THE THEORY OF A GENERAL QUANTUM SYSTEM INTERACTING WITH A LINEAR DISSIPATIVE SYSTEM

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1959

ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to Professor R. P. Feynman for suggesting the problem and for his helpful advice in every phase of the work. He is also grateful to Dr. R. W. Hellwarth for the many stimulating and informative discussions. Thanks are also due to Hughes Aircraft Company for the helpful assistance of the Hughes Staff Doctoral Fellowship which the author held while conducting this research.

ABSTRACT

A formalism has been developed, using Feynman's space-time formulation of non-relativistic quantum mechanics whereby the behavior of a system of interest, which is coupled to other external quantum systems, may be calculated in terms of its own variables only. It is shown that the effect of the external systems in such a formalism can always be included in a general class of functionals (influence functionals) of the coordinates of the system only. The properties of influence functionals for general systems are examined. Then, specific forms of influence functionals representing the effect of definite and random classical forces, linear dissipative systems at finite temperatures, and combinations of these are analyzed in detail. The linear system analysis is first done for perfectly linear systems composed of combinations of harmonic oscillators, loss being introduced by continuous distributions of oscillators. Then approximately linear systems and restrictions necessary for the linear behavior are considered. Influence functionals for all linear systems are shown to have the same form in terms of their classical response functions. In addition, a fluctuation-dissipation theorem is derived relating temperature and dissipation of the linear system to a fluctuating classical potential acting on the system of interest which reduces to the Nyquist-Johnson relation for noise in the case of electric circuits. Sample calculations of transition probabilities for the spontaneous emission of an atom in free space and in a cavity are made. Finally, a theorem is proved showing that within the requirements of linearity all sources of noise or quantum fluctuation introduced by maser type amplification devices are accounted for by a classical calculation of the characteristics of the maser.

CONTENTS

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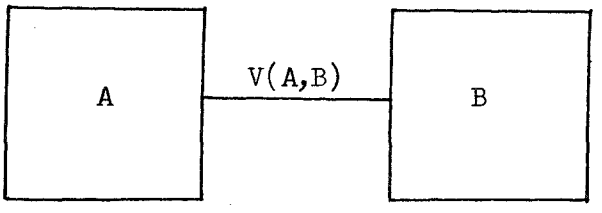
I. Introduction

Many problems arise in quantum mechanics in which several systems are coupled together but one or more of these systems are not of primary interest. These "uninteresting" systems may be regarded as the environment with which the others interact. As an example, let us consider a problem of irreversible quantum statistical mechanics: the relaxation of a system of inverted nuclear spins not coupled to each other but coupled to the crystal lattice in which they are imbedded. The crystal lattice, which acts like a thermal bath, is not of primary interest but its influence on the nuclear spins is the essential cause of the nuclear relaxation. Another example is that of the behavior of an atom in an excited state which interacts with the electromagnetic field in a lossy cavity resonator. Because of the coupling there will be energy exchange between the field and the atom until equilibrium is reached. If, however, the atom were not coupled to any external disturbances, it would simply remain unperturbed in its original excited state. The cavity field, although not of central interest to us, influences the behavior of the atom.

To make the discussion more definite, let us suppose there are two systems, A and B, as shown below, coupled together through some interaction potential which is a function of the parameters of the two systems.



It is desired to compute the expectation value of an observable which is a function of the A variables only. As is well known, the complete problem can be analyzed by taking the Hamiltonian of the complete system, forming the wave equation as follows

$$[H_A + H_B + V(A,B)]\ \psi_{AB} = -\frac{\hbar}{i}\frac{\partial}{\partial t}\psi_{AB} \quad ,$$

and then finding its solution. In general, this is an extremely difficult problem. In addition, when this approach is used, it is not easy to see how to eliminate the coordinates of B and include its effect in an equivalent way when making computations on A . A satisfactory method of formulating such problems as this in a general way was made available by the introduction of the Lagrangian formulation of quantum mechanics by Feynman. He applied the techniques afforded by this method extensively to quantum electrodynamics which is a study of the interaction of matter and the electromagnetic field. For instance, in a problem where several charged particles interact through the electromagnetic field, he found that it was possible to eliminate the coordinates of the field and to include its effect as a delayed interaction between the particles (1),(2).

Initially, the specific subject of interest to us was that of quantum noise in maser type devices. In an effort to locate all the sources of such noise a more general study evolved. The central problem to be considered here is to develop a formalism for finding the effect of a general, unobserved system B on another system (the observed or test system) whose characteristics are of interest, and then to specialize the formalism to cases where various combinations of linear

systems and classical forces act on the observed system. In this case, where the unobserved system is linear, it will be found that parameters such as impedance, which characterize its classical behavior, are also important in determining its quantum effect on the observed system. Since this system may include dissipation, the results have application in a study of irreversible quantum statistical mechanics.

In Section II, a general formulation of the problem is made and certain functionals, called influence functionals, will be defined, which contain the effect of the unobserved systems such as B on the observed system. In Section III the special cases are considered in which the unobserved system is a definite classical force and a random classical force. In Section IV the influence functionals for linear systems at zero temperatures are derived and the use of influence functionals in making calculations is discussed. In addition, the possible forms of influence functionals for linear systems are derived by considering their general properties as discussed in Section II. Section V is devoted to classical forces acting through linear systems, while Section VI considers the effect of finite temperatures of linear systems. Then, in Section VII, the unobserved systems are again assumed to be general but weakly coupled to the observed system. Within the approximation of weak coupling these general systems also behave as if they were linear. Then finally in Section VIII, the results of the analysis are used to prove a general theorem concerning maser noise.

An equivalent approach can be made to the problem using the Hamiltonian formulation of quantum mechanics by making use of the ordered operator calculus developed by Feynman (3). This has some advantages in that many results may be obtained more simply than by Lagrangian

method and nonclassical concepts such as spin may be dealt with readily. This approach has been used to some extent by Fano and is currently being developed by Hellwarth (4),(5).

II. General Formulation; Influence Functional

We shall begin the discussion with a brief introduction to the Lagrangian or space-time approach to quantum mechanics. Since the treatment here is necessarily brief, the reader is referred to the original article published on the subject (1). Secondly, using the techniques thus made available we will define a class of functionals which can be used to describe the effects of a general interaction system in influencing the behavior of a general test system*. These will be called influence functionals and it will be found that they are only functionals of the variable of the test system, the variables of the interaction system being eliminated. We will then be able to express the behavior of the test system in terms of its own coordinates only. Thirdly, we will deduce some general properties of these functionals. Finally, it will be shown that the influence functional represents that part of the propagator of the density matrix of the test system which contains the effects of the interaction system. As such, it is equivalent to general formulations of the same problem, which have been given earlier (4).

Since the influence functional can be formulated for very general systems, it should not be surprising to discover that it may be impossible actually to carry out the process of eliminating the interaction system variables because of the complexity of the mathematical operations required. In principle, however, it can always be done. All of the systems to be considered in detail here are linear, or approximately

*Throughout this work we will designate the test system as the system whose behavior is of interest. Conversely, the interaction system will be the system whose behavior is not of primary interest.

so. As such they have the property that the influence functionals can be evaluated exactly. As a last preliminary remark it should be reiterated that although we shall talk of general test and interaction systems, spin has not been satisfactorily included in the Lagrangian formulation. In fact, the formulation is restricted to cases involving momentum or coordinate operators. Therefore, strictly speaking, systems in which the spin is of importance are not covered by this analysis. However, this has no bearing on the validity of the results since their nature is such that their extension to the case where spins are important can be inferred. The Hamiltonian ordered operator approach to this problem does not suffer this disadvantage.

II.1 Lagrangian Formulation of Quantum Mechanics

Let us suppose that we are considering a single system which has coordinates that are denoted by Q, and that for the time being it is not acted on by any other quantum system. It can be acted on by outside forces, however. The Q system may be very complicated, in which case Q represents all the coordinates in a general way. If at a time $t = \tau$ the value of Q is Q_τ then the amplitude for the system to go from position Q_τ at $t = \tau$ to Q_T at $t = T$ is given by

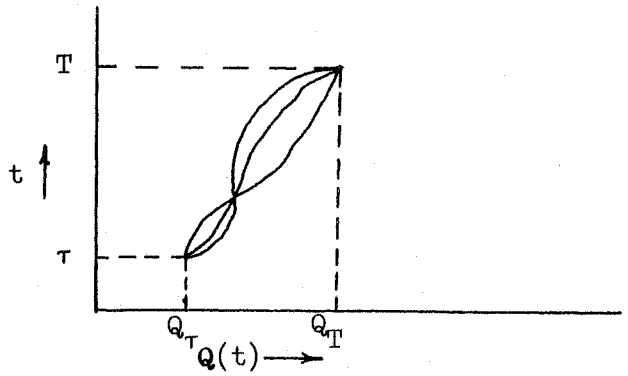
$$K(Q_T,T;Q_\tau,\tau) = \int e^{\frac{i}{\hbar}\int_\tau^T \mathcal{L}(\dot{Q},Q,t)dt} \mathcal{D}Q(t) \quad , \tag{1}$$

an integral which represents the sum over all possible paths in coordinate space from Q_τ to Q_T of the function

$$e^{\frac{i}{\hbar}\int_\tau^T \mathcal{L}(\dot{Q},Q,t)dt}$$

$\mathcal{L}(\dot{Q},Q,t)$ is the classical Lagrangian of the system and

$\int_{\tau}^{T} \mathcal{L}(\dot{Q},Q,t)dt$ is the classical action. For the case that Q is a single linear coordinate of position, this is represented in the diagram below. The magnitude of the amplitude for all paths is equal but the



phase for each path is given by the classical action along that path in units of $\hbar$. Thus, amplitudes for neighboring paths which have large phases tend to cancel. The paths which contribute the greatest amount are those whose amplitudes have stationary phases for small deviations around a certain path. This is the path for which the classical action is at an extremum and is, therefore, the classical path. Remarkably enough, for free particles and harmonic oscillators, the result of the path integration is

$$K(Q_T,T;Q_\tau,\tau) = (\text{Smooth Function})\ e^{\frac{i}{\hbar} S_{c\ell}}$$

where $S_{c\ell}$ is the action evaluated along the classical path. However, for more complicated systems this simple relation does not hold. A discussion of the methods of doing integrals of this type is not appropriate here except for a brief mention of the most basic method. This consists of breaking the action integral into an infinite sum over infinitesimal time increments. Thus, if Q_j corresponds to the value of Q at the time t_j, then the amplitude, equation 1, can be written

$$K(Q_T,T;Q_\tau,\tau)= \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \int \cdots \int \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} \mathcal{L}\left(\frac{Q_{j+1}-Q_j}{\epsilon},Q_j,t_j\right)\epsilon\right] \prod_{j=1}^{N-1} dQ_j \ . \qquad (2)$$

In the above equation $Q_0 \equiv Q_\tau$ and $Q_N \equiv Q_T$. Then the operation written as $\mathcal{D}Q(t)$ becomes a large number of integrals symbolized by $\prod_{j=1}^{N-1} dQ_j$ over all possible values of the variable Q_j*.

If the system at $t = \tau$ is in a state described by $\phi_n(Q_\tau)$ then the wave function at Q_T (i.e., the amplitude to be at Q_T) is given by

$$\psi(Q_T)= \int K(Q_T,T;Q_\tau,\tau)\phi_n(Q_\tau)dQ_\tau = \int\left[\exp \frac{i}{\hbar}\int_\tau^T \mathcal{L}(\dot{Q},Q,t)dt\right] \phi_n(Q_\tau)\mathcal{D}Q(t)dQ_\tau \ . \qquad (3)$$

In the above a single integral sign has been used to express all the integrations represented by $\mathcal{D}Q(t)$ and dQ_τ, a convention which will be employed often. Thus, the mathematical operation

$$\int \exp\left[\frac{i}{\hbar}\int_\tau^T \mathcal{L}(\dot{Q},Q,t)dt\right] \mathcal{D}Q(t)$$

is equivalent to a kernel which can take the wave function from $t = \tau$

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*If there exist vector potential terms or others linear in the velocity, then one must be more careful about the approximation to the action $S(Q_{j+1},Q_j)$ designated here by

$$\epsilon\,\mathcal{L}\left(\frac{Q_{j+1}-Q_j}{\epsilon},Q_j,t_j\right) \ .$$

For a more thorough discussion of the point, see R. P. Feynman (1).

to $t = T$.* It follows that at $t = T$ the probability amplitude that the system is in a state designated by $\phi_m(Q_T)$ is given by

$$\underset{n \to m}{\text{Amp}} = \int \phi_m^*(Q_T)\ \psi(Q_T)\, dQ_T$$

$$= \int \phi_m^*(Q_T) \exp\left[\frac{i}{\hbar}\int_\tau^T \mathcal{L}(\dot{Q},Q,t)\,dt\right]\ \phi_n(Q_\tau)\,\mathcal{D}Q(t)\ dQ_\tau dQ_T \ . \qquad (4)$$

The probability of the transition from $n \to m$ is given by $|\text{Amp}|^2$ and

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*If Q is not acted on by outside forces or by other quantum systems then

$$K(Q_T,T;Q_\tau,\tau) = \sum_n \phi_n(Q_T)\ \phi_n^*(Q_\tau) \exp\left[-\frac{i}{\hbar} E_n(T-\tau)\right]$$

where the $\phi_n(Q)$ are eigenfunctions appropriate to the unperturbed system Q. This can be seen easily by the following. If the state of Q at $t = \tau$ is $\psi(Q_\tau,\tau)$ then it can be expanded as follows:

$$\psi(Q_\tau,\tau) = \sum a_n\ \phi_n(Q_\tau)\ \exp\left[-\frac{i}{\hbar} E_n\tau\right]$$

where $a_n = \int \phi_n^*(Q_\tau)\ \exp\left[\frac{i}{\hbar} E_n\tau\right]\ \psi(Q_\tau,\tau)\,dQ_\tau$.

At $t = T$ we have

$$\psi(Q_T,T) = \sum_n a_n\ \phi_n(Q_T)\ \exp\left[-\frac{i}{\hbar} E_nT\right]$$

$$= \int \sum_n \phi_n(Q_T)\ \phi_n^*(Q_\tau) \exp\left[-\frac{i}{\hbar} E_n(T-\tau)\right]\ \psi(Q_\tau,\tau)\,dQ_\tau$$

$$= \int K(Q_T,T;Q_\tau,\tau)\,\psi(Q_\tau,\tau)\,dQ_\tau \ .$$

This relation will be useful in later calculations.

from equation 4 this can be written

$$P_{n\to m} = \int \phi_m^*(Q_T)\phi_m(Q_T')\exp\left[\frac{i}{\hbar}\left[S(Q)-S(Q')\right]\right] \phi_n^*(Q_\tau')\phi_n(Q_\tau)\,dQ_\tau\,dQ_\tau'\,dQ_T\,dQ_T' \times \mathcal{D}Q(t)\,\mathcal{D}Q'(t) \qquad (5)$$

where $S(Q) = \int_\tau^T \mathcal{L}(\dot{Q},Q,t)dt$, the classical action.

For the next more complicated case let us consider two systems whose coordinates are Q and X .* The systems are coupled by some potential which can be designated as $V(Q,X)$ and incorporated in the total Lagrangian. We assume that when $V = 0$ the states of Q and X can be described by sets of wave functions $\phi_k(Q)$ and $\chi_\ell(X)$ respectively. Then, if at $t = \tau$, Q is in a state $\phi_n(Q_\tau)$ and X is in a state $\chi_i(X_\tau)$, then the amplitude that Q goes from state n to m while X goes from state i to f can be formed in a similar way to that of 4 ,

$$\underset{\substack{n\to m\\ i\to f}}{\text{Amp}} = \int \phi_m^*(Q_T)\chi_f^*(X_T)\exp\left[\frac{i}{\hbar}S(Q,X)\right] \phi_n(Q_\tau)\chi_i(X_\tau)\,dX_\tau\,dQ_\tau\,dX_T\,dQ_T\,\mathcal{D}Q(t)\,\mathcal{D}X(t) \qquad (6)$$

where $S(Q,X)$ represents the classical action of the entire system including both Q and X . The important property of separability afforded by writing the amplitude in this way is now apparent.** For

*Each system will be denoted by the coordinates that characterize it. Where Q or X means specifically a coordinate, it will be so designated by a statement if it is not obvious.

**If system Q represents a harmonic oscillator and the interaction of Q with X were linear and of the form $-\gamma(t,X)\,Q(t)$, then that part of expression 6 which involves the Q variables corresponds to the function G_{mn} defined and used by Feynman to eliminate the electromagnetic field oscillators. See reference (2).

instance, if one wishes to know the effect that the Q system has on X when it undergoes a transition from state n to m, then all of the integrals on the Q variables may be done first. What is left is an expression for $\underset{i \to f}{Amp}$ for X and in terms of X variables only. This property will be examined in more detail in the next section.

The extension of writing transition amplitudes for large numbers of systems is obvious. In principle the order in which the variables are eliminated is always arbitrary. If one has a situation where two systems of interest are coupled together through a third system, then one can eliminate the coordinates of the third system thus expressing the behavior of the two test systems in terms of their own coordinates only. Here it is worth while to mention a well-known property of the Lagrangian and the action. If two systems (Q and X) are coupled by some potential, the Lagrangian for the entire system is written

$$\mathcal{L}(\dot{Q},\dot{X},Q,X,t) = \mathcal{L}_o(\dot{Q},Q,t) + \mathcal{L}_I(Q,X,t) + \mathcal{L}(\dot{X},X,t) \qquad (7)$$

where $\mathcal{L}_o(\dot{Q},Q,t)$ is the Lagrangian for Q with no external influences and $-\mathcal{L}_I(Q,X,t) = V(Q,X,t)$, the potential of interaction between Q and X. $\mathcal{L}(\dot{X},X,t)$ is the Lagrangian for the X system with no external disturbances. The action integral for the complete situation is constructed similarly and it will be represented by

$$S(Q,X) = S_o(Q) + S_I(Q,X) + S(X) \quad .$$

This property will prove to be very useful.

II.2 Definition of Influence Functional

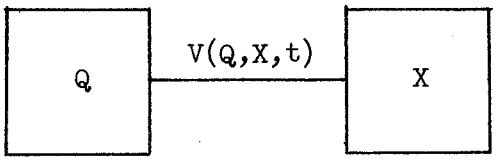
In the types of problems which are under consideration here we will be interested primarily in determining the behavior of one system, when it is being acted on by other quantum mechanical systems, or by classical forces, or both simultaneously. Utilizing the properties of the Lagrangian formulation of quantum mechanics mentioned above, we can now define a functional for taking these external influences into account.

Theorem I. For any Q system acted on by external classical forces and quantum mechanical systems as discussed above, the probability that it makes a transition from state $\psi_n(Q_\tau)$ at $t = \tau$ to $\psi_m(Q_T)$ at $t = T$ can be written

$$P_{n \to m} = \int \psi_m^*(Q_T)\psi_m(Q_T')\exp\left\{\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')\right]\right\}\mathcal{F}(Q,Q')\psi_n^*(Q_\tau')\psi_n(Q_\tau)$$

$$\times \mathcal{D}Q\mathcal{D}Q'dQ_\tau dQ_\tau' dQ_T dQ_T' \qquad (8)$$

where $\mathcal{F}(Q,Q')$ contains all the effects of the external influences on Q, and $S_o(Q) = \int_\tau^T \mathcal{L}_o(\dot{Q},Q,t)dt$, the action of Q without external disturbance. The proof of this is straightforward. Let us examine two coupled systems characterized by coordinates Q and X as represented diagrammatically below. Q will represent the test system and X the



quite general interaction system, perhaps representing all the rest of the universe coupled by a general potential $V(Q,X,t)$ to Q. Assume

Q to be initially $(t = \tau)$ in state $\psi_n(Q_\tau)$ and X to be in state $\chi_i(X_\tau)$. The probability that Q is found in state $\psi_m(Q_T)$ while X is in state $\chi_f(X_T)$ at $t = T$ can be written in the manner discussed above and is

$$P_{\substack{n \to m \\ i \to f}} = \left|\mathrm{Amp}_{\substack{n \to m \\ i \to f}}\right|^2 = \int \psi_m^*(Q_T)\psi_m(Q_T')\chi_f^*(X_T)\chi_f(X_T')$$

$$\times \exp\left\{\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')+S(X)-S(X')+S_I(Q,X)-S_I(Q',X')\right]\right\} \tag{9}$$

$$\times \psi_n^*(Q_\tau')\psi_n(Q_\tau)\chi_i^*(X_\tau')\chi_i(X_\tau)dX_\tau dX_\tau' dX_T dX_T' dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q\mathcal{D}Q'\mathcal{D}X\mathcal{D}X'.$$

The primed variables were introduced when the integrals for each $\mathrm{Amp}_{\substack{n \to m \\ i \to f}}$ were combined. Now if all of expression 9 which involves coordinates other than Q or Q' is separated out and designated as $\mathcal{F}(Q,Q')$, then the following expression is obtained

$$\mathcal{F}(Q,Q') = \int \chi_f^*(X_T)\,\chi_f(X_T')\exp\left\{\frac{i}{\hbar}\left[S(X)-S(X')+S_I(Q,X)-S_I(Q',X')\right]\right\}$$

$$\times\, \chi_i^*(X_\tau')\,\chi_i(X_\tau)dX_\tau dX_\tau' dX_T dX_T' \mathcal{D}X\mathcal{D}X' \quad . \tag{10}$$

This expression when substituted into equation 9 yields the desired form of equation 8. As can be seen, $\mathcal{F}$ is a functional whose form depends upon the physical system X, the coupling between Q and X, and the behavior of X in the time interval between τ and T. Furthermore, explicitly $\mathcal{F}$ is a functional of only $Q(t)$ and $Q'(t)$. In general, it will be different for each system X and the behavior of X during the time interval $\tau < t < T$, at least on a detailed level. However, in

many physical situations weak coupling between systems is involved. We shall see that the approximate influence functional which can be used in this case to represent the effect of the interaction system has a form which is independent of the nature of the interaction system. In such cases, each interaction system has the same effect on Q. This is true because, for instance, if there are two systems A and B which can act on C, and if

$$\mathcal{F}_{A \text{ on } C} = \mathcal{F}_{B \text{ on } C} ,$$

then the effects of A on C are the same as those of B on C. In other situations where several combined systems of the same type (linear) are acting on Q, the same form of $\mathcal{F}$ results. These cases will be considered in detail in later sections. Before this is done there are some properties of $\mathcal{F}$ which can be derived on a general basis.

II.3 General Properties of Influence Functionals

Theorem II. If the physical situation is unsure (as for instance if the type of interaction system X, or the initial or final states are not known precisely) but if the probability of the ℓth situation is w_ℓ and the corresponding influence functional is $\mathcal{F}_\ell$, then the effective $\mathcal{F}$ is given by

$$\mathcal{F}_{\text{eff}} = \sum_\ell w_\ell \mathcal{F}_\ell = \langle \mathcal{F} \rangle . \tag{11}$$

To demonstrate this we write an expression for the probability of transition for the ℓth situation:

$$P_{n \underset{\ell}{\rightarrow} m} = \int \psi_m^*(Q_T)\psi_m(Q_T')\exp\left\{\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')\right]\right\} \mathcal{F}_\ell(Q,Q')\psi_n^*(Q_\tau')\psi_n(Q_\tau)$$

$$\times dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q \mathcal{D}Q' \quad .$$

But the effective $P_{n \underset{\ell}{\rightarrow} m}$ is given by

$$(P_{n\rightarrow m})_{eff} = \sum_\ell w_\ell P_{n \underset{\ell}{\rightarrow} m} = \int \psi_m^*(Q_T)\psi_m(Q_T')\exp\left\{\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')\right]\right\} \sum_\ell w_\ell \mathcal{F}_\ell(Q,Q')$$

$$\times \psi_n^*(Q_\tau')\psi_n(Q_\tau) dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q \mathcal{D}Q' \quad . \qquad (12)$$

Thus, $\mathcal{F}_{eff}$ given by equation 11 is the appropriate expression.

Theorem III. If a number of statistically and dynamically independent partial systems act on Q at the same time and if $\mathcal{F}^{(k)}$ is the influence of the kth system alone, the total influence of all is given by the product of the individual $\mathcal{F}^{(k)}$:

$$\mathcal{F} = \prod_k \mathcal{F}^{(k)} \quad . \qquad (13)$$

The argument here is similar to the ones above. We assume that the initial state of the kth system is $\chi_i^{(k)}(X_\tau^{(k)})$, that its final state is $\chi_f^{(k)}(X_T^{(k)})$, and that the interaction between $X^{(k)}$ and Q is represented by $S_I(X^{(k)},Q)$. The probability that Q makes a transition from $n \rightarrow m$ can be written

$$P_{n \to m} = \int \psi_m^*(Q_T)\psi_m(Q_T')\exp\left[\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')\right]\right]\prod_k\left\{\int \chi_f^{(k)*}(X_T^{(k)})\chi_f^{(k)}(X_T^{(k)'})\right.$$

$$\times \exp\left[\frac{i}{\hbar}\left[S_o(X^{(k)})-S_o(X^{(k)'})+S_I(X^{(k)},Q)-S_I(X^{(k)'},Q')\right]\right]$$

$$\left.\times \chi_i^{(k)*}(X_\tau^{(k)'})\chi_i^{(k)}(X_\tau^{(k)})\,dX_\tau^{(k)}dX_\tau^{(k)'}\,dX_T^{(k)}dX_T^{(k)'}\mathcal{D}X^{(k)}\mathcal{D}X^{(k)'}\right\}\psi_n^*(Q_\tau')\psi_n(Q_\tau)$$

$$\times dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q\mathcal{D}Q' \quad . \tag{14}$$

However, each of the expressions behind the multiplicative sign between the braces is identified as an influence functional. Therefore $P_{n \to m}$ can be written as follows:

$$P_{n \to m} = \int \psi_m(Q_T)\psi_m^*(Q_T')\exp\left\{\frac{i}{\hbar}\left[S_o(Q)-S_o(Q')\right]\right\}\prod_k \mathcal{F}^{(k)}(Q,Q')\,\psi_n^*(Q_\tau')\psi_n(Q_\tau)$$

$$\times \mathcal{D}Q\mathcal{D}Q' dQ_\tau dQ_\tau' dQ_T dQ_T' \quad . \tag{15}$$

Definition. In many cases $\mathcal{F}$ will be of the form $e^{i\Phi(Q,Q')}$. Φ is then called the influence phase. For independent disturbances as considered in Theorem III, the influence phases add. In the event that $i\Phi(Q,Q')$ is a real number we will continue to use the notation Φ; the phase simply becomes imaginary. It will frequently be more convenient to work with Φ rather than $\mathcal{F}$.

Theorem IV. The influence functional has the property that

$$\mathcal{F}^*(Q,Q') = \mathcal{F}(Q',Q) \tag{16}$$

as can be ascertained by inspection of the general expression of equation 10.

Theorem V. In the class of problems in which the final state of the interaction system is arbitrary, which means the final states are to be summed over, then $\mathcal{F}(Q,Q')$ is independent of $Q(t)$ if $Q(t) = Q'(t)$ for all t. All of the problems we will be concerned with here are of this type.

The validity of this statement can be ascertained by observing equation II.10, the general definition of the influence functional. In particular, for the case where the initial and final states of the interaction system X are i and f respectively, as in equation 10, we denote the influence functional by ${}_f\mathcal{F}_i(Q,Q')$. Let us assume we have no interest in the final state of X which means that ${}_f\mathcal{F}_i$ must be summed over all such states. The initial state i can be quite general. Thus, the influence functional for the case of an arbitrary final state is

$$\mathcal{F}_i(Q,Q') = \sum_f {}_f\mathcal{F}_i(Q,Q') \quad .$$

For clarity in finding the result of letting $Q(t) = Q'(t)$ for all t in $\mathcal{F}_i(Q,Q')$ we will write out the expression explicitly from equation 10. It is

$$\mathcal{F}_i(Q,Q) = \int \sum_f \chi_f^*(X_T)\, \chi_f(X_T')\exp\left\{\frac{i}{\hbar}[S(X)-S(X')+S_I(Q,X)-S_I(Q,X')]\right\}$$

$$\times \chi_i^*(X_\tau')\, \chi_i(X_\tau)\, dX_\tau\, dX_\tau'\, dX_T\, dX_T'\, \mathcal{D}X(t)\, \mathcal{D}X'(t) \quad .$$

Since Q appears in the interaction potentials acting on the X and X' variables respectively, it loses its identity as the coordinate of

a quantum system and becomes just a number (which may be, of course, a function of time). Thus, $S_I(Q,X)$ may be interpreted as the action of a classical potential which drives the X system. The above expression becomes then the expression for the probability that X which is in state i initially, is finally in any one of its possible states after being acted on by a classical potential (as, for instance, in equation 5 summed over the final states, m). This result is unity. We have then that $\mathcal{F}(Q,Q') = 1$ and is independent of $Q(t)$.

Corollary. A more restrictive statement of the above theorem can be made. In this same class of problems in which the final states are summed over, if $Q(t) = Q'(t)$ for all $t > t_o$ then $\mathcal{F}(Q,Q')$ is independent of $Q(t)$ for $t > t_o$.

This can be seen in very nearly the same way as for the above theorem. Again let us turn to equation 10 and break up the time integral into two parts, before and after t_o . For $t < t_o$, $Q(t) \neq Q'(t)$, and $\mathcal{F}(Q,Q')$ simply describes the effect of X on Q for the case that X is initially in a state $\chi_i(X_T)$ and at $t = t_o$ is in a state $\chi(X_{t_o})$. For $t > t_o$, $Q(t) = Q'(t)$ and the expression for $\mathcal{F}_{t_o}(Q,Q')$ is

$$\mathcal{F}_{t_o}(Q,Q) = \int \sum \chi_f^*(X_T)\, \chi_f(X_T') \exp\left\{\frac{i}{\hbar}[S(X)-S(X')+S_I(Q,X)-S_I(Q,X')]\right\}$$

$$\times\, \chi^*(X_{t_o}')\, \chi(X_{t_o})\, dX_{t_o}\, dX_{t_o}'\, dX_T\, dX_T'\, \mathcal{D}X(t)\, \mathcal{D}X'(t) \quad .$$

This is the expression for the probability that X has made a transition to any one of its possible states at $t = T$ after being in a

state $\chi(X_{t_o})$ at $t = t_o$ and in the interim $t_o < t < T$ has been acted on by a classical potential. Since this result is unity, we have that $\mathcal{F}_{t_o}(Q,Q) = 1$ and is independent of $Q(t)$ for $t > t_o$. As will be seen later in the specific case of linear systems, this leads to a statement of causality. Notice that the above argument cannot be made if $Q_t = Q'_t$ for $t < t_o$ and $Q_t \neq Q'_t$ for $t > t_o$. This is because the sum over states, which was necessary to make $\mathcal{F}(Q,Q')$ independent of $Q(t)$ is perfomed over final $(t = T)$ and not initial states.

