Online appendix for the paper

A Logical Characterization of the Preferred Models of Logic Programs with Ordered Disjunction

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A Proofs of Section 5

Lemma 1
Let P be a consistent extended logic program. Then the three-valued answer sets of P coincide with the standard answer sets of P.

Proof
By taking $n = 1$ in Definition 10, we get the standard definition of reduct for consistent extended logic programs. □

Lemma 2
Let P be an LPOD and let M be an answer set of P. Then, M is a model of P.

Proof
Consider any rule R in P of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$$

If $R^M_\times = \emptyset$, then $M(B_i) = T$ for some i, $1 \leq i \leq k$. But then, the body of the rule R evaluates to F under M, and therefore M satisfies R. Consider now the case where $R^M_\times$ is nonempty and consists of the following rules:

$$\begin{array}{lcl} C_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ C_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ C_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

We distinguish cases based on the value of $M(A_1, \ldots, A_m)$:

Case 1: $M(A_1, \ldots, A_m) = F$. Then, for some i, $M(A_i) = F$. Then, rule R is trivially satisfied by M.

Case 2: $M(A_1, \ldots, A_m) = F^*$. This implies that $M(C_r) \geq F^*$. We distinguish two subcases. If $r = n$ then $M(C_1 \times \cdots \times C_n) = M(C_1 \times \cdots \times C_r) \geq F^*$ because, by the definition of $P^M_\times$ it is $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and we also know that $M(C_r) \geq F^*$. Thus, in this subcase M satisfies R. If $r < n$, then by the definition of $P^M_\times$, $M(C_r) \neq F^*$; however, we know that $M(C_r) \geq F^*$, and thus $M(C_r) = T$. Thus, in this subcase M also satisfies R.

Case 3: $M(A_1, \ldots, A_m) = T$. Then, for all i, $M(A_i) = T$. Since M is a model of $P^M_\times$, we have $M(C_r) = T$. Moreover, by the definition of $P^M_\times$, $M(C_1) = \cdots = M(C_{r-1}) = F^*$. This implies that $M(C_1 \times \cdots \times C_n) = T$. □

Lemma 3
Let M be a model of an LPOD P. Then, M is a model of $P^M_\times$.

Proof
Consider any rule R in P of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

and assume M satisfies R. If $M(B_i) = T$ for some i, $1 \leq i \leq k$, then no rule is created in $P^M_\times$ for R. Assume therefore that $M(\mathit{not}\, B_1, \ldots, \mathit{not}\, B_k) = T$. By the definition of $P^M_\times$ the following rules have been added to $P^M_\times$:

$$\begin{array}{lcl} C_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ C_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ C_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

where r is the least index such that $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and either $r = n$ or $M(C_r) \neq F^*$. Obviously, the first $r-1$ rules above are satisfied by M. For the rule $C_r \leftarrow A_1, \ldots, A_m$ we distinguish two cases based on the value of $M(A_1, \ldots, A_m)$. If $M(A_1, \ldots, A_m) = F$, then, the rule is trivially satisfied. If $M(A_1, \ldots, A_m) > F$, then, since rule R is satisfied by M and $M(C_r) \neq F^*$, it has to be $M(C_r) = T$. Therefore, the rule $C_r \leftarrow A_1, \ldots, A_m$ is satisfied by M. $\square$

Lemma 4
Every (three-valued) answer set M of an LPOD P, is a $\preceq$-minimal model of P.

Proof
Assume there exists a model N of P with $N \preceq M$. We will show that N is also a model of $P^M_\times$. Since $N \preceq M$, we also have $N \leq M$. Since M is the $\leq$-least model of $P^M_\times$, we will conclude that $N = M$.

Consider any rule R in P of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

Assume that $R^M_\times$ is nonempty. This means that there exists some r, $1 \leq r \leq n$, such that $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and either $r = n$ or $M(C_r) \neq F^*$. Then, $R^M_\times$ consists of the following rules:

$$\begin{array}{lcl} C_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ C_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ C_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

We show that N satisfies the above rules. We distinguish cases based on the value of $M(A_1, \ldots, A_m)$:

Case 1: $M(A_1, \ldots, A_m) = F$. Then, $N(A_1, \ldots, A_m) = F$ and the above rules are trivially satisfied by N.

Case 2: $M(A_1, \ldots, A_m) = F^*$. Then, since $N \preceq M$, it is $N(A_1, \ldots, A_m) \leq F^*$. If $N(A_1, \ldots, A_m) = F$ then N trivially satisfies all the above rules. Assume therefore that $N(A_1, \ldots, A_m) = F^*$. Recall now that $M(C_i) = F^*$ for all i, $1 \leq i \leq r-1$. Moreover, it has to be $M(C_r) \geq F^*$, because otherwise M would not satisfy the rule R. Since $N \preceq M$, it can only be $N(C_i) = F^*$ for all i, $1 \leq i \leq r-1$ and $N(C_r) \geq F^*$, because otherwise N would not be a model of P. Therefore, N satisfies the given rules of $P^M_\times$.

Case 3: $M(A_1, \ldots, A_m) = T$. Then, since $N \preceq M$, it is either $N(A_1, \ldots, A_m) = F$ or $N(A_1, \ldots, A_m) = T$. If $N(A_1, \ldots, A_m) = F$ then N trivially satisfies all the above rules. Assume therefore that $N(A_1, \ldots, A_m) = T$. Recall now that $M(C_i) = F^*$ for all i, $1 \leq i \leq r-1$. Moreover, it has to be $M(C_r) = T$, because otherwise M would not satisfy the rule R. Since $N \preceq M$, it can only be $N(C_i) = F^*$ for all i, $1 \leq i \leq r-1$ and $N(C_r) = T$, because otherwise N would not be a model of P. Therefore, N satisfies the given rules of $P^M_\times$. □

In the proofs that follow, we will use the term *Brewka-model* to refer to that of Definition 2 and *Brewka-reduct* to refer to that of Definition 3 (although, to be precise, this definition of reduct was initially introduced in the paper by Brewka et al. (2004)).

