Queueing Syst (2014) 76:283–308
DOI 10.1007/s11134-013-9361-y

Sample-path analysis of general arrival queueing systems with constant amount of work for all customers

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Received: 26 September 2012 / Revised: 8 February 2013 / Published online: 16 May 2013

Abstract We consider a discrete-time queueing system where the arrival process is general and each arriving customer brings in a constant amount of work which is processed at a deterministic rate. We carry out a sample-path analysis to derive an exact relation between the set of system size values and the set of waiting time values over a busy period of a given sample path. This sample-path relation is then applied to a discrete-time $G/D/c$ queue with constant service times of one slot, yielding a sample-path version of the steady-state distributional relation between system size and waiting time as derived earlier in the literature. The sample-path analysis of the discrete-time system is further extended to the continuous-time counterpart, resulting in a similar sample-path relation in continuous time.

Keywords $G/D/c$ queue · System size · Waiting time · Busy period · Sample-path relation · FCFS

Mathematics Subject Classification (2000) 60K25 · 68M20 · 90B22

1 Introduction

With a great many applications in communication and computer networks, discrete-time queueing systems, including $G/D/c$ queues in particular, have been widely

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studied in the last couple decades. In these systems, time is slotted and servers are assumed to be deterministic in order to model the constant data transmission rate of communication channels. See the monographs [1,15,21,24] for extensive discussions of discrete-time queueing models. In what follows, the service discipline is assumed to be first-come-first-served (FCFS) unless stated otherwise.

In a $G/D/c$ queue, while the arrival process is general (possibly serially correlated and/or nonstationary), service times are assumed constant equal to k slots for some fixed positive integer k. When the arrival process is discrete autoregressive of order 1 (and the queue is usually denoted by $DAR(1)/D/c$), Hwang et al. [11] and Hwang and Sohraby [12] obtained, for the case $c = 1$ (single-server) and $k = 1$ (service times of one slot), the steady-state distributions of waiting time and system size, respectively. Choi et al. [4] later extended these results to the case $c > 1$ (multiserver) and $k = 1$. On the other hand, for a $G/D/c$ queue with $k = 1$ (service times of one slot), Xiong et al. [25] derived an exact steady-state distributional relation between system size and waiting time, which requires no knowledge of the nature of the arrival process. More recently, by introducing the so-called partial system contents Gao et al. [7] have generalised the distributional relation of Xiong et al. [25] to the case $k > 1$ (constant service times of multiple slots). See also [3,2,6,8] for related results under different model assumptions.

In this paper, we propose a new discrete-time queueing system, to be referred to as the discrete-time (c, k)-system and defined in the next section. When either $c = 1$ or $k = 1$, the (c, k)-system is equivalent to the $G/D/c$ queue with constant service times of k slots. But the two systems differ when both $c > 1$ and $k > 1$. For a further comparison between the two systems, see Remark 1 in Sect. 2. In this section, we also carry out a sample-path analysis to derive a simple and exact (sample-path) relation between the set of system size values and the set of waiting time values over a busy period of a given sample path. This sample-path relation is then applied in Sect. 3 to the discrete-time $G/D/c$ queue with constant service times of k slots with either $c = 1$ or $k = 1$, yielding the sample-path version of the distributional relation of Xiong et al. [25]. In Sect. 4, the sample-path analysis of the discrete-time (c, k)-system is extended to the continuous-time counterpart, resulting in a similar sample-path relation. Section 5 contains concluding remarks. There is an appendix containing some technical lemmas and proofs that are needed for deriving the main results in Sect. 2.

We close this section with a brief discussion of some sample-path results on general queueing systems. As remarked on p. 236 of the review article Whitt [22], to express fundamental queueing relations such as Little's law $L = \lambda W$ and its extensions (including $H = \lambda G$ and Miyazawa's rate conservation law), there are two frameworks: a deterministic framework involving individual sample paths and a stationary framework involving steady-state distributions. The deterministic framework (under which a sample-path analysis is performed) is appealing because it requires only elementary arguments and minimal conditions for the fundamental queueing relations to hold. For example, Stidham [19], Heyman and Stidham [9] and Sigman [18] established the sample-path versions of $L = \lambda W$, $H = \lambda G$ and the rate conservation law, respectively. For a comprehensive review of sample-path analysis, the reader is referred to the monograph [5]. See also the more recent review articles [14,20,23] with an emphasis on sample-path analysis. More closely related in spirit to the present study is the work of Sakasegawa and Wolff [16] who showed for the $G/G/1$ queue in continuous time that the empirical distribution of the workload values over a busy period of a given sample path is identical to that of the attained waiting time values over the same period. This sample-path result complements and strengthens Sengupta's [17] invariance relation for the $G/G/1$ queue that the workload and attained waiting time of a customer in service have the same stationary distribution. See [26,27] for related sample-path results in the multiserver case.

2 The discrete-time (c, k)-system

We consider a discrete-time queueing system in which the arrival process of customers is general, each customer brings k units of work to the system, the server completes up to c units of work in each slot, and the service discipline is FCFS. Here, c and k are fixed positive integers. For convenience, we refer to the system as the (discrete-time) (c, k)-system. As an example, suppose that students enter a testing service centre to take a test of k questions. Upon completing the test, they wait for their scores when a grader grades (up to) c questions every half an hour (including a break of 5 min). This example may be viewed as a (c, k)-system where a slot is half an hour, the grader is the server and students are customers. As another example, consider a small resort island where tourists fly to a nearby airport and then take a shuttle bus to a boat station in order to take a shuttle boat to the island. Suppose that a shuttle bus of capacity k leaves the airport as soon as it is fully loaded, and stays at the boat station until each of the k passengers boards a shuttle boat of capacity c (which runs every 20 min). In this example as a (c, k)-system, a slot is 20 min, shuttle boats are the server and shuttle buses are customers.

For a given sample path, let A_n and D_n denote the arrival and departure times of the nth customer C_n, $n = 1, 2, \ldots$, with $A_n \leq D_n$ for all n and $1 = A_1 \leq A_2 \leq \cdots$, implying that the first busy period begins at slot 1 and batch arrivals are allowed. By our convention, $A_n = t_1$ and $D_n = t_2$ means that C_n enters the system at the beginning of slot t_1, and completes service and departs at the end of slot t_2. So C_n's waiting time W_n equals $D_n - A_n + 1$ (the number of slots spent in system by C_n). Note that the server may serve more than one customer in a slot. For example, with $c = 2, k = 3$ and $A_1 = A_2 = 1$, the server completes two units of C_1's work in slot 1 and completes the remaining unit of C_1's work and one unit of C_2's work in slot 2 (so that C_1 departs at the end of slot 2).

Remark 1 It is worthwhile to compare in some detail the (c, k)-system with the (discrete-time) $G/D/c$ queue with constant service times of k slots (which for convenience will be denoted by $G/D_k/c$ here). For the (c, k)-system with $c > 1$, the server may be viewed as a group of c sub-servers where each sub-server completes one unit of work in a slot and sub-servers can serve the same customer in a slot if the customer's work is more than one unit (i.e. $k > 1$). Thus it is readily seen that when either $c = 1$ or $k = 1$, the (c, k)-system is equivalent to the $G/D_k/c$ queue. However, the two systems differ in general when both $c > 1$ and $k > 1$. Note that there is an equivalence relation in the class of all (c, k)-systems. For example, the (c, k)-system with $c = k$ is equivalent to the $(1, 1)$-system by identifying k units of work as a new single unit of work. More generally, for $c' = rc$ and $k' = rk$ with $r > 1$ an integer, by defining a new unit of work as r (original) units of work, it follows that the (c', k')-system is equivalent (reduces) to the (c, k)-system. Such an equivalence relation does not hold for the class of $G/D_k/c$ queues. It may also be of interest to compare the (c, k)-system with a batch service queue where customers are served in batches by a single-server, the size of the batch being either c or the size of the queue, whichever is smaller. The batch service time is assumed to be k slots independent of the batch size. Clearly, the two systems are equivalent if either $c = 1$ or $k = 1$, but not equivalent otherwise. For the special case of $c = k$, if exactly $c(=k)$ customers are in the queue waiting for service, they depart one at a time in the next k slots for the (c, k)-system whereas the customers depart together after k slots in the batch service queue.

Now let N_t and V_t denote, respectively, the number of customers and the amount of work (in work units) in system at t (or more precisely, at the beginning of slot t). Then we have for $t = 1, 2, \ldots,$

$$N_t = |\{n : A_n \le t \le D_n\}| = |\{n : A_n \le t\}| - |\{n : D_n < t\}|, \tag{2.1}$$

$$V_t = (V_{t-1} - c)^+ + k\ |\{n : A_n = t\}| \quad (V_0 := 0), \tag{2.2}$$

$$k(N_t - 1) < V_t \le kN_t \text{ (implying } N_t = \lceil V_t/k \rceil), \tag{2.3}$$

where $|S|$ denotes the cardinality of a set S, $x^+ = \max\{x, 0\}$ and $\lceil x \rceil$ denotes the smallest integer not less than x. We assume $|\{n : A_n \le t\}| < \infty$ for all t, so that $N_t < \infty$ and $V_t < \infty$ for all t.

We also adopt the following definition of the jth busy period, $j = 1, 2, \ldots$. The first busy period begins at $T_1 = 1$ and ends at

$$T_1' = \min\{t \ge T_1 : |\{n : A_n \le t\}| = |\{n : D_n \le t\}|\}.$$

The jth busy period for $j > 1$ begins at

$$T_j = \min\{t > T_{j-1}' : t = A_n \text{ for some } n\}$$

and ends at

$$T_j' = \min\{t \ge T_j : |\{n : A_n \le t\}| = |\{n : D_n \le t\}|\}.$$

Note that there may be no idle period between two consecutive busy periods. As an example, if $A_n = T_1' + 1$ for some n, then $T_2 = T_1' + 1$, i.e. the second busy period immediately follows the first busy period. (In this example, the system empties at the end of slot T_1' and a customer arrives at the beginning of the next slot.)

Remark 2 For a busy period in which n customers enter the system, let (t_i, d_i), $i = 1, \ldots, n$, be the n pairs of arrival and departure times with $t_1 \le t_2 \le \cdots \le t_n$. (So $t_i = A_{\ell+i}$, $i = 1, \ldots, n$, for some $\ell \ge 0$.) Then exactly c units of work is performed in each slot except possibly for the last one when all of the n customers have completed service. It follows that $d_i = t_1 + \lceil ki/c \rceil - 1$, $i = 1, \ldots, n$. Also, $t_{i+1} \le d_i$, $i = 1, \ldots, n-1$.