II.4 Statistical Mechanics

Finally, it is appropriate to point out explicitly the significance of the influence functional in a study of quantum statistical mechanics for the case where summation over the final states of the interaction system is involved. First, we will define the density matrix. Let us assume we have a system Q as before which is in a pure state representable by $\psi^{(i)}(Q,t) = \sum a_n^{(i)}(t)\phi_n(Q)$. The ϕ_n are a set of orthogonal, normalized eigenfunctions. The expectation value of an operator A is, of course, given by

$$\langle A \rangle^{(i)} = \int \psi^{(i)*}(Q,t) A(t) \psi^{(i)}(Q,t) dQ \quad ,$$

where the superscript on the $\langle A \rangle^{(i)}$ is used to denote average over the state i. In practical situations, however, the state of Q is not known precisely. In this case to get the "expected" average, the procedure is to average $\langle A \rangle^{(i)}$ over all the systems in an ensemble each representing a possible state of Q. Then,

$$\langle A \rangle = \frac{1}{N} \sum_{i=1}^{N} \int \psi^{(i)*}(Q,t)\, A(t)\, \psi^{(i)}(Q,t)\, dQ$$

$$= \frac{1}{N} \sum_{i=1}^{N} \sum_{n,n'} \int a_{n'}^{(i)*}(t) a_{n}^{(i)}(t) \phi_{n'}^{*}(Q)\, A\phi_{n}(Q)\, dQ$$

$$= \sum_{n,n'} \left(\frac{1}{N} \sum_{i=1}^{N} a_{n'}^{(i)*}(t) a_{n}^{(i)}(t) \right) A_{n'n} = \sum_{n,n'} \overline{a_{n'}^{*}(t) a_{n}(t)}\, A_{n'n}$$

where $A_{n'n} = \int \phi_{n'}^{*}(Q)\, A\, \phi_{n}(Q)\, dQ$ and where the double bar represents average over the ensemble following the notation of Tolman (6). The quantity $\overline{a_{n'}^{*}(t) a_{n}(t)}$ is designated as $\rho_{n,n'}$, the elements of a matrix ρ the use of which was first introduced by von Neumann (7). Thus we can write

$$\langle A \rangle = \mathrm{Tr}(\rho A) \equiv \sum_{n,n'} \rho_{nn'}\, A_{n'n} \quad . \tag{18}$$

Suppose now that there are two systems involved, Q and X, which have been interacting for times $t < T$. At the time T the two systems are disconnected so that their average states may be represented in terms of the eigenfunctions of the unperturbed systems. If $\phi_{n}(Q_T)$ and $\mu_{m}(X_T)$ represent these eigenfunctions for Q and X respectively, then the total wave function for two representative systems of the ensemble can be written,

$$\Psi(Q_T, X_T) = \psi(Q_T)\, \chi(X_T) = \sum_{n,m} a_{n}(T)\, \phi_{n}(Q_T) b_{m}(T) \mu_{m}(X_T) \tag{19}$$

and the density matrix for the total system is identified as

$$\rho_{nm,n'm'}^{(Q,X)}(T) = \overline{a_{n'}^{*}(T) b_{m'}^{*}(T) a_{n}(T) b_{m}(T)} = \rho_{nn'}^{(Q)}(T)\, \rho_{mm'}^{(X)}(T) \tag{20}$$

Thus the average value of an operator $A(T)\ B(T)$ where A operates on Q and B operates on X is, using equations 19 and 20,

$$\langle A(T)\ B(T)\rangle = \int \psi^*(Q_T)\,\chi^*(X_T)A(T)B(T)\psi(Q_T)\,\chi(X_T)\,dQ_T dX_T \tag{21}$$

$$= \sum_{\substack{m,m'\\ n,n'}} \rho^{(Q,X)}_{nm,n'm'}(T) = \mathrm{Tr}(\rho^{(Q,X)}(T)\ AB) \tag{22}$$

$$= \mathrm{Tr}^{(Q)}\ [A\ \mathrm{Tr}^{(X)}(\rho^{(Q,X)}(T)B)] \tag{23}$$

where $\mathrm{Tr}^{(X)}$ means trace over X variables only.

Suppose now that it is desired to find $\langle A\rangle$. From equation 23 this is found to be

$$\langle A\rangle = \mathrm{Tr}^{Q}\ [A\ \mathrm{Tr}^{X}(\rho^{(Q,X)}(T))] \quad . \tag{24}$$

Thus, $$\rho^{Q}(T) = \mathrm{Tr}^{X}[\rho^{(Q,X)}(T)] \quad (4) \quad . \tag{25}$$

Usually, however, $\rho^{Q}(T)$ is not given and we must find it in terms of the initial value of the density matrix, $\rho^{Q}(\tau)$, i.e., before the interaction between Q and X in terms of an influence functional. Rewriting equation 21 for the case that $B(T) = 1$ we have

$$\langle A(T)\rangle = \int \psi^*(Q_T'')\,\chi^*(X_T')\delta(Q_T-Q_T'')\delta(X-X_T')A(T)\delta(Q_T-Q_T')\,\chi(X_T)\psi(Q_T') \times dQ_T dQ_T' dQ_T'' dX dX_T' \quad . \tag{26}$$

Utilizing the closure property of orthonormal eigenfunctions, i.e.,

$$\sum_n \phi_n^*(Q_T)\phi_n(Q_T') = \delta(Q_T-Q_T') \quad ,$$

and writing down the wave functions of Q and X in terms of their values at the earlier time τ (as in equation 6, for example), we have

$$\langle A(T)\rangle = \int \sum_{n'} \phi^*_{n'}(Q_T)\phi_{n'}(Q'_T)A(T) \sum_{n} \phi_n(Q_T)\phi^*_n(Q''_T)\exp\left[\frac{i}{\hbar}[S_o(Q)-S_o(Q')]\right]$$

$$\times\left\{\delta(X_T-X'_T)\exp\left[\frac{i}{\hbar}[S(X)-S(X')+S_I(Q'',X)-S_I(Q',X')]\right]\chi^*_i(X'_\tau)\,\chi_i(X_\tau)dX_\tau dX'_\tau\right.$$

$$\left.\times\,\mathcal{D}X\mathcal{D}X'\right\}\sum_{a,a'}\rho_{aa'}(\tau)\phi^*_a(Q'_\tau)\phi_a(Q''_\tau)dQ'_\tau dQ''_\tau dQ_T\cdots\mathcal{D}Q'(t) \quad . \qquad (27)$$

The part of equation 27 in the braces is recognized as $\mathcal{J}(Q'',Q')$ for the case that X begins in state i, $\left[\chi^*_i(X'_\tau)\,\chi_i(X_\tau) = \sum \rho_{mm'}(\tau)\mu^*_{m'}(X'_\tau)\mu_m(X_\tau)\right]$, and ends in an arbitrary state. Taking the matrix elements on A(T) we find that equation 27 can be rewritten as follows

$$\langle A(T)\rangle = \sum_{n',n} A_{n'n}(T)\int \phi^*_n(Q_T)\phi_{n'}(Q'_T)\exp\left[\frac{i}{\hbar}[S_o(Q)-S_o(Q')]\right]$$

$$\times\,\mathcal{J}(Q,Q')\sum_{a,a'}\rho_{a,a'}(\tau)\phi^*_{a'}(Q'_\tau)\phi_a(Q_\tau)dQ_T dQ'_T dQ_\tau dQ'_\tau\,\mathcal{D}Q\mathcal{D}Q' \quad . \qquad (28)$$

Thus the coefficient of $A_{n'n}(T)$ is evidently $\rho^{(Q)}_{n,n'}(T)$. It is interesting to point out that in an earlier part of the discussion the sum over the final states in the influence functional was regarded as meaning the final state of the interaction system was arbitrary. Another way of looking at the situation is provided in equation 28 above, where the sum over final states can be regarded as taking the

average of $A(T)\ 1^X$:

$$\langle A(T)\ 1^X \rangle = \mathrm{Tr}(\rho A\ 1^X)$$

where 1^X is the unit operator in the X system and $A(T)$ as before, operates only on the test system.

III. Influence Functionals for Classical Potentials

In this section we will derive specific forms and properties of influence functionals for the effects of classical potentials on the test system. These represent the simplest form of influence functionals and their properties follow directly from the general properties obtained in the previous section. Finally, we will derive the influence functional when the classical potential represents Brownian noise.

III.1 Properties of Influence Functionals for Classical Potentials

The first step is to find the influence functional for a definite classical potential acting on the test system, Q . If the potential energy term in the Lagrangian is of the form $V(Q,t)$, then it can be ascertained readily by referring to Theorem I that

$$\mathcal{F}(Q,Q') = \exp\left\{-\frac{i}{\hbar}\int_{\tau}^{T}\Big[V(Q,t) - V(Q',t)\Big]dt\right\} \tag{1}$$

or equivalently the influence phase is

$$\Phi(Q,Q') = -\frac{1}{\hbar}\int_{\tau}^{T}\Big[V(Q,t) - V(Q',t)\Big]\,dt \quad . \tag{2}$$

The next degree of complication is to have several potentials, $\sum_k V_k(Q,t)$ acting on Q simultaneously. However, since the sum of all these potentials represents an equivalent potential, say $V_T(Q,t) = \sum_k V_k(Q,t)$, then it is obvious that the total influence functional $\mathcal{F}(Q,Q')$ is the product of the individual $\mathcal{F}_k(Q,Q')$. More specifically,

$$\mathcal{F}_T(Q,Q') = \exp\left\{-\frac{i}{\hbar}\int_\tau^T [V_T(Q,t) - V_T(Q',t)]dt\right\}$$

$$= \exp\left\{\sum_k -\frac{i}{\hbar}\int_\tau^T [V_k(Q,t) - V_k(Q't)]\,dt\right\}$$

$$= \prod_k \mathcal{F}_k(Q,Q') \quad , \tag{3}$$

or

$$\Phi(Q,Q') = \sum_k -\frac{1}{\hbar}\int_\tau^T [V_k(Q,t) - V_k(Q',t)]dt = \sum_k \Phi_k(Q,Q') \ . \tag{4}$$

The same result follows directly from Theorem III which gives the total influence functional for several statistically and dynamically independent systems acting on Q . The total influence functional for all the systems (in this case potentials) is the product of the functionals for the individual systems.

Another property of the classical influence functionals is obtained by inspection of equation 1. We notice that for any classical $\mathcal{F}(Q,Q')$ if conditions are such that $Q(t) = Q'(t)$, then $\mathcal{F}(Q,Q')= 1$ and is independent of t for all times that the two variables are equal. It follows that the influence phase is zero for this condition.

Finally, by application of Theorem II we find that if the potential is uncertain but the probability of each $V_\ell(Q,t)$ is w_ℓ then the average functional is given by

$$\langle \mathcal{F}(Q,Q') \rangle = \sum_\ell w_\ell \exp\left\{-\frac{i}{\hbar}\int_\tau^T \left[V_\ell(Q,t) - V_\ell(Q',t)\right]dt\right\} = \sum_\ell w_\ell \ \mathcal{F}_\ell(Q,Q') \ .$$

In the paragraphs following we will assume a probability distribution w_ℓ appropriate to Brownian noise and will be able to derive a specific form for the average influence functional.

III.2 Specific Functionals for Random Potentials

Theorem VI. If the potential has known form $-V(Q)$ but unknown strength $C(t)$ as a function of time so that the total potential is $V(Q,t) = -C(t)\, V(Q)$, then the average influence functional can be written in general form,

$$\langle \tilde{\mathcal{F}} \rangle = \left\langle \exp\left[\frac{i}{\hbar} \int_\tau^T C(t)\, [V(Q) - V(Q')]\, dt \right] \right\rangle \tag{7}$$

or, in Fourier transform notation,

$$\langle \tilde{\mathcal{F}} \rangle = \left\langle \exp\left[\frac{i}{2\pi\hbar} \int_{-\infty}^{\infty} C_\nu \left[V_{-\nu}(Q) - V_{-\nu}(Q') \right] d\nu \right] \right\rangle \tag{8}$$

where

$$C_\nu = \int_{-\infty}^{\infty} C(t) e^{-i\nu t}\, dt \tag{9}$$

and

$$\left[V_\nu(Q) - V_\nu(Q') \right] = \int_\tau^T [V(Q(t)) - V(Q'(t))] e^{-i\nu t}\, dt \quad . \tag{10}$$

If this $C(t)$ is Brownian noise with a power spectrum $\phi(\nu)$ and a correlation function $R(\tau) = \frac{2}{\pi} \int_0^\infty \phi(\nu) \cos \nu\tau\, d\nu$, then the average value of a function such as $\exp\left[i \int_\tau^T C(t) v(t) dt \right]$, is

$$\left\langle \exp\left\{ i\int_{\tau}^{T} C(t)v(t)dt \right\} \right\rangle = \exp\left\{ -\int_{\tau}^{T}\int_{\tau}^{t} R(t-s)v(t)v(s)dsdt \right\} \qquad (11)$$

and it follows that the average influence phase is

$$\langle i\Phi \rangle = -\frac{1}{\hbar^2}\int_{\tau}^{T}\int_{\tau}^{t} R(t-s)[V(Q(t))-V(Q'(t))][V(Q(s))-V(Q'(s))]dsdt \qquad (12)$$

$$= -\frac{1}{\pi\hbar^2}\int_{0}^{\infty} \phi(\nu)\left| V_\nu(Q) - V_\nu(Q')\right|^2 d\nu \quad . \qquad (13)$$

Proof: First it will be shown that for purely Gaussian noise of whatever coupling strength (average magnitude of C) the above expressions apply; second, for noise of a stationary random character, not necessarily Gaussian in nature but very weakly coupled to Q, the process can be represented approximately by the expressions for $\langle \mathcal{F} \rangle$ given above.

A. If the noise is such that each frequency component is random and its distribution is Gaussian, we will show that

$$\left\langle \exp\left\{ \frac{i}{2\pi}\int_{-\infty}^{\infty} C_\nu v_{-\nu} d\nu \right\} \right\rangle = \exp\left\{ -\frac{1}{\pi}\int_{0}^{\infty} \phi(\nu)\left| v_\nu \right|^2 d\nu \right\} ,$$

where it is to be noted that $\int_{\tau}^{T} C(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_\nu v_{-\nu} d\nu$ and $C(t)$ is the random variable.

To find this expectation value the procedure is to multiply

$$\exp\left\{ \frac{i}{2\pi}\int_{-\infty}^{\infty} C_\nu v_{-\nu} d\nu \right\}$$

by the probability for C_ν ; then integrate over all possible "paths", C_ν . Now, for a stationary random function $\phi(\nu)$ is defined by

$$\langle C_\nu C_{\nu'} \rangle = 4\pi\phi(\nu)\delta(\nu+\nu') \quad . \tag{14}$$

However, for doing the path integral in frequency space it is more convenient to break up the path integral into discrete frequency components C_{ν_j} , $C_{\nu_{j+1}}$, etc. such that $\nu_{j+1} - \nu_j = \Delta\nu = \epsilon$; for this case the above relation becomes

$$\langle C_{\nu_j} C_{-\nu_k} \rangle = \frac{4\pi\ \delta_{jk}\ \phi(\nu_j)}{\epsilon} \quad . \tag{15}$$

Since we are dealing with Gaussian noise then the first probability distribution for any frequency component is given by

$$W(C_{\nu_j}) = \sqrt{\frac{\epsilon}{8\pi^2\phi(\nu_j)}} \quad \exp\left[-\frac{C_{\nu_j}\ C_{-\nu_j}}{8\pi\phi(\nu_j)}\ \epsilon\right] \quad . \tag{16}$$

Therefore, we can now write

$$\left\langle \exp\left\{\frac{i}{2\pi}\int_{-\infty}^{\infty} C_\nu v_{-\nu} d\nu\right\}\right\rangle = \lim_{\epsilon \to 0} \prod_{j=-\infty}^{\infty} \exp\left\{\frac{i}{2\pi} C_{\nu_j} v_{-\nu_j}\ \epsilon - \frac{C_{\nu_j}\ C_{-\nu_j}}{8\pi\ \phi(\nu_j)}\ \epsilon\right\} dC_{\nu_j} \tag{17}$$

$$= \lim_{\epsilon \to 0} \prod_{j=1}^{\infty} \iint \exp\left\{\frac{i}{2\pi}[C_{\nu_j} v_{-\nu_j} + C_{-\nu_j} v_{\nu_j}]\epsilon \ - \frac{C_{\nu_j} C_{-\nu_j}}{4\pi\phi(\nu_j)}\epsilon\right\} dC_{\nu_j} dC_{-\nu_j} \tag{18}$$

where in the second expression the form has been changed to an infinite product over positive j . Each of the double integrals indicated above

is self-contained in that it depends on one index only and its form is identical with all the rest. Therefore, only one term need be evaluated. To do this we note that C_{ν_j} is a complex number and can be written

$$C_\nu = A_\nu + iB_\nu$$

and

$$C_{-\nu} = A_\nu - iB_\nu$$

where the index on ν has been dropped for this calculation. Disregarding the constant Jacobian of the transformation from $C_\nu, C_{-\nu}$ to A_ν, B_ν the jth integral can be written

$$\iint \exp\left\{\frac{i}{2\pi}\left[C_\nu v_{-\nu} + C_{-\nu} v_\nu\right]\epsilon - \frac{C_\nu C_{-\nu}}{4\pi\phi(\nu)}\epsilon\right\} dC_\nu dC_{-\nu}$$

$$= \int_{-\infty}^{\infty}\int \exp\left\{-\frac{1}{4\pi\phi(\nu)}(A_\nu^2+B_\nu^2)\epsilon - \frac{i\epsilon}{2\pi} A_\nu(v_\nu+v_{-\nu}) + \frac{\epsilon B_\nu}{2\pi}(v_\nu - v_{-\nu})\right\} dA_\nu dB_\nu \quad . \tag{19}$$

This integral is Gaussian in both A_ν and B_ν and can easily be evaluated by completing the square to give

$$\exp\left\{-\frac{\epsilon\,\phi(\nu)}{4\pi}\left[(v_\nu+v_{-\nu})^2-(v_\nu-v_{-\nu})^2\right]\right\} = \exp\left\{-\frac{v_\nu v_{-\nu}\phi(\nu)\epsilon}{\pi}\right\} \quad .$$

Again the multiplicative constants obtained from the integration process have been dropped. Therefore, to within a normalizing constant

$$\left\langle \exp\left\{\frac{i}{2\pi}\int_{-\infty}^{\infty} C_\nu v_{-\nu} d\nu\right\}\right\rangle = \lim_{\epsilon\to 0} \prod_{j=1}^{\infty} \exp\left\{-\frac{v_{\nu_j} v_{-\nu_j}\phi(\nu_j)\epsilon}{\pi}\right\}$$

$$= \exp\left\{-\frac{1}{\pi}\int_0^\infty \phi(\nu)\,|v_\nu|^2\, d\nu\right\} \quad . \tag{20}$$

That the constant is in fact unity, can be seen by substituting $v_\nu = 0$, $\langle 1 \rangle = 1$. Now, if the substitution is made that

$$v(t) = \frac{1}{\hbar}\left[V(Q(t)) - V(Q'(t)) \right] \quad , \tag{21}$$

then the average influence phase is found to be

$$\langle i\Phi \rangle = - \frac{1}{\pi\hbar^2} \int_0^\infty \phi(\nu) \left| V_\nu(Q) - V_\nu(Q') \right|^2 d\nu \quad .$$

B. The Gaussian behavior of Brownian noise, characterized by the typical Gaussian probability distribution, may be the result of a large noise accumulated by many small independent sources, none of which have a truly Gaussian distribution (provided that each source is small in its contribution compared to the total noise evolved). How that comes about may be seen by the following analysis in which the average influence functional for very weak noisy potentials will be found. We will proceed by again finding an expression for a function of the form

$$\left\langle \exp\left\{ i \int_\tau^T C(t)v(t)dt \right\} \right\rangle$$

where now the average magnitude of $C(t)v(t)$ is very weak but its random character is not necessarily Gaussian in nature. Taking advantage of the fact that $C(t)v(t)$ is small, let us consider the power series expansion of the above function as follows:

$$\left\langle \exp\left\{ i\int_T^T C(t)v(t)dt \right\} \right\rangle = \left\langle 1+i\int_T^T C(t)v(t)dt - \frac{1}{2!}\int_T^T\int_T^T C(t)C(s)v(t)v(s)dsdt + \cdots \right\rangle \tag{22}$$

$$= \left\langle 1 \right\rangle + i\int_T^T \left\langle C(t) \right\rangle v(t)dt - \frac{1}{2!}\int_T^T\int_T^T \left\langle C(t)C(s) \right\rangle v(t)v(s)dsdt - \cdots \tag{23}$$

where the random variable is $C(t)$ as before. Taking the terms one by one we find

$\left\langle 1 \right\rangle = 1$,

$\left\langle C(t) \right\rangle = 0$, since $C(t)$ is random with a distribution centered around $C(t) = 0$,*

$\left\langle C(t)C(s) \right\rangle = R(t-s)$, the correlation function.

If higher order terms than this are considered, for instance, $\left\langle C(t)C(s)C(r) \right\rangle$, more knowledge about the nature of the noise is necessary before definite assertions can be made as to the character of these

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*If $\left\langle C(t) \right\rangle = \overline{C}(t) \neq 0$, the C's can all be changed by $C'(t) = C(t)-\overline{C}(t)$, then R is the correlation function of the fluctuation $C'(t)$, designated by R' here. The $\overline{C}(t)$ acts as an external potential. The expansion becomes

$$\left\langle \exp\left\{ i\int_T^T C(t)v(t)dt \right\} \right\rangle = \left\langle 1 \right\rangle + i\int_T^T \left\langle C'(t) + \overline{C}(t) \right\rangle v(t)dt$$

$$- \frac{1}{2!}\int_T^T\int_T^T \left\langle (C'(t)+C(t))(C'(s)+\overline{C}(s)) \right\rangle v(t)v(s)dsdt + \cdots .$$

All terms which have $\left\langle C'(t) \right\rangle$ disappear, so the above becomes

$$\left\langle \exp\left\{ i\int_T^T C(t)v(t)dt \right\} \right\rangle \approx 1 + i\int_T^T \overline{C}(t)v(t) - \frac{1}{2!}\int_T^T\int_T^T \overline{C}(t)\overline{C}(s)v(t)v(s)dsdt$$

$$- \int_T^T\int_T^t R'(t-s)v(t)v(s)dsdt \approx \exp\left\{ i\int_T^T \overline{C}(t)v(t)dt \right\} \exp\left\{ -\int_T^T\int_T^t R'(t-s)v(t)v(s) \times dsdt \right\} = \exp\left\{ \frac{i}{2\pi\hbar}\int_0^\infty (\overline{C}_\nu v_{-\nu} + \overline{C}_{-\nu} v_\nu)d\nu \right\} \exp\left\{ -\frac{1}{\pi}\int_0^\infty \phi'(\nu) v_\nu v_{-\nu} d\nu \right\} .$$

terms. If it is assumed that all odd terms such as the one above are near zero, as one expects in the case of Brownian noise, then for small average magnitudes of the term $C(t)v(t)$,

$$\left\langle \exp\left\{ i \int_T^T C(t)v(t)dt \right\} \right\rangle \approx 1 - \frac{1}{2} \int_T^T \int_T^T R(t-s)v(t)v(s)dsdt$$

$$\approx \exp\left\{ - \int_T^T \int_T^T R(t-s)v(t)v(s)dsdt \right\} \qquad (24)$$

which was shown to be exactly true in the case of Gaussian noise.

It is of interest to notice that in the event that N such weakly coupled noisy systems are acting on Q then the average influence phase is written very simply in terms of the above form:

$$\langle \phi \rangle_{N \text{ systems}} = -N \int_T^T \int_T^t R(t-s)\, v(t)v(s)dsdt \quad .$$

However, the validity of this result could be questioned on the grounds that in raising the exponential to the Nth power, the errors in the approximation might be compounded into sizeable magnitudes. More clearly stated, if

$$f(a) = 1 + a + C_1 a^2 + C_2 a^3 + \cdots$$

where the C's are of the order of unity, then for sufficiently small values of a one can write $f(a) \approx e^a$. To second order the error incurred is $e^a - f(a) = (\frac{1 - 2C_1}{2})a^2$, therefore an approximation correct to second order would be $e^{a-(1-2C_1)/2\, a^2} \approx f(a)$. Since we have assumed $a \ll 1$ the a^2 in the exponent may be neglected in the

approximation. Now suppose we have $f^N(a)$ to compute. Let us take the expression correct to second order for $f(a)$ and compute $f^N(a)$ as follows,

$$f^N(a) \approx \exp\left[Na - (\frac{1 - 2C_1}{2})Na^2 \right]$$

and the quantity to be examined is the fractional error in representing $f^N(a)$ by e^{Na}

$$\frac{f^N(a)}{e^{Na}} \approx \exp\left[-(\frac{1 - 2C_1}{2})Na^2 \right] \quad .$$

It has been assumed that $a \ll 1$ but Na is not necessarily small. If $Na \sim 1$ and we let N get very large, then the exponent can be expressed

$$\lim_{\substack{Na \sim 1 \\ N \to \infty}} \log(\frac{f^N(a)}{e^{Na}}) \approx -(\frac{1 - 2C_1}{2}) \frac{N^2 a^2}{N} - \cdots \Rightarrow 0 \quad .$$

The next higher order term in the exponent is of the order of Na^3 which approaches zero faster than the second order term. Therefore, in the limit of very small a and very large N the exponential approximation is exactly correct.

IV. Influence Functionals for Linear Systems at Zero Temperatures

There exists a large class of problems in quantum mechanics in which the test system interacts with a linear system. There also may be forces applied to the linear system which in turn transmit these forces in filtered form to the test system. In this section the specific form for influence functionals which are appropriate to linear systems composed entirely of harmonic oscillators will be found. This approach is taken because all linear systems which are lossless and those which contain certain kinds of loss can be represented by combinations of oscillators. In addition, these functionals can be found exactly. In the many cases where dissipation arises from other sources we will show that to the degree that the loss is linear it can also be represented by oscillators. This part of the analysis will be done in Section VII. For simplicity of presentation it is assumed in this section that the interaction system will have zero temperature initially and will have no external forces applied to it. It will be found in later sections that the effects of both temperature and forces can be included in a direct way.

The procedure here will be to consider the interaction system as composed first of one lossless oscillator, then to extend the analysis to arbitrary distributions of oscillators. One outstanding effect of continuous distributions is shown to be that of adding dissipation to the system. Therefore, the form of the influence functionals for dissipative linear systems is also derived, by analyzing lossless oscillators. The use of the influence functional in making a calculation is discussed and second order perturbation expressions are derived for

transition probabilities of a test system when acted on by linear systems and classical potentials. Finally, the general form of functionals for linear systems will be deduced from the general properties of functionals given in Section II.

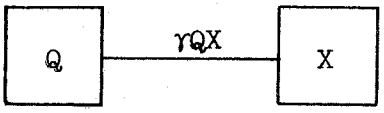
A statement of the theorem to be proved in this section is as follows:

Theorem VII. It is assumed that the test system, Q, is linearly coupled to the linear interaction system, X, so that the total Lagrangian for the system Q,X is

$$\mathcal{L}_{\text{total}} = \mathcal{L}_o(\dot{Q},Q,t) + \mathcal{L}(\dot{X},X,t) + \mathcal{L}_I(\gamma Q,X) \tag{1}$$

where $\mathcal{L}_I(\gamma Q,X) \equiv \gamma QX$.

Diagrammatically, the situation is



The influence phase for the effect of X on Q can then be written

$$\Phi(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu}-Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{Q_{-\nu}(Q_\nu-Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \tag{2}$$

and is found by studying the properties of X alone. Q_ν is the Fourier transform of $\gamma(t)Q(t)$ and $Z(\nu)$ is a function which is derived from the characteristics of X. More exactly, Z_ν is a classical impedance function which relates the reaction of X to an applied force. The rule

is as follows: Z_ν is found by taking the classical system corresponding to X (that is, whose Lagrangian is $\mathcal{L}(\dot{X},X,t)$) and finding the response of the coordinate X to a driving force $f(t)$ which is derived from the potential $-f(t)X(t)$. $f(t)$ is considered to be applied at $t = 0$ subject to the initial conditions that $X(0) = \dot{X}(0) = 0$. Z_ν is defined by the expression

$$Z(\nu) = \frac{f\nu}{i\nu X_\nu} \tag{3}$$

where $f_\nu = \int_0^\infty f(t)\, e^{-i\nu t}dt$ and $X_\nu = \int_0^\infty X(t)\, e^{-i\nu t}dt$. In the time domain equation 2 can be expressed as

$$i\Phi(Q,Q') = -\frac{1}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{t} r_t r_s (Q_t - Q_t')[Q_s F^*(t-s) - Q_s' F(t-s)]dsdt \quad . \tag{4}$$

In the above equation $\mathrm{Im}\, F(t)$, which we will call $B(t)$ is, for $t > 0$, the classical response of X to a force $f(t) = \delta(t)$. $\mathrm{Re}\, F(t)$, which for this zero temperature case we call $A_o(t)$, is the correlation function for the zero point quantum fluctuation of the variable X, a point discussed more at length in Section IV.5. The relations connecting these quantities are then,

$$\begin{aligned} F(t) &= A_o(t) + iB(t) \\ \frac{1}{i\nu Z_\nu} &= \int_0^\infty B(t)e^{-i\nu t}dt \end{aligned} \tag{5a}$$

and the inverse relations

$$A_o(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right)\cos\,\nu t\, d\nu$$

$$B(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right)\sin\,\nu t\, d\nu \qquad (5b)$$

In the case of finite temperatures, the influence phase can be written in the same form as equation 4 except that Re $F(t) = A(t)$, that is, without the subscript o , and a more general relation exists connecting $A(t)$ and $\mathrm{Im}(\frac{1}{i\nu Z_\nu})$, (see Section VI).

IV.1 $\mathcal{F}(Q,Q')$ for Single Lossless Harmonic Oscillator

Consider a test system, Q, which is coupled to a harmonic oscillator whose mass is m, characteristic frequency ω, and displacement coordinate X . The complete Lagrangian for X and Q can be written

$$\mathcal{L}_{total} = \mathcal{L}_o(\dot{Q},Q,t) + \frac{m\dot{X}^2}{2} - \frac{m\omega^2X^2}{2} + \gamma QX \qquad (6)$$

and the total action is written similarly

$$S_{total}(Q,X) = S_o(Q) + \int_\tau^T \left(\frac{m\dot{X}^2}{2} - \frac{m\omega^2X^2}{2} + \gamma QX\right)dt \quad .$$

In the interaction Lagrangian γ is a coupling factor and may or may not be a function of time. If X is assumed to be initially in the ground state (corresponding to zero temperature), then to within a normalizing constant $\chi_i(X) = e^{-m\omega X_\tau^2/2\hbar}$. The final state of X is assumed to be arbitrary which means the final states are to be summed over. Therefore, in equation II.10 the definite state $\chi_f^*(X_T)\,\chi_f(X_T')$ will be replaced by the sum $\sum_n \phi_n^*(X_T)\,\phi_n(X_T') = \delta(X_T - X_T')$.

The $\phi_n(X)$ represent the energy eigenfunctions of the harmonic oscillator. With this information available the influence functional is completely defined and according to the rules already given it can be written as follows:

$$\mathcal{F}_o(Q,Q') = \int \delta(X_T - X'_T) \exp\left[\frac{i}{\hbar}[S(X)-S(X')] + \frac{i}{\hbar}\int_\tau^T \gamma(QX-Q'X')dt\right]$$

$$\times \exp\left[-\frac{m\omega}{2\hbar}(X_\tau^2+X'^2_\tau)\right]\mathcal{D}X(t)\mathcal{D}X(t')dX_\tau dX'_\tau dX_T dX'_T$$

where $S(X) = \int_\tau^T (\frac{m\dot{X}^2}{2} - \frac{m\omega^2 X^2}{2})dt$ and the subscript o on $\mathcal{F}_o(Q,Q')$ indicates zero temperature. The derivation of this expression can be carried out in several ways and in fact has been done previously*. In principle, it is very easy since, as has already been mentioned, for the harmonic oscillator the propagation kernel when a potential is involved, is

$$K(X_T,T;X_\tau,\tau) = \int \exp\left[\frac{i}{\hbar}[S(X)+S_I(Q,X)]\right]\mathcal{D}X(t) = \exp\left[\frac{i}{\hbar}[S(X)+S_I(Q,X)]\right]_{\text{classical}}$$

where the only dependence on X remaining is through its end points X_T and X_τ. Thus,

$$\mathcal{F}(Q,Q') = \int \delta(X_T - X'_T)\ \exp\left[\frac{i}{\hbar}[S(X)-S(X')+S_I(Q,X)-S_I(Q'X')]\right]_{\text{classical}}$$

$$\exp\left[-\frac{m\omega}{2\hbar}(X_\tau^2+X'^2_\tau)\right]\ dX_\tau dX'_\tau dX_T dX'_T \qquad . \qquad (7)$$

*See the equation following Equation 61 in reference (2).