In order to establish Lemmas 5 and 6 we first show the following three propositions.

Proposition A.1
Let P be an LPOD and let M be a three-valued model of P. Then, $N = \mathit{collapse}(M)$ is a Brewka-model of P.

Proof
Consider any rule R of P of the form

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

If there exists $A_i \notin N$ or there exists $B_j \in N$ then then N trivially satisfies R. Assume that $\{A_1, \ldots, A_m\} \subseteq N$ and $\{B_1, \ldots, B_k\} \cap N = \emptyset$. By Definition 15 it follows that $M(A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k) = T$. Since M is a three-valued model of P, it must satisfy R and therefore $M(C_1 \times \cdots \times C_n) = T$. Then, there exists $r \leq n$ such that $M(C_r) = T$ and by Definition 15 we get that $C_r \in N$. Therefore, N satisfies rule R. □

Proposition A.2
Let P be an LPOD and M be a Brewka-model of P. Then, M is also a model of the Brewka-reduct $P^M_\times$.

Proof
Consider any rule R in P of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

and assume M satisfies R. If there exists $B_i \in M$ for some $1 \leq i \leq k$, then no rule is created in the Brewka-reduct for R. Moreover, if for all $i \leq n$, $C_i \notin M$ then also no rule is created in the Brewka-reduct. Assume therefore that $\{B_1, \ldots, B_k\} \cap M = \emptyset$ and there exists $r \leq n$ such that $C_r \in M$ and $\{C_1, \ldots, C_{r-1}\} \cap M = \emptyset$. By the definition of $P^M_\times$ the only rule added to $P^M_\times$ because of R is $C_r \leftarrow A_1, \ldots, A_m$. Since $C_r \in M$ the rule is satisfied by M. □

Proposition A.3
Let P be an LPOD and let M_1, M_2 be three-valued answer sets of P such that $\mathit{collapse}(M_1) = \mathit{collapse}(M_2)$. Then, $M_1 = M_2$.

Proof
Assume, for the sake of contradiction, that $M_1 \neq M_2$. We define:

$$M(A) = \begin{cases} M_1(A) & \text{if } M_1(A) = M_2(A) \\ F & \text{otherwise} \end{cases}$$

It is $M \prec M_1$ and $M \prec M_2$. We claim that M is a model of P. This will lead to contradiction because, by Lemma 4, M_1 and M_2 are $\preceq$-minimal models of P.

Consider any rule R in P of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

If $M(B_i) = T$ for some i, $1 \leq i \leq k$, then M satisfies the rule. Assume therefore that $M(B_i) \neq T$ for all i, $1 \leq i \leq k$. We distinguish cases:

Case 1: $M(A_1, \ldots, A_m) = F$. Then, obviously, M satisfies R.

Case 2: $M(A_1, \ldots, A_m) = F^*$. Then, $M_1(A_1, \ldots, A_m) = F^*$ and $M_2(A_1, \ldots, A_m) = F^*$. Since, by Lemma 2, M_1 and M_2 are models of P it follows that $M_1(C_1 \times \cdots \times C_n) \geq F^*$ and $M_2(C_1 \times \cdots \times C_n) \geq F^*$. First assume that $M_1(C_1 \times \cdots \times C_n) = T$. This implies that there exists $1 \leq r \leq n$ such that $M_1(C_r) = T$ and $M_1(C_i) = F^*$ for all $1 \leq i < r$. Since, by assumption $\mathit{collapse}(M_1) = \mathit{collapse}(M_2)$ it follows that $M_2(C_r) = T$ and therefore $M(C_r) = T$. Moreover, it must be $M_2(C_i) = F^*$ for all $i < r$ because we have already established that $M_2(C_1 \times \cdots \times C_n) \geq F^*$. Therefore, $M(C_i) = F^*$ and $M(C_1 \times \cdots \times C_n) = T$ and M satisfies the rule. Now assume that $M_1(C_1 \times \cdots \times C_n) = F^*$. It is easy to see that the only case is $M_1(C_i) = F^*$ for all $1 \leq i \leq n$. Since M_2 has the same collapse with M_1 it follows that $M_2(C_i) \leq F^*$ and because $M_2(C_1 \times \cdots \times C_n) \geq F^*$ it also follows that $M_2(C_i) = F^*$. By definition of M, $M(C_i) = F^*$ for all $1 \leq i \leq n$ and $M(C_1 \times \cdots \times C_n) = F^*$.

Case 3: $M(A_1, \ldots, A_m) = T$. Then, $M_1(A_1, \ldots, A_m) = T$ and $M_2(A_1, \ldots, A_m) = T$ and therefore $M_1(C_1 \times \cdots \times C_n) = T$ and $M_2(C_1 \times \cdots \times C_n) = T$. This implies that there exists r such that $M_1(C_1) = M_2(C_1) = F^*, \ldots, M_1(C_{r-1}) = M_2(C_{r-1}) = F^*$, and $M_1(C_r) = M_2(C_r) = T$. Therefore, $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and $M(C_r) = T$, which implies that $M(C_1 \times \cdots \times C_n) = T$, and therefore M satisfies R. □

Lemma 5
Let P be an LPOD and M be a three-valued answer set of P. Then, $\mathit{collapse}(M)$ is an answer set of P according to Definition 4.

Proof
Since M is an answer set of P, then by Lemma 2, M is also a model of P. Moreover, by Proposition A.1, $N = \mathit{collapse}(M)$ is a Brewka-model of P. It also follows from Proposition A.2 that N is a model of the Brewka-reduct P^N. It suffices to show that N is also the minimum model of P^N. Assume there exists N' that is a model of P^N and $N' \subset N$. We define M' as

$$M'(A) = \begin{cases} F^* & A \in N \text{ and } A \notin N' \\ M(A) & \text{otherwise} \end{cases}$$

It is easy to see that $M' < M$. We will show that M' is also model of $P^M_\times$ leading to contradiction because we assume that M is the minimum model of $P^M_\times$. Consider first a rule of the form $C_i \leftarrow F^*, A_1, \ldots, A_m$. Since M is an answer set of P it must be $M(C_i) = F^*$. By the definition of M' it follows that $M'(C_i) \geq F^*$ and M' satisfies the rule. Now consider a rule of the form $C_r \leftarrow A_1, \ldots, A_m$. We distinguish cases based on the value of $M(A_1, \ldots, A_m)$.

