four units.

REMARK 6.1: From the above argument it is clear that if $a = axa$ and a^2x is a sum of units, then a is a sum of units.

COROLLARY 6.4: If R is a commutative π-regular ring then every element of R can be written as a sum of four units.

PROOF: $R/\text{Rad } R$ is strongly regular by Lemma 6.11. Hence by Theorem 6.1, for every $\bar{a} = a + \text{Rad } R \in R/\text{Rad } R$,

$$\bar{a} = \bar{u}_1 + \bar{u}_2$$

where $\bar{u}_1 = u_1 + \text{Rad } R$ and $\bar{u}_2 = u_2 + \text{Rad } R$ are units of $R/\text{Rad } R$. Notice that u_1 and u_2 are units of R, since according to Lemma 3.2, units can be lifted modulo Rad R. Thus we have that

$$\bar{a} = (u_1 + u_2) + \text{Rad } R,$$

and so $$a - u_1 - u_2 \in \text{Rad } R.$$

This implies that $a - u_1 - u_2$ is the sum of two units by Lemma 3.1. Say $a - u_1 - u_2 = u_3 + u_4$ for some units u_3 and u_4 of R. Therefore

$$a = u_1 + u + u_3 + u_4$$

is the sum of four units.

THEOREM 6.3: If R is a left (right) self-injective regular ring then R is an even S-ring.

PROOF: By [9, Theorem 3.2] $R \cong A \oplus B$, where A and B are ideals of R such that A is strongly regular and B is generated by idempotents. By Theorem 6.1, A is an even S-ring, and B is an even S-ring by the Assumption. Therefore R is an even S-ring by Lemma 4.5

For information about self-injective rings see [8].

COROLLARY 6.5: If R is a regular ring then R can be embedded into a regular S-ring.

PROOF: The right singular ideal of R is easily verified to be zero, so the complete ring of right quotients of R is a regular self-injective ring [4, pp. 106-107].

CHAPTER VII

QUESTIONS AND COMMENTS

1. We know by Lemma 6.7 that for every idempotent e of a ring R, e R e is regular if R is. In [5, p. 203], the author asks whether e R e must be a (regular) S-ring if R is a (regular) S-ring. The answer to this question is negative. To justify it, we introduce the following example:

EXAMPLE 7.1: Let B be a Boolean ring with more than two elements. We know that B is a regular ring which is not an S-ring. Let $R = B_2$ be the ring of 2 x 2 matrices over B. Then R is a regular S-ring by Lemma 4.6. But,

$$e\,R\,e \cong B$$

where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent element of R. Therefore e R e is a regular ring that is not an S-ring.

Now the question becomes, under what condition is e R e an S-ring? One can state the following immediate result:

REMARK 7.1: If e is an idempotent of an S-ring R and if e commutes with every unit of R, then e R e is an S-ring.

PROOF: If u is a unit of R, then

$$(e\,u\,e)(e\,u^{-1}\,e) = e\,u\,e\,u^{-1}\,e = e\,u\,u^{-1}\,e = e\,1\,e = e,$$

and so e u e is a unit of e R e. Moreover, since R is an S-ring then for every $r \in R$, $r = \Sigma u_i$ where u_i are units of R. This implies that for all $x \in e R e$,

$$x = e(\Sigma u_i)e = \Sigma\ e\ u_i e$$

where $e\ u_i e$ are units of $e R e$. Thus $e R e$ is an S-ring.

2. In the case where R is not an S-ring one can ask the question, what ring theoretic properties are preserved by U?

(a) If R is strongly regular, then U(R) is strongly regular because the quasi-inverse of an element can be chosen to be a unit by Lemma 6.5. In general if R is regular, must U(R) be regular? We do not know the answer to this question.

(b) If R is completely reducible, then by the proof of Theorem 5.2,

$$R \cong R_1 \oplus B$$

where R_1 is an even S-ring and B is the product of at least two 2-element fields, and hence B is a Boolean ring with more than two elements. If 1_B is the identity of B, then the units of B are of the form $(u, 1_B)$. Thus

$$U(R) \cong R_1 \oplus \{0,1\},$$

and it follows that U(R) is a completely reducible ring. Therefore U preserves the property of being completely reducible.

(c) U preserves the property that R/Rad R is completely reducible.