The main results of this section are concerned with the relation between the set of system size values and the set of waiting time values in a busy period. Before stating the results, we need to define a $B(n)$ set and introduce some set notations. An (ordered) set of n pairs of integers $P = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ is said to be a $B(n)$ set (with respect to the (c, k)-system), if $t_1 \le t_2 \le \cdots \le t_n$, $d_i = t_1 + \lceil \frac{ki}{c} \rceil - 1$, $i = 1, \ldots, n$, and $t_{i+1} \le d_i$, $i = 1, \ldots, n-1$. Let

$$\begin{aligned} N_P(t) &= |\{i : t_i \le t \le d_i\}| \quad \text{for } t_1 \le t \le d_n, \\ S_P &= \{N_P(t) : t = t_1, t_1 + 1, \ldots, d_n\}, \\ W_P &= \{d_i - t_i + 1 : i = 1, \ldots, n\}. \end{aligned} \tag{2.4}$$

Remark 3 By Remark 2, for a busy period with n customers, the set of the n pairs of arrival and departure times is a $B(n)$ set. Consider a general $B(n)$ set $P = \{(t_1, d_1), \ldots, (t_n, d_n)\}$. By definition, $\{(t_1, d_1), \ldots, (t_j, d_j)\}$ is a $B(j)$ set for $j < n$. Suppose that the (c, k)-system is empty right before t_1, and in the time interval consisting of slots $t_1, \ldots, d_n$, exactly n customers enter the system at $t_1, \ldots, t_n$. Then $d_i = t_1 + \lceil ki/c \rceil - 1$, $i = 1, \ldots, n$ are the corresponding departure times of the n customers. The condition $t_{i+1} \le d_i$, $i = 1, \ldots, n-1$, implies that the system is not empty before d_n. So the time interval of slots $t_1, \ldots, d_n$ is a busy period, i.e. $t_1 = T_j$ and $d_n = T'_j$ for some j. For $t = t_1, \ldots, d_n$, $N_P(t)$ is the number of customers in system at time t (more precisely, at the beginning of slot t), S_P is the set of $d_n - t_1 + 1$ system size values, and W_P is the set of n waiting time values.

To introduce some set notations, note that in this section the multiplicity of each element α in a set S, denoted $m(\alpha; S)$, is counted. Such a set is sometimes referred to as a multiset (cf. p. 483 of [13]). Denote ν copies of α by $\alpha_{(\nu)}$. For example, $\{\alpha_{(0)}\} = \emptyset$, $\{\alpha_{(0)}, \beta_{(1)}, \gamma_{(2)}\} = \{\beta_{(1)}, \gamma_{(2)}\} = \{\beta, \gamma, \gamma\} \ne \{\beta, \gamma\}$. For n sets $S_1, \ldots, S_n$, let $\uplus_{i=1}^n S_i = S_1 \uplus \cdots \uplus S_n$ denote the set consisting of those elements in at least one of $S_1, \ldots, S_n$ with multiplicities given by

$$m(\alpha; \uplus_{i=1}^n S_i) = \sum_{i=1}^n m(\alpha; S_i),$$

where $m(\alpha; S) := 0$ if $\alpha \notin S$. As an example,

$$\{\alpha_{(2)}, \beta_{(3)}\} \uplus \{\beta, \gamma_{(4)}\} \uplus \{\alpha_{(5)}\} = \{\alpha_{(7)}, \beta_{(4)}, \gamma_{(4)}\}.$$

Now for given positive integers s and δ, we define a set $s^{[\delta]}$ as follows:

(i) If $s \le \delta$, define $s^{[\delta]} := \{1_{(s)}\}$;
(ii) If $s > \delta$, define $s^{[\delta]} := \{\ell_{(\delta-r)}, (\ell+1)_{(r)}\}$, where integers $\ell \ge 1$ and $1 \le r \le \delta$ are uniquely determined by $s = \ell\delta + r$, i.e. $\ell = \lceil \frac{s}{\delta} \rceil - 1$ and $r = s - \left(\lceil \frac{s}{\delta} \rceil - 1\right)\delta$.

Next for a set S of positive integers, define $S^{[\delta]} := \uplus_{s \in S} s^{[\delta]}$. Note that $S^{[1]} = S$.

Remark 4 Observe that the number of elements in $s^{[\delta]}$ (counting multiplicities) is $\min\{s, \delta\}$ and the sum of the elements in $s^{[\delta]}$ equals s. For example, the set $123^{[15]} = \{8_{(12)}, 9_{(3)}\}$ consists of 15 elements the sum of which equals 123. For the set $S = \{2_{(2)}, 5\}$, we have $S^{[1]} = S$, $S^{[2]} = \{1_{(4)}, 2, 3\}$, $S^{[3]} = \{1_{(5)}, 2_{(2)}\}$, $S^{[4]} = \{1_{(7)}, 2\}$, $S^{[\delta]} = \{1_{(9)}\}$ for $\delta \ge 5$. To see the meaning of $s^{[\delta]}$ with $s \ge \delta$, note that there is a unique way to write s as a sum of δ integers which differ by at most 1. These (unique) δ integers form the set $s^{[\delta]}$. (For $s < \delta$, the above interpretation still applies if we change the definition of $s^{[\delta]} = \{1_{(s)}\}$ to $s^{[\delta]} = \{0_{(\delta-s)}, 1_{(s)}\}$.)

Theorem 1 *Let $P = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ be a $B(n)$ set with respect to the discrete-time (c, k)-system. Then $S_P^{[c]} = W_P^{[k]}$.*

Proof We proceed by induction on n. For $P = \{(t_1, d_1)\}$ a $B(1)$ set, we have $d_1 = t_1 + \lceil k/c \rceil - 1$, so $S_P = \{1_{(\nu)}\}$ and $W_P = \{\nu\}$ where $\nu = \lceil k/c \rceil \le k$. Thus, $S_P^{[c]} = W_P^{[k]} = \{1_{(\nu)}\}$. Suppose that $S_Q^{[c]} = W_Q^{[k]}$ for all $B(n)$ sets Q ($n \ge 1$). Let $P = \{(t_i, d_i) : i = 1, \ldots, n+1\}$ be a $B(n+1)$ set. We need to show $S_P^{[c]} = W_P^{[k]}$. Let $Q = \{(t_i, d_i) : i = 1, \ldots, n\}$, which is a $B(n)$ set. By the induction hypothesis, $S_Q^{[c]} = W_Q^{[k]}$. Since P is a $B(n+1)$ set, we have $t_n \le t_{n+1} \le d_n$. For each $j = t_n, t_n + 1, \ldots, d_n$, let

$$P(j) = \{(t_1, d_1), \ldots, (t_n, d_n), (j, d_{n+1})\},$$

which is a $B(n+1)$ set. In particular, $P = P(t_{n+1})$.

We claim (i) $S_{P(d_n)}^{[c]} = W_{P(d_n)}^{[k]}$ and (ii) $S_{P(j+1)}^{[c]} = W_{P(j+1)}^{[k]}$ implies $S_{P(j)}^{[c]} = W_{P(j)}^{[k]}$ for $j = t_n, \ldots, d_n - 1$. (Claim (ii) is vacuous if $t_n = d_n$.) Claims (i) and (ii) follow from Lemmas 1 and 2 in the Appendix, respectively. By Claims (i) and (ii), $S_{P(j)}^{[c]} = W_{P(j)}^{[k]}$ for $j = t_n, \ldots, d_n$, which includes $S_P^{[c]} = W_P^{[k]}$ for $j = t_{n+1}$. This completes the proof. □

As an illustration for Theorem 1, we present in Fig. 1 a busy period of a sample path for the (c, k)-system with $c = 2, k = 3$. In the busy period, there are 7 customers entering the system. In the figure, each circle represents a unit of work and the number inside a circle is the customer i.d. belonging to the set $\{1, 2, \ldots, 7\}$. The (ordered) set of the 7 pairs of arrival and departure times is $P = \{(1, 2), (2, 3), (3, 5), (3, 6), (6, 8), (7, 9), (8, 11)\}$ which is a $B(7)$ set. For this busy period, the sets of system size values and waiting time values are

$$\begin{aligned} S_P &= \{1, 2, 3, 2, 2, 2, 2, 3, 2, 1, 1\} = \{1_{(3)}, 2_{(6)}, 3_{(2)}\}, \\ W_P &= \{2, 2, 3, 4, 3, 3, 4\} = \{2_{(2)}, 3_{(3)}, 4_{(2)}\}. \end{aligned}$$

It is readily verified that $S_P^{[2]} = W_P^{[3]} = \{1_{(17)}, 2_{(2)}\}$ as claimed in Theorem 1.

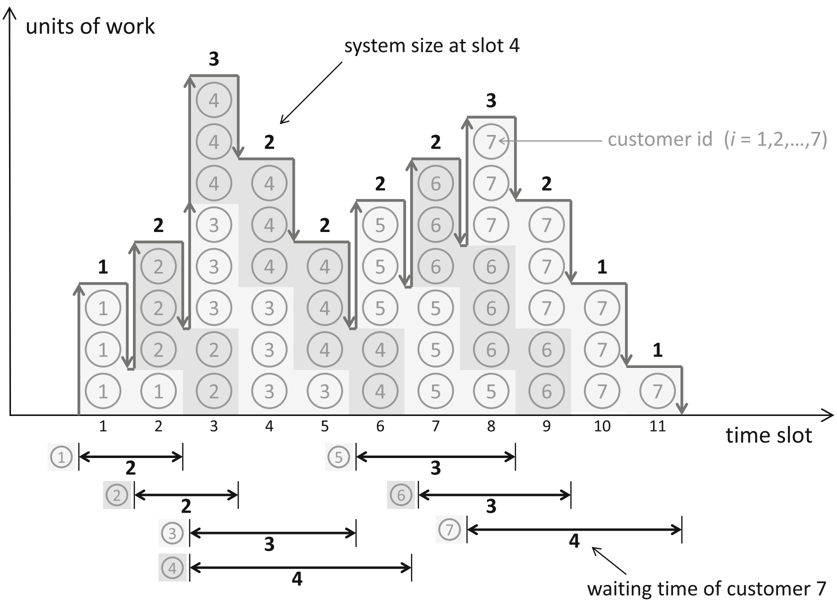

Fig. 1 A busy period of a sample path in the (c, k)-system with $c = 2$ and $k = 3$

Remark 5 Let S and W be, respectively, the set of system size values and the set of waiting time values in a given busy period. By Theorem 1, we have $S^{[c]} = W^{[k]}$. Since $S^{[c]} = S$ for $c = 1$ and $W^{[k]} = W$ for $k = 1$, it follows that in a busy period the set of system size values is identical to the set of waiting time values provided $c = k = 1$. However, when either c or k (or both) is greater than 1, the physical interpretation of $S^{[c]}$ and $W^{[k]}$ is not so clear and requires further investigation.