This represents a series of Gaussian integrals since S is itself quadratic in the X variables. However, the algebra involved in calculating equation 7 is considerable and the extension of 7 to cases involving more complicated combinations of oscillators compounds the difficulties. Therefore, $\mathcal{F}(Q,Q')$ is done by a different method in Appendix I(a) which can be extended more readily to other calculations which we wish to do later. The result of Appendix I(a) is

$$i\Phi_o(Q,Q') = -\frac{1}{2\hbar m\omega}\int_{\tau}^{T}\int_{\tau}^{t} \gamma_t\gamma_s(Q_t-Q_t')(Q_s e^{-i\omega(t-s)} - Q_s' e^{i\omega(t-s)})\,ds\,dt \,. \qquad (8)$$

Thus, $F(t-s)$ in equation 4 corresponds in this case to $e^{+i\omega(t-s)}/m\omega$ and from the definition given above $B(t-s) = \frac{1}{m\omega}\sin\omega(t-s)$. The finite time interval indicated by the limits T and τ can be interpreted as turning the coupling (between Q and X) on at $t = \tau$ and off at $t = T$. However, since the interaction system is to be considered in most cases as part of the steady state environment of Q , it is really more meaningful to extend these limits over an infinite range of time $(\tau \to -\infty,\ T \to +\infty)$. The possibility of allowing X to interact with Q over a finite range of time can be taken care of by giving the coupling factor γ_t the proper time dependence. Rewriting 8 in transform notation (see Appendix I(b)) we have

$$\Phi_o(Q,Q') = \frac{1}{2\pi\hbar}\int_0^{\infty}\left[\frac{Q_\nu'(Q_{-\nu}-Q_{-\nu})}{-m[(\nu-i\epsilon)^2-\omega^2]} + \frac{Q_{-\nu}(Q_\nu-Q_\nu')}{-m[(\nu+i\epsilon)^2-\omega^2]}\right] d\nu \,. \qquad (9)$$

In this equation $\dfrac{1}{-m[(\nu-i\epsilon)^2-\omega^2]} = \displaystyle\int_0^{\infty} \frac{\sin\omega t}{m\omega}\, e^{-i\nu t}\, dt$ which corresponds

to $\frac{1}{i\nu Z_\nu}$ of equation 2* .

Having obtained the expression for the influence phase we now turn to the classical problem of finding the response of $X(t)$ to a driving force, $f(t)$, applied at $t = 0$ with the initial conditions $X(0) = \dot{X}(0) = 0$. Starting with the Lagrangian of the unperturbed oscillator from equation 6, we add to it a potential term $-f(t)X(t)$. This potential has the same form as the coupling potential $-\gamma QX$ used in the quantum calculation. However, it is to be emphasized that the response of X to a force has nothing to do with the system Q outside of the type of coupling involved; therefore, $f(t)$ will symbolize the force in the classical problem. The complete Lagrangian is

$$\mathcal{L}(\dot{X},X,t) = \frac{m\dot{X}^2}{2} - \frac{m\omega^2X^2}{2} + fX \tag{10}$$

and the equation of motion derived from it is, as is well known,

$$m\ddot{X} + m\omega^2X = f \quad . \tag{11}$$

Its solution under the initial conditions stated above is

$$X(t) = \frac{1}{m\omega}\int_0^t f(s)\,\sin\omega(t-s)ds \tag{12}$$

---

* ϵ which occurs in $i\nu Z_\nu$ is a convergence factor which was inserted in taking the Fourier transform $1/m\omega\ 1(t)\sin\omega t$ where $1(t)$ is the unit step function. It must be kept to show the location of the poles with respect to the ν-axis where doing integrations of the type

$$\int_0^\infty \frac{H(\nu)}{i\nu Z(\nu)}\,d\nu \quad .$$

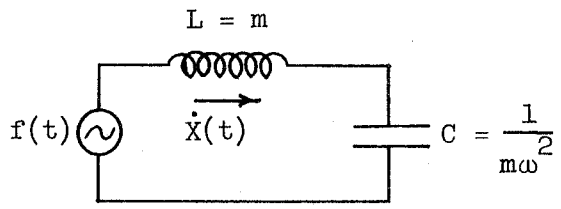
After the calculation the ϵ is set equal to zero. This convergence factor must be used since we are dealing with a lossless system. In practical cases where loss occurs naturally, the poles will automatically be located off the ν-axis.

or alternatively, in terms of Fourier transforms, is

$$X_\nu = \frac{f_\nu}{-m\left[(\nu-i\epsilon)^2-\omega^2\right]} \quad . \qquad (13)$$

Therefore, $B(t-s)$ in this case is a Green's function which yields the response of $X(t)$ to an impulse force $f(s) = \delta(s)$ and its transform yields $1/i\nu Z_\nu$.

Thus a classical calculation of the ratio X_ν/f_ν under quiescent initial conditions yields the proper function for $1/i\nu Z_\nu$ in the influence functional, at least for the case of a single oscillator. The terminology Z_ν was chosen to agree with that widely used in electrical engineering for the case when the harmonic oscillator represents an electric circuit. Thus the circuit corresponding to equation 10 if $X(t)$ represents a charge and $f(t)$ a voltage, is as follows:



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*Since the initial conditions are given for $t = 0$, the transforms correspond to Laplace transforms. There the transformation variable is s and corresponds to our $i\nu+\epsilon$. Thus,

$$X_\nu = \int_0^\infty X(t)e^{-i\nu t}dt$$

$$(\ddot{X})_\nu = \int_0^\infty \ddot{X}e^{i\nu t}dt = \left[\dot{X}(t)e^{-i(\nu-i\epsilon)t} + i(\nu-i\epsilon)X(t)e^{-i(\nu-i\epsilon)t}\right]_0^\infty - (\nu-i\epsilon)^2 X_\nu$$

The products of integration by parts disappear at $t = 0$ because of the quiescent initial conditions and at $t = \infty$ because of the convergence factor. Therefore, $(\ddot{X})_\nu = -(\nu-i\epsilon)^2 X_\nu$.

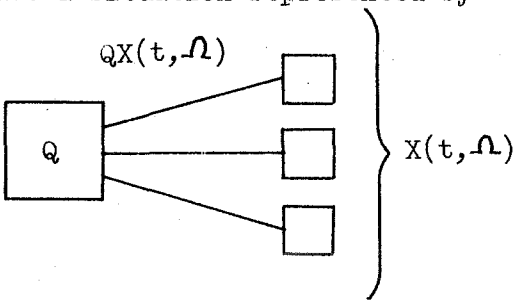
where m represents an inductance, $1/m\omega^2$ a capacitance, with current flow $\dot{X}(t)$. The impedance of the circuit is given by

$$Z_\nu = \frac{f_\nu}{(\dot{X})_\nu} = \frac{f_\nu}{i\nu X_\nu} = \frac{-m[(\nu-i\epsilon)^2-\omega^2]}{i\nu} \quad .$$

This is the same expression for Z_ν as is indicated by equation 13.

IV.2 Distribution of Oscillators. Representation of Loss.

The results of the preceding section are easily extended to the situation where the interaction system is a distribution of oscillators. First, we consider the case of independent oscillators coupled to the test system. It is assumed that there is a distribution of oscillators such that $G(\Omega)d\Omega$ is the weight of oscillators in the range between Ω and $\Omega + d\Omega$. More specifically, $G(\Omega)d\Omega$ is the product of the number of oscillators and the square of their coupling constants in $d\Omega$. Thus, we have a situation represented by



Each oscillator is assumed to be initially in the ground state and finally in an arbitrary state; the coupling is again assumed to be linear. The total action is then given by

$$S(Q,X(\Omega)) = S_o(Q) + \int_T^T \int_0^\infty G(\Omega)\left[\frac{\dot{X}^2}{2} - \frac{\Omega^2 X^2}{2} + QX\right] d\Omega dt \quad . \qquad (14)$$

From the general properties of influence functionals already described

we know that when independent disturbances act on Q the influence functional is a product of the ones for each individual disturbance. Since $\mathcal{F}_o(Q,Q') = \exp[i\Phi_o(Q,Q')]$ for the case of a single oscillator, the total influence phase for the distribution is the sum of the individual phases,

$$\sum \Phi_o(Q,Q') = \int_0^\infty G(\Omega)d\Omega\,\Phi_o(Q,Q') \tag{15}$$

More explicitly,

$$\Phi_o = \frac{1}{2\pi\hbar}\int_0^\infty G(\Omega)d\Omega\int_0^\infty \left\{\frac{Q'_\nu(Q_{-\nu}-Q'_{-\nu})}{-[(\nu-i\epsilon)^2-\Omega^2]} + \frac{Q_{-\nu}(Q_\nu-Q'_\nu)}{-[(\nu+i\epsilon)^2-\Omega^2]}\right\} d\nu \quad . \tag{16}$$

For this case then, the form of equation 2 is obtained if we put

$$\frac{1}{i\nu Z_\nu} = \int_0^\infty \frac{-G(\Omega)}{(\nu-i\epsilon)^2-\Omega^2}\, d\Omega \tag{17}$$

or

$$\frac{1}{Z_\nu} \underset{\epsilon\to 0}{\Longrightarrow} \frac{\pi G(\nu)}{2} - i\nu\int_0^\infty \frac{G(\Omega)}{\nu^2-\Omega^2}\, d\Omega \;^* \quad . \tag{18}$$

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*Equation 18 is obtained from 17 in an obvious way if we make the expansion

$$\lim_{\epsilon\to 0}\frac{1}{(\nu-i\epsilon)^2-\Omega^2} = \lim_{\epsilon\to 0}\left[\frac{\frac{1}{2\Omega}}{\nu-i\epsilon-\Omega} - \frac{\frac{1}{2\Omega}}{\nu-i\epsilon+\Omega}\right]$$

$$= \frac{1}{\nu^2-\Omega^2} + \frac{i}{2\Omega}\lim_{\epsilon\to 0}\frac{\epsilon}{(\nu-\Omega)^2+\epsilon^2} - \frac{i}{2\Omega}\lim_{\epsilon\to 0}\frac{\epsilon}{(\nu+\Omega)^2+\epsilon^2}$$

$$= \frac{1}{\nu^2-\Omega^2} + \frac{i\pi}{2\Omega}\left[\delta(\nu-\Omega)-\delta(\nu+\Omega)\right]$$

Thus the effects of all the oscillators are included in the influence phase through the expression for Z_ν, equation 17. Now, however, because of the continuous distribution of oscillators, Z_ν has a finite real part. We will now show that this real part represents dissipation by arriving at the same impedance function classically.

The procedure is the same as before. We take the part of the Lagrangian from equation 11 having to do with the oscillators, except that the coupling potential $-Q(t)\int_0^\infty G(\Omega)X(\Omega,t)d\Omega$ is replaced by $-f(t)\int_0^\infty G(\Omega)X(\Omega,t)d\Omega$, a classical potential. $X(\Omega,t)$ is the coordinate of the oscillator in the distribution whose frequency is Ω while the total coordinate of the complete linear system with which $f(t)$ is interacting is $\int_0^\infty G(\Omega)X(\Omega,t)d\Omega = X(t)$. It is the relationship between $f(t)$ and $X(t)$ in which we are interested in this classical case:

$$\mathcal{L}(\dot{X}(\Omega),X(\Omega),t)=\int_0^\infty G(\Omega)d\Omega\left[\frac{\dot{X}(\Omega)^2}{2} - \frac{\Omega^2X(\Omega)^2}{2}\right] + f(t)\int_0^\infty G(\Omega)X(\Omega)d\Omega \quad . \tag{19}$$

The equations of motion are the infinite set

$$\int_0^\infty\left[G(\Omega)d\Omega\left[\ddot{X}_\Omega + \Omega^2X_\Omega^2 - f\right]\right] = 0 \quad . \tag{20}$$

However, since the oscillators are not coupled to each other, this relation can be true always only if the integrand is zero. This results in the same equation as before for one oscillator,

$$\ddot{X}(\Omega,t) + \Omega^2 X(\Omega,t) = f(t) \quad . \tag{21}$$

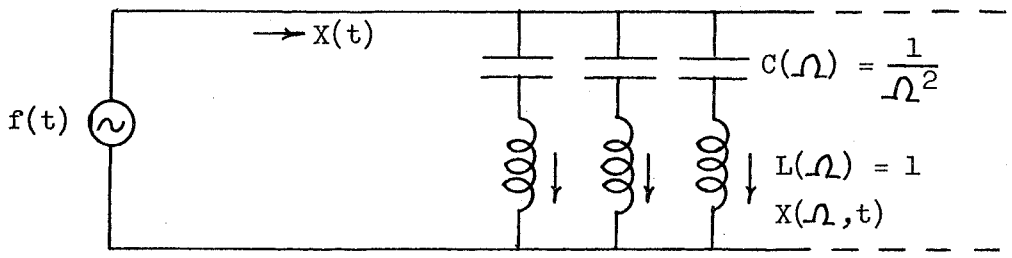
For quiescent initial conditions and for $f(t)$ applied at $t = 0$ we have the same solution as before. In Fourier transforms this solution is expressed

$$\frac{X_\nu(\Omega)}{f_\nu} = -\frac{1}{(\nu-i\epsilon)^2-\Omega^2} = \frac{1}{i\nu Z_\nu(\Omega)} \quad .$$

The relation of the total coordinate X_ν to f_ν is obtained simply*

$$\frac{X_\nu}{f_\nu} = \frac{\int_0^\infty X_\nu(\Omega)G(\Omega)d\Omega}{f_\nu} = -\int_0^\infty \frac{G(\Omega)d\Omega}{(\nu-i\epsilon)^2-\Omega^2} = \frac{1}{i\nu Z_\nu} \quad . \tag{22}$$

*To get a better picture of the situation, let us return to the circuit analogy. Equation 20 is the Lagrangian for an infinite number of resonant circuits in parallel, all driven by the same voltage, $f(t)$. There are $G(\Omega)d\Omega$ circuits with frequencies between Ω and $\Omega + d\Omega$. Each circuit is characterized by a capacitance $1/\Omega^2$, a unit inductance, and a charge flow equal to $X(\Omega,t)$. The total charge flowing through the generator is the sum of the charges in the individual circuits, i.e., $X(t) = \int_0^\infty G(\Omega)X(\Omega,t)d\Omega$.



Comparing this with equation 17 it is seen that again the same expression for Z_ν is obtained in the quantum and classical cases. In addition, since Z_ν is now identified with a classical impedance, the real part represents resistance while the imaginary part corresponds to reactance. Therefore, dissipation has been introduced into the influence functional for cases where the loss is representable by a continuous distribution of oscillators. To demonstrate this in a concrete case the spontaneous emission probability of a particle in free space has been computed using the influence function method (Appendix III). This example demonstrates a particular case in quantum mechanics where loss arises from a distribution of oscillators.

The relationship

$$\frac{1}{i\nu Z_\nu} = \int_0^\infty B(t)\, e^{-i\nu t} dt$$

has already been established during the course of the derivation of the influence phase for the single oscillator. Now the inverse relation between $F(t)$ and $1/i\nu Z_\nu$ can be written for the zero temperature case. In the time domain the influence phase for the distribution of oscillators is

$$\Phi(Q,Q') = -\frac{1}{2\hbar}\int_0^\infty \frac{G(\Omega)}{\Omega}\, d\Omega \int_{-\infty}^{\infty}\int_{-\infty}^{t} (Q_t - Q_t')(Q_s e^{-i\Omega(t-s)} - Q_s' e^{i\Omega(t-s)})\, ds\, dt \ .$$

Comparing this with equation 4 it is evident that

$$F(t) = \int_0^\infty \frac{G(\Omega)}{\Omega}\, e^{i\Omega t}\, d\Omega \ .$$

However, it has just been shown in equation 18 that

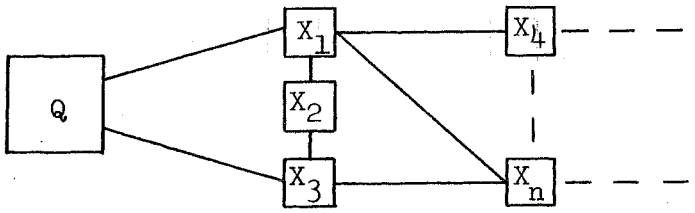
$$-\mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right) = \frac{\pi G(\nu)}{2\nu} \quad .$$

Therefore, it can be immediately written that

$$F(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right) e^{i\nu t}\, d\nu$$

as was given in equations 5b.

The results above can now be extended by a simple argument to include all linear systems composed entirely of distributions of oscillators. To do this it need only be shown that the general system can be reduced to a distribution of oscillators independently coupled to the test system, which was the situation just considered. To be more definite, suppose there exists a test system Q , coupled to an assemblage of oscillators which are also interconnected with each other. For instance, the situation might be as shown below, where each of the X_n components of the total interaction system could also represent a system of oscillators.



However, it is well known* that such a linear system may be represented by an equivalent set of oscillators (the normal modes of the total system) independently coupled to Q .**

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* This point is considered more fully in Section V on classical forces.

**The fact that one or more of the X_n might represent continuous distributions of oscillators need not be bothersome since in principle they represent the behavior of the total system in terms of its infinite set of normal modes.

Or, said another way, the classical representation of the Lagrangian in normal modes finds new linear combinations of the X_n which makes the total Lagrangian, except for the coupling, a sum of individual quadratic forms with no cross terms. But, this same transformation of variables can be made on the expression for $\mathcal{F}(Q,Q')$ (see equation II.10). The effect of this transformation is to change the $\mathcal{D}X$ volume by a numerical factor, since the transformation is linear.* Thus, in effect, we get the sum of independent systems in the quantum mechanical case also. From this argument it is concluded that the results above regarding a distribution of independent oscillators coupled to a test system, apply to any linear interaction system. Therefore, it has been found that the influence functional for all linear systems has exactly the same form $e^{+i\Phi_o(Q,Q')}$ where $\Phi_o(Q,Q')$ is a quadratic functional of the Q and Q'. $\Phi_o(Q,Q')$ is adapted to a particular linear system only through the classical response of that linear system to a force. Thus, the procedure for finding the influence functional for a linear system has been reduced to a classical problem, as was outlined at the beginning of this section. The fact that eliminating the coordinate of an oscillator always yields an influence functional which is quadratic in the potential applied to that oscillator, is a basic property of linear systems. For example, where the coupling Lagrangian is linear between an oscillator of coordinate X and another system of coordinate Q, the elimination of the X coordinate yields an influence

*The only result of such a numerical factor would be to change the normalization of $\mathcal{F}(Q,Q')$. However, we already know that for the case that the final states of the interaction system are summed over $\mathcal{F}(Q,Q) = 1$. Therefore the normalization of $\mathcal{F}(Q,Q')$ is not changed by the transformation and thus is not dependent upon the coordinates chosen to represent the interaction system.

phase which is quadratic in Q as has already been shown. If Q were the coordinate of another oscillator coupled to P, then elimination of the Q coordinates would yield an influence phase quadratic in P, etc. This can be understood mathematically by observing that the Lagrangian for all the oscillators with linear coupling is always quadratic. Doing the path integral to eliminate a coordinate is basically a process of completing the square and performing Gaussian integrals. This process of completing the square also yields quadratic terms. It is therefore not surprising that the influence phase for any linear system should be always of the same quadratic form.

It is to be emphasized that the analysis so far presented has been concerned entirely with systems whose complete behavior can be described by combinations of lossless oscillators at zero temperature. The only example of such a system is the electromagnetic field in free space. In all other physical situations linear behavior is an approximation to the actual behavior. However, this approximation may be very good over a wide range of operating conditions. In a later section the problem of approximately linear systems will be considered in detail. The results will be found to be the same as for perfect oscillators to the extent that linear behavior is realized.

IV.3 Use of Influence Functionals

At this point in the analysis we need to consider how influence functionals are to be used in the calculation of a problem. For clarity the discussion will be specialized to a particular problem but the principle is valid more generally. Suppose we wish to know the probability that a test system Q makes a transition from an initial state

$\phi_n(Q_\tau)\exp\left[-\frac{i}{\hbar}E_n\tau\right]$ to a final state $\phi_m(Q_T)\exp\left[-\frac{i}{\hbar}E_mT\right]$ when coupled to an interaction system. The formal expression for this probability is, from Section II,

$$P_{n\to m}=\int\phi_m^*(Q_T)\phi_m(Q_T')\exp\left\{\frac{i}{\hbar}[S_o(Q)-S_o(Q')]\right\}\mathcal{F}(Q,Q')\phi_n^*(Q_\tau')\phi_n(Q_\tau)dQ_\tau\cdots\mathcal{D}Q'(t). \tag{23}$$

This is formally exact but except in special cases it cannot be evaluated exactly. Furthermore, to obtain any specific answers to the problem the characteristics of Q must be known as well as knowing the influence functional. However, by using perturbation theory we may find general expressions for transition probabilities to as many orders as desired. For example, if the interaction system is a linear system at zero temperature, we know that $\mathcal{F}_o(Q,Q')$ is of the form $e^{i\Phi_o(Q,Q')}$. The perturbation expansion is obtained by writing $e^{i\Phi_o(Q,Q')}$ in terms of a power series and evaluating the path integral corresponding to each term in the expansion. In many cases the coupling between Q and the interaction system is small enough that only a few terms in the expansion are necessary. In Appendix II the procedure for finding the perturbation expansion is described and in addition specific expressions are found up to second order in the potentials involved for the several forms of influence functionals already derived. To illustrate the types of expressions which might be obtained we suppose that the coupling potential between Q and the linear system described above is of the form $XV(Q)$ where $V(Q)$ is very small but that they are coupled for a very long interval of time T. The approximate expression for equation 23 is as follows:

$$P_{\substack{n \to m \\ n \neq m}} = \frac{2T\left|V_{mn}\right|^2}{\hbar\, \nu_{nm}} \mathrm{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) \qquad \text{for} \quad \nu_{nm} > 0 \tag{A2.20}$$

$$= 0 \qquad \text{for} \quad \nu_{nm} < 0$$

and

$$P_{n \to n} = 1 - \sum_k \frac{2T\left|V_{nk}\right|^2}{\hbar\, \nu_{nk}} \mathrm{Re}\left(\frac{1}{Z_{\nu_{nk}}}\right) \quad \text{for all } k \text{ such that } \nu_{nk} > 0 \tag{A2.21}$$

where

$$Z_{\nu_{nk}} = Z_\nu\Big|_{\nu=\nu_{nk}}$$

and

$$V_{nk} = \int \phi_n^*(Q) V(Q) \phi_k(Q)\, dQ \quad .$$

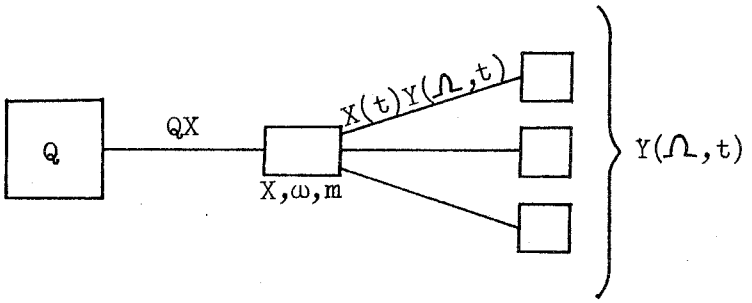
Equation A2.20 is interpreted as saying that Q may make only transitions to states of lower energy which follows from the fact that the interaction system is itself initially at zero temperature. Equation A2.21 is the probability for Q to remain in its initial state. This, of course, is unity minus the probability of transitions to all possible other states. In this case the only possibilities are the states lower in energy. These, then, are expressions for the probability of spontaneous emission of the test system into the linear interaction system and it is to be noticed that they are derived from the zero temperature influence functional $\mathcal{F}_o(Q,Q')$, equation 2.

Calculation of transition probabilities represents only one piece of information that one might desire to know about a test system. For instance, it is more usually desired to find the expectation value of

an operator in the test system. To calculate this one needs to know the density matrix describing the test system when it is coupled to an interaction system. The exact expression for the required density matrix is given in Section II.4. Again in the general case, one runs into the difficulty of making an exact calculation and is forced to make calculations using perturbation theory. The same procedure of expanding the influence functional into a power series and performing the required path integrations yields the perturbation expressions.

IV.4 Influence Functional for an Oscillator Damped by a Distribution.

It has already been established that the influence of any linear system comprised of oscillators can be found from a knowledge of the impedance of the system. Therefore, this example is not needed to establish the validity of the principle. However, the way in which loss is represented by a distribution of oscillators is perhaps easier to visualize by the physical picture afforded in this example than in the case of Section IV.2 where a very general distribution of oscillators was considered. The situation here can be thought of as a group of atoms, comprising the test system, Q, located within a resonant cavity. The cavity walls are lossless except for one or more holes through them which couple the electromagnetic field in free space outside to the field inside of the cavity. The free space field will be taken care of by a distribution of oscillators as before. The problem is therefore one of finding the influence of an oscillator coupled to a distribution $G(\Omega)$ defined as before, on a test system. The diagram below depicts the situation:



The Lagrangian for the complete system is written

$$\mathcal{L}(\dot{Q},Q,\dot{X},X,\dot{Y}(\Lambda),Y(\Lambda),t) =$$

$$\mathcal{L}_o(\dot{Q},Q,t)+ \frac{m\dot{X}^2}{2} - \frac{m\omega^2X^2}{2} + QX + \int_0^\infty G(\Lambda)\left[\frac{\dot{Y}^2}{2} - \frac{\Lambda^2Y^2}{2} + XY\right] d\Lambda \quad . \qquad (24)$$

The influence of the distribution on X has been determined already. In the time integral form it is of the form of equation 8 integrated over the distribution of oscillators. Knowing this, which we call $\mathcal{F}(X,X')$, then

$$\mathcal{F}(Q,Q') = \int \delta(X_T - X_T')\,\mathcal{F}(X,X')\exp\left\{\frac{i}{\hbar}[S(X)-S(X')+\int_\tau^T (QX-Q'X')dt]\right.$$

$$\left. - \frac{m\omega}{2\hbar}(X_\tau^2 + X_\tau'^2)\right\} dX_\tau \cdots \mathcal{D}X(t) \quad , \qquad (25)$$

where $S(X)$ is the action of the unperturbed oscillator. This derivation is carried out in Appendix V by doing the path integral directly as indicated in equation 25. As was anticipated, the form of the influence functional is the same as found previously (equation 2). The expression for Z_ν is (from equation A5.17)

$$Z_\nu = \frac{\pi G(\nu)}{2\nu^2} + \frac{i}{\nu}\left[m(\nu^2-\omega^2) - \int_0^\infty \frac{G(\Lambda)d\Lambda}{\nu^2-\Lambda^2}\right] . \qquad (26)$$

Again, the distribution of oscillators produces a dissipative term, here

recognized as $\pi G(\nu)/2\nu^2$

To compute the corresponding classical case the procedure is the same as before. Thus, taking the Lagrangian for the linear system driven by a classical force $f(t)$ we obtain

$$\mathcal{L}(\dot{X},X,\dot{Y}_{\Omega},Y_{\Omega},t) =$$

$$\frac{m\dot{X}^2}{2} - \frac{m\omega^2X^2}{2} + fX + \int_0^\infty G(\Omega)\left[\frac{\dot{Y}^2}{2} - \frac{\Omega^2Y^2}{2} + XY\right] d\Omega \quad . \tag{27}$$

The equations of motion are in transform notation,

$$m\left[(\nu-i\epsilon)^2 - \omega^2\right] X_\nu + \int_0^\infty G(\Omega)\, Y_\nu(\Omega)\, d\Omega = -f_\nu$$

$$\left[(\nu-i\epsilon)^2 - \Omega^2\right] Y_\nu = -X_\nu \quad . \tag{28}$$

As before, $f(t)$ is applied at $t = 0$ and quiescent initial conditions prevail on $X(t)$ and $Y(\Omega,t)$. Solving these equations for the ratio f_ν/X_ν again yields equation 26. Notice that the finite resistance $\pi G(\nu)/2\nu^2$ permits us to drop the convergence factors because the poles of $1/Z_\nu$ do not lie on the ν axis as they do in the case of no loss. Again we find that it is the classical impedance which appears

in the influence phase.*

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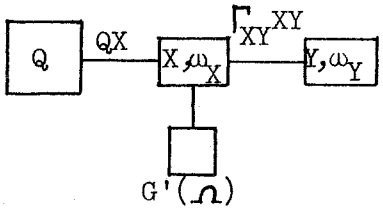
*It is interesting to note that one can find the influence functional for two oscillators in series as follows:



The impedance Z_ν can be deduced easily from the results of calculation in the quantum case by letting

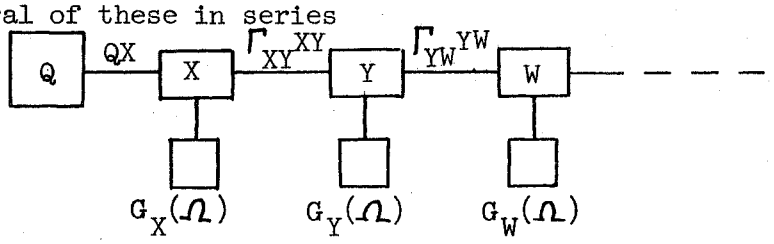
$$G(\Omega) = G'(\Omega) + \frac{\Gamma_{XY}^2}{m_Y}\,\delta(\Omega-\omega_Y) \quad .$$

From equation 26,

$$Z'_\nu = \frac{\pi G'(\nu)}{2\nu^2} + \frac{im_X}{\nu}(\nu^2-\omega_X^2) - \frac{i}{\nu}\left[\frac{\Gamma_{XY}^2}{m_Y(\nu^2-\omega_Y^2)}\right] - \frac{i}{\nu}\int_0^\infty \frac{G'(\Omega)\,d\Omega}{\nu^2-\Omega^2}$$

$$= Z_\nu + \frac{\Gamma_{XY}^2/\nu^2}{\left[\frac{im}{\nu}(\nu^2-\omega_Y^2)\right]} = Z_\nu + \frac{\Gamma_{XY}^2/\nu^2}{Z_Y(\nu)} \quad .$$

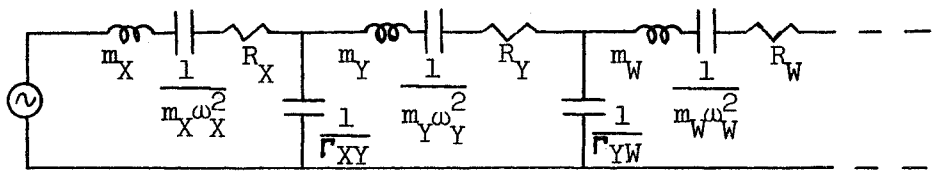
Here Γ_{XY} is a coupling constant corresponding in circuit theory to capacitive coupling between two resonant circuits. Similarly, if there are several of these in series



then

$$Z_\nu = Z_X(\nu) + \cfrac{\Gamma_{XY}^2/\nu^2}{Z_Y(\nu) + \cfrac{\Gamma_{YW}^2/\nu^2}{Z_W(\nu)+\cdots}}$$

corresponding to



IV.5 Form of Influence Functionals for Linear Systems and Classical Forces Deduced from Properties of Influence Functionals

So far, we have found the influence phase for classical potentials, uncertain classical potentials, and linear systems at zero temperature. By studying equations III.1, III.12, and 4, we see that the general form for the influence functional in which all three of these were acting on Q is

$$\mathcal{F}(Q,Q') = \exp\left\{\int_T^T i\alpha(t)(Q(t)-Q'(t))dt - \int_T^T\int_T^t A(t-s)(Q_t-Q_t')(Q_s-Q_s')dsdt + \int_T^T\int_T^t iB(t-s)(Q_t-Q_t')(Q_s+Q_s')dsdt\right\} \tag{27}$$

where $A(t-s) + iB(t-s) \equiv F(t-s)$.