Case 1: $M(A_1, \ldots, A_m) = F$. Then, since $M' < M$ it is $M'(A_1, \ldots, A_m) = F$ and the rule is trivially satisfied.

Case 2: $M(A_1, \ldots, A_m) = F^*$. Then, $M(A_i) \geq F^*$ and there exists A_i such that $M(A_i) = F^*$. It follows that $A_i \notin N$ and therefore $M'(A_i) = M(A_i) = F^*$. Moreover, by the construction of M', for all A_i we have $M'(A_i) \geq F^*$ and therefore $M'(A_1, \ldots, A_m) = F^*$. Since M is a model of $P_\times^M$, $M(C_r) \geq F^*$. Again, by the construction of M' we have $M'(C_r) \geq F^*$ and the rule is satisfied.

Case 3: $M(A_1, \ldots, A_m) = T$. By the construction of $P_\times^M$ the rule $C_r \leftarrow A_1, \ldots, A_m$ is a result of a rule in P of the form

$$C_1 \times \cdots \times C_r \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

and it must be $M(C_i) = F^*$ for all $i \leq r-1$ and $M(B_j) \leq F^*$ for all $1 \leq j \leq k$. It follows that $\{C_1, \ldots, C_{r-1}\} \cap N = \emptyset$ and $\{B_1, \ldots, B_k\} \cap N = \emptyset$. Moreover, since M is a model of $P_\times^M$ we get that $M(C_r) = T$ and it follows that $C_r \in N$. By the construction of the Brewka-reduct, there exists a rule $C_r \leftarrow A_1, \ldots, A_m$ in P^N. We distinguish two cases. If $\{A_1, \ldots, A_m\} \subseteq N'$ then $C_r \in N'$ because N' is a model of P^N. It follows by the construction of M' that $M'(C_r) = M(C_r) = T$ and M' satisfies the rule. Otherwise, there exists l, $1 \leq l \leq m$ such that $A_l \notin N'$. Notice also that $\{A_1, \ldots, A_m\} \subseteq N$, so $A_l \in N$. Therefore, $M'(A_l) = F^*$ and $M'(A_1, \ldots, A_m) \leq F^*$. Moreover, since $C_r \in N$, we have $M'(C_r) \geq F^*$ that satisfies the rule. $\square$

Lemma 6
Let N be an answer set of P according to Definition 4. There exists a unique three-valued interpretation M such that $N = \mathit{collapse}(M)$ and M is a three-valued answer set of P.

Proof
We construct iteratively a set of literals that must have the value F^* in M. Let $\mathcal{F}^n$ be the sequence:

$$\begin{aligned}
\mathcal{F}^0 &= \emptyset \\
\mathcal{F}^{n+1} &= \{C_j \mid (C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots \mathit{not}\, B_k) \in P \\
&\qquad \text{and } \{B_1, \ldots, B_k\} \cap N = \emptyset \\
&\qquad \text{and } \{C_1, \ldots, C_j\} \cap N = \emptyset \\
&\qquad \text{and } \{A_1, \ldots, A_m\} \subseteq N \cup \mathcal{F}^n\} \\
\mathcal{F}^\omega &= \cup_{n<\omega} \mathcal{F}^n
\end{aligned}$$

We construct M as

$$M(A) = \begin{cases} F & A \notin N \text{ and } A \notin \mathcal{F}^\omega \\ F^* & A \notin N \text{ and } A \in \mathcal{F}^\omega \\ T & A \in N \end{cases}$$

First we prove that M is a model of $P_\times^M$. Consider first any rule of the form $C_i \leftarrow F^*, A_1, \ldots, A_m$. By the construction of $P_\times^M$, such a rule exists because $M(C_i) = F^*$; therefore M satisfies this rule. Now consider any rule of the form $C_r \leftarrow A_1, \ldots, A_m$. Such a rule was produced by a rule R in P of the form

$$C_1 \times \cdots \times C_r \times \cdots \times C_n \leftarrow A_1, \ldots, A_n, \ldots, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k.$$

By the construction of $P^M_\times$ it follows that $M(C_i) = F^*$ for all $i < r$. Therefore $C_i \notin N$ and also $C_i \in \mathcal{F}^\omega$ for all $i < r$. Moreover, it must be $M(B_j) \leq F^*$ for all $1 \leq j \leq k$, so $\{B_1, \ldots, B_k\} \cap N = \emptyset$. We distinguish cases based on the value of $M(A_1, \ldots, A_m)$.

Case 1: If $M(A_1, \ldots, A_m) = F$ then the rule is trivially satisfied by M.

Case 2: If $M(A_1, \ldots, A_m) = F^*$ then for some A_i, $M(A_i) = F^*$. By the construction of M, it follows that $A_i \in \mathcal{F}^\omega$. It follows by the definition of $\mathcal{F}^\omega$ that $C_r \in \mathcal{F}^\omega$ and therefore $M(C_r) \geq F^*$.

Case 3: If $M(A_1, \ldots, A_m) = T$ then $\{A_1, \ldots, A_m\} \subseteq N$ and since N is an answer set according to Definition 4 it follows that N is a model of P. It follows that there exists a least $j \leq n$ such that $C_j \in N$. Since we have already established that for all $i < r$, $C_i \notin N$ it must be $r \leq j \leq n$. But, if $r < j$ then $C_r \notin N$ and by the construction of M it must be $M(C_r) = F^*$. If $M(C_r) = F^*$, then, by the construction of $P^M_\times$, the rule for C_r should be of the form $C_r \leftarrow F^*, A_1, \ldots, A_m$. So, it must $j = r$ and $C_r \in N$. Therefore, $M(C_r) = T$ and M satisfies the rule.