We now apply Theorem 1 to a sample path with the arrival and departure times A_i, D_i of the ith customer $C_i, i = 1, 2, \ldots$. Recall that N_t and V_t denote, respectively, the number of customers and the amount of work (in work units) in system at time t. Recall also that $[T_j, T_j']$ denotes the jth busy period. Let $I_j = \{i : T_j \le A_i \le T_j'\}$, the set of the indices of customers entering the system in the jth busy period. Then $P_j := \{(A_i, D_i) : i \in I_j\}$ is a $B(|I_j|)$ set. By (2.4), for $T_j \le t \le T_j'$, $N_{P_j}(t) = \{i \in I_j : A_i \le t \le D_i\}$, which agrees with N_t (cf. (2.1)). Also,

$$
\begin{aligned}
S_{P_j} &= \{N_{P_j}(t) : T_j \le t \le T_j'\} = \{N_t : T_j \le t \le T_j'\},\\
W_{P_j} &= \{D_i - A_i + 1 : i \in I_j\}.
\end{aligned}
$$

Thus S_{P_j} and W_{P_j} are, respectively, the set of system size values and the set of waiting time values in the jth busy period. For $\tau > 0$, let $\mathbb{S}_\tau = \{N_t : N_t > 0 \text{ and } 1 \le t \le \tau\}$, the set of *positive* system size values in $\{N_1, \ldots, N_\tau\}$, and $\mathbb{W}_\tau = \{D_i - A_i + 1 : A_i \le \tau\}$, the set of waiting time values of customers entering the system at or before time τ.

With $T'_0 := 0$, let $\nu = \max\{j : T'_j \le \tau\} \ge 0$, the number of busy periods ending at or before τ. Then $T'_\nu \le \tau < T'_{\nu+1}$, and

$$\mathbb{S}_\tau = \uplus_{j=1}^{\nu} S_{P_j} \uplus \{N_t : T_{\nu+1} \le t \le \tau\}, \tag{2.5}$$

$$\mathbb{W}_\tau = \uplus_{j=1}^{\nu} W_{P_j} \uplus \{D_i - A_i + 1 : T_{\nu+1} \le A_i \le \tau\}. \tag{2.6}$$

If $V_\tau = 0$, then the system is empty at τ, implying $\tau < T_{\nu+1}$. If $0 < V_\tau \le c$, the system empties at the end of slot τ, implying $\tau = T'_\nu < T_{\nu+1}$. So $V_\tau \le c$ if and only if $\tau < T_{\nu+1}$. If $V_\tau \le c$, then it follows from (2.5), (2.6) and $\tau < T_{\nu+1}$ that

$$\mathbb{S}_\tau = \uplus_{j=1}^{\nu} S_{P_j} \text{ and } \mathbb{W}_\tau = \uplus_{j=1}^{\nu} W_{P_j}.$$

By Theorem 1,

$$\mathbb{S}_\tau^{[c]} = \uplus_{j=1}^{\nu} S_{P_j}^{[c]} = \uplus_{j=1}^{\nu} W_{P_j}^{[k]} = \mathbb{W}_\tau^{[k]}. \tag{2.7}$$

If $V_\tau > c$ (i.e. $\tau \ge T_{\nu+1}$), let $i_1 = \min\{i : A_i = T_{\nu+1}\}$, since there is (at least) a customer arriving at the beginning of the $(\nu+1)$th busy period. Let $i_2 = \max\{i : A_i \le \tau\}$. Then $\tau \ge T_{\nu+1}$ implies $i_1 \le i_2$. Let $Q = \{(A_i, D_i) : i_1 \le i \le i_2\}$, which is a $B(i_2 - i_1 + 1)$ set. (Note that $Q \subset P_{\nu+1}$.) We have

$$\begin{aligned} W_Q &= \{D_i - A_i + 1 : i_1 \le i \le i_2\} \\ &= \{D_i - A_i + 1 : T_{\nu+1} \le A_i \le \tau\}, \end{aligned}$$

so by (2.6)

$$\mathbb{W}_\tau = \uplus_{j=1}^{\nu} W_{P_j} \uplus W_Q. \tag{2.8}$$

Noting that $T_{\nu+1} = A_{i_1}$ and $N_Q(t) = N_t$ for $T_{\nu+1} \le t \le \tau$, we have

$$\begin{aligned} S_Q &= \{N_Q(t) : T_{\nu+1} \le t \le D_{i_2}\} \\ &= \{N_t : T_{\nu+1} \le t \le \tau\} \uplus \{N_Q(t) : \tau < t \le D_{i_2}\}, \end{aligned}$$

which together with (2.5) yields

$$\mathbb{S}_\tau \uplus \{N_Q(t) : \tau < t \le D_{i_2}\} = \uplus_{i=1}^{\nu} S_{P_j} \uplus S_Q. \tag{2.9}$$

The set $\{N_Q(t) : \tau < t \le D_{i_2}\}$ can be expressed in terms of V_τ as follows. At (the beginning of) each of slots $\tau+1, \tau+2, \ldots$, the amount of work remaining (in work units) among customers $C_i, i_1 \le i \le i_2$, equals $V_\tau - c, V_\tau - 2c, \ldots$, so that $N_Q(\tau+1) = \lceil \frac{V_\tau - c}{k} \rceil$, $N_Q(\tau+2) = \lceil \frac{V_\tau - 2c}{k} \rceil, \ldots$ (cf. (2.3)). It follows that

$$\{N_Q(t) : \tau < t \le D_{i_2}\} = \left\{ \left\lceil \frac{V_\tau - ic}{k} \right\rceil : 1 \le i \le \left\lceil \frac{V_\tau}{c} \right\rceil - 1 \right\}.$$

By (2.9),

$$\mathbb{S}_\tau \uplus \left\{ \left\lceil \frac{V_\tau - ic}{k} \right\rceil : 1 \le i \le \left\lceil \frac{V_\tau}{c} \right\rceil - 1 \right\} = \uplus_{j=1}^{\nu} S_{P_j} \uplus S_Q. \tag{2.10}$$

By (2.8), (2.10) and Theorem 1,

$$\begin{aligned} \mathbb{S}_\tau^{[c]} \uplus \left\{ \left\lceil \frac{V_\tau - ic}{k} \right\rceil : 1 \le i \le \left\lceil \frac{V_\tau}{c} \right\rceil - 1 \right\}^{[c]} &= \uplus_{j=1}^{\nu} S_{P_j}^{[c]} \uplus S_Q^{[c]} \\ &= \uplus_{j=1}^{\nu} W_{P_j}^{[k]} \uplus W_Q^{[k]} \\ &= \mathbb{W}_\tau^{[k]}. \end{aligned} \tag{2.11}$$

By (2.7) and (2.11), we have the following result.

Theorem 2 *For* $\tau > 0$,

$$\mathbb{W}_\tau^{[k]} = \mathbb{S}_\tau^{[c]} \uplus \left\{ \left\lceil \frac{V_\tau - ic}{k} \right\rceil : 1 \le i \le \left\lceil \frac{V_\tau}{c} \right\rceil - 1 \right\}^{[c]},$$

where the second set on the right-hand side is empty if $V_\tau \le c$. *Equivalently, for* $\ell = 1, 2, \ldots,$

$$\sum_{i=1}^{\Lambda(\tau)} m(\ell; (D_i - A_i + 1)^{[k]}) = \sum_{t=1}^{\tau} m(\ell; N_t^{[c]}) + \sum_{i=1}^{\lceil V_\tau / c \rceil - 1} m\left(\ell; \left\lceil \frac{V_\tau - ic}{k} \right\rceil^{[c]}\right),$$

where $\Lambda(\tau) := \max\{i : A_i \le \tau\}$ *and* $0^{[c]} := \emptyset$.

3 Applications to $G/D/c$ queues

Xiong et al. [25] investigated in great detail the $G/D/c$ queue with constant service times of one slot which is equivalent to the discrete-time (c, k)-system with $k = 1$. Assuming that the system reaches a steady-state (which implies that the mean number λ of arrivals per slot is less than c), they derived the relation (cf. Eq. (19) in [25])

$$\lambda F_{\mathrm{w}}\{\ell\} = \sum_{i=0}^{c-1} (i F_{\mathrm{s}}\{(\ell - 1)c + i\} + (c - i) F_{\mathrm{s}}\{\ell c + i\}), \quad \ell = 1, 2, \ldots, \tag{3.1}$$

where F_{w} and F_{s} denote the steady-state waiting time and system size distributions, respectively. Note that no knowledge of the exact nature of the arrival process is required in order to derive (3.1). As a simple application of Theorem 2, we will derive the sample-path version of (3.1) without assuming that the system is in steady-state.

Fix a sample path ω in which customers enter the system at $1 = A_1 \le A_2 \le \cdots$ and depart at $D_1 \le D_2 \le \cdots$. Recall $\Lambda(\tau) = \max\{i : A_i \le \tau\}$, the number of customers entering the system at or before τ. Assume the following limits exist for the sample path ω:

$$\lambda^{(\omega)} = \lim_{\tau\to\infty} \Lambda(\tau)/\tau (0 < \lambda^{(\omega)} < c),$$
$$F_{\mathrm{w}}^{(\omega)}\{\ell\} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{D_i - A_i + 1 = \ell\}} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{W_i = \ell\}}, \quad \ell = 1, 2, \ldots,$$
$$F_{\mathrm{s}}^{(\omega)}\{\ell\} = \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} 1_{\{N_t = \ell\}}, \quad \ell = 0, 1, \ldots,$$

where $W_i = D_i - A_i + 1$, and 1_E denotes the indicator of an event E, which equals 1 or 0 according to whether or not E occurs.

By Theorem 2, for $\ell = 1, 2, \ldots,$

$$\sum_{i=1}^{\Lambda(\tau)} m(\ell; W_i^{[k]}) = \sum_{t=1}^{\tau} m(\ell; N_t^{[c]}) + \sum_{i=1}^{\lceil V_\tau/c\rceil - 1} m\left(\ell; \left\lceil \frac{V_\tau - ic}{k} \right\rceil^{[c]}\right).$$

Since

$$\sup_{0<\tau<\infty} \sum_{i=1}^{\lceil V_\tau/c\rceil - 1} m\left(\ell; \left\lceil \frac{V_\tau - ic}{k} \right\rceil^{[c]}\right) \le k \sum_{j=1}^{\infty} m(\ell; j^{[c]})$$
$$= k \sum_{j=c(\ell-1)+1}^{c(\ell+1)-1} m(\ell; j^{[c]}) = kc^2 < \infty,$$

we have

$$\lim_{\tau\to\infty} \frac{1}{\tau} \left\{ \sum_{i=1}^{\Lambda(\tau)} m(\ell; W_i^{[k]}) - \sum_{t=1}^{\tau} m(\ell; N_t^{[c]}) \right\} = 0. \tag{3.2}$$

Note that

$$m(\ell; j^{[\alpha]}) = \begin{cases} r, & \text{for } j = (\ell-1)\alpha + r, r = 1, \ldots, \alpha - 1, \\ \alpha - r, & \text{for } j = \ell\alpha + r, r = 0, 1, \ldots, \alpha - 1, \\ 0, & \text{for } j \le (\ell-1)\alpha \text{ or } j \ge (\ell+1)\alpha. \end{cases} \tag{3.3}$$