The exponent is writtenall in terms of Q for simplicity although when the potentials in Q are not linear (as $V(Q)$), the same general form exists, except that it is written in terms of $V(Q)$. We now observe that there are other possible combinations of the Q,Q' variables not represented here such as terms in (Q_t+Q_t'), $(Q_t+Q_t')(Q_s+Q_s')$. To see if such terms are possible, let us form a hypothetical functional containing all possible forms up to second order in Q .

$$\mathcal{F}(Q,Q') = \exp\left\{\int_T^T\left[i\alpha(t)(Q_t-Q_t')+\beta(t)(Q_t+Q_t')\right]dt - \int_T^T\int_T^t A(t-s)(Q_t-Q_t')(Q_s-Q_s')dsdt + \int_T^T\int_T^t\left[iB(t-s)(Q_t-Q_t')(Q_s+Q_s')+iC(t-s)(Q_t+Q_t')(Q_s-Q_s')+D(t-s)(Q_t+Q_t')(Q_s+Q_s')\right] \times dsdt\right\} \tag{28}$$

That the coefficients of the Q's inside the double integrals should be functions of $(t-s)$ is evident since the functional should not depend on the absolute time. We now will try to eliminate terms in the exponent by using the general properties of $\mathcal{F}(Q,Q')$ given in Section II. First, we know $\mathcal{F}(Q,Q') = \mathcal{F}^*(Q',Q)$. This implies that all the functions $\alpha,\beta,A,B,C,$ and D are real. Next, we know that $\mathcal{F}(Q,Q') = 1$ if $Q'(t) = Q(t)$. Hence β,D are zero. This leaves only one term which we did not have before, that of $C(t-s)$. Now we apply the corollary to Theorem V which says that if $Q_t = Q'_t$ for $t > t_o$ then $\mathcal{F}(Q,Q')$ is independent of Q for $t > t_o$. This statement is obviously true for the $\alpha(t)$ and $A(t-s)$ terms. Consider now the $B(t-s)$ term. For $t > t_o$, $Q_t - Q'_t = 0$ and therefore this term is also legitimate. As for the $C(t-s)$ term, let us consider $t > t_o$, but $s < t_o$. Then $Q_s - Q'_s \neq 0$. Furthermore, $Q_t + Q'_t = 2Q_t \neq 0$. Therefore, $C(t-s)$ must also be zero. The fact that $C(t-s) = 0$ is actually a statement of causality, i.e., that the effect due to an applied force cannot precede the time the force was applied. To see this, let us change the limits of integration on this term

$$\int_T^T \int_T^t C(t-s)(Q_t+Q'_t)(Q_s-Q'_s)dsdt = \int_T^T \int_S^T C(t-s)(Q_t+Q'_t)(Q_s-Q'_s)dtds \quad .$$

Interchanging s for t in the last integral yields

$$\int_T^T \int_t^T C(s-t)(Q_t-Q'_t)(Q_s+Q'_s)dsdt \quad .$$

Now the integrand is of the same form as that of the $B(t-s)$ term. However, for a fixed t the integration over s is over the range of

$s > t$. This amounts to a sum over the future rather than a sum over past histories of the variable Q .

The conclusion to be drawn from all this is that there are three possible types of terms up to second order in Q and Q' when definite classical forces, indefinite classical forces, and linear systems act on Q . Terms of this type have already been derived during the course of our analysis. Therefore, there are no major types of phenomena which have not been noticed. In the light of the above discussion we would expect the effects of additional phenomena, if they are described by terms of second order or less in Q and Q', to be contained in one or more of the three forms of exponents shown in equation 27. For instance, in the case of a linear system at finite temperature, it will be found that the effect of temperature is to change the effective value of $A(t-s)$ in the exponent of equation 27 from its minimum value, $A_o(t-s)$, which occurs at zero temperature (see equation 5b). It should be pointed out that although $A(t-s)$ occurs in a term which has the form of an uncertain classical potential acting on the test system, at zero temperature one must be careful about this interpretation. For the existence of a random classical potential implies a random fluctuation of the variables of the interaction system which could induce transitions in the test system either upwards or downwards in energy. However, if the interaction system is already in its lowest state it can only induce downward transitions in the test system so that the term in $A_1(t-s)$ by itself is not sufficient. Thus, as has already been found, the exponent of the zero temperature influence functional contains two terms, one in $A_o(t-s)$ and the other in $B(t-s)$--which are related through

equations 5a,b. Together they give the whole picture, i.e., that there is a zero point, random fluctuation of the variables of the interaction system but that this fluctuation can induce only those transitions in the test system which give up energy to the interaction system.

V. Influence Functionals for Classical Forces Acting through Linear Systems

If a classical force is applied to a linear interaction system which in turn is coupled to a test system, the effect of the interaction system is to modify the character of the force applied to the test system. In this section we will find the exact form for the influence functional of this effective force. More precisely, the theorem is as follows:

Theorem VIII. If a linear system is coupled to $Q(t)$ through one of its coordinates $X(t)$ and if a classical force $C(t)$ is coupled to another coordinate $Y(t)$, then $\Phi(Q,Q')$ representing the effect of both the linear system and the force is

$$\Phi(Q,Q') = \Phi_o(Q,Q') + \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{C_\nu(Q_{-\nu}-Q'_{-\nu})}{(i\nu z_\nu)} + \frac{C_{-\nu}(Q_\nu-Q'_\nu)}{(-i\nu z_{-\nu})}\right] d\nu \qquad (1)$$

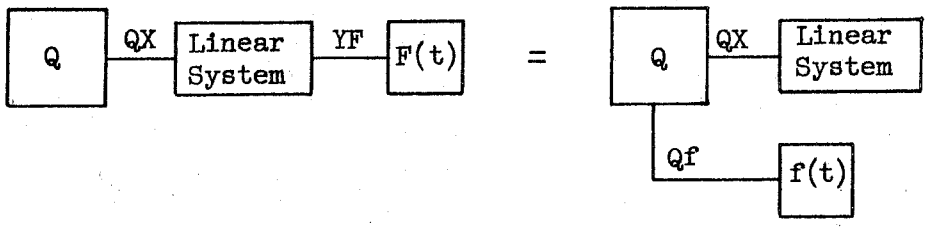
where z_ν is a transfer impedance function which modifies the effect of C_ν on Q. It is found by computing the classical response of the coordinate X to the force C with all other potentials acting on the linear system (including those due to coordinates of external systems such as Q) set equal to zero. The result of the calculation yields $i\nu Z_\nu = C_\nu/X_\nu$. Alternatively, in the time domain,

$$\Phi(Q,Q') = \Phi_o(Q,Q') + \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{t}(Q_t-Q'_t)b(t-s)C_s\,ds\,dt \qquad (2)$$

where

$$\frac{1}{i\nu z_\nu} = \int_0^\infty b(t)e^{-i\nu t}dt \quad .*$$

In the form of a diagram the theorem can be stated



where $f(t) = \int_{-\infty}^{t} F(s)b(t-s)ds$. It will be convenient to work in the frequency domain.

First we recall the influence phase for a classical force acting directly on Q . From this expression we will be able to identify the character of the force acting on Q in more complicated expressions. If the potential is of the form $-C(t)Q(t)$ we have, from Section III

$$\Phi(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[C_\nu(Q_{-\nu}-Q'_{-\nu}) + C_{-\nu}(Q_\nu - Q'_\nu) \right] d\nu \qquad (3)$$

where $C_\nu = \int_T^T C(t)\ e^{-i\nu t}\ dt$.

V.1 Classical Potential and Q Coupled to the Same Coordinate

Before developing the general situation we consider the simpler situation where Q is coupled to a linear system through the potential $-QX$ and a force $F(t)$ is applied to the same system through the potential $-FX$. The Lagrangian for the complete system is written

*The notation $z_\nu, b(t-s)$ was chosen to avoid confusion with Z_ν and $B(t-s)$ which are the impedance and response function respectively of the linear system as seen by the test system.

$$\mathcal{L}(\text{system}) = \mathcal{L}_o(\dot{Q},Q,t) + (F+Q)X + \mathcal{L}(\dot{X},\dot{Y},\cdots X,Y,\cdots t) \qquad (4)$$

where $X,Y,\cdots$ represent all the coordinates of the linear system. If $F = 0$,

$$\Phi(Q,Q') \equiv \Phi_o(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu}-Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{Q_{-\nu}(Q_\nu-Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \quad . \qquad (5)$$

If $F \neq 0$ it is evident from 3 that the required influence phase can be found by replacing Q_ν by $Q_\nu + F_\nu$ and Q'_ν by $Q'_\nu + F_\nu$. Notice that F_ν does not carry the prime notation since it is not a coordinate. If this substitution is made in 5, we have

$$\Phi(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{(Q'_\nu+F_\nu)(Q_{-\nu}-Q'_{-\nu})}{(i\nu Z_\nu)} + \frac{(Q_{-\nu}+F_{-\nu})(Q_\nu-Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu$$

$$= \Phi_o(Q,Q') + \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{F_\nu}{i\nu Z_\nu}(Q_{-\nu}-Q'_{-\nu}) + \frac{F_{-\nu}(Q_\nu-Q'_\nu)}{(-i\nu Z_{-\nu})}\right] d\nu \quad . \qquad (6)$$

As might be expected the total effect of the linear system and the driving force consists of two separate terms, one describing the effect of the linear system alone, and the other describing the effect of the driving force. Comparison of equations 2 and 6 shows that the effective force applied to Q is in transform language $F_\nu/i\nu Z_\nu$ and further, shows that F_ν is modified by $1/i\nu Z_\nu$, the classical impedance function of the interaction system. In this special example where F and Q are both coupled to the same coordinate, Z_ν is both the correct impedance to be used in $\Phi_o(Q,Q')$, i.e., that impedance seen by the test system, and is the transfer impedance z_ν which modifies F_ν. This is not true generally as we shall see in

the next section. In addition, it is interesting to observe that no unexpected quantum effects appear because of the addition of a force to the interaction system. The only effect of the interaction system is to modify the characteristics of $F(t)$ in an entirely classical way.

V.2 Classical Forces Acting through a General Linear System

Having obtained an idea of the type of results to expect in the above simplified analysis we now proceed to the more general case. Let the N coordinates of the interaction system be represented by X_i, $i = 1 \cdots N$. Its coupling to the test system, $Q(t)$, and to the driving force $C(t)$ is given by the potentials $-X_n Q$ and $-X_k C$, respectively. Thus we are assuming for simplification in writing that Q is coupled only to the variable X_n and the force $C(t)$ is applied just to the variable X_k. Again the interaction system is assumed to be composed entirely of harmonic oscillators. The Lagrangian is

$$\mathcal{L}(\text{system}) = \mathcal{L}(\dot{Q},Q,t) + \sum_{i,j} [\tfrac{1}{2}(T_{ij}\dot{X}_i\dot{X}_j - V_{ij}X_iX_j)] + X_nQ + X_kC \quad . \tag{7}$$

It is well known in the theory of linear systems that new coordinates may be defined by means of a linear transformation of the X_i. These new coordinates will be chosen as the eigenvectors, Y_ℓ, of the interaction system (8). Thus,

$$X_i = \sum_{\ell=1}^{n} a_{i\ell} Y_\ell \qquad i = 1,2,\cdots n \quad .$$

Assuming the $a_{i\ell}$ to be properly normalized, the Lagrangian may be rewritten as follows

$$\mathcal{L}(\dot{Q},Q,\dot{Y}_i,Y_i,t)=\mathcal{L}_0(\dot{Q},Q,t)+\sum_\ell\left[\frac{1}{2}(\dot{Y}_\ell^2-\omega_\ell^2 Y_\ell^2)+Y_\ell(a_{n\ell}Q+a_{k\ell}C)\right] \quad . \tag{8}$$

Since these are now independent oscillators coupled to Q the influence phase can be written down immediately,

$$\Phi(Q,Q')=\sum_\ell \Phi_\ell(Q,Q')=\sum_\ell \frac{1}{2\pi\hbar}\int_0^\infty \left[Q'_\nu(Q_{-\nu}-Q'_{-\nu})\left(\frac{a_{n\ell}^2}{i\nu Z_\ell(\nu)}\right)+Q_{-\nu}(Q_\nu-Q'_\nu)\left(\frac{a_{n\ell}^2}{-i\nu Z_\ell(-\nu)}\right)\right]d\nu$$

$$+\sum_\ell \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{a_{n\ell}a_{k\ell}C_\nu}{i\nu Z_\ell(\nu)}(Q_{-\nu}-Q'_{-\nu})+\frac{a_{n\ell}a_{k\ell}C_{-\nu}}{-i\nu Z_\ell(-\nu)}(Q_\nu-Q'_\nu)\right]d\nu \tag{9}$$

where

$$\frac{1}{i\nu Z_\ell(\nu)}=\left.\frac{Y_\ell(\nu)}{a_{n\ell}Q_\nu}\right|_{C_\nu=0}\equiv\left.\frac{Y_\ell(\nu)}{a_{k\ell}C_\nu}\right|_{Q_\nu=0} \quad , \tag{10}$$

calculated classically.

This can be written in the form of equation 1 if we make the correspondence

$$\frac{1}{i\nu Z_\nu}=\sum_\ell \frac{a_{n\ell}^2}{i\nu Z_\ell(\nu)}$$

and

$$\frac{1}{i\nu z_\nu}=\sum_\ell \frac{a_{n\ell}a_{k\ell}}{i\nu Z_\ell(\nu)} \tag{11}$$

Using equations 10 and 11 we now wish to show that $\frac{1}{i\nu Z_\nu}$ and $\frac{1}{i\nu z_\nu}$ are equivalent to $X_n(\nu)/Q_\nu$ and $X_n(\nu)/C_\nu$ respectively. Thus

$$\frac{1}{i\nu Z_\nu}=\sum_\ell \frac{a_{n\ell}^2}{i\nu Z_\ell(\nu)}=\sum_\ell a_{n\ell}^2\left(\frac{Y_\ell(\nu)}{a_{n\ell}Q_\nu}\right)\Bigg|_{C_\nu=0}=\sum_\ell \left.\frac{a_{n\ell}Y_\ell(\nu)}{Q_\nu}\right|_{C_\nu=0}=\left.\frac{X_n(\nu)}{Q_\nu}\right|_{C_\nu=0} \tag{12}$$

and

$$\frac{1}{i\nu z_\nu} = \sum_\ell \frac{a_{n\ell}a_{k\ell}}{i\nu Z_\ell(\nu)} = \sum_\ell a_{n\ell}a_{k\ell}\left.\frac{Y_\ell(\nu)}{a_{k\ell}C_\nu}\right|_{Q_\nu=0} = \left.\frac{\sum_\ell a_{n\ell}Y_\ell(\nu)}{C_\nu}\right|_{Q_\nu=0} \doteq \left.\frac{X_n(\nu)}{C_\nu}\right|_{Q_\nu=0} . \tag{13}$$

Equation 12 is a mathematical expression of the argument used earlier to find the appropriate impedance function to be used in the influence functional for a linear system acting on Q. The additional information obtained here in this regard is that when other forces are present they are to be set equal to zero when this computation is made. Equation 13 states the new result that the transfer impedance which modifies C_ν in its effect on the test system is to be found by computing the ratio of C_ν to the coordinate $X_n(\nu)$ to which the test system is coupled. Again the other forces acting on the system are to be set equal to zero when this classical computation is made. The total force acting on the test system when several forces are acting on the interaction system is simply the sum of these forces each modified by the appropriate transfer impedance determined in the above described manner.

V.3 Random Potential Acting through a Linear System

To conclude the analysis of classical potentials, we will now find the average influence functional for an uncertain force acting through a linear system on a test system. As in Section III we assume the force has a power spectrum given by

$$\phi(\nu) = \frac{\langle C_\nu C_{\nu'}\rangle}{4\pi\delta(\nu+\nu')}$$

and that the probability for the amplitude of any frequency component is random. The problem is then to find

$$\left\langle \mathcal{F}(Q,Q') \right\rangle = \left\langle \mathcal{F}_o(Q,Q') \exp\left\{ \frac{i}{2\pi\hbar} \int_0^\infty \left[\frac{C_\nu}{(i\nu z_\nu)} (Q_{-\nu} - Q'_{-\nu}) + \frac{C_{-\nu}}{(-i\nu z_{-\nu})} (Q_\nu - Q'_\nu) \right] d\nu \right\} \right\rangle$$

$$= \mathcal{F}_o(Q,Q') \left\langle \exp\left\{ \frac{i}{2\pi\hbar} \int_{-\infty}^\infty \frac{C_\nu}{(i\nu z_\nu)} (Q_{-\nu} - Q'_{-\nu})\, d\nu \right\} \right\rangle . \qquad (14)$$

However, the coefficient of $\mathcal{F}_o(Q,Q')$ is just the same form as the expression evaluated in Section III. Thus from equation III.8 we have

$$\left\langle \mathcal{F}(Q,Q') \right\rangle = \mathcal{F}_o(Q,Q') \exp\left\{ -\frac{1}{\pi\hbar^2} \int_0^\infty \frac{\phi(\nu)}{\nu^2 |z_\nu|^2} \left| Q_\nu - Q'_\nu \right|^2 d\nu \right\} . \qquad (15)$$

We find that the effective power spectrum of the force acting on the test system $\phi'(\nu)$ is given by

$$\phi'(\nu) = \frac{1}{4\pi\delta(\nu+\nu')} \left\langle \frac{C_\nu C_{\nu'}}{\nu^2 |z_\nu|^2} \right\rangle = \frac{\phi(\nu)}{\nu^2 |z_\nu|^2} . \qquad (16)$$

The interpretation of this result is that the power spectrum of the effective force acting on the test system is simply the power spectrum of the original force filtered by the transfer characteristics of the linear system. This can be summarized by the following: If C_ν is the uncertain force (in Fourier transform notation) applied to the linear interaction system and its power spectrum is $\phi(\nu) = \frac{1}{4\pi\delta(\nu+\nu')} \times \langle C_{+\nu} C_{+\nu'} \rangle$, the effective force applied to the linear system is $C_\nu' = C_\nu / i\nu z_\nu$ and its power spectrum is $\phi'(\nu) = \frac{\langle C_{+\nu} C_{+\nu'} \rangle}{4\pi\delta(\nu+\nu')} \frac{1}{|i\nu z_\nu|^2} = \phi(\nu) / |i\nu z_\nu|^2$

VI. Linear Systems at Finite Temperatures

In the previous two sections the influence functionals for linear systems at zero temperatures were developed through the analysis of oscillators. In this section this analysis will be extended to include the effects of finite temperatures. We have already established the forms of influence functionals which are possible for linear systems in an argument which utilized the general properties discussed in Section II. Each of these forms has already occurred in the analyses of classical potentials, random potentials, and zero temperature linear systems. Therefore, the results to be expected here are one or more of the forms already obtained.

The discussion is begun again with a single oscillator as our linear system, for simplicity. From this, the extension to distributions of oscillators is immediate as it was in Section IV. The complete problem is set up in the same way as for zero temperature except that the initial state of the oscillator is not simply the ground state or any definite eigenstate. The effect of temperature is to make the initial state uncertain and it is properly represented by a sum over all states weighted by the Boltzmann factor $e^{-\beta E_n}$ where $\beta = 1/kT$, T being the temperature in this case. The final state is again arbitrary; therefore, a formal expression for the influence functional is

$$\mathcal{F}(Q,Q') = \sum_n \int \delta(X_T - X'_T) \exp\left\{\frac{i}{\hbar}\left[S(X)-S(X')+\int_\tau^T (QX-Q'X')\,dt\right]\right\}$$

$$\exp\left\{-\beta E_n\right\} \phi_n^*(X'_\tau)\phi_n(X_\tau)\,dX_\tau \cdots \mathcal{D}X'(t) \qquad (1)$$

where, of course, the ϕ_n represent the energy eigenfunctions of the oscillator unperturbed by external forces. The first problem is to find a closed form for the expression $\sum_n \phi_n(X_t)\phi_n^*(X_t')e^{-\beta E_n}$. This can be done by noticing that its form is identical with the kernel which takes a wave function from one time to another if we make the correspondence that β represents an imaginary time interval. If the times involved are t_2 and t_1, this kernel is

$$K_o(2,1) = \sum_n \phi_n(X_2)\phi_n^*(X_1)\exp\left\{-\frac{i}{\hbar} E_n(t_2 - t_1)\right\} \tag{2}$$

$$= \exp\left\{\frac{i}{\hbar} S(X)_{c\ell}\right\} \text{ for the harmonic oscillator , and}$$

where the subscript o indicates the absence of external forces. For the harmonic oscillator the expression for S is easily obtained in terms of the initial and final positions X_1 and X_2 (2). Thus,

$$S_{c\ell} = \frac{m\omega}{2 \sin \omega(t_2 - t_1)}\left[(X_1^2 + X_2^2)\cos \omega(t_2 - t_1) - 2X_1X_2\right] . \tag{3}$$

Utilizing equations 2 and 3 and making the correspondence $\beta = i(t_2 - t_1)/\hbar$, $X_1 = X$, and $X_2 = X'$, we find that

$$\sum \phi_n(X_\tau)\phi_n^*(X_\tau')e^{-\beta E_n} = \exp\left\{-\frac{m\omega}{2\hbar \sinh(\beta\hbar\omega)}\left[(X_\tau^2 + X_\tau'^2)\cosh \beta\hbar\omega - 2X_\tau X_\tau'\right]\right\} . \tag{4}$$

Using this expression for the initial state of the oscillator, we now will manipulate the form of $\mathcal{F}(Q,Q')$ to coincide with the expression A1.3 in Appendix I . Therefore,

$$\sum \phi_n(X_\tau)\phi_n^*(X'_\tau)e^{-\beta E_n} = \exp\left\{-\frac{m\omega}{4\hbar}\left[x_\tau^2\left(\frac{\cosh \beta\hbar\omega - 1}{\sinh \beta\hbar\omega}\right)+x'^2_\tau\left(\frac{\cosh \beta\hbar\omega + 1}{\sinh \beta\hbar\omega}\right)\right]\right\} \tag{5}$$

where the change of variables $x_\tau = X_\tau + X'_\tau$ and $x'_\tau = X_\tau - X'_\tau$ has been made to produce equation 5 from equation 4. Furthermore, making use of the identity

$$\frac{\cosh(\beta\hbar\omega) - 1}{\sinh(\beta\hbar\omega)} = \frac{\sinh(\beta\hbar\omega)}{\cosh(\beta\hbar\omega)+ 1} \quad ,$$

the right side of equation 5 can be written

$$\exp\left\{-\frac{m\omega}{4\hbar}\left(\frac{x_\tau^2}{a} + ax'^2_\tau\right)\right\} \tag{6}$$

where $$a = \frac{\cosh(\beta\hbar\omega) + 1}{\sinh(\beta\hbar\omega)} \quad . \tag{7}$$

The influence functional of equation 1 can be rewritten now as

$$\mathcal{F}(q,q') = \int \delta(x_T')\exp\left\{\frac{i}{\hbar}\left[S(x+x')-S(x-x')+\int_\tau^T\left(\frac{qx'}{2}+\frac{q'x}{2}\right)dt\right]\right\}$$

$$\times \exp\left\{-\frac{m\omega}{4\hbar}\left(\frac{x_\tau^2}{a} + ax'^2_\tau\right)\right\} dx_\tau \cdots \mathcal{D}x'(t) \tag{8}$$

where the additional substitution $Q = q+q'$ and $Q' = q-q'$ has been made. Equation 8 is exactly the same influence functional as is indicated by equation A1.3 in Appendix I(a), the evaluation of which has been carried through. From A1.27

$$\mathcal{F}(Q,Q') = \mathcal{F}_o(Q,Q')\exp\left\{-\frac{1}{\pi\hbar^2}\int_0^\infty \left|Q_\nu - Q'_\nu\right|^2\left(\frac{\pi\hbar}{4m\omega}\right)\right.$$

$$\left.\left[(a-1)\delta(\nu-\omega)+(a+1)\delta(\nu+\omega)\right] d\nu\right\} \quad .$$

Substituting $a = \frac{\cosh(\beta\hbar\omega)+1}{\sinh \beta\hbar\omega} = \coth(\frac{\beta\hbar\omega}{2})$ and noting that $\int_0^\infty \delta(\nu+\omega)d\nu = 0$, we get finally the influence phase

$$i\Phi(Q,Q') = i\Phi_o(Q,Q') - \frac{1}{\pi\hbar^2}\int_0^\infty \frac{\pi\hbar/2m\omega}{e^{\beta\hbar\omega}-1}\,\delta(\nu-\omega)\left|Q_\nu - Q'_\nu\right|^2 d\nu \quad . \qquad (9)$$

Thus, the influence phase is made up of two terms, the first of which is the effect of the oscillator at zero temperature. The second is recognized as having the same form which was found for an uncertain classical potential with a Gaussian distribution. Therefore, the effect of finite temperature is to introduce a noisy potential acting on Q at the frequency of the original oscillator. The power spectrum of the noise produced by the finite temperature is

$$\phi(\nu) = \frac{\pi\hbar}{2m\nu(e^{\beta\hbar\nu}-1)}\,\delta(\nu-\omega) \quad . \qquad (10)$$

To indicate more clearly the relationship of $\phi(\nu)$ to the characteristics of the linear system it is instructive to extend this expression to the case of a distribution of oscillators $G(\omega)$ all at the same finite temperature. The resulting influence phase is

$$i\Phi(Q,Q') = i\int_0^\infty \Phi_o(Q,Q')G(\omega)d\omega - \frac{1}{\pi\hbar^2}\int_0^\infty \frac{\pi\hbar G(\nu)}{2\nu}\,\frac{\left|Q_\nu - Q'_\nu\right|^2}{(e^{\beta\hbar\nu}-1)}\,d\nu \qquad (11)$$

where in the distribution m has been set equal to unity. The first term is again the influence phase for zero temperature, while the second term again has the form of a noisy potential whose power spectrum is

$$\phi(\nu) = \frac{\pi\hbar G(\nu)}{2\nu(e^{\beta\hbar\nu}-1)} \quad .$$

Recalling the analysis of the distribution of oscillators of Section IV.2, it is found for equation IV.18 that $\pi G(\nu)/2 = \mathrm{Re}(1/Z_\nu)$. Therefore, the power spectrum can be written

$$\phi(\nu) = \frac{\hbar\, \mathrm{Re}(\frac{1}{Z_\nu})}{\nu(e^{\beta\hbar\nu} - 1)} \quad . \tag{12}$$

In the time domain we have from equation A1.17

$$i\dot{\Phi} = i\dot{\Phi}_o - \frac{1}{\hbar^2}\int_0^\infty \frac{d\nu G(\nu)\hbar}{\nu(e^{\beta\hbar\nu} - 1)} \int_{-\infty}^{\infty}\int_{-\infty}^{t} (Q_t - Q_t')(Q_s - Q_s')\cos\,\nu(t-s)\,ds\,dt \quad . \tag{13}$$

Comparing this with equation III.12 for random classical forces we see that the correlation function of the noise due to the finite temperature is

$$R(t-s) = \int_0^\infty \frac{\hbar\, G(\nu)\, \cos\,\nu(t-s)}{\nu(e^{\beta\hbar\nu} - 1)}\, d\nu$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\hbar\, \mathrm{Im}(\frac{1}{i\nu Z_\nu})\, \cos\,\nu(t-s)}{(e^{\beta\hbar\nu} - 1)}\, d\nu \tag{14}$$

Finally, if we write equation 13 in terms of $F(t-s)$ as, for instance in equation IV.4, we find from equation A1.16b that

$$F(t) = \int_0^\infty \frac{G(\nu)\, \coth\,(\frac{\beta\hbar\nu}{2})}{\nu} \cos\,\nu t\, d\nu + i\int_0^\infty \frac{G(\nu)}{\nu} \sin\,\nu t\, d\nu \quad .$$

Thus,

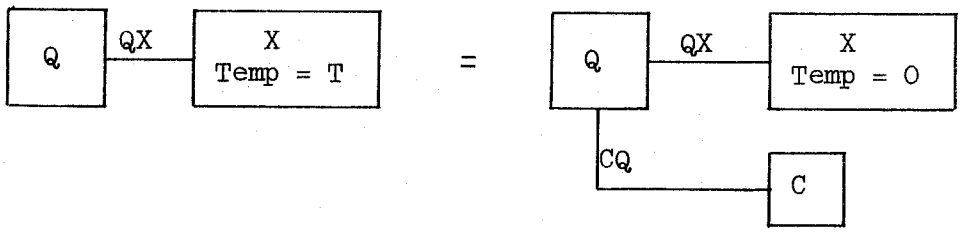
$$A(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right)\coth\left(\frac{\beta\hbar\nu}{2}\right)\cos\nu t\, d\nu$$

and (15)

$$B(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{i\nu Z_\nu}\right)\sin\nu t\, d\nu$$

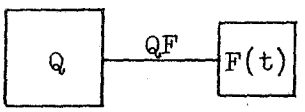
which are the more general counterparts of equations IV.5b. Notice however, that only the relation for $A(t)$ changes with temperature.

Theorem IX. If a linear interaction system is initially at a finite temperature, the resulting effect is the same, as far as the test system is concerned, as if the linear system were at zero temperature and in addition, a random classical potential were connected independently to the test system. The power spectrum of the random potential is given by equation 12 and is related both to the temperature and to the dissipative part of the impedance of the linear system. The theorem stated in terms of a diagram is



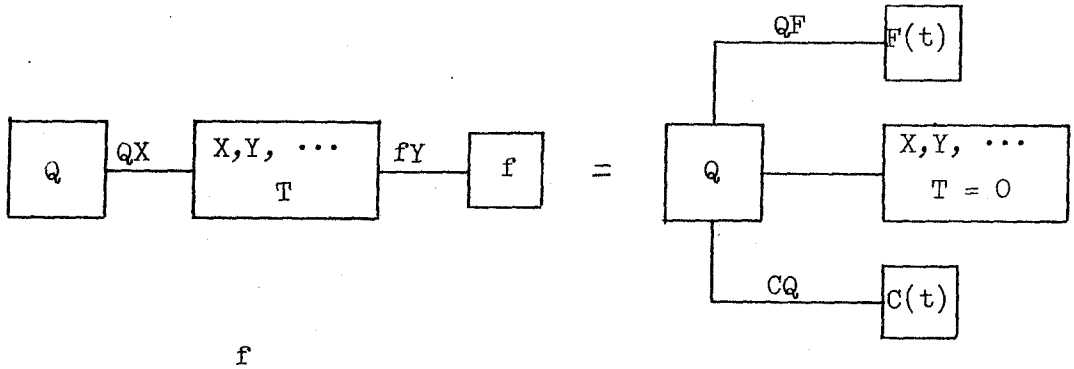
where $\phi(\nu) = \dfrac{\langle C_\nu C_{\nu'}\rangle}{4\pi\delta(\nu+\nu')} = \dfrac{\mathrm{Re}\left(\frac{1}{Z_\nu}\right)}{\nu(e^{\beta\hbar\nu}-1)}$. This fluctuation dissipation theorem has a content which is different from those stated by Callen and Welton (9), Kubo (10), and others. It represents still another generalization of the Nyquist theorem which relates noise and resistance in electric circuits (11). These previously stated fluctuation dissipation theorems related the fluctuations of some variable in

an isolated system, which is initially at thermal equilibrium, to the dissipative part of the impedance of the isolated system. This would be equivalent to relating $\langle Q^2 \rangle$ when the test system is at equilibrium $(F = 0)$ to the dissipative part of F_ν/Q_ν where F_ν is a classical force acting on Q .