<u>PROOF:</u> From Lemma 3.2 it follows that for any ring R

$$U(R/\text{Rad } R) = U(R)/\text{Rad } R \qquad (*)$$

Now, let R/Rad R be completely reducible. This implies that U(R/Rad R), is completely reducible by (6). Since completely reducible rings are semiprimitive [4, p.68], thus

$$\text{Rad } U(R/\text{Rad } R) = 0$$

$$\Rightarrow \text{Rad}(U(R)/\text{Rad } R) = 0 \qquad \text{by } (*)$$

$$\Rightarrow U(R) \subset \text{Rad } R$$

$$\Rightarrow \text{Rad } U(R) \subset \text{Rad } R$$

Moreover, Rad R $\subset$ Rad U(R) by Lemma 3.2. Hence we have that

$$\text{Rad } R = \text{Rad } U(R)$$

so $\quad U(R/\text{Rad } R) = U(R)/\text{Rad } R = U(R)/\text{Rad } U(R)$

Therefore $\quad U(R)/\text{Rad } U(R)$ is completely reducible.

(d) U preserves the property of being semiperfect, and hence left or right perfect.

<u>PROOF</u>: Assume that idempotents modulo Rad R can be lifted. Therefore for every $v \in U(R)$ there exists $e = e^2 \in R$ such that

$$e - v \in \text{Rad } R.$$

Since Rad R $\subset$ Rad U(R), thus $e - v \in$ Rad U(R). This also implies that $e \in U(R)$. Hence, idempotents can be lifted modulo Rad U(R). Now it follows from (c) that U preserves the property of being semiperfect.

3. If R is a regular S-ring and I is an ideal of R, then is End(I) an S-ring? Notice it is known that End(I) is a regular ring, if R is regular.

4. It is noteworthy that whenever one is able to show that a regular ring is an S-ring, there is a bound on the number of units required to represent the elements of the ring.

(a) If R is a regular self-injective ring, then one must examine the proofs of [8, Lemma 5 and Theorem 2] to see that a bound is available.

(b) If D is a division ring other than the two-element field, then any element of D_n can be written as a sum of two units.

PROOF: Let A be an element of D_n. Then

$$A = PBQ$$

where P and Q are products of elementary matrices and B is diagonal with entries equal to 0 or 1. Thus P and Q are units and B is idempotent. Notice that D contains a unit u such that u + 1 is a unit. Hence uI is a unit of D_n such that uI + I is a unit and uI commutes with B. It follows from the proof of Lemma 6.13 that B is a sum of two units. Therefore A is a sum of two units.

One can see that for the most interesting regular S-rings the bound 2 is available.

The above questions and comments are left to the reader's interest.

REFERENCES

[1] FAITH, C. and UTUMI, Y., "Maximal quotient rings", Proc. Math. Soc. 16 (1956), 1084-1089.

[2] GILLMAN, L. and JERSON, M., Rings of Continuous Functions, Van Nostrand, Princeton, N.J., 1960.

[3] KANDO, T., "Strong regularity in arbitrary rings", Nagoya Math. J., 4, (1952), 51-53.

[4] LAMBEK, J., Lectures on Rings and Modules, Chelsea, New York, N.Y., 1976.

[5] RAPHAEL, R., "Rings which are generated by their units", J. Algebra, 28, (1974), 199-205.

[6] SKORNYAKOV, L., Complemented Modular Latices and Regular Rings, Oliver and Boyd, London, 1964.

[7] STORRER, H., "Epimorphismen von kommutativen Ringen", Comment. Math. Helv., 43, (1968), 378-401.

[8] UTUMI, Y., "On continuous regular rings and semisimple self-injective rings," Canad. J. Math., 12, (1960), 597-605.

[9] UTUMI, Y., "On rings of which any one-sided quotient rings are two-sided," Proc. Amer. Math. Soc., 14, (1963), 141-147.

[10] VON NEUMANN, J., Continuous Geometry, Princeton Univ. Press, Princeton, N.J., 1960.

[11] WOLFSON, K., "An ideal-theoretic characterization of the ring of all linear transformations", Amer. J. Math., 75, 358-386.

[12] ZARISKI, O. and SAMUEL, P., "Commutative Algebra", Vol. I, Van Nostrand, Princeton, N.J., 1958.

[13] ZELINSKY, D., "Every linear transformation is a sum of non-singular ones," Proc. Amer. Math. Soc., 5, (1954), 627-630.