By (3.3) with $\alpha = k$, we have

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{i=1}^{\Lambda(\tau)} m(\ell;\, W_i^{[k]}) = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{i=1}^{\Lambda(\tau)}\left\{\sum_{r=0}^{k-1}(r1_{\{W_i=(\ell-1)k+r\}} + (k-r)1_{\{W_i=\ell k+r\}})\right\}$$
$$= \lim_{\tau\to\infty}\frac{\Lambda(\tau)}{\tau}\sum_{r=0}^{k-1}\left(\frac{r}{\Lambda(\tau)}\sum_{i=1}^{\Lambda(\tau)}1_{\{W_i=(\ell-1)k+r\}}\right.$$
$$\left.+\frac{k-r}{\Lambda(\tau)}\sum_{i=1}^{\Lambda(\tau)}1_{\{W_i=\ell k+r\}}\right)$$
$$= \lambda^{(\omega)}\sum_{r=0}^{k-1}\left(rF_{\mathrm{w}}^{(\omega)}\{(\ell-1)k+r\} + (k-r)F_{\mathrm{w}}^{(\omega)}\{\ell k+r\}\right).$$

By (3.3) with $\alpha = c$, we have

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau} m(\ell;\, N_t^{[c]}) = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}\left\{\sum_{r=0}^{c-1}\left(r1_{\{N_t=(\ell-1)c+r\}} + (c-r)1_{\{N_t=\ell c+r\}}\right)\right\}$$
$$= \lim_{\tau\to\infty}\sum_{r=0}^{c-1}\left(r\tau^{-1}\sum_{t=1}^{\tau}1_{\{N_t=(\ell-1)c+r\}}\right.$$
$$\left.+(c-r)\tau^{-1}\sum_{t=1}^{\tau}1_{\{N_t=\ell c+r\}}\right)$$
$$= \sum_{r=0}^{c-1}\left(rF_{\mathrm{s}}^{(\omega)}\{(\ell-1)c+r\} + (c-r)F_{\mathrm{s}}^{(\omega)}\{\ell c+r\}\right).$$

It then follows from (3.2) that for $\ell = 1, 2, \ldots,$

$$\lambda^{(\omega)}\sum_{r=0}^{k-1}\left(rF_{\mathrm{w}}^{(\omega)}\{(\ell-1)k+r\} + (k-r)F_{\mathrm{w}}^{(\omega)}\{\ell k+r\}\right)$$
$$= \sum_{r=0}^{c-1}\left(rF_{\mathrm{s}}^{(\omega)}\{(\ell-1)c+r\} + (c-r)F_{\mathrm{s}}^{(\omega)}\{\ell c+r\}\right). \qquad (3.4)$$

For $k = 1$, (3.4) reduces to

$$\lambda^{(\omega)}F_{\mathrm{w}}^{(\omega)}\{\ell\} = \sum_{r=0}^{c-1}\left(rF_{\mathrm{s}}^{(\omega)}\{(\ell-1)c+r\} + (c-r)F_{\mathrm{s}}^{(\omega)}\{\ell c+r\}\right),\quad \ell = 1, 2, \ldots,$$

which is the sample-path version of (3.1). For $c = 1$, (3.4) reduces to

$$F_{\rm s}^{(\omega)}\{\ell\} = \lambda^{(\omega)} \sum_{r=0}^{k-1} \left(r F_{\rm w}^{(\omega)}\{(\ell-1)k + r\} + (k-r) F_{\rm w}^{(\omega)}\{\ell k + r\} \right), \qquad (3.5)$$

which is a sample-path distributional relation between system size and waiting time for the discrete-time $G/D/1$ queue with constant service times of k slots.

For the single-server case ($c = 1$), (3.1) reduces to a simple formula

$$F_{\rm s}\{\ell\} = \lambda F_{\rm w}\{\ell\}, \quad \ell = 1, 2, \ldots, \qquad (3.6)$$

which indicates a particularly close connection between the stationary distributions of system size and waiting time in this special case. In fact, this relation has been presented in Humblet et al. [10] (p. 85) for an $nD/D/1$ queue where the arrival process is a superposition of n deterministic sources. (The assumption on the arrival process is not necessary since (3.1) holds for general arrival processes.) To gain further insights into (3.6) from the sample-path point of view, by Theorem 1 with $c = k = 1$, we have $S_P = W_P$ for any $B(n)$ set P. It follows that for a busy period of a sample path, the set of system size values is identical to that of waiting time values. (This implies implicitly that the number of customers entering the system during a busy period equals the length of the busy period.) Consequently, when the system is in steady-state, the conditional distribution of system size given that the system is not empty is the same as the distribution of waiting time. Since λ is the (long-run) fraction of time that the system is not empty, we get $F_{\rm s}\{\ell\}/\lambda = F_{\rm w}\{\ell\}$, $\ell = 1, 2, \ldots$, which is (3.6).

4 The continuous-time (c, k)-system

In this section, we consider the following continuous-time analogue of the discrete-time (c, k)-system. The arrival process of customers is general, each customer brings k units of work to the system, the service rate is c units of work per unit time, and the service discipline is FCFS. Here, c and k are fixed positive values (not necessarily integers). Again we are concerned with the relation between the set of system size values and the set of waiting time values in a busy period. The results and their proofs are similar to (and simpler than) the discrete-time counterparts. In particular, the result in Theorem 3 below depends on c and k only through $k' := k/c$. The main difference between the continuous- and discrete-time cases is that the (multi)set of system size values (which are positive integers) in a busy period in continuous time is uncountable, which will be characterised via the amount of time the system size equals ℓ in the busy period for $\ell = 1, 2, \ldots$.

Remark 6 It is indicated in Remark 1 that there is an equivalence relation for the class of discrete-time (c, k)-systems. A similar equivalence relation also holds for the class of continuous-time (c, k)-systems. Specifically, for $c^* = rc$ and $k^* = rk$ with $r > 0$, by defining a new unit of work as r (original) units of work, it is readily seen that the (continuous-time) (c^*, k^*)-system is equivalent to the (c, k)-system. In particular, the (c, k)-system is equivalent to the $(1, k')$-system where $k' = k/c$. It follows easily that the (c, k)-system is equivalent to the (continuous-time) $G/D/1$ queue with deterministic service time $k' = k/c$.

Now for a given sample path in the continuous-time (c, k)-system, let $\mathscr{A}_n < \mathscr{D}_n$ be the arrival and departure times of the nth customer C_n, $n = 1, 2, \ldots$, with $0 = \mathscr{A}_1 \le \mathscr{A}_2 \le \cdots$, indicating that the first busy period begins at time 0 and batch arrivals are allowed. Note that $\mathscr{D}_1 = k'(= k/c)$; $\mathscr{D}_2 = 2k'$ if $\mathscr{A}_2 < \mathscr{D}_1$, or $= \mathscr{A}_2 + k'$ if $\mathscr{A}_2 \ge \mathscr{D}_1$. In general, for $n > 1$, $\mathscr{D}_n = \mathscr{D}_{n-1} + k'$ if $\mathscr{A}_n < \mathscr{D}_{n-1}$, or $= \mathscr{A}_n + k'$ if $\mathscr{A}_n \ge \mathscr{D}_{n-1}$. So $\mathscr{D}_n = \max\{\mathscr{D}_{n-1}, \mathscr{A}_n\} + k'$ for $n > 1$. Note that $\mathscr{D}_n - \mathscr{A}_n \ge k'$ with equality holding if and only if customer C_n finds the system empty upon arrival. Let $\mathscr{N}_t$ and $\mathscr{V}_t$ denote, respectively, the number of customers and the amount of work (in work units) in system at time t. Then

$$\mathscr{N}_t = \lceil \mathscr{V}_t / k \rceil \text{ and } \mathscr{N}_t = |\{n : \mathscr{A}_n \le t < \mathscr{D}_n\}|. \tag{4.1}$$

At each arrival epoch, $\mathscr{V}_t$ jumps up with the jump size equal to k times the number of arrivals at the epoch. Between jumps, $\mathscr{V}_t$ decreases linearly at the rate of c whenever $\mathscr{V}_t > 0$. The first busy period begins at $\mathscr{T}_1 = 0$ and ends at $\mathscr{T}_1' = \min\{t > \mathscr{T}_1 : \mathscr{N}_t = 0\}$. The jth busy period for $j > 1$ begins at

$$\mathscr{T}_j = \min\{t > \mathscr{T}_{j-1}' : t = \mathscr{A}_n \text{ for some } n\},$$

and ends at

$$\mathscr{T}_j' = \min\{t > \mathscr{T}_j : \mathscr{N}_t = 0\}.$$

For a busy period in which n customers enter the system, let (t_i, d_i), $i = 1, \ldots, n$, be the corresponding n pairs of arrival and departure times with $t_1 \le t_2 \le \cdots \le t_n$. Then $d_i = t_1 + k'i$, $i = 1, \ldots, n$, and $t_{i+1} \le d_i$, $i = 1, \ldots, n-1$. This motivates the following definition of a $\mathscr{B}(n)$ set. An (ordered) set of n pairs of numbers $P = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ is said to be a $\mathscr{B}(n)$ set (with respect to the continuous-time (c, k)-system) if $t_1 \le t_2 \le \cdots \le t_n$, $d_i = t_1 + k'i$, $i = 1, \ldots, n$, and $t_{i+1} \le d_i$, $i = 1, \ldots, n-1$. Let

$$\mathscr{N}_P(t) = |\{1 \le i \le n : t_i \le t < d_i\}|, t_1 \le t < d_n, \tag{4.2}$$

$$\mathscr{S}_P(\ell) = \mu(\{t_1 \le t < d_n : \mathscr{N}_P(t) = \ell\}), \quad \ell = 1, 2, \ldots, \tag{4.3}$$

$$\mathscr{W}_P = \{d_i - t_i : i = 1, \ldots, n\}, \tag{4.4}$$

where $\mu(S)$ denotes the Lebesgue measure of a (measurable) subset S of the real line. It will be convenient to define $\mathscr{N}_P(t) = 0$ for $t \ge d_n$. As an illustration for $\mathscr{N}_P(t)$, $\mathscr{S}_P(\ell)$ and $\mathscr{W}_P$ defined above, Fig. 2 presents the sample path of a $\mathscr{B}(5)$ set with respect to the continuous-time (c, k)-system with $c = 1$ and $k = 4$ where $t_2 = t_3 = t_1 + 2$, $t_4 = t_1 + 7$, $t_5 = t_1 + 13$ and $d_i = t_1 + 4i$, $i = 1, \ldots, 5$.