However, Theorem IX describes the effect of an external quantum system at thermal equilibrium on a test system and separates it into two effects, a zero point quantum term, which cannot be classified as pure noise, and a random potential term. Using this influence functional approach we can find $\langle Q^2 \rangle$ due not only to the internal fluctuations of Q but also due to the effect of X .

The fact that all the derivations so far have been exact, which is a consequence of dealing only with systems made up of distributions of oscillators, brings up two interesting aspects of the theory. The first one is that Z_ν does not depend on the temperature, only on the distribution of oscillators. Yet in any real finite system, one would expect that the temperature of a system does affect its impedance somewhat. The second aspect is that when a force $C(t)$ of any magnitude whatever, is applied to the linear system, its temperature does not change although it is obvious that if Z_ν has a finite real part the linear system must absorb energy from $C(t)$. For clarity, the influence functional for



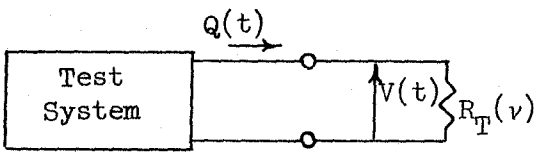
where $F_\nu = \frac{f_\nu}{i\nu z_\nu}$ and $C(t)$ is the random force with a power spectrum given above, is

$$i\Phi(Q,Q') = i\Phi_o(Q,Q') + \frac{i}{2\pi\hbar}\int_{-\infty}^{\infty} \frac{f_\nu(Q_{-\nu}-Q'_{-\nu})}{i\nu z_\nu}\, d\nu - \frac{1}{\pi\hbar^2}\int_0^\infty \phi(\nu)\left|Q_\nu - Q'_\nu\right|^2 d\nu \ .$$

As is seen, the addition of $f(t)$ does not change Z_ν or T. However, the fact that Z_ν is independent of temperature is made more plausible when it is observed that the energy level spacings between states of a harmonic oscillator are equal. Furthermore, matrix elements of a linear coordinate say, X, only exist between adjacent states. Therefore, transitions excited in X due to an applied force involve the same emitted or absorbed energies regardless of the initial state of X. The fact that the temperature of the linear system does not depend on any force applied to it has its explanation in the way in which dissipation is represented, i.e., through distributions of oscillators. These distributions contain infinite numbers of oscillators and thus an infinite amount of energy would be required to change their average energy state. All of this discussion is just another way of saying that systems of oscillators are exactly linear in their behavior.

All the results so far suggest that any linear system can be

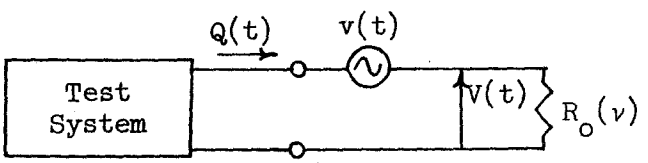
handled by the same rules that have been developed for systems of oscillators. This will be developed fully in the next section. However, we will assume this to be true now and conclude this section by applying the theorem just derived to obtain Nyquist's result for noise from a resistor. Take as an example an arbitrary circuit as the test system connected to a resistor $R_T(\nu)$ at a temperature T. The resistor comprises the interaction system.



The interaction between the test system and the resistor is characterized by a charge $Q(t)$ flowing through the test system and resistor and a voltage $V(t)$ across the terminals. Let us associate $Q(t)$ with the coordinate of the test system and $V(t)$ with the coordinate of $R(\nu)$. The interaction part of the Lagrangian is symbolically represented by $-Q(t)V(t)$ since the current voltage relationship in $R(\nu)$ is opposite to that of a generator. The quantity $i\nu Z_\nu$ appearing in the influence functional is given by

$$\frac{Q_\nu}{-V_\nu} = -\frac{1}{i\nu R_\nu} = i\nu Z_\nu \quad , \text{ or } \quad \mathrm{Re}\left(\frac{1}{Z_\nu}\right) = \nu^2 R_\nu \quad .$$

Then Theorem IX tells us that this situation may be replaced by



a resistor at zero temperature and a random classical voltage whose power spectrum is

$$\phi(\nu) = \frac{\not{h}\nu R_\nu}{e^{\beta\not{h}\nu} - 1} \quad . \tag{17}$$

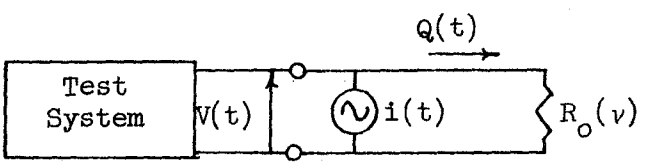
The mean value of this voltage is

$$\langle v^2(t)\rangle = \frac{2}{\pi}\int_0^\infty \phi(\nu)d\nu = \int_0^\infty \frac{2\not{h}\nu R_\nu}{\pi(e^{\beta\not{h}\nu}-1)}\, d\nu \quad . \tag{18}$$

For high temperature $\beta \ll 1$, and we find that over the frequency range where $\beta\not{h}\nu \ll 1$, $\phi(\nu) = kTR_\nu$. If R_ν is constant over the frequency range of interest whose limits are ν_2 and ν_1 , then as a result of the noise power in this range

$$\langle v^2(t)\rangle = 4kTR(f_2 - f_1) \tag{19}$$

where f is the circular frequency. This is the famous result for noise from a resistor due to Nyquist. Notice that the noise voltage generator must be placed in series with the resistor as a consequence of interacting directly with the coordinate of the test system, which in this case is $Q(t)$, the charge flowing in the circuit. If the coordinates were chosen such that $Q(t)$ were the coordinate of the resistor and $V(t)$ that of the test system, then the noise generator would become a current source interacting with the voltage $V(t)$ so that the situation would be



where $\langle i^2 \rangle = \frac{4kT}{R} (f_2 - f_1)$ in the high temperature limit.

It is worth mentioning but obvious from the derivation, that if there were many sources of dissipation coupled to the test system, each at different temperatures, then there would be a fluctuating potential associated with each source of dissipation with a power spectrum characteristic to the temperature involved. The case of different temperatures represents a non-equilibrium condition in that the hot resistors are always giving up energy to the cold ones. However, when the resistors are represented by distributions of oscillators the temperatures do not change because of the infinite energy content of each one, as discussed before.

VII. Weakly Coupled Systems

We are now faced with the problem of finding influence functionals whose behavior is in some sense linear but whose total behavior is not representable by systems of perfect oscillators. There are many examples of this. The concepts of resistance, electric and magnetic polarizations, etc. are basic quantitities which characterize the classical electrical behavior of matter. However, for an accurate description of this behavior, these quantities can be constant, independent of the applied electromagnetic field only in the range of approximation that the magnitude of this applied field does not become too large. We will consider here the analysis of such systems and the approximations which make linear analysis valid. Insofar as linear behavior is obtained, the results of this section are basically the same as those obtained in previous sections with regard to finding influence functionals. However, it is interesting to see the same principles come out of the analysis another way. In addition, it will be found that expressions corresponding to Z_ν which appear in the influence functional are actually closed forms which can be used to compute such quantities as the conductivity from a knowledge of the unperturbed quantum characteristics of a system. These expressions have been derived before by several authors but it is interesting to find that they appear in the influence functional quite naturally. The results will then be applied to the case of a beam of non-interacting particles passing near a test system such as a cavity. This analysis naturally lends itself to a discussion of noise in beam type maser amplifiers. In Appendix IV, the results of this and Section IV are used to compute the spontaneous emission of a particle in a cavity.

VII.1 Interaction Systems with Coupling Potentials of the Form -V(Q)U(P)

Theorem X. If a general interaction system, P , is coupled to a test system Q so that the interaction potential is small and of the form -V(Q)U(P), then the effect of the test system is that of a sum of oscillators whose frequencies correspond to the possible transitions of the interaction system. Therefore, to the extent that second order perturbation theory yields sufficient accuracy, the effect of an interaction system is that of a linear system.

To show this, we shall first assume the interaction system to be in an eigenstate $\phi_a(P_\tau)\exp\left\{-\frac{i}{\hbar}E_a\tau\right\}$ at the beginning of the interaction and in an arbitrary state at the end of the interaction consistent with the usual procedure we have followed. Also, for convenience in writing the interaction potential will be assumed -U(P)Q . The influence functional is then

$$\mathcal{F}(Q,Q') = \int \delta(P_T-P'_T)\exp\left\{\frac{i}{\hbar}\left[S_o(P)-S_o(P')+\int_\tau^T (QU-Q'U')dt\right]\right\}$$

$$\times \phi_a^*(P'_\tau)\phi_a(P_\tau)dP_\tau \cdots \mathcal{D}P'(t) \tag{1}$$

where in the above we have written U for U(P) and U' for U(P'). Since the magnitude of the interaction is assumed to be small, the perturbation approach can be used to good advantage. Thus, expanding the interaction part of the exponent and keeping terms to second order in Q only,

$$\mathcal{F}(Q,Q')=\int \delta(P_T-P'_T)\exp\left[\frac{i}{\hbar}\left[S(P)-S(P')\right]\right]\left\{1+\int_\tau^T (QU-Q'U')dt\right.$$

$$\left.+\left(\frac{i}{\hbar}\right)^2\int_\tau^T\int_\tau^t (Q_tU_t-Q'_tU'_t)(Q_sU_s-Q'_sU'_s)dsdt\right\}\phi_a^*(P'_\tau)\phi_a(P_\tau)dP_\tau \cdots \mathcal{D}P'(t) \ . \tag{2}$$

The evaluation of this is done in an entirely straightforward manner.* The complete expression for 1 is then,

$$\mathcal{F}(Q,Q') = 1 + \frac{iU_{aa}}{\hbar}\int_T^T (Q_t - Q_t')dt - \frac{b\,(U_{ab}U_{ba})}{\hbar^2}$$

$$\times \int_T^T\int_T^t (Q_t - Q_t')\left[Q_s \exp[-i\omega_{ba}(t-s)] - Q_s' \exp[i\omega_{ba}(t-s)]\right]dsdt \qquad (3)$$

where $U_{ab} = \int \phi_a^*(P)U(P)\phi_b(P)dP$, and $\omega_{ba} = (E_b - E_a)/\hbar$.

All terms which involve U_{aa} will be disregarded. This is because U_{aa} represents the average value of the operator $U(P)$ in an eigenstate of the interaction system alone. Even if it is not zero, it will be noted from equation 2 that terms involving U_{aa} can be written

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*A typical term in equation 2 is as follows:

$$\int \delta(P_T - P_T')\exp\left\{\frac{i}{\hbar}[S(P) - S(P')]\right\}\left(\frac{i}{\hbar}\right)^2 \int_T^T\int_T^t Q_t U_t Q_s U_s dsdt$$

$$\times \phi_a^*(P_\tau')\phi_a(P_\tau)dP_t \cdots \mathcal{D}P'(t) \quad .$$

Taking the time integrations outside the path integral and replacing the path integrations by propagating kernels, this becomes

$$\left(\frac{i}{\hbar}\right)^2 \int_T^T\int_T^t Q_t Q_s dsdt \int \delta(P_T - P_T')K^*(P_T', P_\tau')K(P_T, P_t)U_t K(P_t, P_s)U_s K(P_s, P_\tau)$$

$$\times \phi_a^*(P_\tau')\phi_a(P_\tau)dP_\tau \cdots dP_T' \quad .$$

Remembering that $K(P_T, P_t) = \sum \phi_n(P_T)\phi_n^*(P_t)\exp\left\{-\frac{i}{\hbar}E_n(T-t)\right\}$, this expression becomes simply,

$$-\sum_b \frac{U_{ab}U_{ba}}{\hbar^2}\int_T^T\int_T^t Q_t Q_s \exp\left[-i\omega_{ba}(t-s)\right] dsdt \quad .$$

$$1 + \frac{iU_{aa}}{\hbar}\int_{\tau}^{T}(Q_t - Q_t')dt + \frac{1}{2!}\left[\frac{iU_{aa}}{\hbar}\int_{\tau}^{T}(Q_t - Q_t')dt\right]^2 + \cdots$$

$$= \exp\left[\frac{i}{\hbar}\int_{\tau}^{T} U_{aa}(Q_t - Q_t')dt\right] \quad .$$

When $\mathcal{F}(Q,Q')$ is used to make a calculation on the test system, this term has the effect of adding a constant potential $V(Q(t)) = -U_{aa}Q(t)$ to the unperturbed test system. Disregarding U_{aa}, the influence functional then becomes

$$\mathcal{F}(Q,Q') \approx 1 - \frac{\sum_b U_{ab}U_{ba}}{\hbar^2}\int_{-\infty}^{\infty}\int_{-\infty}^{t} \gamma_t\gamma_s(Q_t - Q_t')(Q_s \exp\left[-i\omega_{ba}(t-s)\right]$$

$$- Q_s' \exp\left[i\omega_{ba}(t-s)\right]) \qquad (4)$$

where the limits have been extended and the factors $\gamma_t\gamma_s$ inserted to allow finite coupling time if necessary. If the strength of coupling is sufficiently weak then equation 4 can be rewritten

$$\mathcal{F}(Q,Q') \approx \exp\left\{-\sum_b \frac{|U_{ab}|^2}{\hbar^2}\int_{-\infty}^{\infty}\int_{-\infty}^{t} \gamma_t\gamma_s(Q_t - Q_t')\left(Q_s \exp\left[-i\omega_{ba}(t-s)\right]\right.\right.$$

$$\left.\left. - Q_s' \exp\left[i\omega_{ba}(t-s)\right]\right)\right\} \quad . \qquad (5)$$

In this form we recognize $\mathcal{F}(Q,Q')$ as that describing the effect of a sum of harmonic oscillators independently connected to the test

system each of whose "weights" is $\frac{U_{ba}U^*_{ba}}{\hbar^2}$.* The complete response function for the system of oscillators is

$$B_a(t-s) = \text{Im } F_a(t-s) = \sum_b \frac{2U_{ab}U_{ba}}{\hbar} \sin \omega_{ba}(t-s)$$

where the subscript a on B_a refers to the initial eigenstate. According to previous definition, the mass of each individual oscillator is identified by $m = \frac{\hbar}{2|U_{ba}|^2 \omega_{ba}}$ and its characteristic frequency by ω_{ba} . Therefore, to the extent that second order perturbation theory yields satisfactory accuracy, any system may be considered as a sum of harmonic oscillators. This is equivalent in classical mechanics to the theory of the motion of a particle having small displacements around an equilibrium position. Its motion, to a first approximation, is also that of a harmonic oscillator, if the first effective term in a power series expansion of the potential around that equilibrium position is quadratic in the displacement.

In this part of the discussion we should again point out the motivation for writing the approximate influence functional, equation 4, in terms of the approximate exponential of equation 5, apart from the obvious advantage of making the form agree with that of exactly

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*It is interesting to examine the total weight of all the oscillators. This is

$$\sum_b \frac{U_{ab}U_{ba}}{\hbar^2} = \frac{(U^2)_{aa}}{\hbar^2} \quad .$$

When U(P) is the coordinate, P , of a harmonic oscillator and a is the ground state, then

$$\sum_b \frac{U_{ab}U_{ba}}{\hbar^2} = \frac{1}{\hbar^2}\left(\frac{\omega m}{\pi\hbar}\right)^{1/2} \int P^2 \exp\left[-\frac{m\omega}{\hbar} P^2\right] dP = \frac{1}{2\hbar m\omega} \quad .$$

This is the coefficient or weight of a single harmonic oscillator as we have already discovered.

linear systems. Frequently, we deal with a test system which is influenced by another system which is actually made up of a large number of very small systems. Examples of such an interaction system would be a beam of atoms or the electrons in a metal. Although the expression for the influence functional for any one of the sub-systems is only good to second order, their individual effects are so small that this accuracy is very good and equation 4 or 5 is equally valid. However, when the sum of the effects of the sub-systems is not small, then the two forms above do not describe the situation equally insofar as the composite effect of the interaction system is concerned. We know that when these sub-systems are dynamically and statistically independent, the total influence functional is simply a product of the individual ones. In such a case the influence functional obtained by using equation 5 as follows

$$\mathcal{F}(Q,Q') \approx \exp\left\{ i \sum_k \Phi_k(Q,Q') \right\}$$

yields much greater accuracy than that obtained from equation 4 where we would find

$$\mathcal{F}(Q,Q') \approx 1 + i \sum_k \Phi_k(Q,Q') \quad ,$$

$\Phi_k(Q,Q')$ being the influence phase for the kth sub-system. This is recognized as a variation of the problem in Section III where it was shown that if $f(a) \approx 1 + a$ for small a, then $f^N(a)$ is very accurately represented by e^{Na} where Na is not small. Here we wish to show that

$$G(x) = \lim_{N \text{ large}} \prod_{k=1}^{N} (1 + x_k) \Rightarrow \exp\left[\sum_k x_k \right]$$

where the x_k are small but not necessarily equal to each other, and where the total sum $\sum_k x_k$ is finite. Rewriting the expression for $G(x)$ we have

$$G(x) = 1 + \sum_{k=1}^{N} x_k + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} x_j x_k + \frac{1}{6} \sum_{\substack{j,k,\ell=1 \\ j \neq k \neq \ell}}^{N} x_j x_k x_\ell + \cdots$$

$$= 1 + \sum_{k=1}^{N} x_k + \frac{1}{2!} \sum_{j,k=1}^{N} x_j x_k (1-\delta_{jk}) + \frac{1}{3!} \sum_{j,k,\ell=1}^{N} x_j x_k x_\ell$$

$$\times (1-\delta_{jk}-\delta_{k\ell}-\delta_{j\ell}+2\delta_{jk}\delta_{k\ell}) + \cdots .$$

As N is allowed to get very large the contribution of the terms involving quantities such as δ_{jk} becomes less significant. For instance, in the third term

$$\sum_{j,k=1}^{N} (x_j x_k)(1-\delta_{jk}) \sim (Nx)^2 - \frac{(Nx)^2}{N}$$

and for very large N only the leading term in this sum is important. Thus, we have the result that

$$G(x) \underset{\text{large } N}{\approx} 1 + \sum_{k=1}^{N} x_k + \frac{1}{2!}\left(\sum_{k=1}^{N} x_k\right)^2 + \frac{1}{3!}\left(\sum_{k=1}^{N} x_k\right)^3$$

$$\approx \exp\left[\sum_k x_k \right] .$$

The expression given by equation 5 has additional implications which are not immediately apparent from the analogy with the harmonic

oscillator. No assumption was made as to whether the initial state was necessarily the lowest of the system. Therefore, ω_{ba} could be either positive or negative. Suppose for a moment that the interaction system has only two states ($\phi_a(P)$ and $\phi_b(P)$) and that the initial state, a , is the lower one ($\omega_{ba} > 0$) . It is obvious that the only effect it can have on the test system Q is to absorb energy from Q . However, if the initial state, a , is the upper state (corresponding to $\omega_{ba} < 0$) then the interaction system can only give up energy to the test system. It can do this in two ways, through spontaneous emission or through coherent emission due to some coherent driving force exerted on it by the test system. So for the case $\omega_{ba} > 0$ we expect the influence functional to show that the interaction system has the effect of a cold system characterized by a dissipative impedance (or positive resistance.) Conversely, for $\omega_{ba} < 0$ it is expected that $\mathcal{F}(Q,Q')$ will be characterized by a negative resistance and a random potential due to the spontaneous emission transitions. The influence functional for either case is given by the result for the harmonic oscillator, in Appendix I(b). It is only necessary to make the correspondence $|U_{ba}|^2/\hbar = 1/2m\omega$.

$$\Phi(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu}-Q'_{-\nu})}{(i\nu Z_{ba,\nu})} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu Z_{ba,-\nu})}\right] - \frac{1}{\pi\hbar^2}\int_0^\infty |Q_\nu - Q'_\nu|^2 \times \pi |U_{ba}|^2 \delta(\nu+\omega_{ba})d\nu \qquad (6)$$

where $$i\nu Z_{ba,\nu} = -\frac{\hbar}{2\omega_{ba}|U_{ba}|^2}\left[(\nu - i\epsilon)^2 - \omega_{ba}^2\right] \quad . \qquad (7)$$

First of all we notice that the sign of $Z_{ba,\nu}$ changes with that of

ω_{ba} and therefore its dissipative part can be positive or negative as was argued above. Secondly, for $\omega_{ba} < 0$ there is a random potential acting on Q whose power spectrum is given by $\phi(\nu) = \pi |U_{ba}|^2 \delta(\nu+\omega_{ba}$ Of course when $\omega_{ba} > 0$, the integral involving this term disappears indicating the noise potential does not exist.

In a real physical situation it is not likely that the interaction system will be in a definite state initially. To extend the above results to a more general case we assume that the initial state is described as a sum over states weighted by a density matrix $\rho(\tau)$ which is diagonal in the energy eigenstates of the system. For example, if the system is initially in temperature equilibrium, $\rho = e^{-\beta\mathcal{H}}/T_r(e^{-\beta\mathcal{H}})$ where $\mathcal{H}$ is the Hamiltonian operator such that $\rho_{mn} = \delta_{mn} \exp(-\beta E_n)/\sum_n \exp(\beta E_n)$. The influence functional becomes

$$\mathcal{F}(Q,Q') = \int \delta(P_T - P'_T) \exp\left[\frac{i}{\hbar}[S(P) - S(P') + \int_\tau^T (UQ - U'Q')dt]\right]$$

$$\sum_a \rho_{aa} \phi^*_a(P'_\tau)\phi_a(P_\tau) dP_\tau \cdots \mathcal{D}P'(t) \quad . \tag{8}$$

Within the limit of small coupling, then we can simply extend the influence phase of equation 5 by summing over all initial states weighted by the initial density matrix ρ_{aa} . If this is done, we obtain the usual form of the influence phase, equation 4, with a response function $B(t-s)$ given by

$$B(t-s) = \sum_{a,b} \frac{2\rho_{aa} U_{ab} U_{ba}}{\hbar} \sin \omega_{ba}(t-s) \quad . \tag{9}$$

Again $\Phi(Q,Q')$ is the phase for a sum of oscillators, each of whose

weights is $\rho_{aa}|U_{ab}|^2/\hbar^2$.* In Section IV it was shown that $B(t-s)$ was the classical response of the linear system to an impulse of force applied to the coordinate $U(P)$. However, the expression above is in a form which is familiar to us only when we think of the interaction system, P , as consisting of a sum of oscillators. In this connection $B(t-s)$ is the total classical response of the oscillators describing the system P to an impulse of force applied to the "coordinate" $U(P)$.** To obtain a more direct interpretation of equation 9 we now calculate the linear classical response $B(t-s)$ of the interaction system to an applied impulse of force in terms of its unperturbed quantum characteristics. In so doing we will show that equation 9 is indeed this expression. Therefore, we will again have the result that the influence functional for a general, linear interaction system is formed simply from a knowledge of its classical characteristics just as in the case of systems of perfect oscillators. To calculate the response function $B(t-s)$ we calculate the expected value of $U(P)$ as a function of time after a potential $-f(t)\,U(P)$ is applied at $t = 0$. Thus

$$\langle U[P(t)]\rangle = \int \psi^*(P_t)U(P_t)\psi(P_t)dP_t$$

where $\psi(P_t)$ represents the state of system P at t . Using the

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*It is interesting to notice that the relative populations of any two levels may be described by an effective temperature $T_e = 1/k\beta_e$. For instance if the probabilities of occupation of states a and b are ρ_{aa} and ρ_{bb} respectively, we use the definition $\rho_{aa}/\rho_{bb}=\exp \beta_e(E_a-E_b)$. If $\rho_{aa}= 0$ this is described by setting $T_e=0+$, meaning to approach zero from the positive side. Similarly if the two states were inverted $\rho_{bb}=0$ and $T_e=0-$. This device has been used widely in the description of such situations.

**$U(P)$ may be regarded as a coordinate which is a function of other coordinates P in terms of which we choose to describe the interaction system.

path integral representation for the development of a wave function with time, as outlined in Section II, this can be written

$$\langle U(P_t)\rangle = \int U_t \delta(P_t - P_t') \exp\left\{\frac{i}{\hbar}\left[S_o(P) - S_o(P') + \int_0^t f(s)(U_s - U_s')ds\right]\right\} \times \psi^*(P_o')\psi(P_o)dP_o \cdots \mathcal{D}P'(t) \quad . \tag{10}$$

The initial state will be an average state described by a density matrix diagonal in the energy representation so that

$$\psi^*(P_o')\psi(P_o) = \sum_a \rho_{aa}\phi_a^*(P_o')\phi_a(P_o) \quad .$$

Assuming $f(s)$ to be small in magnitude, equation 10 can be written to first order

$$\begin{aligned}\langle U_t\rangle &= \sum_a \rho_{aa}\int U_t \delta(P_t - P_t')\exp\left\{\frac{i}{\hbar}\left[S_o(P) - S_o(P')\right]\right\}\left\{1 + \frac{i}{\hbar}\int_0^t f_s U_s ds\right. \\ &\quad \left. - \frac{i}{\hbar}\int_0^t f_s U_s' ds\right\} \phi_a^*(P_o')\phi_a(P_o)dP_o \cdots \mathcal{D}P' \\ &= \sum_a \rho_{aa}\left[U_{aa} + \sum_b \frac{iU_{ab}U_{ba}}{\hbar}\int_0^t f_s \exp\left[-i\omega_{ba}(t-s)\right]\right. \\ &\quad \left. - \sum_b \frac{iU_{ab}U_{ba}}{\hbar}\int_0^t f_s \exp\left[i\omega_{ba}(t-s)\right] ds\right] \quad .\end{aligned} \tag{11}$$

Again assuming $U_{aa} = 0$, equation 11 becomes

$$\langle U_t\rangle = \sum_{a,b} \frac{2\rho_{aa}U_{ab}U_{ba}}{\hbar}\int_0^t f_s \sin\omega_{ba}(t-s)ds \quad . \tag{12}$$

For $f(s) = \delta(s)$, then the classical response function is*

$$B(t) = \sum_{a,b} \frac{2\rho_{aa} U_{ab} U_{ba}}{\hbar} \sin \omega_{ba} t \tag{13}$$

which is identical with the response function found in the influence functional.

Since we have found expressions identifiable as classical response functions and impedances, it remains to show that associated with the dissipative part of the impedance is a noise potential. The impedance is simply obtained from

$$\frac{1}{i\nu Z_\nu} = \int_0^\infty B(t) e^{-i\nu t} dt = - \sum_{a,b} \frac{2\rho_{aa} \omega_{ba} |U_{ba}|^2}{\hbar[(\nu - i\epsilon)^2 - \omega_{ba}^2]} \quad . \tag{14}$$

To obtain the power spectrum it is only necessary to sum the influence phase of equation 6 over all initial states weighted by ρ_{aa} . Thus we find

$$\phi(\nu) = \sum_{a,b} \pi \rho_{aa} |U_{ab}|^2 \delta(\nu + \omega_{ba}) \tag{15}$$

and we now wish to relate this to the real part of $1/Z_\nu$. From equation 14 it is found that

$$\mathrm{Re}\left(\frac{1}{Z_\nu}\right) = \sum_{a,b} \frac{\pi |U_{ba}|^2}{\hbar} \nu \left[\rho_{bb} - \rho_{aa}\right] \delta(\nu + \omega_{ba}) \quad . \tag{16}$$

Rewriting $\phi(\nu)$

*It should be noted that implicit in the use of first order perturbation theory to obtain 13, is the fact that for this relation for the response function to hold as a steady state description of the linear system, the initial distribution must not be significantly disturbed by the application of the driving force.

$$\phi(\nu) = \sum_{a,b} \pi \left| U_{ab} \right|^2 \left(\frac{\rho_{bb} - \rho_{aa}}{\frac{\rho_{bb}}{\rho_{aa}} - 1} \right) \delta(\nu + \omega_{ba}) \quad . \tag{17}$$

If the average initial state of the interaction system is one of temperature equilibrium, then $\rho_{aa} = e^{-\beta E_a} / \sum_n e^{-\beta E_n}$ and $\frac{\rho_{bb}}{\rho_{aa}} = e^{\beta(E_a - E_b)} \equiv e^{\beta \hbar \omega_{ab}}$. Taking advantage of the characteristics of $\delta(\nu+\omega_{ba})$ so that ν can replace ω_{ab}, from equations 16 and 17,

$$\phi(\nu) = \left[\sum_{a,b} \frac{\pi \left| U_{ba} \right|^2}{\hbar} \nu (\rho_{bb} - \rho_{aa}) \delta(\nu + \omega_{ba}) \right] \frac{\hbar}{\nu \left[e^{\beta \hbar \nu} - 1 \right]} \quad *$$

$$= \frac{\hbar \, \mathrm{Re}(\frac{1}{Z_\nu})}{\nu \left[e^{\beta \hbar \nu} - 1 \right]} \quad . \tag{18}$$

Therefore, again we find that the power spectrum is related to the dissipative parts of the impedance when the initial state is one of temperature equilibrium. The power spectrum given in the form of equation 17 again illustrates the origin of thermal noise and identifies it as being just another aspect of spontaneous emission. Pound has also discussed this (12). The only contribution to the noise power spectrum is through the possible downward transitions of each possible

*Notice that if P were a two-level system initially in the lower state, then $\rho_{bb}=0$ and $T_e=0+$. In this case $\phi(\nu)=0$ which agrees with the required result for $\nu > 0$ (equation 6) for $\omega_{ba} > 0$). If initially P were in the upper state then $\rho_{aa}=0$ and $T_e=0-$ yielding $\phi(\nu) = -\frac{\hbar}{\nu} \mathrm{Re}(1/Z_\nu)$ agreeing with equation 6 for $\omega_{ba} < 0$. This is the power spectrum of the so-called spontaneous emission noise from an inverted two-level system.

state, a , weighted by the statistical factor, ρ_{aa} . It may be disturbing that $Re(\frac{1}{Z_\nu})$ contains singular forms such as $\delta(\nu+\omega_{ba})$. However, the infinite sums over the distribution of states of which it is a coefficient can be replaced by integrals over densities of states in most practical situations and as part of an integrand $\delta(\nu+\omega_{ba})$ is not unrealistic. We will now find the influence functionals for situations where the coupling potential is more general than has been handled up to now in this section.