Therefore, we have established that M is a model of $P^M_\times$. It remains to show that M is the $\leq$-least model of $P^M_\times$. Assume now that there exists M' that is a model of $P^M_\times$ and $M' < M$. Let $N' = \mathit{collapse}(M')$. We distinguish two cases.

Case 1: $N' = N$ and thus M' differs from M only on some atoms C_r such that $M'(C_r) = F$ and $M(C_r) = F^*$. First, by the construction of M, if $M(C_r) = F^*$ then $C_r \in \mathcal{F}^\omega$. We show by induction on n that for every $C_r \in \mathcal{F}^n$, $M'(C_r) \geq F^*$. This leads to contradiction and therefore M is minimal.

Induction base: $n = 0$: the statement is satisfied vacuously.

Induction step: $n = n_0 + 1$: Every atom $C_r \in \mathcal{F}^{n_0+1}$ must occur in a head of a rule in P. such that $\{C_1, \ldots, C_{r-1}\} \cap N = \emptyset$ and therefore $\{C_1, \ldots, C_r\} \subseteq \mathcal{F}^{n_0+1}$. It follows then that $M(C_i) = F^*$ for $1 \leq i \leq r$. By the construction of $P^M_\times$, for every atom $C_r \in \mathcal{F}^{n_0+1}$ there must be a rule in $P^M_\times$ either of the form $C_r \leftarrow F^*, A_1, \ldots, A_m$ or of the form $C_r \leftarrow A_1, \ldots, A_m$. Moreover, since $C_r \in \mathcal{F}^{n_0+1}$ it follows that $\{A_1, \ldots, A_m\} \subseteq N \cup \mathcal{F}^{n_0}$. Therefore, by the induction hypothesis, $M(A_1, \ldots, A_m) = M'(A_1, \ldots, A_m) \geq F^*$. Since M' is also a model of $P^M_\times$ it must satisfy those rules thus $M'(C_r) \geq F^*$.

Case 2: $N' \subset N$. We show that N' is a model of P^N leading to contradiction because, by definition, N is the minimum model of P^N. Consider a rule R of the form $C_r \leftarrow A_1, \ldots, A_m$ in P^N. The rule R has been produced by a rule in P of the form:

$$C_1 \times \cdots \times C_r \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

such that $\{C_1, \ldots, C_{r-1}\} \cap N = \emptyset$ and $C_r \in N$.

If there exists $A_i \notin N$ then also $A_i \notin N'$ and the rule is trivially satisfied by N'. Assume, on the other hand, that $\{A_1, \ldots, A_n\} \subseteq N$. It follows, by the definition of M, that $M(A_1, \ldots, A_m) = T$, $M(C_i) = F^*$ for $i < r$ and $M(C_r) = T$. Therefore, there exist a rule in $P^M_\times$ of the form $C_r \leftarrow A_1, \ldots, A_m$. If $M'(A_1, \ldots, A_m) = F$ or $M'(A_1, \ldots, A_m) = F^*$ then there exists $A_i \notin N'$ and N' again satisfies the rule. If $M'(A_1, \ldots, A_m) = T$ then since M' is a model of $P^M_\times$ it follows that $M'(C_r) = T$. Since N' is the collapse of M' it is $\{A_1, \ldots, A_m\} \subseteq N'$ and $C_r \in N'$. Therefore, N' satisfies the rule R in P^N.

The uniqueness of M follows directly from Proposition A.3. $\square$

B Proofs of Section 6

In order to establish Theorem 1, we show two lemmas (which essentially establish the left-to-right and the right-to-left directions of the theorem, respectively).

Lemma B.1
Let P be an LPOD program and let M be an answer set of P. Then, M is a $\preceq$-minimal model of P and M is solid.

Proof
Since M is an answer set of P, then, by Lemma 2, M is a model of P. Moreover, M is solid because our definition of answer sets does not involve the value T^*. It remains to show that it is minimal with respect to the $\preceq$ ordering. Assume, for the sake of contradiction, that there exists a model N of P with $N \prec M$. By Lemma 4, M is (three-valued) $\preceq$-minimal. Therefore, N can not be solid. We first show that N can not be a model of the reduct $P^M_\times$. Assume for the sake of contradiction that N is a model of $P^M_\times$. We construct the following interpretation N':

$$N'(A) = \begin{cases} F^*, & \text{if } N(A) = T^* \\ N(A), & \text{otherwise} \end{cases}$$

We claim that N' must also be a model of $P^M_\times$. Consider first a rule of the form $C \leftarrow F^*, A_1, \ldots, A_m$. Since N is a model of $P^M_\times$, it is $N(C) \geq F^*$. By the definition of N', it is $N(C) \geq F^*$ and therefore N' satisfies this rule. Consider now a rule of the form $C \leftarrow A_1, \ldots, A_m$ in $P^M_\times$. We show that N' also satisfies this rule. We perform a case analysis:

Case 1: $N(A_1, \ldots, A_m) = F$. Then, $N'(A_1, \ldots, A_m) = F$ and N' trivially satisfies the rule.

Case 2: $N(A_1, \ldots, A_m) = F^*$. Then, $N'(A_1, \ldots, A_m) = F^*$. Moreover, $N(C) \geq F^*$ because N is a model of $P^M_\times$. By the definition of N', it is $N'(C) \geq F^*$, and therefore N' satisfies the rule.

Case 3: $N(A_1, \ldots, A_m) = T^*$. Then, $N'(A_1, \ldots, A_m) = F^*$. Moreover, $N(C) \geq T^*$ because N is a model of $P^M_\times$. By the definition of N', it is $N'(C) \geq F^*$, and therefore N' satisfies the rule.

Case 4: $N(A_1, \ldots, A_m) = T$. Then, $N'(A_1, \ldots, A_m) = T$. Moreover, $N(C) = T$ because N is a model of $P^M_\times$. By the definition of N', it is $N'(C) = T$, and therefore N' satisfies the rule.

Therefore, N' must also be a model of $P^M_\times$. Moreover, by definition, N' is solid and $N' < M$. This contradicts the fact that, by construction, M is the $\leq$-least model of $P^M_\times$. In conclusion, N can not be a model of $P^M_\times$.