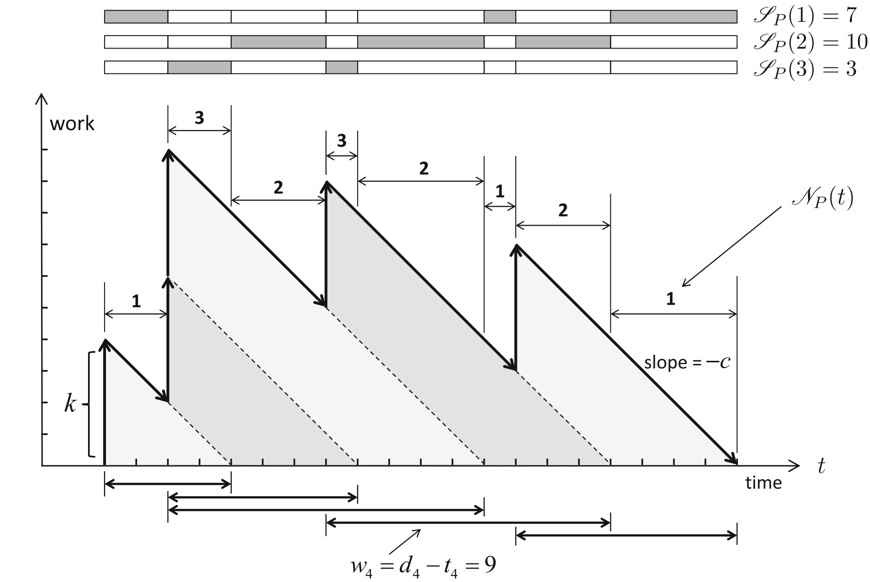

Fig. 2 Sample path of a $\mathscr{B}(5)$ set P with respect to the continuous-time (c, k)-system with $c = 1$ and $k = 4$

Remark 7 For a busy period with n customers, the set of n pairs of arrival and departure times is a $\mathscr{B}(n)$ set. Consider a general $\mathscr{B}(n)$ set $P = \{(t_1, d_1), \dots, (t_n, d_n)\}$. By definition, $\{(t_1, d_1), \dots, (t_j, d_j)\}$ is a $\mathscr{B}(j)$ set for $1 \le j < n$. Suppose that the system is empty right before t_1, and n customers enter the system at $t_1, \dots, t_n$. Then $d_i = t_1 + k'i$, $i = 1, \dots, n$, are the corresponding departure times. For $t_1 \le t < d_n$, $\mathcal{N}_P(t)$ is the number of customers in system at t, $\mathscr{S}_P(\ell)$ is the amount of time (in $[t_1, d_n]$) exactly ℓ customers are in system, and $\mathscr{W}_P$ is the set of n waiting time values.

For a positive number δ and a set S consisting of numbers greater than or equal to δ, we define $S^{<\delta>}(\ell)$, $\ell = 1, 2, \dots$, as follows.

For each $s \in S$, if $s = \delta$, let

$$s^{\langle\delta\rangle}(\ell) = \begin{cases} \delta, & \text{for } \ell = 1, \\ 0, & \text{for } \ell > 1. \end{cases}$$

If $s > \delta$, let

$$s^{\langle\delta\rangle}(\ell) = \begin{cases} s - \delta(\lceil s/\delta \rceil - 1), & \text{for } \ell = \lceil s/\delta \rceil, \\ \delta\lceil s/\delta \rceil - s, & \text{for } \ell = \lceil s/\delta \rceil - 1, \\ 0, & \text{for } \ell \ne \lceil s/\delta \rceil, \lceil s/\delta \rceil - 1. \end{cases}$$

Define $S^{\langle\delta\rangle}(\ell) = \sum_{s \in S} s^{\langle\delta\rangle}(\ell)$, $\ell = 1, 2, \dots$.

Remark 8 To see the meaning of $s^{\langle\delta\rangle}$, note that there is a unique sequence $(a_1, a_2, \dots)$ such that (i) $a_\ell \ge 0$ for all ℓ, (ii) $a_i a_j > 0$ implies $|i - j| \le 1$, (iii) $\sum_{\ell=1}^{\infty} a_\ell = \delta$

and (iv) $\sum_{\ell=1}^{\infty} \ell a_\ell = s$. Then $a_\ell = s^{\langle\delta\rangle}(\ell)$ for all ℓ. A similar interpretation of $s^{[\delta]}$ is given in Remark 4.

Theorem 3 *Let $P = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ be a $\mathscr{B}(n)$ set with respect to the continuous-time (c, k)-system. Then $\mathscr{S}_P \equiv \mathscr{W}_P^{\langle k/c\rangle}$, i.e. $\mathscr{S}_P(\ell) = \mathscr{W}_P^{\langle k/c\rangle}(\ell)$, $\ell = 1, 2, \ldots$.*

Proof We proceed by induction on n. For $P = \{(t_1, d_1)\}$, a $\mathscr{B}(1)$ set, since $\mathscr{N}_P(t) = 1$ for $t_1 \le t < d_1$, $\mathscr{S}_P(1) = d_1 - t_1 = k/c = k'$ and $\mathscr{S}_P(\ell) = 0$ for $\ell > 1$. Also, $\mathscr{W}_P = \{d_1 - t_1\} = \{k'\}$, implying that $\mathscr{W}_P^{\langle k'\rangle}(1) = k'$ and $\mathscr{W}_P^{\langle k'\rangle}(\ell) = 0$ for $\ell > 1$. So $\mathscr{S}_P \equiv \mathscr{W}_P^{\langle k'\rangle}$.

Now suppose that $\mathscr{S}_Q \equiv \mathscr{W}_Q^{\langle k'\rangle}$ for all $\mathscr{B}(n)$ sets Q $(n \ge 1)$. Let $P = \{(t_1, d_1), \ldots, (t_n, d_n), (t_{n+1}, d_{n+1})\}$ be a $\mathscr{B}(n+1)$ set. We need to show $\mathscr{S}_P \equiv \mathscr{W}_P^{\langle k'\rangle}$. Let $Q = \{(t_1, d_1), \ldots, (t_n, d_n)\}$, which is a $\mathscr{B}(n)$ set. By the induction hypothesis, $\mathscr{S}_Q \equiv \mathscr{W}_Q^{\langle k'\rangle}$. Noting that $t_n \le t_{n+1} \le d_n$, let $I \in \{1, \ldots, n+1\}$ be such that $d_{I-1} \le t_{n+1} < d_I$ where $d_0 := t_1$. (If $I = n+1$, we must have $t_{n+1} = d_n$ since P being a $\mathscr{B}(n+1)$ set implies $t_{n+1} \le d_n$.) It is readily shown that

$$\mathscr{N}_P(t) = \mathscr{N}_Q(t) = |\{1 \le i \le n : t_i \le t < d_i\}|, \quad \text{for } t_1 \le t < t_{n+1}, \tag{4.5}$$

$$\mathscr{N}_P(t) = \mathscr{N}_Q(t) + 1 = \begin{cases} n - I + 2, & \text{for } t_{n+1} \le t < d_I, \\ n - j + 1, & \text{for } d_j \le t < d_{j+1}, \quad j = I, \ldots, n, \end{cases} \tag{4.6}$$

where $\mathscr{N}_Q(t) := 0$ for $t \ge d_n$. By (4.5) and (4.6), if $I \le n$,

$$\mathscr{S}_P(\ell) - \mathscr{S}_Q(\ell) = \begin{cases} d_I - t_{n+1}, & \text{for } \ell = n - I + 2, \\ t_{n+1} - d_{I-1}, & \text{for } \ell = n - I + 1, \\ 0, & \text{for } \ell \ne n - I + 1, n - I + 2. \end{cases} \tag{4.7}$$

If $I = n + 1$ (implying $t_{n+1} = d_n$),

$$\mathscr{S}_P(\ell) - \mathscr{S}_Q(\ell) = \begin{cases} k', & \text{for } \ell = 1, \\ 0, & \text{for } \ell > 1. \end{cases} \tag{4.8}$$

Since $\mathscr{W}_P = \mathscr{W}_Q \uplus \{d_{n+1} - t_{n+1}\}$, we have

$$\mathscr{W}_P^{\langle k'\rangle}(\ell) - \mathscr{W}_Q^{\langle k'\rangle}(\ell) = (d_{n+1} - t_{n+1})^{\langle k'\rangle}(\ell), \quad \ell = 1, 2, \ldots. \tag{4.9}$$

Noting that $d_{n+1} - d_I < d_{n+1} - t_{n+1} \le d_{n+1} - d_{I-1}$, if $I \le n$,

$$(d_{n+1} - t_{n+1})^{\langle k'\rangle}(\ell) = \begin{cases} d_I - t_{n+1}, & \text{for } \ell = n - I + 2, \\ t_{n+1} - d_{I-1}, & \text{for } \ell = n - I + 1, \\ 0, & \text{for } \ell \ne n - I + 1, n - I + 2. \end{cases} \tag{4.10}$$

If $I = n + 1$ (i.e. $t_{n+1} = d_n$),

$$(d_{n+1} - t_{n+1})^{\langle k' \rangle}(\ell) = \begin{cases} k', & \text{for } \ell = 1, \\ 0, & \text{for } \ell > 1. \end{cases} \tag{4.11}$$

It follows from $\mathscr{S}_Q \equiv \mathscr{W}_Q^{\langle k' \rangle}$ and (4.7)–(4.11) that $\mathscr{S}_P \equiv \mathscr{W}_P^{\langle k' \rangle}$. This completes the proof. □

Remark 9 Note that the continuous-time (c, k)-system can be approximated by a discrete-time $(1, k^*)$-system for a suitably chosen (large) k^*. More precisely, let time be divided into slots of (small) length Δ. Then the server completes $c\Delta$ units of work per slot. Define a D-unit of work to be $c\Delta$ units of work (D standing for discrete), so the server completes one D-unit of work per slot. Each customer brings to the system k units of work which is approximately k^* D-units of work where $k^* = \lceil k/(c\Delta) \rceil = \lceil k'/\Delta \rceil$. Furthermore, since Δ is small, the arrival time $\mathscr{A}_i$ of customer C_i is (approximately) slot A_i where $A_i = \lceil \mathscr{A}_i/\Delta \rceil$. This shows that the continuous-time (c, k)-system and the discrete-time $(1, k^*)$-system are approximately the same provided Δ is sufficiently small. While it is easier to prove Theorem 3 directly as we have done, one can in fact establish Theorem 3 by making use of Theorem 1 along with a limiting argument as $\Delta \to 0$.