VII.2 Interaction Systems Characterized by Vector Coupling

Until now the coupling between the interaction and test systems has simply been represented as the product of two scalar quantities which were functions of the coordinates of the two systems respectively. The case of an oscillator coupled to a test system was even more restricted; to find the influence functional exactly, the coupling was required to be linear in the oscillator coordinate as,

$$V_I(Q,X) = -XV(Q) \quad .$$

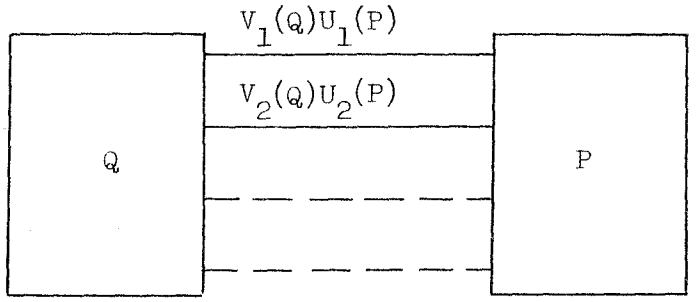
However, where approximations appropriate to weakly coupled systems can be made as in the preceding section, the potential can be of the slightly more general form

$$V_I(Q,P) = V(Q)U(P)$$

where this time P is the interaction system coordinate. However, let us now consider the case that the coupling potential between the test and interaction systems is a sum of potentials of the type already discussed. By this we mean that the potential is of the form,

$$V_I(Q,P) = \sum_\alpha V_\alpha(Q)U_\alpha(P) \quad , \tag{19}$$

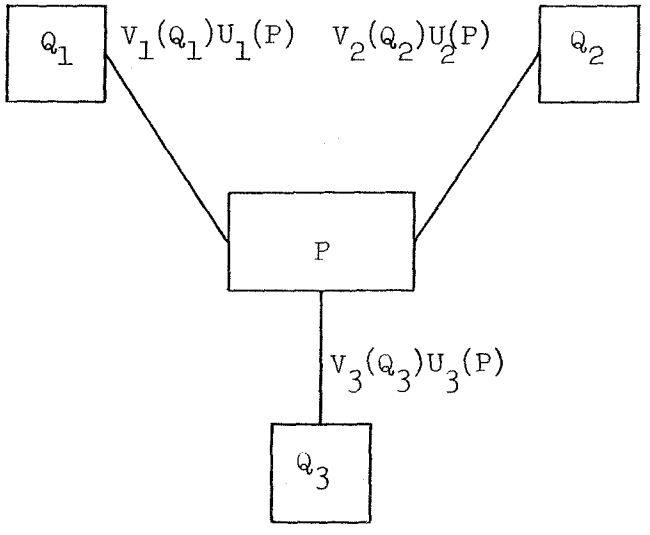
a situation which is depicted in the diagram below.



A simple example of this is the situation where the two systems are coupled through several of their respective coordinates as follows:

$$V_I(Q,P) = \sum_\alpha Q_\alpha P_\alpha \quad .$$

Alternatively, the problem may be such that there are several test systems which are coupled to each other through a common interaction system:



In this case, the coordinates Q_1, Q_2, and Q_3 could be considered, for purposes of finding the influence functional, as independent

coordinates of a composite test system which was coupled to the interaction system through a potential of the form of equation 19.

There are many examples of physical situations in which the coupling potentials are of the form of equation 19. To illustrate the type of problem in a more specific way, we will discuss a particular case. Let us consider a problem such as would arise in a study of quantum electrodynamics where several atoms, composing test systems, interact with each other through the radiation field. For the purposes of this discussion we are only interested in the coupling between the test systems and the radiation field of coordinate q, which represents the interaction system, and in showing that it is of the form of equation 19 so the terms of the Lagrangian not applying to the interaction will be neglected. If $\underline{r}_{mn}$ and e_{mn} represent the position vector and charge of the nth particle in the atom labeled m respectively, then the coupling potential between test and interaction systems is

$$V_I(r_m,q) = -\sum_{m,n} e_{mn}\,\dot{\underline{r}}_{mn}\cdot \underline{A}^{tr}(\underline{r}_{mn})$$

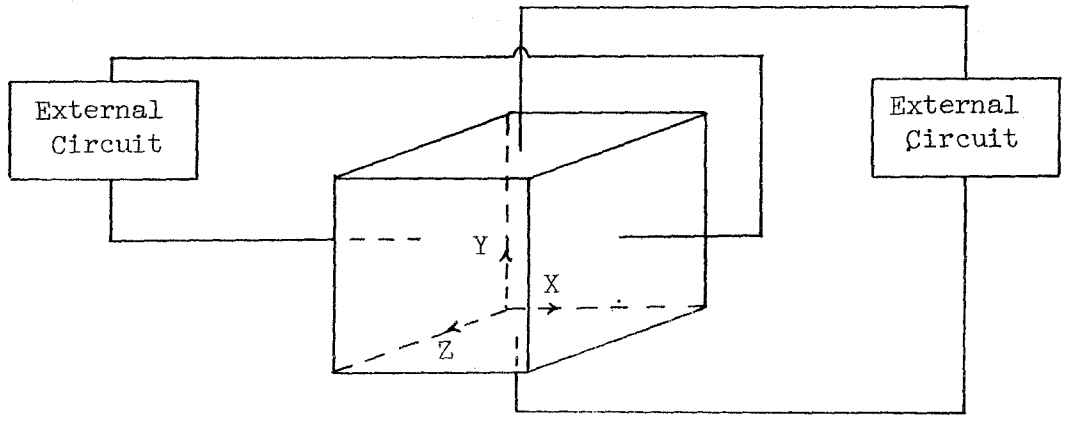
$$= -\sum_{\substack{m,n \\ \alpha=x,y,z}} e_{mn}(\dot{\underline{r}}_{mn})_\alpha\, A^{tr}_\alpha(\underline{r}_{mn}) \tag{20}$$

In the preceding equation the subscript α denotes one of the three components of the vector to which it is attached. It is well known that the transverse part of the electromagnetic field, $\underline{A}^{tr}(\underline{r})$, can be expanded in plane waves as follows (13)

$$\underline{A}^{tr}(q(t),\underline{r}) = \sqrt{8\pi c^2}\sum_{\underline{k}} \Big[\underline{e}_\lambda(q^{(1)}_{\underline{k}}\cos(\underline{k}\cdot\underline{r}) + q^{(3)}_{\underline{k}}\sin(\underline{k}\cdot\underline{r})) + \underline{e}_\ell(q^{(2)}_{\underline{k}}\cos(\underline{k}\cdot\underline{r}) + q^{(4)}_{\underline{k}}\sin(\underline{k}\cdot\underline{r}))\Big] \tag{21}$$

where $\underline{e}_\lambda$ and $\underline{e}_\ell$ are polarization vectors orthogonal with each other and with the propagation vector $\underline{k}$. The $q_{\underline{k}}^{(1,2,3,4)}$ may be considered the coordinates of the harmonic oscillators of the radiation field comprising the interaction system. Equations 20 and 21 together, yield a coupling potential of the form of equation 19.

In these cases, previously derived forms for influence functionals are not applicable and additional analysis is necessary. However, instead of deriving the form in very general terms we will take a specific example which will serve to illustrate another physical situation which is applicable to the discussion and which will yield some rather interesting results. The general relation is easily deduced from the results thus attained. Let us consider the effect of a material characterized by a tensor conductivity $\sigma_{\alpha\beta}$ on a test system which interacts with the material through an electric field. To make the situation definite, let the interaction system consist of a cubical block of resistive material each of the opposite sides of which is connected to an external circuit. In the figure below, connections are only shown to four of the six sides for clarity. The coordinates of



the test system, which is composed of the external circuits, are the three components of the electric field, $\underline{E}$. The resistor itself is assumed to be an electrically neutral, solid material which is composed of a large number of interacting charged particles. Not all of the particles in the resistor will necessarily contribute to the effect of the resistor on the test system, but since we are not making a quantitative computation of the influence functional here, that need not concern us. The assumptions made are, 1) that the entire coupling between resistor and the test systems is through the coupling potential

$$V_I(\underline{E},\underline{X}) = \sum_i e_i \underline{X}_i \cdot \underline{E} \ ,$$

where the summation is over all particles in the field and, 2) that the electric field $\underline{E}$ is not a function of position in the resistor. For ease of writing we will represent $\sum_i e_i \underline{X}_i$ by an equivalent sum $e\underline{X}$ and in addition the product $\underline{X}\cdot\underline{E}$ will be written $X_\alpha E^\alpha$ employing the Einstein summation rule. X_α , of course, refers to one of the components of the vector $\underline{X}$. Thus, the interaction potential is

$$V_I(E,X) = eX_\alpha E^\alpha \ ,$$

which obviously has the form of equation 19. It is understood that E^α and X_α are functions of time although the notation will be omitted. The influence functional can be formulated in the same way as that given by equation 1

$$\mathcal{F}(E,E') = \int \delta(\underline{X}_T - \underline{X}'_T)\exp\left\{\frac{i}{\hbar}\left[S(\underline{X}) - S(\underline{X}') - e\int_\tau^T (E^\alpha X_\alpha - E'^\alpha X'_\alpha)dt\right]\right\}$$

$$\times \sum_a \rho_{aa}\, \phi_a^*(\underline{X}'_\tau)\phi_a(\underline{X}_\tau) d^3X_\tau \cdots \mathcal{D}^3X'(t)$$

$$\approx \int \delta(\underline{X}_T - \underline{X}'_T)\exp\left\{\frac{i}{\hbar}\left[S(\underline{X}) - S(\underline{X}')\right]\right\} \left[1 - \frac{ie}{\hbar}\int_\tau^T (E_t^\alpha X_{\alpha,t} - E_t'^\alpha X'_{\alpha,t})dt\right.$$

$$\left. - \left(\frac{e^2}{\hbar^2}\right)\int_\tau^T\int_\tau^t (E_t^\alpha X_{\alpha,t} - E_t'^\alpha X'_{\alpha,t})(E_s^\beta X_{\beta,s} - E_s'^\beta X'_{\beta,s})dsdt\right] \sum_a \rho_{aa}\phi_a^*(\underline{X}'_\tau)\phi_a(X_\tau)$$

$$\times d^3X_\tau \cdots \mathcal{D}^3X'(t) \qquad (22)$$

In the above equations $\phi_a(\underline{X})$ represents one of a set of orthogonal eigenfunctions which describes the resistor in a state unperturbed by external influences such as the electric field $\underline{E}(t)$ and ρ_{aa} is an element of the initial density matrix diagonal in the representation of the $\phi_a(\underline{X})$. The procedure for evaluating this is exactly similar to that which was outlined in Section VII.1, and has no need for further discussion here. If the evaluation is done, the influence functional can be written

$$i\Phi(E,E') = -\frac{1}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{t} (E_t^\alpha - E_t^{\alpha'})\left[E_s^\beta F^*_{\alpha\beta}(t-s) - E_s'^\beta F_{\alpha\beta}(t-s)\right] dsdt \qquad (23)$$

where $$F_{\alpha\beta}(t) = \sum_{a,b} \frac{2e^2\rho_{aa}(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar} e^{i\omega_{ba}t} \quad . \qquad (24).$$

According to the rules previously derived we would expect that the response function which yields $\langle X_\alpha(t)\rangle$ as a function of an impulse of

field applied in the β direction at $t = 0$ is

$$B_{\alpha\beta}(t) = \text{Im } F_{\alpha\beta}(t) = \frac{i}{2}\left[F^*_{\alpha\beta}(t) - F_{\alpha\beta}(t)\right]$$

or, in terms of equation 24,

$$B_{\alpha\beta}(t) = \sum_{a,b} \frac{ie^2\rho_{aa}(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar} e^{-i\omega_{ba}t} + \text{complex conj.} \qquad (25)$$

It is easy to verify this result by computing $eX_\alpha(t)$ as a function of an applied field $E_\beta(t)$ using first order perturbation theory. This is exactly the same procedure as was followed in the previous section for the problem under consideration there (see equations 10 through 13) and the algebra need not be repeated. However, the computation yields the result of equation 25 as expected. If the average value of current $\langle j_\alpha(t)\rangle$ in the resistor is desired instead of $\langle eX_\alpha(t)\rangle$ as a function of an applied impulse of voltage, this is easily obtained as follows

$$\langle j_\alpha(t)\rangle = \langle e\dot{X}_\alpha(t)\rangle = \sum_{a,b} \frac{e^2\rho_{aa}\omega_{ba}(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar} e^{-i\omega_{ba}t} + \text{complex conj.}$$

The impedance $Z_{\alpha\beta}(\nu)$ or admittance $Y_{\alpha\beta}(\nu)$ obtained from equation 25 is

$$\frac{1}{Z_{\alpha\beta}(\nu)} = Y_{\alpha\beta}(\nu) \equiv \frac{i\nu(\langle eX_\alpha(t)\rangle)_\nu}{E_\beta(\nu)} = \int_0^\infty B_{\alpha\beta}(t)e^{-i\nu t}dt$$

$$= \sum_{a,b} \frac{e^2\rho_{aa}}{\hbar}\left[\frac{i\nu(X_\alpha)_{ab}(X_\beta)_{ba}}{\nu - i\epsilon + \omega_{ba}} - \frac{i\nu(X_\beta)_{ab}(X_\alpha)_{ba}}{\nu - i\epsilon - \omega_{ba}}\right]$$

$$= \sum_{a,b} \frac{i\nu e^2(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar(\nu - i\epsilon + \omega_{ba})}(\rho_{aa} - \rho_{bb}) \qquad (26)$$

and if we restrict the resistor to unit volume $\mathrm{Re}\, Y_{\alpha\beta}(\nu)$ is the conductivity of the material while $\mathrm{Im}\, Y_{\alpha\beta}(\nu)$ relates the current due to the induced electric polarization to the applied field $\underline{E}(t)$. The results, equations 25 and 26, check with those obtained by Kubo (10), Nakano (14), Feynman (15), and others. For completeness and because its form is interesting, we write the above influence phase (equation 23) in transform notation. The form is derived in a similar way to that done for the harmonic oscillator (Appendix I(b)) but is more complicated in this case and the work is done in Appendix VI. This expression is

$$i\tilde{\Phi}(E,E') = \frac{i}{2\pi\hbar}\int_0^\infty \left[\frac{E'^{\beta}_{\nu}(E^{\alpha}_{-\nu}-E'^{\alpha}_{-\nu})}{i\nu Z_{\alpha\beta}(\nu)} + \frac{E^{\beta}_{-\nu}(E^{\alpha}_{\nu}-E'^{\alpha}_{\nu})}{-i\nu Z_{\alpha\beta}(-\nu)} \right] d\nu$$

$$- \frac{1}{\pi\hbar^2}\int_0^\infty (E^{\alpha}_{\nu}-E'^{\alpha}_{\nu})(E^{\beta}_{-\nu}-E'^{\beta}_{-\nu})\, \phi_{\alpha\beta}(\nu)\, d\nu \qquad (27)$$

where

$$\phi_{\alpha\beta}(\nu) = \frac{\hbar\left[\frac{1}{Z_{\alpha\beta}(\nu)} + \frac{1}{Z^{*}_{\beta\alpha}(\nu)} \right]}{2\nu\left[e^{\beta\hbar\nu} - 1 \right]} \quad . \qquad (28)$$

It is seen that $\phi_{\alpha\beta}(\nu)$ which is a sort of tensor power spectrum when written in this form is not simply related to $1/Z_{\alpha\beta}(\nu)$ for $\alpha \neq \beta$. In the case $\alpha = \beta$, $\phi_{\alpha\alpha}(\nu)$ reduces to the forms obtained for the simpler systems considered earlier (see equation 18). Therefore, if the required tensor impedance properties of a material are known, the influence functional may be formed using equations 23 and 24 or equations 27 and 28. In this case the classical quantities which one needs to know are

$$\frac{i\nu(\langle eX_{\alpha}\rangle)_{\nu}}{E_{\beta}(\nu)} = \frac{1}{Z_{\alpha\beta}(\nu)} \quad .$$

If they are not known, they may be calculated using the closed forms of equations 24, 25, or 26. The calculation of such forms is, however, normally a very difficult task.

In conclusion it is again found that the parameters which are significant in forming the influence functionals for interaction systems which are characterized by the more complicated type of coupling considered here are the classical response functions and the corresponding impedances. In addition, this formulation of the influence functional has yielded closed expressions which may be used to compute these response functions if the unperturbed quantum characteristics of the interaction system are known. As a last remark it may be again noted that the generalization of the above expressions to the interaction potential of equation 19,

$$V_I = \sum_\alpha V_\alpha(Q)\, U_\alpha(P) \equiv V^\alpha(Q)\, U_\alpha(P)$$

is direct. To illustrate this we can write the influence phase for this case from equation 23 which was derived for the case of the coupling potential $eE^\alpha X_\alpha$. It is only necessary to substitute $V^\alpha(Q)$ for E^α and $U_\alpha(P)$ for eX_α. Thus,

$$i\Phi(Q,Q') = -\frac{1}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{t}(V_t^\alpha - V_t'^\alpha)\left[V_s^\beta F^*_{\alpha\beta}(t-s) - V_s'^\beta F_{\alpha\beta}(t-s)\right] ds\,dt \qquad (29)$$

where

$$V_t^\alpha = V^\alpha\left[Q(t)\right]$$

and

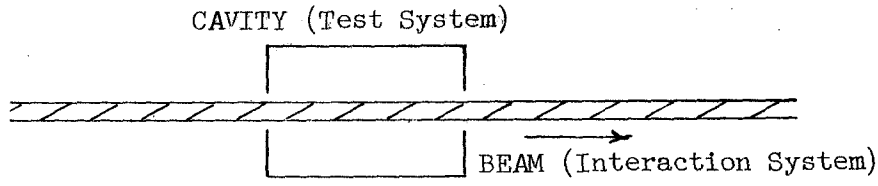
$$F_{\alpha\beta}(t) = \sum_{a,b}\frac{2\rho_{aa}(U_\alpha)_{ab}(U_\beta)_{ba}}{\hbar}e^{i\omega_{ba}t} \quad . \qquad (30)$$

The other expressions for $B_{\alpha\beta}(t)$ and $Z_{\alpha\beta}(\nu)$ may be found similarly.

VII.3 Beam of Particles Interacting with a Cavity

Consider a beam of non-interacting, identical particles which interact weakly with a resonant cavity as might occur, for instance, in a gas maser. We assume that the beam is not necessarily in temperature equilibrium but that the initial state of the particles entering the cavity would be properly represented by a density matrix diagonal in the energy representation. Such a situation would occur if the beam were prepared by passing it through a beam separator whose function would be to eliminate certain particles from the beam depending on their energy levels. For the purposes of simplifying the analysis we assume the molecules to be two-level quantum systems and that before they enter the cavity all of them are in the lower state or all in the upper state. It is easy to extend these results to the case where the beam is mixed with a certain fraction in the upper state and a certain fraction in the lower state initially. Since the beam is assumed to be composed of non-interacting particles, we can consider the total beam as composed of two independent beams appropriate to the two possible initial states of the constituent particles. The influence phase for the complete beam is simply the sum of the influence phases for the two beams. In addition, we assume that the beam is characterized by a spatial density such that the number of particles passing a given point along the beam in a time dt is Ndt and if t_o is the time a molecule passes a reference point inside the cavity, $\gamma(t-t_o)$ describes the coupling between the molecule and the cavity. Thus, the beam is a univelocity beam. Again, in a real case

where the beam is characterized by a distribution of velocities, the total beam may be split up into many univelocity beams. The total influence phase is simply a sum of those for each component beam.



Let us call the coordinates of the cavity Q, representing the test system and the coordinates of a particle in the beam P. The interaction between beam and particles is given by $\mathcal{L}_I(Q,P) = \gamma(t-t_o)QP$. Under these circumstances the influence functional for the effect of the beam on the cavity can be written down immediately:

$$i\Phi_B(Q,Q') = -N\left|\frac{P_{ab}}{\hbar}\right|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{t} ds dt \left[\int_{-\infty}^{\infty} \gamma(t-t_o)\gamma(s-t_o)dt_o\right](Q_t-Q_t') \times (Q_s e^{-i\omega_{ba}(t-s)} - Q_s' e^{i\omega_{ba}(t-s)}) \quad (31)$$

For the integral involving the coupling parameters,

$$\int_{-\infty}^{\infty} \gamma(t-t_o)\gamma(s-t_o)dt_o = \int_{-\infty}^{\infty} \gamma(\xi)\gamma(\xi-t+s)d\xi \equiv \Gamma(t-s) \quad . \quad (32)$$

Therefore, equation 31 becomes

$$i\Phi_B(Q,Q') = -N\left|\frac{P_{ab}}{\hbar}\right|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{t} \Gamma(t-s)(Q_t-Q_t')(Q_s e^{-i\omega_{ba}(t-s)} - Q_s' e^{i\omega_{ba}(t-s)})\,ds\,dt \tag{33}$$

From this we can identify the response function of the beam as

$$B(t-s) = \frac{2N\left|P_{ab}\right|^2}{\hbar}\,\Gamma(t-s)\,\sin\omega_{ba}(t-s) \quad . \tag{34}$$

In the same way that was done in Appendix I for the harmonic oscillator this influence functional can be converted to transform notation and yields the usual expression, equation IV.2, where from equation 34 we find

$$\left(\frac{1}{i\nu Z_\nu}\right)_{\text{beam}} = \frac{iN\left|P_{ab}\right|^2}{\hbar}\,\left(\Gamma_{\nu+\omega_{ba}} - \Gamma_{\nu-\omega_{ba}}\right) \tag{35}$$

where

$$\Gamma_{\nu+\omega_{ba}} = \int_{-\infty}^{\infty} 1(r)\Gamma(r)\, e^{-i(\nu+\omega_{ba})r}\, dr \tag{36}$$

and with a power spectrum

$$\phi(\nu) = \frac{N\left|P_{ab}\right|^2}{2}\left[\Gamma_{\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}}\right] \quad . \tag{37}$$

Previously we have found that for a linear system initially in the ground state the power spectrum is zero over the range of positive ν thus indicating a zero noise potential due to the linear system. However, $\phi(\nu)$ is not necessarily zero for $\nu > 0$ in equation 37 in the case that the beam is initially in its lowest state $(\omega_{ba} > 0)$. This is because no restrictions were placed on the time variation of $\gamma(t-t_o)$. However, in practical situations the coupling between a

cavity and a particle in a beam passing through the cavity varies adiabatically so that for all practical purposes $\phi(\nu)$ is really zero for $\nu > 0$. To point this out more clearly, let us assume that $\gamma(t-t_o) = 1(t-t_o)\ 1(t_o+\tau_o-t)$, that is, the coupling is turned on at t_o and off at $t_o+\tau_o$, the time of transit in the cavity being τ_o. We find by evaluating equation 32 that

$$\Gamma(t-s) = (s-t+\tau_o)\ 1(s-t+\tau_o)$$

and from this we can find

$$\Gamma_{\nu+\omega_{ba}} = \int_{-\infty}^{\infty} (\tau_o - r)\ 1(\tau_o - r)\ 1(r)\ e^{-i(\nu+\omega_{ba})r}\, dr$$

$$= \frac{1 - i\tau_o(\nu-\omega_{ba}) - e^{-i(\nu+\omega_{ba})\tau_o}}{(\nu+\omega_{ba})^2} \quad .$$

Therefore, from equation 37,

$$\phi(\nu) = 2N \left|P_{ab}\right|^2 \frac{\sin^2\left(\frac{\nu+\omega_{ba}}{2}\right)\tau_o}{(\nu+\omega_{ba})^2} \quad . \tag{38}$$

The maximum of $\phi(\nu)$ occurs at $\nu = -\omega_{ba}$ and is $\left[N\left|P_{ab}\right|^2/2\right]\tau_o^2$ whereas the maximum value $\phi(\nu)$ can have for $\nu > 0$ is $\left[N\left|P_{ab}\right|^2/2\right](4/\omega_{ab}^2)$. The ratio of the two is

$$\frac{\text{Max. of } \phi(\nu)}{\text{Max. of } \phi(\nu) \text{ for } \nu > 0} = \left(\frac{\omega_{ba}\tau_o}{2}\right)^2 \quad .$$

For ammonia molecules at a temperature of $T = 290^{\circ}K$, the average velocity $v \approx 6\times10^{4}$ cm/sec. For a microwave cavity of 10 cm length, $\tau_o \approx 2\times10^{-4}$ sec. For the 3-3 line of ammonia $\omega_{ba} \approx 1.5\times10^{11}$ radians/sec.

Therefore, for this case $(\omega_{ba}\tau_o/2)^2 \approx 2\times10^{14}$. From this, we can conclude even in this case of suddenly turning the coupling on and off, that the $\phi(\nu)$ is negligible for $\nu > 0$.

Examination of $\mathrm{Re}(1/Z_\nu)$ shows terms of the same form as those just discussed,

$$\mathrm{Re}(1/Z_\nu) = \frac{1}{2}\left[\frac{1}{Z_\nu} + \left(\frac{1}{Z_\nu}\right)^*\right] = \frac{\nu N\left|P_{ab}\right|^2}{2\hbar}\left[\Gamma_{\nu+\omega_{ba}} - \Gamma_{\nu-\omega_{ba}} - \Gamma_{-\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}}\right]. \tag{39}$$

By the same type of argument as above, the terms in $\Gamma_{\nu-\omega_{ba}}$ can be neglected when $\nu > 0$, $\omega_{ba} < 0$ and conversely, when $\nu > 0$, $\omega_{ba} > 0$ the terms in $\Gamma_{\nu+\omega_{ba}}$ can be neglected. Since this is the case, we can write

$$\phi_\nu \approx \frac{\hbar\,\mathrm{Re}\left(\frac{1}{Z_\nu}\right)}{\nu\left[e^{-\beta_e\hbar\,\omega_{ba}} - 1\right]} \tag{40}$$

where $\beta_e = 1/kT_e$, describes the relative initial populations of the two states. Therefore, we conclude that in most practical cases the power spectrum can be written in the form of equation 40 and for all practical purposes it will be exactly correct. In cases where the transients cannot be neglected for very low ω_{ba} and very short transit times, however, $\phi(\nu)$ is not so simply related to $\mathrm{Re}\left(\frac{1}{Z_\nu}\right)$ and must be written in the form given by equation 37.

It is to be noted that $\phi(\nu)$ of equation 40 is not precisely of the form for the Nyquist relation because of the appearance of $e^{-\beta\hbar\omega_{ba}}$ in the denominator rather than $e^{\beta\hbar\nu}$. This is a consequence of the finite coupling time between each part of the beam and the

cavity which results in a non-equilibrium condition. If the coupling times were infinite, the expression for $\mathrm{Re}(1/Z_\nu)$ would contain forms such as $\delta(\nu+\omega_{ba})$, a situation discussed earlier, so that the Nyquist form then results. However, when the coupling time is long as in masers,

$$\Gamma_{\nu+\omega_{ba}} + \Gamma_{-\nu-\omega_{ba}} \approx 4\pi\delta(\nu+\omega_{ba})$$

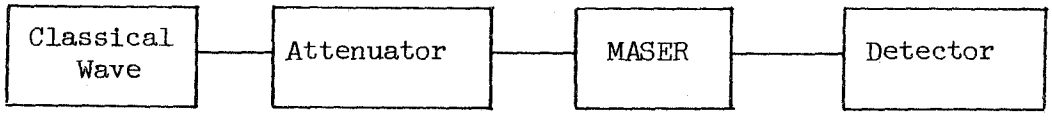
so that the true Nyquist relation may be used with negligible error.

VIII. Sources of Noise in Masers

Having developed the theory of linear systems in detail we are now in a position to discuss the sources of noise in quantum-mechanical devices such as masers. The subject of maser noise has been explored by many authors (12),(16),(17),(18). The details of the treatment given the subject differ, but the principles are essentially the same. The amplifiers are considered as operating at signal levels high enough (classical) that a signal entering a maser may be considered as a group of photons whose number is large enough that the amplification process increases the signal in a continuous fashion. The sources of noise were found to be those derived from the thermal noise arising from the sources of dissipation, and those derived from spontaneous emission from the "active" quantum material. They go further and define an effective temperature of the active quantum system so that the noise it produces is related to the negative resistance of the active materials. Our analysis of linear systems has also shown that these same sources of noise exist. In a subsequent article Townes et al (19), discussed the characteristics of maser type devices when the signal levels entering the maser were very small so that one could talk of relatively few photons per second entering the maser. From the results of the analysis it was concluded that in the case of a very small number of photons there would be a large fluctuation in the output of the maser even without thermal and spontaneous emission sources of noise. The mean square fluctuation of the output signal from large, ideal amplification was found to be proportional to $1/n$ where the input signal consisted of n photons/sec entering the

cavity. However, since this effect depended upon the signal level it was not considered to be noise in the usual sense. The results of our analysis shows that this last source of quantum fluctuation does not exist, at least insofar as the maser device produces it. In the following discussion this assertion will be stated more precisely and its proof will be demonstrated.

To make the discussion definite, let us suppose that we have a beam type maser amplifier in which all participating systems used meet the requirement of linearity. There may be one or more beams interacting with various electromagnetic resonators which can be coupled together in any way desired. The output of the maser is connected to a detector of some sort which perhaps consists of a resistor in which the current is to be measured. To the input of the maser system we now apply an incoming classical signal of large magnitude and of frequency ω through an attenuator whose value of attenuation may be varied at will. Diagrammatically we have:



In practice such a situation could arise if the classical wave originates from a distant antenna with a very large magnitude of output, so large that all quantum effects in the wave are effectively obscured. The long distance would then play the role of the attenuator.

Theorem XI. Suppose that the classical wave were attenuated by a large amount so that only a few photons/sec were entering the maser. The only uncertainty in the signal in the output of the maser caused by the maser itself arises from those sources of noise which can be arrived at by a classical calculation of the characteristics of the maser. There is no extra quantum fluctuation introduced by the maser into the output signal due to the small number of quanta entering the maser.

It is true that the amplitude of the signal output from the maser might itself be so small that it is still on a quantum level. In this case the detector output would be uncertain due to inherently small magnitude of the signal from the maser. However, if this is the case, we may put as many amplifiers in series as necessary to bring the output signal back to a classical level. When this is done the signal applied to the detector consists of the original signal modified by the transfer characteristics of the maser system (and attenuator) and noise signals which arise from all the possible sources computed classically.* The proof of this assertion is not difficult. We divide up the total system into a test system, here the detector, and an interaction system which consists of the maser, attenuator, and classical signal, C(t). Then, to find the effect of the interaction system on the detector we need only to look at the influence functional. However, we already know that it can be written as follows:

*The only error from this procedure results from a situation such as encountered when a portion of the circuit such as the beam has finite coupling times as discussed in Section VII.3 .

$$\mathcal{F}(Q,Q') = \exp\left\{\frac{i}{2\pi\hbar}\int_0^\infty \left[\frac{Q_\nu'(Q_{-\nu}-Q_\nu)}{i\nu Z_\nu} + \frac{Q_{-\nu}(Q_\nu-Q_\nu')}{-i\nu Z_{-\nu}}\right] d\nu \right.$$

$$\left. + \frac{i}{2\pi\hbar}\int_{-\infty}^\infty \frac{C_\nu}{i\nu z_\nu}(Q_{-\nu}-Q_{-\nu})d\nu - \frac{1}{\pi\hbar^2}\int_0^\infty |Q_\nu-Q_\nu'|^2 \sum_i \phi_i(\nu)d\nu\right\}$$

where $\phi_i(\nu)$ represents the power spectrum of the noise from the i^{th} source, z_ν is the classical transfer characteristic of the complete interaction system*, and Z_ν is the impedance of the maser system as seen by the detector. All the terms in the influence functional are familiar in view of the derivations which have been presented previously. The first term in the exponent of the influence functional is recognized as describing a linear system at zero temperature (Section IV), the linear system in this case being the maser. This term describes the spontaneous emission of the detector back into the maser, a situation which was discussed in Section IV.3. It was further discussed (Section VII.1) that this spontaneous emission could be thought of as resulting from a noise generator created by the detector (test system) acting on the interaction system in the usual classical way, i.e., whose power spectrum was related to the dissipative part of the detector impedance and to the temperature by the Nyquist relation

$$\phi(\nu) = \frac{\hbar\nu R_\nu}{e^{\hbar\nu/kT}-1}$$

where R_ν is the detector resistance and T its temperature. The second term in the above composite influence functional is easily

*In this example the complete interaction system would be everything in between the source of the classical wave and the detector.

interpreted and is simply the effect of a classical voltage, related to the input voltage by the classical transfer characteristic of the maser, acting on the detector (see Section V). The last term represents the effect on the detector of random noise voltages associated with the various classical noise sources in the maser, (Sections VI, VII). Both positive and negative resistances are such noise sources. In either case the power spectrum of the noise from a particular resistance is computed from the same relation as given above

$$\phi(\nu) = \frac{\not{h}\nu R_\nu}{e^{\not{h}\nu/kT} - 1}$$

where T is the temperature of the reactance R_ν. If R_ν is negative the effective temperature of R_ν will also be negative, always giving a positive power spectrum. Therefore, if we were to compute the current in the detector due to the interaction system (maser) using the influence functional, we would find components of current due to 1) the noise voltage generated by the detector itself, the power spectrum of which is related to the resistance of the detector by the generalized Nyquist relation given above, 2) a classical voltage related to the input voltage $C(t)$ by the classical transfer characteristic of the maser, and 3) random noise voltages associated with the various classical noise sources (resistances) in the maser. Therefore, the maser simply acts as a classical amplifier with sources of noise which can be predicted from considerations of its classical characteristics.