We now show that N can not be a model of P. As we showed above, N is not a model of $P^M_\times$, and consequently there exists a rule in $P^M_\times$ that is not satisfied by N. Such a rule in $P^M_\times$ must have resulted due to a rule R of the following form in P:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$$

According to the definition of $P^M_\times$, for all i, $1 \leq i \leq k$, $M(not\, B_i) = T$, and since $N \prec M$, it is also $N(not\, B_i) = T$. Moreover, there exists some $r \leq n$ such that $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and either $r = n$ or $M(C_r) \neq F^*$. Since $N \prec M$, it is $N(C_i) \leq F^*$ for all i, $1 \leq i \leq r-1$. Consider now the rule that is not satisfied by N in $P^M_\times$. If it is of the form $C_i \leftarrow F^*, A_1, \ldots, A_m$, i, $1 \leq i \leq r-1$, then $N(A_1, \ldots, A_m) > F$ and $N(C_i) = F$. This implies that $N(C_1 \times \cdots \times C_n) = F$ and therefore N does not satisfy the rule R. If the rule that is not satisfied by N in $P^M_\times$ is of the form $C_r \leftarrow A_1, \ldots, A_m$, then $N(C_r) < N(A_1, \ldots, A_m)$ and therefore, since $N(C_i) \leq F^*$ for all i, $1 \leq i \leq r-1$, it is:

$$N(C_1 \times \cdots \times C_n) < N(A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k)$$

Thus, N is not a model of P. □

Lemma B.2
Let P be an LPOD program and let M be a $\preceq$-minimal model of P and M is solid. Then, M is an answer set of P.

Proof
First observe that, by Lemma 3, M is also a model of $P^M_\times$. We demonstrate that M is actually the $\leq$-least model of $P^M_\times$. Assume, for the sake of contradiction, that N is the $\leq$-least model of $P^M_\times$. Then, N will differ from M in some atoms A such that $N(A) < M(A)$. We distinguish two cases. In the first case all the atoms A such that $N(A) < M(A)$ have $M(A) \leq F^*$. In the second case there exist at least one atom A such that $M(A) > F^*$.

In the first case it is easy to see that $N \prec M$. We demonstrate that N is also model of P leading to contradiction since M is $\preceq$-minimal. Assume that N is not a model of P. Then, there exists in P a rule R of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

such that $N(C_1 \times \cdots \times C_n) < N'(A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k)$. Notice that this implies that $N(\mathit{not}\, B_1, \ldots, \mathit{not}\, B_k) = M(\mathit{not}\, B_1, \ldots, \mathit{not}\, B_k) = T$. Therefore, $N(C_1 \times \cdots \times C_n) < N(A_1, \ldots, A_m)$. We distinguish cases based on the value of $N(A_1, \ldots, A_m)$:

Case 1: $N(A_1, \ldots, A_m) = F$. This case leads immediately to contradiction because N trivially satisfies R.

Case 2: $N(A_1, \ldots, A_m) > F$. Then, $N(A_1, \ldots, A_m) = M(A_1, \ldots, A_m)$. Since M is a model of P, it is $M(C_1 \times \cdots \times C_n) \geq M(A_1, \ldots, A_m) > F$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(C_1) = \cdots = M(C_{r-1}) = F^*$ and $M(C_r) \geq F^*$. By the definition of the reduct, the rule $C_r \leftarrow A_1, \ldots, A_m$ exists in $P^M_\times$. Since N is a model of $P^M_\times$, we get that $N(C_r) > F$. Moreover, N should also satisfy the rules $C_i \leftarrow F^*, A_1, \ldots, A_m$ for $1 \leq i \leq r-1$. Since $N(C_i) \leq M(C_i)$ and $N(C_r) = M(C_r)$ we get $N(C_1) = \cdots = N(C_{r-1}) = F^*$ and $N(C_r) = M(C_r)$. Therefore $N(C_1 \times \cdots C_n) = M(C_1 \times \cdots C_n)$ and $N(C_1 \times \cdots C_n) \geq N(A_1, \ldots, A_m)$ (contradiction).

In the second case we construct the following interpretation N':

$$N'(A) = \begin{cases} T^*, & \text{if } M(A) = T \text{ and } N(A) \in \{F, F^*\} \\ F^*, & \text{if } M(A) = F^* \\ N(A), & \text{otherwise} \end{cases}$$

It is easy to see that $N' \prec M$. We demonstrate that N' is a model of P, which will lead to a contradiction (since we have assumed that M is $\preceq$-minimal).

Assume N' is not a model of P. Then, there exists in P a rule R of the form:

$$C_1 \times \cdots \times C_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

such that $N'(C_1 \times \cdots \times C_n) < N'(A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k)$. Notice that this implies that $N'(not\, B_1, \dots, not\, B_k) = N(not\, B_1, \dots, not\, B_k) = M(not\, B_1, \dots, not\, B_k) = T$. Therefore, $N'(C_1 \times \dots \times C_n) < N'(A_1, \dots, A_m)$. We distinguish cases based on the value of $N'(A_1, \dots, A_m)$: *Case 1:* $N'(A_1, \dots, A_m) = F$. This case leads immediately to contradiction because N' trivially satisfies R.

Case 2: $N'(A_1, \dots, A_m) = F^*$. Then, by the definition of N', $M(A_1, \dots, A_m) = F^*$. Since M is a model of P, it is $M(C_1 \times \dots \times C_n) \geq F^*$. This implies that either $M(C_1) = \dots = M(C_n) = F^*$ or there exists $r \leq n$ such that $M(C_1) = \dots = M(C_{r-1}) = F^*$ and $M(C_r) = T$. By the definition of N', we get in both cases $N'(C_1 \times \dots \times C_n) \geq F^*$ (contradiction).