To apply Theorem 3 to a sample path with the arrival and departure times $\mathscr{A}_i, \mathscr{D}_i$ of the ith customer $C_i, i = 1, 2, \ldots,$ recall that $\mathscr{N}_t$ and $\mathscr{V}_t$ denote, respectively, the number of customers and the amount of work in system at time t. Recall also that $[\mathscr{T}_j, \mathscr{T}_j']$ denotes the jth busy period. Let $\mathscr{I}_j = \{i : \mathscr{T}_j \le \mathscr{A}_i < \mathscr{T}_j'\}$, the set of the indices of customers entering the system in the jth busy period. Then $\mathscr{P}_j := \{(\mathscr{A}_i, \mathscr{D}_i) : i \in \mathscr{I}_j\}$ is a $\mathscr{B}(|\mathscr{I}_j|)$ set. By (4.2), for $\mathscr{T}_j \le t < \mathscr{T}_j'$, $\mathscr{N}_{\mathscr{P}_j}(t) = |\{i \in \mathscr{I}_j : \mathscr{A}_i \le t < \mathscr{D}_i\}|$, which agrees with $\mathscr{N}_t$ (cf. (4.1)). Also, for $\ell = 1, 2, \ldots,$

$$\mathscr{S}_{\mathscr{P}_j}(\ell) = \mu(\{\mathscr{T}_j \le t < \mathscr{T}_j' : \mathscr{N}_{\mathscr{P}_j}(t) = \ell\}) = \mu(\{\mathscr{T}_j \le t < \mathscr{T}_j' : \mathscr{N}_t = \ell\}),$$
$$\mathscr{W}_{\mathscr{P}_j} = \{\mathscr{D}_i - \mathscr{A}_i : i \in \mathscr{I}_j\}.$$

For $\tau > 0$, let $\mathbf{S}_\tau(\ell) = \mu(\{0 \le t \le \tau : \mathscr{N}_t = \ell\})$, $\ell = 1, 2, \ldots,$ and $\mathbf{W}_\tau = \{\mathscr{D}_i - \mathscr{A}_i : \mathscr{A}_i \le \tau\}$. With $\mathscr{T}_0' := 0$, let $\nu = \max\{j : \mathscr{T}_j' \le \tau\} \ge 0$, the number of busy periods ending at or before τ. Then $\mathscr{T}_\nu' \le \tau < \mathscr{T}_{\nu+1}'$, and

$$\mathbf{S}_\tau(\ell) = \sum_{j=1}^{\nu} \mathscr{S}_{\mathscr{P}_j}(\ell) + \mu(\{\mathscr{T}_{\nu+1} \le t \le \tau : \mathscr{N}_t = \ell\}), \quad \ell = 1, 2, \ldots, \tag{4.12}$$

$$\mathbf{W}_\tau = \uplus_{j=1}^{\nu} \mathscr{W}_{\mathscr{P}_j} \uplus \{\mathscr{D}_i - \mathscr{A}_i : \mathscr{T}_{\nu+1} \le \mathscr{A}_i \le \tau\}. \tag{4.13}$$

If $\mathscr{V}_\tau = 0$ (implying $\mathscr{N}_\tau = 0$), then $\tau < \mathscr{T}_{\nu+1}$, so that

$$\mathbf{S}_\tau \equiv \sum_{j=1}^{\nu} \mathscr{S}_{\mathscr{P}_j} \text{ and } \mathbf{W}_\tau = \uplus_{j=1}^{\nu} \mathscr{W}_{\mathscr{P}_j},$$

from which and Theorem 3, it follows that

$$\mathbf{S}_\tau(\ell) = \sum_{j=1}^{\nu} \mathscr{S}_{\mathscr{P}_j}(\ell) = \sum_{j=1}^{\nu} \mathscr{W}_{\mathscr{P}_j}^{\langle k' \rangle}(\ell) = \mathbf{W}_\tau^{\langle k' \rangle}(\ell), \quad \ell = 1, 2, \ldots. \tag{4.14}$$

If $\mathscr{V}_\tau > 0$, then $\tau \geq \mathscr{T}_{\nu+1}$. Let $\iota_1 = \min\{i : \mathscr{A}_i = \mathscr{T}_{\nu+1}\}$ and $\iota_2 = \max\{i : \mathscr{A}_i \leq \tau\}$. Consider $Q = \{(\mathscr{A}_i, \mathscr{D}_i) : \iota_1 \leq i \leq \iota_2\}$ which is a $\mathscr{B}(\iota_2 - \iota_1 + 1)$ set. (Note that $Q \subset \mathscr{P}_{\nu+1}$ and no (t, d) in Q has $t > \tau$.) Since $\mathscr{W}_Q = \{\mathscr{D}_i - \mathscr{A}_i : \iota_1 \leq i \leq \iota_2\} = \{\mathscr{D}_i - \mathscr{A}_i : \mathscr{T}_{\nu+1} \leq \mathscr{A}_i \leq \tau\}$, we have by (4.13)

$$\mathbf{W}_\tau = \uplus_{j=1}^{\nu} \mathscr{W}_{\mathscr{P}_j} \uplus \mathscr{W}_Q. \tag{4.15}$$

Noting that $\mathscr{T}_{\nu+1} = \mathscr{A}_{\iota_1}$ and $\mathscr{N}_Q(t) = \mathscr{N}_t$ for $\mathscr{T}_{\nu+1} \leq t \leq \tau$, we have

$$\begin{aligned}
\mathscr{S}_Q(\ell) &= \mu(\{\mathscr{T}_{\nu+1} \leq t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\}) \\
&= \mu(\{\mathscr{T}_{\nu+1} \leq t \leq \tau : \mathscr{N}_t = \ell\}) + \mu(\{\tau < t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\}),
\end{aligned}$$

which together with (4.12) yields

$$\mathbf{S}_\tau(\ell) + \mu(\{\tau < t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\}) = \sum_{j=1}^{\nu} \mathscr{S}_{\mathscr{P}_j}(\ell) + \mathscr{S}_Q(\ell). \tag{4.16}$$

Here $\mu(\{\tau < t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\})$ can be expressed in terms of $\mathscr{V}_\tau$ as follows. Write $\mathscr{V}_\tau = qk + r$ where

$$q := \lceil \mathscr{V}_\tau / k \rceil - 1 \text{ and } 0 < r := \mathscr{V}_\tau - k(\lceil \mathscr{V}_\tau / k \rceil - 1) \leq k. \tag{4.17}$$

Note that no (t, d) in Q has $t > \tau$. In the case that no arrivals occur after time τ, $\mathscr{V}_t$ $(t \geq \tau)$ decreases linearly at the rate of c whenever $\mathscr{V}_t > 0$, so

(i) for $t \in (\tau, \tau + r/c)$, we have $qk < \mathscr{V}_t \leq (q+1)k$ and $\mathscr{N}_t = q + 1$;
(ii) for $t \in [\tau + r/c + (q-n)k', \tau + r/c + (q-n+1)k')$ with $n = q, q-1, \ldots, 1$, we have $(n-1)k < \mathscr{V}_t \leq nk$ and $\mathscr{N}_t = n$. (Part (ii) is vacuous if $q = 0$.)

It follows that

$$\mu(\{\tau < t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\}) = \begin{cases} k', & \text{for } \ell = 1, \ldots, q, \\ r/c, & \text{for } \ell = q + 1, \\ 0, & \text{for } \ell > q + 1. \end{cases} \tag{4.18}$$

By (4.15)–(4.18) and Theorem 3,

$$\begin{aligned}\mathbf{W}_\tau^{\langle k'\rangle}(\ell) - \mathbf{S}_\tau(\ell) &= \sum_{j=1}^{\nu} \mathscr{W}_{\mathscr{P}_j}^{\langle k'\rangle}(\ell) + \mathscr{W}_Q^{\langle k'\rangle}(\ell) - \mathbf{S}_\tau(\ell)\\ &= \sum_{j=1}^{\nu} \mathscr{S}_{\mathscr{P}_j}(\ell) + \mathscr{S}_Q(\ell) - \mathbf{S}_\tau(\ell)\\ &= \mu(\{\tau < t < \mathscr{D}_{\iota_2} : \mathscr{N}_Q(t) = \ell\})\\ &= \begin{cases} k', & \text{for } \ell = 1, \ldots, \lceil \mathscr{V}_\tau/k\rceil - 1,\\ \mathscr{V}_\tau/c - k'(\lceil \mathscr{V}_\tau/k\rceil - 1), & \text{for } \ell = \lceil \mathscr{V}_\tau/k\rceil,\\ 0, & \text{for } \ell > \lceil \mathscr{V}_\tau/k\rceil. \end{cases} \end{aligned} \tag{4.19}$$

(The case $\ell = 1, \ldots, \lceil \mathscr{V}_\tau/k\rceil - 1$ is vacuous if $0 < \mathscr{V}_\tau \le k$.) Combining (4.14) for $\mathscr{V}_\tau = 0$ and (4.19) for $\mathscr{V}_\tau > 0$ yields the following result.

Theorem 4 (i) *If* $\mathscr{V}_\tau = 0$, $\mathbf{S}_\tau(\ell) = \mathbf{W}_\tau^{\langle k'\rangle}(\ell)$, $\ell = 1, 2, \ldots$.
(ii) *If* $0 < \mathscr{V}_\tau \le k$,

$$\mathbf{W}_\tau^{\langle k'\rangle}(\ell) - \mathbf{S}_\tau(\ell) = \begin{cases} \mathscr{V}_\tau/c, & for\ \ell = 1,\\ 0, & for\ \ell > 1. \end{cases}$$

(iii) *If* $\mathscr{V}_\tau > k$,

$$\mathbf{W}_\tau^{\langle k'\rangle}(\ell) - \mathbf{S}_\tau(\ell) = \begin{cases} k/c, & for\ \ell = 1, \ldots, \lceil \mathscr{V}_\tau/k\rceil - 1,\\ \{\mathscr{V}_\tau - k(\lceil \mathscr{V}_\tau/k\rceil - 1)\}/c, & for\ \ell = \lceil \mathscr{V}_\tau/k\rceil,\\ 0, & for\ \ell > \lceil \mathscr{V}_\tau/k\rceil. \end{cases}$$

In Sect. 3, we derived a sample-path distributional relation between system size and waiting time for the discrete-time (c, k)-system. We now apply Theorem 4 to show a similar result for the continuous-time (c, k)-system. Fix a sample path ω in which customer C_i arrives and departs at $\mathscr{A}_i$ and $\mathscr{D}_i$, respectively, with waiting time $\mathscr{W}_i := \mathscr{D}_i - \mathscr{A}_i$, $i = 1, 2, \ldots$. Let $\Lambda(\tau) = \max\{i : \mathscr{A}_i \le \tau\}$, the number of customers arriving at or before τ. Assume the following limits exist for the sample path ω:

$$\begin{aligned}\lambda^{(\omega)} &= \lim_{\tau\to\infty} \Lambda(\tau)/\tau, \quad 0 < \lambda^{(\omega)} < 1/k',\\ \mathscr{F}_{\mathrm{W}}^{(\omega)}(x) &= \lim_{n\to\infty} \frac{1}{n}|\{1 \le i \le n : \mathscr{D}_i - \mathscr{A}_i \le x\}|\\ &= \lim_{n\to\infty} \frac{1}{n}|\{1 \le i \le n : \mathscr{W}_i \le x\}|, \quad 0 < x < \infty,\\ \mathscr{F}_{\mathrm{S}}^{(\omega)}\{\ell\} &= \lim_{\tau\to\infty} \frac{1}{\tau}\mathbf{S}_\tau(\ell) = \lim_{\tau\to\infty} \frac{1}{\tau}\mu(\{0 \le t \le \tau : \mathscr{N}_t = \ell\}), \quad \ell = 1, 2, \ldots.\end{aligned}$$

By Theorem 4, we have

$$0 \leq \mathbf{W}_\tau^{\langle k' \rangle}(\ell) - \mathbf{S}_\tau(\ell) = \sum_{i=1}^{\Lambda(\tau)} \mathscr{W}_i^{\langle k' \rangle}(\ell) - \mathbf{S}_\tau(\ell) \leq k',$$

which implies

$$\mathscr{F}_{\mathrm{s}}^{(\omega)}(\ell) = \lim_{\tau \to \infty} \frac{1}{\tau} \mathbf{S}_\tau(\ell) = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{i=1}^{\Lambda(\tau)} \mathscr{W}_i^{\langle k' \rangle}(\ell). \tag{4.20}$$

We claim that for $\ell = 1, 2, \ldots,$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathscr{W}_i^{\langle k' \rangle}(\ell) = \int_{(\ell-1)k'}^{(\ell+1)k'} (k' - |t - \ell k'|) \mathrm{d}\mathscr{F}_{\mathrm{w}}^{(\omega)}(t), \tag{4.21}$$

which together with (4.20) implies

$$\mathscr{F}_{\mathrm{s}}^{(\omega)}(\ell) = \lambda^{(\omega)} \int_{(\ell-1)k'}^{(\ell+1)k'} (k' - |t - \ell k'|) \mathrm{d}\mathscr{F}_{\mathrm{w}}^{(\omega)}(t). \tag{4.22}$$

This sample-path distributional relation between system size and waiting time is the continuous-time counterpart of the discrete-time result (3.5).