APPENDIX I(a)

Proof of Theorem VII for the Case of a Single Harmonic Oscillator

It was shown in Section IV.1 that the influence functional for the effect of a single harmonic oscillator on a test system can be written as follows:

$$\mathcal{F}(Q,Q') = \int \delta(X_T - X'_T) \exp\left\{ \frac{i}{\hbar} \int_\tau^T \left[\frac{m}{2}(\dot{X}^2 - \dot{X}'^2) - \frac{m\omega^2}{2}(X^2 - X'^2) + QX - Q'X' \right] dt \right.$$
$$\left. - \frac{m\omega}{2\hbar}(X_\tau^2 + X_\tau'^2) \right\} \mathcal{D}X(t)\mathcal{D}X'(t)\, dX_\tau dX'_\tau dX_T dX'_T \qquad \text{(A1.1)}$$

where dX_τ is an integration over all possible values of X_τ, etc. To evaluate this we introduce the variables

$$\begin{aligned} x &= X + X' & q &= Q + Q' \\ x' &= X - X' & q' &= Q - Q' \end{aligned} \qquad \text{(A1.2)}$$

If these substitutions are made

$$\mathcal{F}(q,q') = \int \delta(x'_T) \exp\left[\frac{i}{\hbar} \int_\tau^T \left\{ \frac{m\dot{x}\dot{x}'}{2} - \frac{m\omega^2 xx'}{2} + \frac{qx'}{2} + \frac{q'x}{2} \right\} dt - \frac{m\omega x_\tau^2}{4\hbar a} - \frac{m\omega a x_\tau'^2}{4\hbar} \right]$$
$$\times \mathcal{D}x(t)\mathcal{D}x(t)\, dx_\tau dx'_\tau dx_T dx'_T \qquad \text{(A1.3)}$$

In anticipation of a later derivation it will be noted that the constant a has been introduced as a coefficient of x_τ^2 and $x_\tau'^2$ respectively. For the harmonic oscillator initially in the ground state we will set $a = 1$ in the final result. Whenever some mathematical operation yields merely a constant multiplying $\mathcal{F}$ it will be the practice to ignore it. This makes the writing of $\mathcal{F}$ functionals less complicated and at the end

the constant can be recovered through normalization, if necessary. This was done between Al.1 and Al.3 since the change in variables of integration should be accompanied by the Jacobian of the transformation.

Integrating the first term in the exponent of the integrand by parts yields the result

$$\mathcal{J}(q,q') = \int \delta(x'_T)\, \exp\Big[-\frac{m\omega}{4\hbar}\Big(\frac{x_\tau^2}{a} + a x'^2_\tau\Big) + \frac{im}{2\hbar}(x_T\dot{x}_T - x_\tau\dot{x}'_\tau) - \frac{i}{\hbar} \times \int_\tau^T \Big\{ x\Big(\frac{m}{2}\ddot{x}' + \frac{m\omega^2}{2}x' - \frac{q'}{2}\Big) - \frac{qx'}{2}\Big\} dt \Big] \mathcal{D}x(t)\mathcal{D}x'(t)\, dx_\tau dx'_\tau dx_T dx'_T \,. \tag{Al.4}$$

The first part of this expression to be calculated is the integral over $x(t)$, $x'(t)$ paths and for this the only part of the integrand which is affected is the time integral in the exponent. Considering the $x(t)$ paths first and calling this integral $I_{x,x'}$ we have

$$I_{x,x'} = \int \exp\left\{ -\frac{i}{\hbar}\int_\tau^T \Big[xg(x') - \frac{qx'}{2}\Big] dt \right\} \mathcal{D}x(t)\mathcal{D}x'(t) \tag{Al.5}$$

where $$g(x') = \frac{m}{2}\ddot{x}'(t) + \frac{m\omega^2}{2}x'(t) - \frac{q'(t)}{2} \tag{Al.6}$$

To evaluate this expression we break up the time integral into a sum over N increments of time $\Delta t = \epsilon$ so that $N\epsilon = T - \tau$. At a time $t = k\epsilon$ the value of x is denoted by $x_k = x(t_k)$. Therefore, the path integral in x,x' becomes

$$I_{x,x'} = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int_{-\infty}^{\infty}\cdots\int \exp\left\{ -\frac{i}{\hbar}\sum_{k=1}^{N-1}\Big[x_k g(x'_k) - \frac{q_k x'_k}{2}\Big]\epsilon \right\} \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N-1} dx'_k\Big). \tag{Al.7}$$

The integral over each x_k yields the result $\delta(g'_k)$ and

$$I_{x,x'} = \lim_{\substack{N \to \infty \\ \epsilon \to 0}} \prod_{k=1}^{N-1} \int (\text{const})\ \delta[g(x'_k)]\ \exp[-\frac{i}{2\hbar} q_k x'_k \epsilon]\ dx'_k \quad . \tag{A1.8}$$

Since the time interval may be made infinitesimally small, the result of the path integration on x is seen to be a δ functional of the path denoted by $g[x'(t)]$. The solution $g[x'(t)] = 0$ is simply the classical solution and the function of $x'(t)$ for which this equation is satisfied will be denoted by $\overline{x}'(t)$. Then we can immediately write for A1.8

$$I_{x,x'} = \lim_{\substack{N \to \infty \\ \epsilon \to 0}} \prod_{k=1}^{N-1} (\text{const})\ \exp[-\frac{i}{2\hbar} q_k \overline{x'_k} \epsilon] = \exp\left[-\frac{i}{2\hbar}\int_\tau^T q(t)\overline{x'}(t)dt\right] \quad . \tag{A1.9}$$

Substituting this result into the expression for $\mathcal{F}$ given by equation A1.4 we obtain the following:

$$\mathcal{F}(q,q') = \int \delta(x'_T)\exp\left\{-\frac{m\omega}{4\hbar}\left(\frac{x_\tau^2}{a} + a x_\tau'^2\right) + \frac{im}{2\hbar}(x_T\dot{x}'_T - x_\tau\dot{x}'_\tau) - \frac{i}{2\hbar}\int_\tau^T q(t)\overline{x'}(t)dt\right\}$$

$$\times\ dx_\tau dx'_\tau dx_T dx'_T \quad . \tag{A1.10}$$

The integrals on dx_τ and dx_T can be done next. Integration on dx_τ is a Gaussian integral and can be done by completing the square. The dx_T integration yields a δ-function of $\dot{x}'_T$. Therefore,

$$\mathcal{F}(q,q') = \int \delta(x'_T)\delta(\dot{x}'_T)\exp\left[-\frac{m\omega a}{4\hbar}\left(x_\tau'^2 + \frac{x_\tau'^2}{\omega^2}\right) + \frac{i}{2\hbar}\int_\tau^T q(t)\overline{x'}(t)dt\right] dx'_\tau dx'_T \tag{A1.11}$$

The two functions $\delta(x'_T)$ and $\delta(\dot{x}'_T)$ can be interpreted as boundary conditions on the classical solution of $g[x'(t)] = 0$, a second order linear differential equation. These conditions are that $x'_T = \dot{x}'_T = 0$. The result of doing the above integral will be to substitute for x'_τ, $\dot{x}'_\tau$, and $\overline{x'}(t)$ the solutions which satisfy these conditions. We anticipate the result by taking as the solution

$$\overline{x'} = -\int_t^T \frac{q'(s)}{m\omega} \sin \omega(t-s)ds + x'_T \cos \omega(T-t) - \frac{\dot{x}'_T}{\omega} \sin \omega(T-t) . \quad (A1.12)$$

Secondly, we change the variable of integration (using A1.12) from dx'_τ to $d\dot{x}'_T$ and find from A1.12 that,

$$dx'_\tau = \cos \omega(T-\tau)d\dot{x}'_T \quad .$$

Substituting A1.12 into A1.11 and changing dx_τ to $d\dot{x}'_T$ we find that

$$\mathcal{F}(q,q') = \int \delta(x'_T)\delta(\dot{x}'_T) \exp\left\{ -\frac{m\omega a}{4\hbar}\left(x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2}\right) + \frac{i}{2\hbar}\int_\tau^T q(t) \times \right.$$

$$\left. \left[-\int_t^T \left[\frac{q'(s)}{m\omega} \sin \omega(t-s)ds + x'_T \cos \omega(T-t) + \frac{\dot{x}'_T}{\omega} \sin \omega(T-t) \right] dt \right] \right\} dx'_T d\dot{x}'_T$$

The result of the integration is

$$\mathcal{F}(q,q') = \exp\left\{ -\frac{m\omega a}{4\hbar}\left(x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2}\right) - \frac{i}{2m\omega\hbar}\int_\tau^T\int_t^T q(t)q'(s)\sin \omega(t-s)dsdt \right\} . \quad (A1.13)$$

To get this in a more usable form it is best to express the exponent in terms of double integrals:

$$x'_\tau = -\int_\tau^T \frac{q'(s)}{m\omega} \sin \omega(\tau-s)ds \quad \text{and} \quad \dot{x}'_\tau = \int_\tau^T \frac{q'(s)}{m} \cos \omega(\tau-s)ds \ .$$

Therefore,

$$x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2} = \int_\tau^T\int_\tau^T \frac{q'(t)q'(s)}{m^2\omega^2}[\sin \omega(T-s)\sin \omega(T-t) + \cos \omega(T-s)\cos \omega(T-t)]\,dsdt$$

$$= \int_\tau^T\int_\tau^T \frac{q'(t)q'(s)}{m^2\omega^2} \cos \omega(t-s)dsdt \qquad \text{(A1.14)}$$

Substituting this in A1.13 the influence phase ($\mathcal{F} = e^{i\Phi}$) is

$$i\,\Phi(q,q') = -\frac{a}{4\hbar m\omega}\int_\tau^T\int_\tau^T q'(t)q'(s) \cos \omega(t-s)dsdt +$$

$$-\frac{i}{2\hbar m\omega}\int_\tau^T\int_\tau^T q(t)q'(s) \sin \omega(t-s)dsdt \qquad \text{(A1.15)}$$

Since t and s enter into the first integral symmetrically it can be rewritten as indicated by

$$\int_\tau^T\int_\tau^T E(t,s) = 2\int_\tau^T\int_\tau^t E(t,s)dsdt$$

The second integral can be rewritten as follows:

$$\int_\tau^T\int_\tau^T E(t,s)dsdt = \int_\tau^T\int_\tau^t E(s,t)dsdt \ .$$

Rewriting A1.15:

$$i\,\Phi = -\frac{i}{2\hbar\, m\omega}\int_T^T\int_T^t [aq'(t)q'(s)\cos\omega(t-s) - iq'(t)q(s)\sin\omega(t-s)]dsdt$$

Substituting Q,Q' for q,q' we can write

$$i\,\Phi = -\frac{1}{2\hbar}\int_T^T\int_T^t (Q_t - Q_t')[Q_s F^*(t-s) - Q_s' F(t-s)]dsdt \qquad \text{(A1.16a)}$$

where

$$F(t-s) = \frac{a}{m\omega}\cos(t-s) + \frac{i}{m\omega}\sin\omega(t-s) \qquad \text{(A1.16b)}$$

or,

$$i\,\Phi = -\frac{1}{2\hbar\, m\omega}\int_T^T\int_T^t (Q_t - Q_t')[Q_s e^{-i\omega(t-s)} - Q_s' e^{i\omega(t-s)}]$$

$$+\frac{1-a}{2\hbar m\omega}\int_T^T\int_T^t (Q_t - Q_t')(Q_s - Q_s')\cos\omega(t-s)dsdt \qquad \text{(A1.17)}$$

For an oscillator initially in the ground state $a = 1$ and

$$i\,\Phi = -\frac{1}{2\hbar\, m\omega}\int_T^T\int_T^t (Q_t - Q_t')[Q_s e^{-i\omega(t-s)} - Q_s' e^{i\omega(t-s)}]dsdt \qquad \text{(A1.18)}$$

APPENDIX I(b)

In this section the preceding results will be put in Fourier transform notation. It is slightly more convenient to work in terms of q and q', so we will start with an expression of the form of equation A1.16 and change variables from Q,Q' to q,q' :

$$i\,\Phi = \frac{1}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{t}[iq'(t)q(s)\,B(t-s) - q'(t)q'(s)\,A_o(t-s)]dsdt \tag{A1.19}$$

$$= \frac{1}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[iq'(t)q(s)1(t-s)B(t-s) - q'(t)q'(s)1(t-s)A_o(t-s)]dsdt$$

$$\text{where } B(t-s) = \frac{1}{m\omega}\sin\omega(t-s) = \text{Im } F(t-s) \tag{A1.20}$$

$$A_o(t-s) = \frac{a}{m\omega}\cos\omega(t-s) = \text{Re } F(t-s)$$

and $1(t-s)$ = a unit step function. The subscript o pertains to this zero temperature case and is attached to $A_o(t)$ because the relations connecting $A(t)$ and $B(t)$ given at the end of this appendix are only valid at zero temperature. If the above influence phase acts only over a finite interval of time as in A1.16, this can be taken care of in the above expression A1.19 by replacing q_t, q_s by $\gamma_t q_t, \gamma_s q_s$ where γ_t is a coupling factor. Both terms in A1.19 are of the following form which can be evaluated using the Faltung theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(t)\,P(t-s)\,R(s)dsdt = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} N_{-\nu}P_{\nu}R_{\nu}d\nu \quad . \tag{A1.21}$$

Applying we obtain

$$i\,\Phi = +\frac{1}{4\pi\hbar}\int_{-\infty}^{\infty}(iq'_{-\nu}q_{\nu}B_{\nu} - q'_{-\nu}q'_{\nu}A_{\nu})d\nu \tag{A1.22}$$

where

$$q_\nu = \int_{-\infty}^{\infty} q(t)\, e^{-i\nu t}\, dt \quad ,$$

$$B_\nu = \int_{-\infty}^{\infty} 1(t)\, B(t)\, e^{-i\nu t}\, dt \quad , \quad \text{etc.}$$

Changing the interval of integration to include positive frequencies only, and by adding and subtracting the term $iq'_\nu\, q'_{-\nu}(B_\nu - B_{-\nu})$, Al.21 becomes

$$i\Phi = + \frac{1}{4\pi\hbar}\int_0^{\infty}\left[iq'_{-\nu}(q_\nu - q'_\nu)B_\nu + iq'_\nu(q_{-\nu} + q'_{-\nu})B_{-\nu} - q'_\nu q'_{-\nu}(A_{o_\nu} + A_{o_{-\nu}} - iB_\nu + iB_{-\nu})\right] d\nu \tag{Al.23}$$

Next, the A_{o_ν}, B_ν need to be calculated:

$$A_{o_\nu} + A_{o_{-\nu}} = \frac{a}{m\omega}\int_{-\infty}^{\infty} 1(t)\, \cos\omega t(e^{-i\nu t} + e^{i\nu t})dt$$

$$= \frac{a}{2m\omega}\int_{-\infty}^{\infty} (e^{i\omega t} + e^{-i\omega t})e^{-i\nu t}dt = \frac{\pi a}{m\omega}\left[\delta(\nu+\omega) + \delta(\nu-\omega)\right] \tag{Al.24}$$

Similarly,

$$iB_\nu - iB_{-\nu} = \frac{i}{m\omega}\int_{-\infty}^{\infty} 1(t)\sin\omega t(e^{-i\nu t} - e^{i\nu t})dt = -\frac{\pi}{m\omega}\left[\delta(\nu+\omega) - \delta(\nu-\omega)\right] \tag{Al.25}$$

and

$$B_\nu = \frac{1}{m\omega}\int_0^{\infty} \sin\omega t\, e^{-i\nu t}dt = -\frac{1}{m\left[(\nu - i\epsilon)^2 - \omega^2\right]} \quad . \tag{Al.26}$$

The ϵ is an infinitesimal introduced as a convergence factor and is

kept throughout the calculation to indicate the location of the poles. The addition of ϵ is equivalent to adding an infinitesimal amount of loss to the oscillator. Substituting the results for A_ν, B_ν back in equation A1.23, we find

$$i\,\Phi = \frac{i}{4\pi\hbar}\int_0^\infty \left[\frac{q_{-\nu}(q_\nu - q'_\nu)}{-m\left[(\nu - i\epsilon)^2 - \omega^2\right]} + \frac{q'_\nu(q_{-\nu} + q'_{-\nu})}{-m\left[(\nu + i\epsilon)^2 - \omega^2\right]}\right] d\nu +$$

$$- \frac{1}{\pi\hbar^2}\int_0^\infty q'_\nu\, q'_{-\nu}\left(\frac{\pi\hbar}{4m\omega}\right)\left[(a-1)\delta(\nu-\omega) + (a+1)\delta(\nu+\omega)\right] d\nu \,. \qquad \text{(A1.27)}$$

Substituting Q, Q' for q, q' and setting $a = 1$, we have

$$i\,\Phi = \frac{i}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu} - Q'_{-\nu})}{i\nu\, Z_\nu} + \frac{Q_{-\nu}(Q_\nu - Q'_\nu)}{(-i\nu\, Z_{-\nu})}\right] d\nu +$$

$$- \frac{1}{\pi\hbar^2}\int_0^\infty \left|Q_\nu - Q'_\nu\right|^2 \left[+\frac{\pi\hbar}{2m\omega}\,\delta(\nu+\omega)\right] d\nu \qquad \text{(A1.28)}$$

For cases zero of temperature the last term is always zero since the integration is only over positive frequencies and $\omega > 0$. It will be noted, however, that the last term has the same form as that produced by a noisy classical potential with a power spectrum $\phi(\nu) = +\pi\hbar/2m\omega \times \delta(\nu+\omega)$. In more general terms this is, from A1.23,

$$\phi(\nu) = \frac{\hbar}{4}\left(A_{o_\nu} + A_{o_{-\nu}} - iB_\nu + iB_{-\nu}\right) \qquad \text{(A1.29)}$$

It is developed in the text that $B(t)$ is the classical response function for the linear system being analyzed. Here, of course, the system

is a single lossless harmonic oscillator. Moreover, $A_o(t)$ is, within a factor of $\frac{\hbar}{2}$, the correlation function of the zero point quantum fluctuation of the oscillator variables. Relations connecting them can be deduced as follows. From equation A1.25 we have

$$-\mathrm{Im}(B_\nu) = \frac{\pi}{2m\omega}\left[\delta(\nu-\omega) - \delta(\nu+\omega)\right] \quad . \tag{A1.25}$$

Thus, we can write

$$B(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}\,(B_\nu)\sin\,\nu t\,d\nu = \frac{\sin\,\omega t}{m\omega} \quad . \tag{A1.29}$$

Similarly,

$$A_o(t) = -\frac{2}{\pi}\int_0^\infty \mathrm{Im}(B_\nu)\cos\,\nu t\,d\nu = \frac{\cos\,\omega t}{m\omega} \quad . \tag{A1.30}$$

Both of the above two relations can be readily ascertained by referring to equation A1.20 for the case that $a = 1$. It is easy to see that A_{o_ν} and B_ν are related by a dispersion relation which can easily be obtained from the second of the above relations, equation A1.30,

$$A_{o_\nu} = \int_0^\infty A_o(t)e^{-i\nu t}d\nu = -\frac{2}{\pi}\int_0^\infty\int_0^\infty \mathrm{Im}(B_\omega)\cos\,\omega t\; e^{-i\nu t}d\omega dt$$

$$= \frac{2i}{\pi}\int_0^\infty \mathrm{Im}(B_\omega)\,\frac{\nu}{\left[(\nu-i\epsilon)^2-\omega^2\right]}\,d\omega \tag{A1.31}$$

From equation A1.29 we can find the usual dispersion relation connecting the real and imaginary parts of an analytic function,

$$B_\nu = \int_0^\infty B(t)\ e^{-i\nu t} dt = -\frac{2}{\pi} \int_0^\infty \int_0^\infty \mathrm{Im}(B_\omega)\ \sin \omega t\ e^{-i\nu t} dt\ d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \mathrm{Im}(B_\omega)\ \frac{\omega}{[(\nu - i\epsilon)^2 - \omega^2]}\ d\omega \quad . \qquad \text{(A1.32)}$$

Observation of equation A1.24 for $a = 1$ shows us that equations A1.29 through A1.32 can be rewritten in terms of A_{o_ν} by replacing $\mathrm{Im}\ B_\nu$ by $\mathrm{Re}\ A_\nu$.

APPENDIX II

a. Statement of the Problem

Having found the influence functional for the effect of an interaction system on a test system, it is useful to have perturbation expansions available for use in calculations. There are three forms of influence functionals which are especially useful in describing the effects of linear systems on test systems. They are $\mathcal{J}(Q,Q')$ for a) linear systems at zero temperature, b) a known classical potential, and c) an uncertain classical potential. Second order transition probabilities will be found for all three of these. The first will be done in detail using the path integral formulation to show how perturbation theory can be done by that method. In addition it is interesting to see how the analysis shows that for a zero temperature interaction system, transitions of the test system are only allowed to states of lower energy than the initial one.

In a general case where the initial and final states of the test system are definite. these states can be expressed as

$$\psi_i(Q_\tau) = \sum_n a_n(\tau)\, \phi_n(Q_\tau)$$

and (A2.1)

$$\psi_f(Q_T) = \sum_m b_m(T)\, \phi_m(Q_T)$$

where T and τ represent the initial and final times respectively, and the $\phi_n(Q)$ represent a complete set of orthonormal eigenfunctions of Q. However, for the purposes here the initial and final states will be considered eigenstates since the calculations are considerably

simplified and the extension to cases of more complicated initial and final states is straightforward. Therefore, we will take

$$\psi_i(Q_\tau) = \phi_n(Q_\tau) e^{-\frac{i}{\hbar} E_n \tau}$$
$$\psi_f(Q_T) = \phi_m(Q_T) e^{-\frac{i}{\hbar} E_n T} \quad . \tag{A2.2}$$

Since the influence functionals of interest here are exponential in form, the $n \to m$ transition probability for Q is

$$P_{n\to m} = \int \phi_m^*(Q_T)\phi_m(Q_T') e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]} \; e^{i\Phi(Q,Q')} \; \phi_n^*(Q_\tau')\phi_n(Q_\tau)$$
$$\times dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q \mathcal{D}Q' \quad .$$

If the magnitude of Φ is small enough, it will be sufficient to expand $\mathcal{F}(Q,Q')$ in a power series and keep only the first few terms. Thus,

$$P_{n\to m} = \int \phi_m^*(Q_T)\phi_m(Q_T') e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]} \left[1 + i\Phi + \frac{(i\Phi)^2}{2!} + \cdots\right]$$
$$\times \phi_n^*(Q_\tau')\phi_n(Q_\tau) dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q \mathcal{D}Q' \quad . \tag{A2.3}$$

$$= P_o + P_1 + P_2 + \cdots P_k + \cdots \tag{A2.4}$$

where P_k refers to the term in order k of the potentials involved. In this appendix we will not be concerned with orders greater than 2 . Inspection of equations A2.3 and A2.4 shows that the zeroth order term P_o does not depend upon Φ and can be calculated immediately. Thus, we have

$$P_o = \int \phi_m^*(Q_T)\phi_m(Q_T')e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]}\,\phi_n^*(Q_\tau')\phi_n(Q_\tau)\,dQ_\tau\,dQ_\tau'\,dQ_T\,dQ_T'\,\mathcal{D}Q\,\mathcal{D}Q',$$

$$= \left| \int \phi_m^*(Q_T)e^{\frac{i}{\hbar}S_o(Q)}\,\phi_n(Q_\tau)\,dQ_\tau\,dQ_T\,\mathcal{D}Q \right|^2$$

$$= \left| \int \phi_m^*(Q_T)K(Q_T,T;Q_\tau,\tau)\,\phi_n(Q_\tau)\,dQ_\tau\,dQ_T \right|^2$$

$$= \left| \int \phi_m^*(Q_T) \sum_\ell \phi_\ell(Q_T)\phi_\ell^*(Q_\tau)e^{-\frac{i}{\hbar}E_\ell(T-\tau)}\,\phi_n(Q_\tau)\,dQ_\tau\,dQ_T \right|^2$$

$$= \left| \sum_\ell \delta_{m\ell}\,\delta_{\ell n} \right|^2 = \delta_{mn} \quad . \tag{A2.5}$$

This is simply the $n \to m$ transition probability for Q with no perturbation applied. Now, we will proceed to calculate higher order terms involving Φ .

b. Linear System at Zero Temperature Acting on Q

If the coupling potential between Q and a zero temperature linear interaction system is $-\gamma QX$ the appropriate influence phase has been shown to be

$$i\,\Phi_o = \frac{i}{2\pi\hbar}\int_0^\infty \left[\frac{Q_\nu'(Q_{-\nu} - Q_{-\nu}')}{(i\nu\,Z_\nu)} + \frac{Q_{-\nu}(Q_\nu - Q_\nu')}{(-i\nu\,Z_{-\nu})}\right] d\nu \quad . \tag{A2.6}$$

A more convenient form for this calculation is

$$i\,\Phi_o = -\int_0^\infty \frac{Q_\nu Q_{-\nu}}{2\pi\hbar\nu Z_{-\nu}}\,d\nu + \int_0^\infty \frac{Q_\nu' Q_{-\nu}}{2\pi\hbar\nu Z_\nu}\,d\nu$$

$$+ \int_0^\infty \frac{Q_{-\nu}Q_\nu'}{2\pi\hbar\nu Z_{-\nu}}\,d\nu - \int_0^\infty \frac{Q_\nu' Q_{-\nu}'}{2\pi\hbar\nu\,Z_\nu}\,d\nu \tag{A2.7}$$

Since Φ is already quadratic in Q, then $P_1 = 0$ and we can go directly to the calculation of P_2. Thus, from A2.3

$$P_2 = \int \phi_m^*(Q_T)\phi_m(Q_T')e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]}(i\Phi)\phi_n^*(Q_\tau')\phi_n(Q_\tau)dQ_\tau dQ_\tau' dQ_T dQ_T' \mathcal{D}Q \mathcal{D}Q' . \tag{A2.8}$$

In the evaluation of A2.8 four path integrations must be carried out corresponding to the four terms of A2.7. These will be designated $P_2^{1,2,3,4}$ respectively. Further observation of these two equations shows that $P_2^1 = (P_2^4)^*$ and $P_2^2 = (P_2^3)^*$. Therefore, only two integrals need be calculated in detail. For P_2^1 we have

$$P_2^1 = \int \phi_m^*(Q_T)\phi_m(Q_T')e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]}\left\{-\int_0^\infty \frac{d\nu}{2\pi\hbar\nu Z_{-\nu}} Q_{-\nu}Q_\nu\right\}$$

$$\times \phi_n^*(Q_\tau')\phi_n(Q_\tau)dQ_\tau \cdots \mathcal{D}Q' . \tag{A2.9}$$

To evaluate this we replace $Q_{-\nu}$ and Q_ν by their inverse transforms, and find

$$Q_{-\nu}Q_\nu = \int_{-\infty}^{\infty}\int \gamma_t\gamma_s Q_t Q_s e^{i\nu(t-s)}dsdt = \int_{-\infty}^{\infty}\int_{-\infty}^{t} \gamma_t\gamma_s Q_t Q_s (e^{i\nu(t-s)} + e^{-i\nu(t-s)})dsdt$$

The time ordering of Q_t and Q_s so that $t > s$ is necessary for purposes of doing the path integral. Changing the order of integration on ν, t, and s, P_2^1 can be written

$$P_2^1 = -\int_0^\infty \frac{d\nu}{2\pi\hbar\nu Z_{-\nu}} \int_{-\infty}^{\infty}\int_{-\infty}^{t} dsdt\ \gamma_t\gamma_s\left[e^{i\nu(t-s)} + e^{-i\nu(t-s)}\right]\int \phi_m^*(Q_T)\phi_m(Q_T')$$

$$\times e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]} Q_t Q_s \phi_n^*(Q_\tau')\phi_n(Q_\tau)dQ \cdots \mathcal{D}Q' .$$

Representing the integral operation on ν, t, and s by $I(\nu,t,s)$, P_2^1 can be written more clearly as

$$P_2^1 = I(\nu,t,s)\int \phi_m^*(Q_T)e^{\frac{i}{\hbar}S_o(Q)}\, Q_t Q_s \phi_n(Q_\tau)dQ_\tau dQ_T \mathcal{D}Q \int \phi_m(Q_T')e^{-\frac{i}{\hbar}S_o(Q')}$$

$$\times \phi_n^*(Q_\tau')dQ_\tau' dQ_T' \mathcal{D}Q' \quad . \qquad (A2.10)$$

As in the calculation of P_o we take advantage of the fact that

$$\int e^{\frac{i}{\hbar}S_o(Q)}\mathcal{D}Q = K(Q_T,Q_\tau) = \int K(Q_T,Q_t)\, K(Q_t,Q_\tau)dQ_t$$

and that

$$K(Q_r,Q_s) = \sum \phi_n(Q_r)\phi_n(Q_s)e^{-\frac{i}{\hbar}E_n(r-s)} \quad .$$

Equation A2.10 can be written

$$P_2^1 = I(\nu,t,s)\sum_{j,k,\ell}\int \phi_m^*(Q_T)\phi_j(Q_T)e^{-\frac{i}{\hbar}E_j(T-t)}\phi_j^*(Q_t)Q_t\phi_k(Q_t)$$

$$\times e^{-\frac{i}{\hbar}E_k(t-s)}\phi_k^*(Q_s)Q_s\phi_\ell(Q_s)e^{-\frac{i}{\hbar}E_\ell(s-\tau)}\phi_\ell^*(Q_\tau)\phi_n(Q_\tau)dQ_\tau dQ_s dQ_t dQ_T$$

$$\times \int \phi_m(Q_T')\sum_p \phi_p^*(Q_T')e^{\frac{i}{\hbar}E_p(T-\tau)}\phi_p(Q_\tau')\phi_n(Q_\tau')dQ_\tau dQ_T \quad .$$

This can be evaluated readily to yield

$$P_2^1 = \delta_{mn}\sum_k \left|Q_{mk}\right|^2 I(\nu,t,s)e^{i\nu_{mk}(t-s)}$$

where $Q_{mk} = \int \phi_m^*(Q)\, Q\, \phi_k(Q)\, dQ$ and $\nu_{mk} = \frac{E_m - E_k}{\hbar}$.