Case 3: $N'(A_1, \dots, A_m) = T^*$. Then, by the definition of N', $M(A_1, \dots, A_m) = T$. Since M is a model of P, it is $M(C_1 \times \dots \times C_n) = T$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(C_1) = \dots = M(C_{r-1}) = F^*$ and $M(C_r) = T$. By the definition of N', we get that $N'(C_1 \times \dots \times C_n) \geq T^*$ (contradiction).

Case 4: $N'(A_1, \dots, A_m) = T$. Then, by the definition of N', $N(A_1, \dots, A_m) = T$ and $M(A_1, \dots, A_m) = T$. Since M is a model of P, it is $M(C_1 \times \dots \times C_n) = T$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(C_1) = \dots = M(C_{r-1}) = F^*$ and $M(C_r) = T$. By the definition of the reduct, the rule $C_r \leftarrow A_1, \dots, A_m$ exists in $P^M_\times$. Since N is a model of $P^M_\times$, we get that $N(C_r) = T$. Thus, $N'(C_1) = \dots = N'(C_{r-1}) = F^*$ and $N'(C_r) = T$, and therefore $N'(C_1 \times \dots \times C_n) = T$ (contradiction). □

Theorem 1
Let P be an LPOD. Then, M is a three-valued answer set of P iff M is a consistent $\preceq$-minimal model of P and M is solid.

Proof
Immediate from Lemma B.1 and Lemma B.2. □

C Proofs of Section 7

Lemma 7
Let P be a consistent disjunctive extended logic program. Then, the answer sets of P according to Definition 20, coincide with the standard answer sets of P.

Proof
By taking $n = 1$ in Definition 19, we get the standard definition of reduct for consistent disjunctive extended logic programs. □

Lemma 8
Let P be a DLPOD program and let M be an answer set of P. Then, M is a model of P.

Proof
Consider any rule R in P of the form:

$$\mathcal{C}_1 \times \dots \times \mathcal{C}_n \leftarrow A_1, \dots, A_m, not\, B_1, \dots, not\, B_k$$

If $R^M_\times = \emptyset$, then $M(B_i) = T$ for some i, $1 \leq i \leq k$. But then, the body of the rule R

evaluates to F under M, and therefore M satisfies R. Consider now the case where $R_{\times}^M$ is nonempty and consists of the following rules:

$$\begin{array}{lll} \mathcal{C}_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ \mathcal{C}_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ \mathcal{C}_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

We distinguish cases based on the value of $M(A_1, \ldots, A_m)$:

Case 1: $M(A_1, \ldots, A_m) = F$. Then, for some i, $M(A_i) = F$. Then, rule R is trivially satisfied by M.

Case 2: $M(A_1, \ldots, A_m) = F^*$. This implies that $M(\mathcal{C}_r) \geq F^*$. We distinguish two subcases. If $r = n$ then $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_r) \geq F^*$ because, by the definition of $P_{\times}^M$ it is $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and we also know that $M(\mathcal{C}_r) \geq F^*$. Thus, in this subcase M satisfies R. If $r < n$, then by the definition of $P_{\times}^M$, $M(\mathcal{C}_r) \neq F^*$; however, we know that $M(\mathcal{C}_r) \geq F^*$, and thus $M(\mathcal{C}_r) = T$. Thus, in this subcase M also satisfies R.

Case 3: $M(A_1, \ldots, A_m) = T$. Then, for all i, $M(A_i) = T$. Since M is a model of $P_{\times}^M$, we have $M(\mathcal{C}_r) = T$. Moreover, by the definition of $P_{\times}^M$, $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$. This implies that $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = T$. □

Lemma 9
Let M be a model of a DLPOD P. Then, M is a model of $P_{\times}^M$.

Proof
Consider any rule R in P of the form:

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

and assume M satisfies R. If $M(B_i) = T$ for some i, $1 \leq i \leq k$, then no rule is created in $P_{\times}^M$ for R. Assume therefore that $M(\mathit{not}\, B_1, \ldots, \mathit{not}\, B_k) = T$. By the definition of $P_{\times}^M$ the following rules have been added to $P_{\times}^M$:

$$\begin{array}{lll} \mathcal{C}_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ \mathcal{C}_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ \mathcal{C}_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

where r is the least index such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and either $r = n$ or $M(\mathcal{C}_r) \neq F^*$. Obviously, the first $r-1$ rules above are satisfied by M. For the rule $\mathcal{C}_r \leftarrow A_1, \ldots, A_m$ we distinguish two cases based on the value of $M(A_1, \ldots, A_m)$. If $M(A_1, \ldots, A_m) = F$, then, the rule is trivially satisfied. If $M(A_1, \ldots, A_m) > F$, then, since rule R is satisfied by M and $M(\mathcal{C}_r) \neq F^*$, it has to be $M(\mathcal{C}_r) = T$. Therefore, the rule $\mathcal{C}_r \leftarrow A_1, \ldots, A_m$ is satisfied by M. □

Lemma 10
Every answer set M of a DLPOD P, is a $\preceq$-minimal model of P.

Proof
Assume there exists a model N of P with $N \preceq M$. We will show that N is also a model of $P^M_\times$. Since $N \preceq M$, we also have $N \leq M$. Since M is the $\leq$-least model of $P^M_\times$, we will conclude that $N = M$.

Consider any rule R in P of the form:

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$$

Assume that $R^M_\times$ is nonempty. This means that there exists some r, $1 \leq r \leq n$, such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and either $r = n$ or $M(\mathcal{C}_r) \neq F^*$. Then, $R^M_\times$ consists of the following rules:

$$\begin{array}{lcl} \mathcal{C}_1 & \leftarrow & F^*, A_1, \ldots, A_m \\ & \cdots & \\ \mathcal{C}_{r-1} & \leftarrow & F^*, A_1, \ldots, A_m \\ \mathcal{C}_r & \leftarrow & A_1, \ldots, A_m \end{array}$$

We show that N satisfies the above rules. We distinguish cases based on the value of $M(A_1, \ldots, A_m)$:

Case 1: $M(A_1, \ldots, A_m) = F$. Then, $N(A_1, \ldots, A_m) = F$ and the above rules are trivially satisfied by N.