It remains to prove the claim (4.21). Fix an $\ell \geq 1$, and divide the interval $((\ell - 1)k', (\ell + 1)k']$ into $2R$ subintervals each of length k'/R, i.e. $J_r := ((\ell - 1)k' + (r - 1)k'/R, (\ell - 1)k' + rk'/R]$, $r = 1, \ldots, 2R$. Note that $\mathscr{W}_i^{\langle k' \rangle}(\ell) = 0$ if $\mathscr{W}_i \leq (\ell - 1)k'$ or $\mathscr{W}_i \geq (\ell + 1)k'$. If $\mathscr{W}_i \in J_r$, then

$$(r - 1)k'/R \leq \mathscr{W}_i^{\langle k' \rangle}(\ell) \leq rk'/R, \quad \text{for } r = 1, \ldots, R,$$
$$(2R - r)k'/R \leq \mathscr{W}_i^{\langle k' \rangle}(\ell) \leq (2R - r + 1)k'/R, \quad \text{for } r = R + 1, \ldots, 2R.$$

It follows that for $i = 1, 2, \ldots,$

$$L_{i,R} \leq \mathscr{W}_i^{\langle k' \rangle}(\ell) \leq L_{i,R} + k'/R, \tag{4.23}$$

where

$$L_{i,R} := \sum_{r=1}^{R} \frac{k'(r-1)}{R} 1_{\{\mathscr{W}_i \in J_r\}} + \sum_{r=R+1}^{2R} \frac{k'(2R - r)}{R} 1_{\{\mathscr{W}_i \in J_r\}}.$$

Then

$$\frac{1}{n}\sum_{i=1}^{n}\mathscr{W}_i^{\langle k'\rangle}(\ell) \ge \frac{1}{n}\sum_{i=1}^{n}L_{i,R} = \sum_{r=1}^{R}\frac{k'(r-1)}{R}\left[\frac{1}{n}\sum_{i=1}^{n}1_{\{\mathscr{W}_i\in J_r\}}\right] + \sum_{r=R+1}^{2R}\frac{k'(2R-r)}{R}\left[\frac{1}{n}\sum_{i=1}^{n}1_{\{\mathscr{W}_i\in J_r\}}\right], \tag{4.24}$$

which converges as $n \to \infty$ to

$$L_R^* := \sum_{r=1}^{R}\frac{k'(r-1)}{R}\mathscr{F}_{\mathrm{w}}^{(\omega)}(J_r) + \sum_{r=R+1}^{2R}\frac{k'(2R-r)}{R}\mathscr{F}_{\mathrm{w}}^{(\omega)}(J_r), \tag{4.25}$$

where

$$\mathscr{F}_{\mathrm{w}}^{(\omega)}(J_r) := \mathscr{F}_{\mathrm{w}}^{(\omega)}((\ell-1)k' + rk'/R) - \mathscr{F}_{\mathrm{w}}^{(\omega)}((\ell-1)k' + (r-1)k'/R).$$

So, by (4.23)–(4.25),

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathscr{W}_i^{\langle k'\rangle}(\ell) \ge L_R^*.$$

Noting that a standard argument in calculus yields

$$\lim_{R\to\infty} L_R^* = \int_{(\ell-1)k'}^{(\ell+1)k'} (k' - |t-\ell k'|)\mathrm{d}\mathscr{F}_{\mathrm{w}}^{(\omega)}(t) =: H(\ell),$$

we have $\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathscr{W}_i^{\langle k'\rangle}(\ell) \ge H(\ell)$. Similarly using the upper bound in (4.23) yields $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathscr{W}_i^{\langle k'\rangle}(\ell) \le H(\ell)$. This proves the claim (4.21).

5 Concluding remarks

We introduced a discrete-time queueing system called the (c, k)-system where the arrival process is general and each arriving customer brings k units of work to the system which is processed by a single-server at a deterministic rate of c units of work per slot. It was indicated that an equivalence relation holds for the class of all (c, k)-systems. It was also pointed out that the (c, k)-system is equivalent to the $G/D/c$ queue with constant service times of k slots when either $c = 1$ or $k = 1$, but the two systems differ when both $c > 1$ and $k > 1$.

By a detailed sample-path analysis together with the notion of multiset, we derived an exact sample-path relation between the set of system size values and the set of waiting time values over a busy period of a given sample path. When $c = k = 1$, this relation simply implies that the multiset of system size values is identical to the multiset of waiting time values over a busy period. However, when either c or k (or both) is greater than 1, the physical interpretation of this relation is not so clear and requires further investigation.

This sample-path relation was applied to the discrete-time $G/D/c$ queue with constant service times of k slots with either $c = 1$ or $k = 1$, yielding the sample-path version of the steady-state distributional relation between system size and waiting time in Xiong et al. [25]. When c and k are both greater than 1, the $G/D/c$ queue with constant service times of k slots is harder to analyze than the (c, k)-system. It remains an open problem to derive a sample-path distributional relation between system size and waiting time for this more challenging case.

We also considered the continuous-time analogue of the discrete-time (c, k)-system and carried out a similar sample-path analysis to derive a sample-path relation between system size and waiting time in continuous time.

Acknowledgments The authors are grateful to the two reviewers for helpful comments which led to an improved presentation. The first author acknowledges support from the National Science Council of Taiwan. The second author acknowledges the support from Academia Sinica, Taiwan for his visits to the Institute of Statistical Science.

6 Appendix

6.1 Lemmas and proofs

In this appendix, we state and prove a few lemmas that are needed for the proof of Theorem 1 in Sect. 2.

Lemma 1 *For the discrete-time* (c, k)*-system, let* $Q = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ *be a* $B(n)$ *set, and let* $P = \{(t_1, d_1), \ldots, (t_n, d_n), (t_{n+1}, d_{n+1})\}$ *with* $t_{n+1} = d_n$ *and* $d_{n+1} = t_1 + \lceil k(n+1)/c \rceil - 1$ *(which is a* $B(n+1)$ *set). If* $S_Q^{[c]} = W_Q^{[k]}$*, then* $S_P^{[c]} = W_P^{[k]}$.

Proof Consider the setting where the (c, k)-system is empty right before t_1, and $n+1$ customers $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ enter the system at $t_1, \ldots, t_{n+1}$, respectively. Since P is a $B(n+1)$ set, the $\mathcal{C}_i$ depart at $d_i = t_1 + \lceil ki/c \rceil - 1$, $i = 1, \ldots, n+1$, and exactly c units of work is performed at each of the slots $t_1, \ldots, d_{n+1} - 1$. Let $\ell = N_Q(d_n) = |\{i \le n : d_i = d_n\}|$, the number of those in $\mathcal{C}_1, \ldots, \mathcal{C}_n$ who are in system at d_n (and also depart at the end of slot d_n), and $v =$ the number of units of work remaining for these ℓ customers at (the beginning of) slot d_n. We have (cf. (2.3)),

$$(\ell - 1)k + 1 \le v \le \min\{\ell k, c\} \text{ and } \ell = \lceil v/k \rceil. \tag{6.1}$$

Also $N_P(t) = N_Q(t)$ for $t_1 \le t < d_n$, since $t_{n+1} = d_n$. We consider the three cases $(c \ge 1, k = 1)$, $(c = 1, k \ge 1)$ and $(c \ge 2, k \ge 2)$ separately.

Case (i): $c \ge 1, k = 1$. By (6.1), $\ell = \lceil v/k \rceil = v$. Write $n = qc + r$ with $q = \lceil n/c \rceil - 1 \ge 0$ and $0 < r \le c$. With $k = 1$, the total number of work units for customers

$\mathcal{C}_1, \ldots, \mathcal{C}_n$ is n. Since exactly c units of work is performed at $t \in \{t_1, \ldots, d_n - 1\}$ and since $\mathcal{C}_n$ departs at d_n, we must have $r = v = \ell$. Two subcases are considered below.

Subcase (i.1): $0 < r < c$. In this case, the total number of work units (including the work of $\mathcal{C}_{n+1}$) at d_n equals $r + 1 \le c$, so that $d_{n+1} = d_n$. It follows that

$$\begin{aligned} S_Q &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{r\}, \\ S_P &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{r+1\}, \\ W_P &= W_Q \uplus \{1\}. \end{aligned}$$

Since $\{r\}^{[c]} = \{1_{(r)}\}$ and $\{r+1\}^{[c]} = \{1_{(r+1)}\}$ and $S_Q^{[c]} = W_Q^{[k]} = W_Q$,

$$S_P^{[c]} = S_Q^{[c]} \uplus \{1\} = W_Q \uplus \{1\} = W_P.$$

Subcase (i.2): $r = c$. The total number of work units (including the work of $\mathcal{C}_{n+1}$) at d_n equals $c + 1$, so that $d_{n+1} = d_n + 1$. It follows that

$$\begin{aligned} S_Q &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{c\}, \\ S_P &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{1, c+1\}, \\ W_P &= W_Q \uplus \{2\}. \end{aligned}$$

Since $\{1, c+1\}^{[c]} = \{1_{(c)}, 2\}$ and $\{c\}^{[c]} = \{1_{(c)}\}$ and $S_Q^{[c]} = W_Q$,

$$S_P^{[c]} = S_Q^{[c]} \uplus \{2\} = W_Q \uplus \{2\} = W_P.$$

This completes the proof for case (i).