Case 2: $M(A_1, \ldots, A_m) = F^*$. Then, since $N \preceq M$, it is $N(A_1, \ldots, A_m) \leq F^*$. If $N(A_1, \ldots, A_m) = F$ then N trivially satisfies all the above rules. Assume therefore that $N(A_1, \ldots, A_m) = F^*$. Recall now that $M(\mathcal{C}_i) = F^*$ for all i, $1 \leq i \leq r-1$. Moreover, it has to be $M(\mathcal{C}_r) \geq F^*$, because otherwise M would not satisfy the rule R. Since $N \preceq M$, it can only be $N(\mathcal{C}_i) = F^*$ for all i, $1 \leq i \leq r-1$ and $N(\mathcal{C}_r) \geq F^*$, because otherwise N would not be a model of P. Therefore, N satisfies the given rules of $P^M_\times$.

Case 3: $M(A_1, \ldots, A_m) = T$. Then, since $N \preceq M$, it is either $N(A_1, \ldots, A_m) = F$ or $N(A_1, \ldots, A_m) = T$. If $N(A_1, \ldots, A_m) = F$ then N trivially satisfies all the above rules. Assume therefore that $N(A_1, \ldots, A_m) = T$. Recall now that $M(\mathcal{C}_i) = F^*$ for all i, $1 \leq i \leq r-1$. Moreover, it has to be $M(\mathcal{C}_r) = T$, because otherwise M would not satisfy the rule R. Since $N \preceq M$, it can only be $N(\mathcal{C}_i) = F^*$ for all i, $1 \leq i \leq r-1$ and $N(\mathcal{C}_r) = T$, because otherwise N would not be a model of P. Therefore, N satisfies the given rules of $P^M_\times$. $\square$

Theorem 2
Let P be a DLPOD. Then, M is an answer set of P iff M is a consistent $\preceq$-minimal model of P and M is solid.

The proof of the above theorem follows directly by the following two lemmas.

Lemma C.1
Let P be an DLPOD and let M be an answer set of P. Then, M is a consistent $\preceq$-minimal model of P and M is solid.

Proof
Since M is an answer set of P, then, by Lemma 8, M is a model of P. Moreover, M is solid because our definition of answer sets does not involve the value T^*. It remains to show that it is minimal with respect to the $\preceq$ ordering. Assume, for the sake of contradiction, that there exists a model N of P with $N \prec M$. By Lemma 10, M is (three-valued) $\preceq$-minimal. Therefore, N can not be solid. We first show that N can not be a model

of the reduct $P_{\times}^{M}$. Assume for the sake of contradiction that N is a model of $P_{\times}^{M}$. We construct the following interpretation N':

$$N'(A) = \begin{cases} F^*, & \text{if } N(A) = T^* \\ N(A), & \text{otherwise} \end{cases}$$

We claim that N' must also be a model of $P_{\times}^{M}$. Consider first a rule of the form $\mathcal{C} \leftarrow F^*, A_1, \ldots, A_m$. Since N is a model of $P_{\times}^{M}$, it is $N(\mathcal{C}) \geq F^*$. By the definition of N', it is $N(\mathcal{C}) \geq F^*$ and therefore N' satisfies this rule. Consider now a rule of the form $\mathcal{C} \leftarrow A_1, \ldots, A_m$ in $P_{\times}^{M}$. We show that N' also satisfies this rule. We perform a case analysis:

Case 1: $N(A_1, \ldots, A_m) = F$. Then, $N'(A_1, \ldots, A_m) = F$ and N' trivially satisfies the rule.

Case 2: $N(A_1, \ldots, A_m) = F^*$. Then, $N'(A_1, \ldots, A_m) = F^*$. Moreover, $N(\mathcal{C}) \geq F^*$ because N is a model of $P_{\times}^{M}$. By the definition of N', it is $N'(\mathcal{C}) \geq F^*$, and therefore N' satisfies the rule.

Case 3: $N(A_1, \ldots, A_m) = T^*$. Then, $N'(A_1, \ldots, A_m) = F^*$. Moreover, $N(\mathcal{C}) \geq T^*$ because N is a model of $P_{\times}^{M}$. By the definition of N', it is $N'(\mathcal{C}) \geq F^*$, and therefore N' satisfies the rule.

Case 4: $N(A_1, \ldots, A_m) = T$. Then, $N'(A_1, \ldots, A_m) = T$. Moreover, $N(\mathcal{C}) = T$ because N is a model of $P_{\times}^{M}$. By the definition of N', it is $N'(\mathcal{C}) = T$, and therefore N' satisfies the rule.

Therefore, N' must also be a model of $P_{\times}^{M}$. Moreover, by definition, N' is solid and $N' < M$. This contradicts the fact that, by construction, M is the $\leq$-least model of $P_{\times}^{M}$. In conclusion, N can not be a model of $P_{\times}^{M}$.

We now show that N can not be a model of P. As we showed above, N is not a model of $P_{\times}^{M}$, and consequently there exists a rule in $P_{\times}^{M}$ that is not satisfied by N. Such a rule in $P_{\times}^{M}$ must have resulted due to a rule R of the following form in P:

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \leftarrow A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k$$

According to the definition of $P_{\times}^{M}$, for all i, $1 \leq i \leq k$, $M(\mathit{not}\, B_i) = T$, and since $N \prec M$, it is also $N(\mathit{not}\, B_i) = T$. Moreover, there exists some $r \leq n$ such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and either $r = n$ or $M(\mathcal{C}_r) \neq F^*$. Since $N \prec M$, it is $N(\mathcal{C}_i) \leq F^*$ for all i, $1 \leq i \leq r-1$. Consider now the rule that is not satisfied by N in $P_{\times}^{M}$. If it is of the form $\mathcal{C}_i \leftarrow F^*, A_1, \ldots, A_m$, i, $1 \leq i \leq r-1$, then $N(A_1, \ldots, A_m) > F$ and $N(\mathcal{C}_i) = F$. This implies that $N(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = F$ and therefore N does not satisfy the rule R. If the rule that is not satisfied by N in $P_{\times}^{M}$ is of the form $\mathcal{C}_r \leftarrow A_1, \ldots, A_m$, then $N(\mathcal{C}_r) < N(A_1, \ldots, A_m)$ and therefore, since $N(\mathcal{C}_i) \leq F^*$ for all i, $1 \leq i \leq r-1$, it is:

$$N(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) < N(A_1, \ldots, A_m, \mathit{not}\, B_1, \ldots, \mathit{not}\, B_k)$$

Thus, N is not a model of P. $\square$

Lemma C.2

Let P be an DLPOD and let M be a consistent $\preceq$-minimal model of P and M is solid. Then, M is an answer set of P.

Proof
First observe that, by Lemma 9, M is also a model of $P_{\times}^{M}$. We demonstrate that M is actually the $\leq$-least model of $P_{\times}^{M}$. Assume, for the sake of contradiction, that N is the $\leq$-least model of $P_{\times}^{M}$. Then, N will differ from M in some atoms A such that $N(A) < M(A)$. We distinguish two cases. In the first case all the atoms A such that $N(A) < M(A)$ have $M(A) \leq F^*$. In the second case there exist at least one atom A such that $M(A) > F^*$.

In the first case it is easy to see that $N \prec M$. We demonstrate that N is also model of P leading to contradiction since M is $\preceq$-minimal. Assume that N is not a model of P. Then, there exists in P a rule R of the form:

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$$

such that $N(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) < N'(A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k)$. Notice that this implies that $N(not\, B_1, \ldots, not\, B_k) = M(not\, B_1, \ldots, not\, B_k) = T$. Therefore, $N(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) < N(A_1, \ldots, A_m)$. We distinguish cases based on the value of $N(A_1, \ldots, A_m)$:

Case 1: $N(A_1, \ldots, A_m) = F$. This case leads immediately to contradiction because N trivially satisfies R.

Case 2: $N(A_1, \ldots, A_m) > F$. Then, $N(A_1, \ldots, A_m) = M(A_1, \ldots, A_m)$. Since M is a model of P, it is $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) \geq M(A_1, \ldots, A_m) > F$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and $M(\mathcal{C}_r) \geq F^*$. By the definition of the reduct, the rule $\mathcal{C}_r \leftarrow A_1, \ldots, A_m$ exists in $P_{\times}^{M}$. Since N is a model of $P_{\times}^{M}$, we get that $N(\mathcal{C}_r) > F$. Moreover, N should also satisfy the rules $\mathcal{C}_i \leftarrow F^*, A_1, \ldots, A_m$ for $1 \leq i \leq r-1$. Since $N(\mathcal{C}_i) \leq M(\mathcal{C}_i)$ and $N(\mathcal{C}_r) = M(\mathcal{C}_r)$ we get $N(\mathcal{C}_1) = \cdots = N(\mathcal{C}_{r-1}) = F^*$ and $N(\mathcal{C}_r) = M(\mathcal{C}_r)$. Therefore $N(\mathcal{C}_1 \times \cdots \mathcal{C}_n) = M(\mathcal{C}_1 \times \cdots \mathcal{C}_n)$ and $N(\mathcal{C}_1 \times \cdots \mathcal{C}_n) \geq N(A_1, \ldots, A_m)$ (contradiction).

In the second case we construct the following interpretation N':

$$N'(A) = \begin{cases} T^*, & \text{if } M(A) = T \text{ and } N(A) \in \{F, F^*\} \\ F^*, & \text{if } M(A) = F^* \\ N(A), & \text{otherwise} \end{cases}$$

It is easy to see that $N' \prec M$. We demonstrate that N' is a model of P, which will lead to a contradiction (since we have assumed that M is $\preceq$-minimal).

Assume N' is not a model of P. Then, there exists in P a rule R of the form:

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$$

such that $N'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) < N'(A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k)$. Notice that this implies that $N'(not\, B_1, \ldots, not\, B_k) = N(not\, B_1, \ldots, not\, B_k) = M(not\, B_1, \ldots, not\, B_k) = T$. Therefore, $N'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) < N'(A_1, \ldots, A_m)$. We distinguish cases based on the value of $N'(A_1, \ldots, A_m)$:

Case 1: $N'(A_1, \ldots, A_m) = F$. This case leads immediately to contradiction because N' trivially satisfies R.

Case 2: $N'(A_1, \ldots, A_m) = F^*$. Then, by the definition of N', $M(A_1, \ldots, A_m) = F^*$. Since M is a model of P, it is $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) \geq F^*$. This implies that either $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_n) = F^*$ or there exists $r \leq n$ such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and $M(\mathcal{C}_r) = T$. By the definition of N', we get in both cases $N'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) \geq F^*$ (contradiction).

Case 3: $N'(A_1, \ldots, A_m) = T^*$. Then, by the definition of N', $M(A_1, \ldots, A_m) = T$. Since

M is a model of P, it is $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = T$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and $M(\mathcal{C}_r) = T$. By the definition of N', we get that $N'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) \geq T^*$ (contradiction).

Case 4: $N'(A_1, \dots, A_m) = T$. Then, by the definition of N', $N(A_1, \dots, A_m) = T$ and $M(A_1, \dots, A_m) = T$. Since M is a model of P, it is $M(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = T$. This implies that there exists some r, $1 \leq r \leq n$, such that $M(\mathcal{C}_1) = \cdots = M(\mathcal{C}_{r-1}) = F^*$ and $M(\mathcal{C}_r) = T$. By the definition of the reduct, the rule $\mathcal{C}_r \leftarrow A_1, \dots, A_m$ exists in $P_{\times}^M$. Since N is a model of $P_{\times}^M$, we get that $N(\mathcal{C}_r) = T$. Thus, $N'(\mathcal{C}_1) = \cdots = N'(\mathcal{C}_{r-1}) = F^*$ and $N'(\mathcal{C}_r) = T$, and therefore $N'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n) = T$ (contradiction). □

References

Brewka, G., Niemelä, I., and Syrjänen, T. 2004. Logic programs with ordered disjunction. *Comput. Intell. 20,* 2, 335–357.