Case (ii): $c = 1, k \ge 1$. Since $c = 1$, all of $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ must have departed before d_n, and $\mathcal{C}_n$ has just one unit of work remaining at d_n. So $\ell = v = 1$. When $\mathcal{C}_{n+1}$ enters the system at $t_{n+1} = d_n$, it takes k additional slots to complete the work of $\mathcal{C}_{n+1}$. So $d_{n+1} = d_n + k$. It follows that

$$\begin{aligned} S_Q &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{1\}, \\ S_P &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{1_{(k)}, 2\}, \\ W_P &= W_Q \uplus \{k+1\}. \end{aligned}$$

Since $\{k+1\}^{[k]} = \{1_{(k-1)}, 2\}$ and $S_Q = S_Q^{[c]} = W_Q^{[k]}$,

$$\begin{aligned} W_P^{[k]} &= W_Q^{[k]} \uplus \{1_{(k-1)}, 2\} \\ &= S_Q \uplus \{1_{(k-1)}, 2\} \\ &= \{N_Q(t) : t_1 \le t < d_n\} \uplus \{1\} \uplus \{1_{(k-1)}, 2\} \\ &= S_P, \end{aligned}$$

completing the proof for case (ii).

Case (iii): $c \geq 2, k \geq 2$. We further consider the following three subcases.

Subcase (iii.1): $d_{n+1} = d_n$. Necessarily, $c \geq v + k$ (the amount of work at d_n, including the work of $\mathcal{C}_{n+1}$), which together with (6.1) implies $c > \ell k \geq 2\ell$. Also we have $W_P = W_Q \uplus \{1\}$, $S_Q = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{\ell\}$, and $S_P = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{\ell + 1\}$. Since $\ell + 1 \leq 2\ell < c$, we have $\{\ell\}^{[c]} = \{1_{(\ell)}\}$, $\{\ell + 1\}^{[c]} = \{1_{(\ell+1)}\}$, so that

$$S_P^{[c]} = S_Q^{[c]} \uplus \{1\} = W_Q^{[k]} \uplus \{1\} = W_P^{[k]}.$$

Subcase (iii.2): $d_{n+1} = d_n + 1$. Necessarily, $v + k \leq 2c$, which together with (6.1) implies $2c \geq v + k > \ell k \geq 2\ell$, i.e. $c > \ell$. Then $W_P = W_Q \uplus \{2\}$ (implying $W_P^{[k]} = W_Q^{[k]} \uplus \{1_{(2)}\}$),

$$S_Q = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{\ell\}, S_P = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{1, \ell + 1\},$$

from which and $c > \ell$ it follows that

$$S_P^{[c]} = S_Q^{[c]} \uplus \{1_{(2)}\} = W_Q^{[k]} \uplus \{1_{(2)}\} = W_P^{[k]}.$$

Subcase (iii.3): $d_{n+1} \geq d_n + 2$. Necessarily $\ell = 1$. Since $\mathcal{C}_{n+1}$ is the only customer in system after d_n and since exactly c units of work is performed at each of the slots $d_n + 1, \ldots, d_{n+1} - 1$, we have $k \geq c(d_{n+1} - d_n - 1) + 1 \geq d_{n+1} - d_n + 1$. So $W_P = W_Q \uplus \{d_{n+1} - d_n + 1\}$ (implying $W_P^{[k]} = W_Q^{[k]} \uplus \{1_{(d_{n+1}-d_n+1)}\}$),

$$S_Q = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{\ell\}, S_P = \{N_Q(t) : t_1 \leq t < d_n\} \uplus \{\ell + 1, 1_{(d_{n+1}-d_n)}\}.$$

It follows that

$$S_P^{[c]} = S_Q^{[c]} \uplus \{1_{(d_{n+1}-d_n+1)}\} = W_Q^{[k]} \uplus \{1_{(d_{n+1}-d_n+1)}\} = W_P^{[k]}.$$

This completes the proof for case (iii). The proof of Lemma1 is complete. □

Lemma 2 *For the discrete-time* (c, k)*-system, let* $Q = \{(t_1, d_1), \ldots, (t_n, d_n)\}$ *be a* $B(n)$ *set with* $t_n < d_n$*. Let* j *satisfy* $t_n \leq j < j + 1 \leq d_n$*, and let* $P = \{(t_1, d_1), \ldots, (t_n, d_n), (j, d_{n+1})\}$ *and* $P' = \{(t_1, d_1), \ldots, (t_n, d_n), (j+1, d_{n+1})\}$ *(with* $d_{n+1} = t_1 + \lceil (n + 1)k/c \rceil - 1$*), which are both* $B(n + 1)$ *sets. Then*

$$S_P^{[c]} = W_P^{[k]} \textit{ if and only if } S_{P'}^{[c]} = W_{P'}^{[k]}.$$

Proof Consider the setting where the (c, k)-system is empty right before t_1, and $n + 1$ customers $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ enter the system at $t_1, \ldots, t_n, t_{n+1}$, respectively, where $t_{n+1} = j$ or $j + 1$. Since P and P' are both $B(n + 1)$ sets, the $\mathcal{C}_i$ depart at $d_i = t_i + \lceil ki/c \rceil - 1$, $i = 1, \ldots, n+1$, and exactly c units of work is done at each of the slots $t_1, \ldots, d_{n+1} - 1$. Let $\ell = N_Q(j)$, the number of those in $\mathcal{C}_1, \ldots, \mathcal{C}_n$ who are in system at j, and $v =$ the number of units of work remaining at j for the ℓ customers $\mathcal{C}_{n-\ell+1}, \ldots, \mathcal{C}_n$. Since none of $\mathcal{C}_{n-\ell+2}, \ldots, \mathcal{C}_n$ begins service before time j, we have $(\ell - 1)k + 1 \leq v \leq \ell k$, implying $\ell = \lceil v/k \rceil$ (cf. (2.3)). On the other hand, with $t_{n+1} = j$, $v+k$ units of work (including $\mathcal{C}_{n+1}$'s work) needs to be completed at d_{n+1}, so $(d_{n+1} - j)c + 1 \le v + k \le (d_{n+1} - j + 1)c$, implying $d_{n+1} - j + 1 = \lceil \frac{v+k}{c} \rceil$. We have shown

$$\ell + 1 = \left\lceil \frac{v}{k} \right\rceil + 1 = \left\lceil \frac{v+k}{k} \right\rceil \text{ and } d_{n+1} - j + 1 = \left\lceil \frac{v+k}{c} \right\rceil. \tag{6.2}$$

Since $\mathcal{C}_{n+1}$ departs at d_{n+1} for both cases $t_{n+1} = j$ and $t_{n+1} = j + 1$, the waiting time of $\mathcal{C}_{n+1}$ is $d_{n+1} - j + 1$ for the former case and $d_{n+1} - j$ for the latter. As the waiting times of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ do not depend on t_{n+1}, we have

$$W_P \uplus \{d_{n+1} - j\} = W_{P'} \uplus \{d_{n+1} - j + 1\}. \tag{6.3}$$

Also, $\mathcal{C}_{n+1}$ begins service at the same time for the two cases $t_{n+1} = j$ and $t_{n+1} = j+1$. In particular, for $t_{n+1} = j$, $\mathcal{C}_{n+1}$ cannot begin service at $j (< d_n)$. It follows that

$$N_P(t) = N_{P'}(t) \quad \text{for } t \ne j,\ N_P(j) = \ell + 1,\ N_{P'}(j) = \ell.$$

So,

$$S_P \uplus \{\ell\} = S_{P'} \uplus \{\ell + 1\}. \tag{6.4}$$

By (6.2)–(6.4),

$$S_P \uplus \left\{ \left\lceil \frac{v+k}{k} \right\rceil - 1 \right\} = S_{P'} \uplus \left\{ \left\lceil \frac{v+k}{k} \right\rceil \right\}, \tag{6.5}$$

$$W_P \uplus \left\{ \left\lceil \frac{v+k}{c} \right\rceil - 1 \right\} = W_{P'} \uplus \left\{ \left\lceil \frac{v+k}{c} \right\rceil \right\}. \tag{6.6}$$

By Lemma 3 below, $\lceil \lceil (v+k)/k \rceil / c \rceil = \lceil \lceil (v+k)/c \rceil / k \rceil$, which is denoted by ρ. We consider $\rho = 1$ and $\rho > 1$ separately.

Case (i): $\rho = 1$. We have $\lceil (v+k)/k \rceil \le c$ and $\lceil (v+k)/c \rceil \le k$, so

$$\left\{ \left\lceil \frac{v+k}{k} \right\rceil \right\}^{[c]} = \left\{ \left\lceil \frac{v+k}{k} \right\rceil - 1 \right\}^{[c]} \uplus \{1\}, \tag{6.7}$$

$$\left\{ \left\lceil \frac{v+k}{c} \right\rceil \right\}^{[k]} = \left\{ \left\lceil \frac{v+k}{c} \right\rceil - 1 \right\}^{[k]} \uplus \{1\}, \tag{6.8}$$

By (6.5)–(6.8),

$$S_P^{[c]} = S_{P'}^{[c]} \uplus \{1\} \text{ and } W_P^{[k]} = W_{P'}^{[k]} \uplus \{1\}.$$

It follows that $S_P^{[c]} = W_P^{[k]}$ if and only if $S_{P'}^{[c]} = W_{P'}^{[k]}$.

Case (ii): $\rho > 1$. It is readily seen that $\{\lceil \frac{v+k}{k} \rceil\}^{[c]}$ has one more copy of ρ and one fewer copy of $\rho - 1$ than $\{\lceil \frac{v+k}{k} \rceil - 1\}^{[c]}$, and $\{\lceil \frac{v+k}{c} \rceil\}^{[k]}$ has one more copy of ρ and one fewer copy of $\rho - 1$ than $\{\lceil \frac{v+k}{c} \rceil - 1\}^{[k]}$. It follows from (6.5) and (6.6) that

$$S_P^{[c]} \uplus \{\rho - 1\} = S_{P'}^{[c]} \uplus \{\rho\} \text{ and } W_P^{[k]} \uplus \{\rho - 1\} = W_{P'}^{[k]} \uplus \{\rho\},$$

which implies that $S_P^{[c]} = W_P^{[k]}$ if and only if $S_{P'}^{[c]} = W_{P'}^{[k]}$. This completes the proof. □

Lemma 3 *For positive integers i, j and ℓ, we have*

$$\lceil \lceil \ell/i \rceil / j \rceil = \lceil \lceil \ell/j \rceil / i \rceil.$$

Proof Let $\rho = \lceil \ell/i \rceil$. Then $\ell \le i\rho$, and $\lceil \ell/j \rceil \le \lceil i\rho/j \rceil \le i\lceil \rho/j \rceil$, implying $\lceil \lceil \ell/j \rceil / i \rceil \le \lceil \rho/j \rceil = \lceil \lceil \ell/i \rceil / j \rceil$. By symmetry, $\lceil \lceil \ell/i \rceil / j \rceil \le \lceil \lceil \ell/j \rceil / i \rceil$. This completes the proof. □

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