P_2^1 can be put in a form convenient for later calculation by the following:

$$I(\nu,t,s)e^{i\nu_{mk}(t-s)} = -\int_0^\infty \frac{d\nu}{2\pi\hbar\nu Z_{-\nu}} \int_{-\infty}^{\infty}\int_{-\infty}^{t} dsdt\ \gamma_s\gamma_t$$

$$\times\left[e^{i(\nu_{mk}+\nu)(t-s)} + e^{i(\nu_{mk}-\nu)(t-s)}\right]$$

Applying the convolution theorem in the form

$$\int_{-\infty}^{\infty}\int F(t)\ G(s)\ H(t-s)dsdt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\eta\ G_{-\eta}\ H_{-\eta}\ d\eta$$

where $G_\eta = \int_{-\infty}^{\infty} G(t)\ e^{-i\eta t}\ dt$, we have

$$I(\nu,t,s)e^{i\nu_{mk}(t-s)} = -\int_0^\infty\int_{-\infty}^{\infty} \frac{d\nu\ d\eta}{(2\pi)^2\hbar\nu Z_{-\nu}}\ \gamma_\eta\gamma_{-\eta}\left[\frac{i}{\eta+\nu_{mk}+\nu+i\epsilon} + \frac{i}{\eta+\nu_{mk}-\nu+i\epsilon}\right] \tag{A2.11}$$

The ϵ is a convergence factor introduced in taking the transform

$$\int_{-\infty}^{\infty} 1(r)e^{i(\eta+\nu_{mk}+\nu)r}\ dr = \frac{e^{i(\eta+\nu_{mk}+\nu)r}}{i(\eta+\nu_{mk}+\nu+i\epsilon)}\Bigg|_0^\infty = \frac{i}{\eta+\nu_{mk}+\nu+i\epsilon}\ .$$

Thus, the expression for P_2^1 becomes

$$P_2^1 = \delta_{mn}\sum_k \left|Q_{mk}\right|^2 \int_{-\infty}^{\infty}\int_0^\infty \frac{d\nu d\eta}{(2\pi)^2\hbar\nu Z_{-\nu}}\ \gamma_\eta\gamma_{-\eta}\left[-\frac{i}{\nu+(\nu_{mk}+\eta)+i\epsilon} + \frac{i}{\nu-(\nu_{mk}+\eta)-i\epsilon}\right]$$

The expression for P_2^2 can be evaluated in an entirely similar way:

$$P_2^2 = \int \phi_m^*(Q_T)\phi_m(Q_T')e^{\frac{i}{\hbar}[S_o(Q) - S_o(Q')]}\left\{\int_0^\infty \frac{Q_{-\nu}Q_\nu'}{2\pi\hbar\nu Z_\nu}\, d\nu\right\}\phi_n^*(Q_\tau')\phi_n(Q_\tau)dQ_\tau \cdots \mathcal{D}Q'$$

$$= \int_0^\infty \frac{d\nu}{2\pi\hbar\nu Z_\nu} \iint_{-\infty}^{\infty} dsdt\ \gamma_s\gamma_t e^{i\nu(t-s)} \int \phi_m^*(Q_T)e^{\frac{i}{\hbar}S_o(Q)} Q_t\phi_n(Q_\tau)dQ_\tau dQ_T \mathcal{D}Q$$

$$\times \int \phi_m(Q_T')e^{-\frac{i}{\hbar}S_o(Q')}\ Q_s'\phi_n^*(Q_\tau')dQ_\tau' dQ_T' \mathcal{D}Q' \quad . \tag{A2.12}$$

Note that Q_t and Q_s' need not be ordered in time since they occur in different path integrals. Equation A2.12 when evaluated becomes

$$P_2^2 = \int_0^\infty \frac{d\nu}{2\pi\hbar\nu Z_\nu} \int_{-\infty}^{\infty} dsdt\ \gamma_s\gamma_t e^{i\nu(t-s)}\ e^{i\nu_{mn}(t-s)} \left|Q_{mn}\right|^2$$

$$= \left|Q_{mn}\right|^2 \int_0^\infty \frac{\left|\gamma_{\nu+\nu_{mn}}\right|^2}{2\pi\hbar\nu Z_\nu}\, d\nu \tag{A2.13}$$

where

$$\gamma_{\nu+\nu_{mn}} = \int_{-\infty}^{\infty} \gamma_t e^{-i(\nu+\nu_{mn})t}\, dt \quad .$$

Since we already know that $P_2^1 = (P_2^4)^*$ and $P_2^2 = (P_2^3)^*$, to second order

$$P_{n\to m} = \delta_{mn} + \sum_{\ell=1}^{4} P_2^\ell = \delta_{mn}\left\{1 + \sum_k \left|Q_{mk}\right|^2 \int_{-\infty}^{\infty} \frac{d\eta\gamma_\eta\gamma_{-\eta}}{(2\pi)^2\hbar}\right.$$

$$\left.\times \int_0^\infty d\nu\left[\frac{1}{\nu Z_{-\nu}}\left(-\frac{i}{\nu+(\nu_{mk}+\eta)+i\epsilon} + \frac{i}{\nu-(\nu_{mk}+\eta)-i\epsilon}\right) + \frac{1}{\nu Z_\nu}\left(\frac{i}{\nu+(\nu_{mk}+\eta)-i\epsilon} - \frac{i}{\nu-(\nu_{mk}+\eta)+i\epsilon}\right)\right]\right\}$$

$$+ \left|Q_{mn}\right|^2 \int_0^\infty d\nu\, \frac{\left|\gamma_{\nu+\nu_{mn}}\right|^2}{\pi\hbar\nu}\, \mathrm{Re}\left(\frac{1}{Z_\nu}\right) \quad . \tag{A2.14}$$

It remains to simplify the bracketed term inside the braces. In doing this let us write the pertinent part of equation A2.14 which is to be evaluated:

$$I = \int_0^\infty d\nu \left[\frac{1}{i\nu Z_{-\nu}} \left(\frac{1}{\nu+(\nu_{mk}+\eta)+i\epsilon} - \frac{1}{\nu-(\nu_{mk}+\eta)-i\epsilon} \right) \right.$$

$$\left. + \frac{1}{i\nu Z_{\nu}} \left(- \frac{1}{\nu+(\nu_{mk}+\eta)-i\epsilon} + \frac{1}{\nu-(\nu_{mk}+\eta)+i\epsilon} \right) \right] . \qquad \text{(A2.15)}$$

To do this we must resort to contour integration. There are two cases involved $\nu_{mk}+\eta > 0$ and $\nu_{mk}+\eta < 0$.

Case I. $\nu_{mk}+\eta > 0$

Since the integration is over positive ν , in the limit as $\epsilon \to 0$, the sign in front of the ϵ in the first and third terms is arbitrary. Changing the signs and changing the interval of integration on these terms from $+\nu$ to $-\nu$ on these same terms, we have

$$I = + \int_{-\infty}^{0} \frac{d\nu}{i\nu Z_{+\nu}[\nu-(\nu_{mk}+\eta)+i\epsilon]} - \int_0^\infty \frac{d\nu}{i\nu Z_{-\nu}[\nu-(\nu_{mk}+\eta)-i\epsilon]}$$

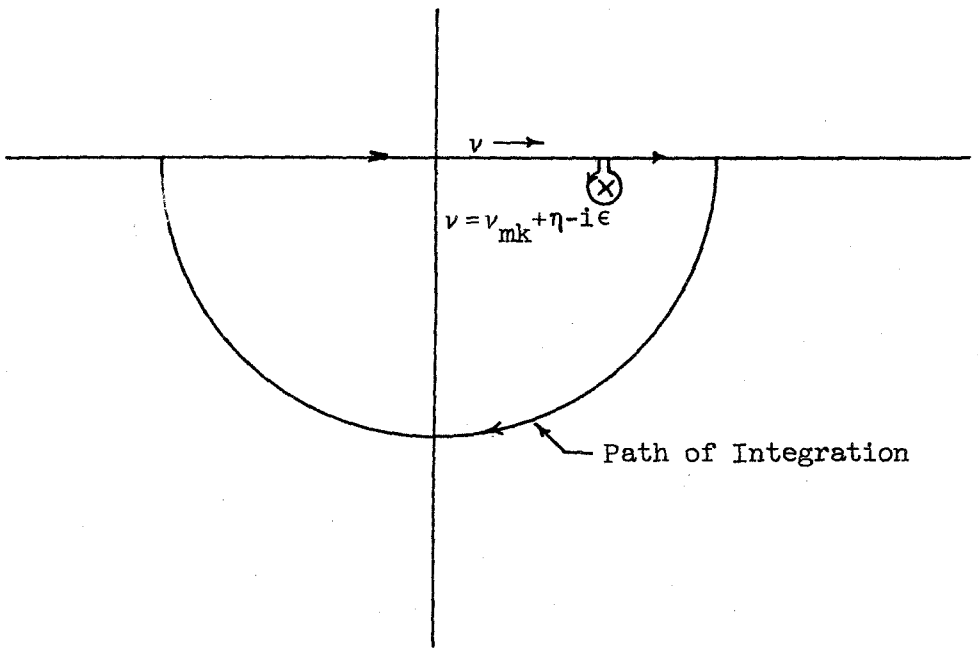
$$- \int_{-\infty}^{0} \frac{d\nu}{i\nu Z_{-\nu}[\nu-(\nu_{mk}+\eta)-i\epsilon]} + \int_{-\infty}^{0} \frac{d\nu}{i\nu Z_{+\nu}[\nu-(\nu_{mk}+\eta)+i\epsilon]} .$$

Combining terms this becomes,

$$I = + \int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_{\nu}[\nu-(\nu_{mk}+\eta)+i\epsilon]} - \int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_{-\nu}[\nu-(\nu_{mk}+\eta)-i\epsilon]} . \qquad \text{(A2.16)}$$

Since Z_ν represents the impedance of a passive linear system, the

poles of $1/Z_\nu$ must lie above the ν axis. Conversely, the pole of $\frac{1}{\nu-(\nu_{mk}+\eta)+i\epsilon}$ lies just below the ν axis. To evaluate the first term of A2.16, the contour will be closed below the ν axis, thus



Using Cauchy's theorem we have

$$\int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_\nu[\nu-(\nu_{mk}+\eta)+i\epsilon]} + \lim_{r\to 0}\int_0^{2\pi} \frac{d\nu}{i\nu Z_\nu[\nu-(\nu_{mk}+\eta)-i\epsilon]}\Bigg|_{\nu=\nu_{mk}+\eta+i\epsilon+re^{i\theta}}$$

$$+ \lim_{R\to\infty}\int_0^{-\pi} \frac{d\nu}{i\nu Z_\nu[\nu-(\nu_{mk}+\eta)-i\epsilon]}\Bigg|_{\nu=Re^{i\theta}} = 0$$

or

$$\int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_\nu[\nu-(\nu_{mk}+\eta)+i\epsilon]} = -\frac{2\pi}{(\nu_{mk}+\eta)Z_{\nu_{mk}+\eta}} - \frac{1}{\nu Z_\nu}\Bigg|_{\nu\to\infty}.$$

It is shown in the theory of linear systems that in general a linear passive impedance function Z_ν may be expressed in terms of the quotient of two polynomials in $(i\nu)$, i.e., $R(i\nu)/S(i\nu)$. It is further shown that the highest powers of ν occurring in $R(i\nu)$ and $S(i\nu)$ may differ by no more than unity (20). Therefore, if $1/\nu Z_\nu \Big|_{\nu\to\infty}$ exists, it will be equal to an imaginary constant, say ib_∞.

The second integral of A2.16 is the complex conjugate of the first. The poles of the integrand lie on the opposite side of the ν axis so the contour will be closed in the upper half plane. We have then the result

$$+\int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_{-\nu}[\nu-(\nu_{mk}+\eta)-i\epsilon]} = \frac{2\pi}{(\nu_{mk}+\eta)Z_{-\nu_{mk}-\eta}} - \frac{1}{\nu Z_{-\nu}}\Bigg|_{\nu\to\infty} .$$

The complete expression for I is (for $\nu_{mk}+\eta>0$)

$$I = -\frac{2\pi}{(\nu_{mk}+\eta)}\left[\frac{1}{Z_{\nu_{mk}+\eta}} + \frac{1}{Z_{-\nu_{mk}-\eta}}\right] = -\frac{4\pi}{\nu_{mk}+\eta}\,\mathrm{Re}\left(\frac{1}{Z_{\nu_{mk}+\eta}}\right) \qquad \text{(A2.17)}$$

where the quantities

$$\left(\frac{1}{\nu Z_\nu} + \frac{1}{\nu Z_{-\nu}}\right)_{\nu\to\infty} = ib_\infty - ib_\infty = 0 .$$

Case II. $\nu_{mk}+\eta<0$

In equation A2.15 we again juggle signs. In this case the signs on the $i\epsilon$ in the second and fourth terms are arbitrary. Changing these signs, changing the variable of integration from $+\nu$ to $-\nu$ on

these same terms, and combining, yields

$$I = \int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_{-\nu}[\nu+(\nu_{mk}+\eta)+i\epsilon]} - \int_{-\infty}^{\infty} \frac{d\nu}{i\nu Z_{\nu}[\nu+(\nu_{mk}+\eta)-i\epsilon]} \cdot$$

In the first integral all the poles lie below the ν axis. Closing the contour in the upper half plane yields only the result of integration around the circle at infinity, i.e., $\lim_{\nu\to\infty} \frac{1}{\nu Z_{-\nu}}$. Again, the second integral is just the complex conjugate of the first, and closing it in the lower half plane yields $\lim_{\nu\to\infty} \frac{1}{\nu Z_{\nu}}$. Therefore,

$$I = \lim_{\nu\to\infty} \left(\frac{1}{\nu Z_{\nu}} + \frac{1}{\nu Z_{-\nu}}\right) = 0 \qquad \text{(A2.18)}$$

as explained before.

Equation A2.14 for the probability now can be written to second order,

$$P_{n\to m} = \delta_{mn}\left\{1 - \sum_k \left|Q_{mk}\right|^2 \int_{-\nu_{mk}}^{\infty} \frac{d\eta \gamma_{\eta}\gamma_{-\eta}}{\pi\hbar(\nu_{mk}+\eta)} \mathrm{Re}\left(\frac{1}{Z_{\nu_{mk}+\eta}}\right)\right\} + \left|Q_{mn}\right|^2$$

$$\times \int_0^{\infty} d\nu \frac{\left|\gamma_{\nu+\nu_{mn}}\right|^2}{\pi\hbar\nu} \mathrm{Re}\left(\frac{1}{Z_{\nu}}\right) = \delta_{mn}\left\{1 - \sum_k \left|Q_{nk}\right|^2 \int_0^{\infty} d\nu \frac{\left|\gamma_{\nu+\nu_{kn}}\right|^2}{\pi\hbar\nu} \mathrm{Re}\left(\frac{1}{Z_{\nu}}\right)\right\}$$

$$+ \left|Q_{mn}\right|^2 \int_0^{\infty} d\nu \frac{\left|\gamma_{\nu+\nu_{mn}}\right|^2}{\pi\hbar\nu} \mathrm{Re}\left(\frac{1}{Z_{\nu}}\right) \qquad \text{(A2.19)}$$

where in the latter expression the change in variables denoted by $\eta+\nu_{km} = \nu$ was made.

One special case which is useful is that of very long coupling times. Let us assume that the coupling is as follows:

$$\gamma(t) = 0 \quad \text{for} \quad t < -T/2 \ , \ t > T/2$$
$$\gamma(t) = 1 \quad \text{for} \quad T/2 < t < T/2 \quad .$$

Then,

$$\left| \gamma_{\nu+\nu_{mn}} \right|^2 = \left| \int_{-T/2}^{T/2} e^{i(\nu+\nu_{mn})t} \, dt \right|^2 = 4 \frac{\sin^2(\nu+\nu_{mn})T/2}{(\nu+\nu_{mn})^2} \quad .$$

If T is very long, then

$$\lim_{T \to \infty} \frac{\sin^2(\frac{\nu+\nu_{mn}}{2})T}{(\frac{\nu+\nu_{mn}}{2})^2} = > 2\pi T \, \delta(\nu+\nu_{mn}) \quad .$$

Then, substituting in A2.19 and evaluating the integrals

$$P_{\substack{n \to m \\ n \neq m}} = \frac{2T|Q_{mn}|^2}{\hbar \nu_{nm}} \mathrm{Re}(\frac{1}{Z_{\nu_{nm}}}) \quad \text{for} \quad \nu_{nm} > 0$$

$$= 0 \quad \text{for} \quad \nu_{nm} < 0 \quad . \tag{A2.20}$$

$$P_{\substack{n \to m \\ n = m}} = 1 - \sum_k \frac{2T|Q_{nk}|^2}{\hbar \nu_{nk}} \mathrm{Re}(\frac{1}{Z_{\nu_{nk}}}) \quad \text{for all} \quad k \tag{A2.21}$$

such that $\nu_{nk} > 0$.

c. Classical Potential Acting on Test System

If a known classical potential $\gamma(t)C(t)Q(t)$ acts on the test system the influence phase has been shown to be

$$\Phi = -\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} C_{-\nu}(Q_\nu - Q'_\nu)\,d\nu \qquad (A2.22)$$

where the coupling term can be included in either $C_{-\nu}$ or Q_ν. In this case Φ is of the first order in the potentials; therefore, the expansion of $e^{i\Phi}$ appropriate to the second order transition probability is

$$e^{i\Phi} \approx 1 + i\Phi + \frac{(i\Phi)^2}{2!}$$

$$= 1 - \frac{i}{2\pi\hbar}\int_{-\infty}^{\infty} C_{-\nu}(Q_\nu - Q'_\nu)\,d\nu - \frac{1}{2(2\pi\hbar)^2}\iint_{-\infty}^{\infty} C_{-\nu}C_{-\eta}(Q_\nu - Q'_\nu)(Q_\eta - Q'_\eta)\,d\nu\,d\eta \,. \qquad (A2.23)$$

The calculation indicated by equations A2.23 and A2.3 may be carried out in a way similar to that done in part b. However, the effect of such a classical potential on a quantum system has been done many times in the Hamiltonian formulation of quantum mechanics. Therefore, it is more direct to adapt these expressions to a form more suitable to our uses.

If we assume that the perturbation γCQ above is turned off before the time of observation on the test system, then the second order probability of the Q system from a state n to a state $m (n \neq m)$, is (21)

$$P_{n \to m} \approx \left| \frac{i}{\hbar}\int_{-\infty}^{\infty} [\gamma(t)C(t)Q(t)]_{mn}\, e^{i\nu_{mn}t}\, dt \right|^2 \qquad (A2.24)$$

where $\nu_{mn} = \dfrac{E_m - E_n}{\hbar}$ and $[\gamma(t)C(t)Q(t)]_{mn} = \gamma(t)C(t)\int \phi_m^*[Q(t)] \times Q(t)\phi_n[Q(t)]\,dQ(t)$.

If we write $C(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{+\nu}\, e^{i\nu t} d\nu$ and take the indicated matrix element of Q we have

$$P_{n\to m} \approx \left| \frac{Q_{mn}}{2\pi\hbar} \int\int_{-\infty}^{\infty} C_{+\nu}\gamma(t) e^{i(\nu+\nu_{mn})t}\, dt d\nu \right|^2$$

$$= \left| \frac{Q_{mn}}{2\pi\hbar} \right|^2 \left| \int_{-\infty}^{\infty} C_{+\nu}\gamma_{-\nu-\nu_{mn}} d\nu \right|^2 \qquad \text{(A2.25)}$$

where $\gamma_{-\nu-\nu_{mn}} = \int_{-\infty}^{\infty} \gamma(t) e^{i(\nu+\nu_{mn})t}\, dt$.

It is important to notice that C_ν, in transform language, is the force acting directly on the test system as indicated by the influence phase in equation A2.22. Suppose, however, that a classical force, say $E(t)$ acted on Q through a linear system X, such that the coupling potential between Q and X is $\gamma(t)Q(t)X_1(t)$ and that between the force and X is $E(t)X_2(t)$, then the influence phase is from V.1

$$\Phi = + \frac{1}{2\pi\hbar} \int_0^{\infty} \left[\frac{E_{+\nu}(Q_{-\nu}-Q'_{-\nu})}{i\nu z(\nu)} + \frac{E_{-\nu}(Q_\nu - Q'_\nu)}{-i\nu z(-\nu)} \right] d\nu \qquad \text{(A2.26)}$$

The transform force acting on Q is $\frac{E_\nu}{i\nu z(\nu)}$ and the expression for $P_{n\to m}$ corresponding to A2.25 is

$$P_{n\to m} \approx \left| \frac{Q_{mn}}{2\pi\hbar} \right|^2 \left| \int_{-\infty}^{\infty} \frac{E_{+\nu}}{\nu z(\nu)} \gamma_{-\nu-\nu_{mn}} d\nu \right|^2 . \qquad \text{(A2.27)}$$

It is again worth mentioning that if the potential in Q were more complicated, say $V(Q)$, then the only modification of the above

expressions would be to replace Q_{mn} by V_{mn} .

d. Gaussian Random Potential Acting on Test System

If an uncertain potential of the form $\gamma(t)C(t)Q(t)$ is coupled to the test system, the average influence phase has been found to be

$$i\,\bar{\Phi} = -\frac{1}{\pi\hbar^2}\int_0^\infty \phi(\nu)\,|Q_\nu - Q'_\nu|^2\,d\nu$$

where $\phi(\nu)$ is the power spectrum of the noisy force $C(t)$. In expanded form

$$i\,\bar{\Phi} = -\frac{1}{\pi\hbar^2}\int_0^\infty \phi(\nu)\,(Q_\nu Q_{-\nu} - Q_\nu Q'_{-\nu} - Q_{-\nu}Q'_\nu + Q'_\nu Q'_{-\nu})d\nu \quad . \qquad (A2.29)$$

Since the potentials in $\bar{\Phi}$ are already of second order, no higher powers of $\bar{\Phi}$ than the first need be included for the second order perturbation expansion. The calculation indicated by equation A2.29, along with equation A2.3, could be done using the same methods as in part b. However, it is much shorter to utilize the results of part c. If a known classical force $C(t)$ acts on Q , the transition probability

$$\underset{n \neq m}{P_{n \to m}} \approx \left|\frac{Q_{mn}}{2\pi\hbar}\right|^2 \left|\int_{-\infty}^{\infty} C_{-\nu}\gamma_{\nu+\nu_{mn}}\right|^2$$

$$= \left|\frac{Q_{mn}}{2\pi\hbar}\right|^2 \int_{-\infty}^{\infty}\int C_{-\nu}C_{-\eta}\gamma_{\nu+\nu_{mn}}\gamma_{-\eta-\nu_{mn}}\,d\nu d\eta \quad . \qquad (A2.30)$$

If the potential is uncertain, however, we must deal with average probabilities. Thus, for a noisy force

$$\underset{n\to m}{\langle P\rangle} = \left|\frac{Q_{mn}}{2\pi\hbar}\right|^2 \left\langle \int\int_{-\infty}^{\infty} C_{-\nu} C_{+\eta} \gamma_{\nu+\nu_{mn}} \gamma_{-\eta-\nu_{mn}} \, d\nu d\eta \right\rangle$$

$$= \left|\frac{Q_{mn}}{2\pi\hbar}\right|^2 \int\int_{-\infty}^{\infty} \left\langle C_{-\nu} C_{+\eta} \right\rangle \gamma_{\nu+\nu_{mn}} \gamma_{-\eta-\nu_{mn}} \, d\nu d\eta \quad . \tag{A2.31}$$

But $\left\langle C_{-\nu} C_{+\eta} \right\rangle = 4\pi\phi(\nu)\,\delta(\nu-\eta)$, therefore

$$\underset{n\to m}{P} = \left|\frac{Q_{mn}}{2\pi\hbar}\right|^2 4\pi \int\int_{-\infty}^{\infty} \phi(\nu)\delta(\nu-\eta)\gamma_{\nu+\nu_{mn}} \gamma_{-\eta-\nu_{mn}} \, d\nu d\eta$$

$$= \frac{\left|Q_{mn}\right|^2}{\pi\hbar^2} \int_{-\infty}^{\infty} \phi(\nu) \left|\gamma_{\nu+\nu_{mn}}\right|^2 d\nu \quad . \tag{A2.32}$$

One special case which is frequently useful is that of a coupling potential $\gamma(t)$ which is turned on for $t<-T/2$, is unity for $-T/2<t<+T/2$, and is off for $t>T/2$ where T is very large. Doing this calculation we find

$$\underset{n\to m}{P} \approx \frac{2T\left|Q_{mn}\right|^2}{\hbar^2} \phi(\nu_{mn}) \quad . \tag{A2.33}$$

APPENDIX III

a. Influence Phase for Effect of Free Space on an Atom

As an example of the tools developed so far, we will now take the case of an atom in free space and will calculate the influence phase for the effect of free space. To do the problem it is assumed that the atom is made up of a system of particles of mass m_n, charge e_n, and position $\underline{r}_A + \underline{x}_n$ where $\underline{r}_A$ is the position of the center of charge of the atom. If the transverse part of the radiation field in the box is expanded into a series of plane waves each representing independent harmonic oscillations (13), then the non-relativistic Lagrangian for the complete system consisting of the atom and the field in the box can be written (2)

$$\mathcal{L}(\dot{X},X,\dot{q}_{\underline{k}},q_{\underline{k}},t) = \mathcal{L}_A + \sum_n \frac{e_n \dot{\underline{X}}_n}{c} \cdot \underline{A}^{tr}(\underline{r}_A + \underline{x}_n) + \frac{1}{2} \sum_k \sum_{r=1}^{4} [(\dot{q}_{\underline{k}}^{(r)})^2 - c^2 k^2 (q_{\underline{k}}^{(r)})^2] \qquad (A3.1)$$

where $\mathcal{L}_A$ is the Lagrangian of the atom unperturbed by outside forces and

$$\underline{A}^{tr}(\underline{x}) = (8\pi c^2)^{1/2} \sum_{\underline{k}} [\underline{e}_\lambda (q_{\underline{k}}^{(1)} \cos(\underline{k}\cdot\underline{x}) + q_{\underline{k}}^{(3)} \sin(\underline{k}\cdot\underline{x}) + \underline{e}_\ell (q_{\underline{k}}^{(2)} \cos(\underline{k}\cdot\underline{x}) + q_{\underline{k}}^{(4)} \sin(\underline{k}\cdot\underline{x})] \qquad (A3.2)$$

Here $\underline{e}_\lambda$ and $\underline{e}_\ell$ are two mutually orthogonal polarization vectors, each orthogonal to the propagation vector $\underline{k}$. If the box is large enough so that one can talk of an integral over $\underline{k}$ instead of a sum

as we will want to do very soon, then $\sum_{\underline{k}} \Rightarrow \frac{1}{2}\int \frac{d^3\underline{k}}{(2\pi)^3}$, if the integral is normalized to unit volume. Now we assume that the radiation field of the box is constant over the particle, i.e., that $\underline{A}(\underline{r}_A + \underline{x}_n) \approx \underline{A}(\underline{r}_A)$ for all particles in the atom. This is the dipole approximation.* In addition, even though $\underline{e}_\lambda \cdot \underline{e}_\ell = 0$, which fixes their relative orientations, their absolute directions in a plane perpendicular to $\underline{k}$ are still arbitrary. Let us choose the direction $\underline{e}_\ell$ so that

$$\underline{e}_\ell \cdot \sum_n e_n \underline{x}_n = 0 \quad . \tag{A3.3}$$

Then, combining A3.1, A3.2, and A3.3,

$$\mathcal{L}(\dot{X},X,\dot{q}_{\underline{k}},q_{\underline{k}},t) = \mathcal{L}_A + \frac{1}{2}\sum_{\underline{k}}\sum_{r=1,3}\left[(\dot{q}_{\underline{k}}^{(r)})^2 - c^2k^2(q_{\underline{k}}^{(r)})^2\right]$$
$$+ \left(\sum_n \frac{e_n \dot{\underline{x}}_n}{c}\right)\cdot(8\pi c^2)^{1/2}\sum_{\underline{k}} \underline{e}_\lambda\left[(q_{\underline{k}}^{(1)}\cos(\underline{k}\cdot\underline{r}_A)\right.$$
$$\left. + q_{\underline{k}}^{(3)}\sin(\underline{k}\cdot\underline{r}_A)\right] \quad . \tag{A3.4}$$

The box is now assumed to be very large and the summation will be replaced by an integral as suggested above. Then, rearranging terms

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*This is equivalent to taking $\underline{A}(\underline{k}_A+\underline{x}_n)$, expanding it in a series of $\underline{k}\cdot\underline{x}_n$ since this is assumed small, and keeping only those terms which keep the interaction term of the Lagrangian linear. Since the interaction is of the form $e_n\underline{x}_n\cdot\underline{A}(\underline{k}_A+\underline{x}_n)$ for the nth particle, then $\underline{A}$ can only contain constant terms.

$$\mathcal{L}(\dot{X},X,\dot{q}_{\underline{k}},q_{\underline{k}},t) = \mathcal{L}_A + \int \frac{d^3\underline{k}}{16\pi^3}\left[\frac{1}{2}(\dot{q}_{\underline{k}}^{(1)})^2 - \frac{k^2c^2}{2}(q_{\underline{k}}^{(1)})^2\right.$$

$$\left. + \left(\sum_n \frac{e_n \dot{\underline{X}}_n \cdot \underline{e}_\lambda}{c}\right)\sqrt{8\pi c^2}\, q_{\underline{k}}^{(1)} \cos(\underline{k}\cdot\underline{r}_A)\right] + \int \frac{d^3\underline{k}}{16\pi^3}\left[\frac{1}{2}(\dot{q}_{\underline{k}}^{(3)})^2\right.$$

$$\left. - \frac{k^2c^2}{2}(q_{\underline{k}}^{(3)})^2 + \left(\sum_n \frac{e_n \dot{\underline{X}}_n \cdot \underline{e}_\lambda}{c}\right)\sqrt{8\pi c^2}\, q_{\underline{k}}^{(3)} \sin(\underline{k}\cdot\underline{r}_A)\right] \quad . \qquad \text{(A3.5)}$$

Since the Lagrangian for the field has been expressed as a distribution of oscillators, the action formed from this Lagrangian is of the form of equation IV.6. The influence phase can then be written in the same form as equation IV.8. To facilitate writing the influence phase we notice that $\sum_n e_n \underline{x}_n^{(t)}$ is the dipole moment operator, $\underline{\mu}$. The influence phase is then

$$\Phi(\mu,\mu') = \frac{1}{2\pi\hbar}\int \frac{d^3\underline{k}}{16\pi^3}\int_0^\infty 8\pi \cos^2(\underline{k}\cdot\underline{r}_A)\left\{\frac{(\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_\nu\left[(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_{-\nu} - (\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_{-\nu}\right]}{-\left[(\nu-i\epsilon)^2 - k^2c^2\right]} + \right.$$

$$\left. + \frac{(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_{-\nu}\left[(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_\nu - (\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_\nu\right]}{-\left[(\nu+i\epsilon)^2 - k^2c^2\right]}\right\} d\nu + \frac{1}{2\pi\hbar}\int \frac{d^3\underline{k}}{16\pi^3}\int_0^\infty 8\pi \sin^2(\underline{k}\cdot\underline{r}_A)$$

$$\times\left\{\frac{(\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_\nu\left[(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_{-\nu} - (\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_{-\nu}\right]}{-\left[(\nu-i\epsilon)^2 - k^2c^2\right]} + \frac{(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_{-\nu}\left[(\dot{\underline{\mu}}\cdot\underline{e}_\lambda)_\nu - (\dot{\underline{\mu}}'\cdot\underline{e}_\lambda)_\nu\right]}{-\left[(\nu+i\epsilon)^2 - k^2c^2\right]}\right\} d\nu \qquad \text{(A3.6)}$$

The two terms can be combined since they are identical except for the $\sin^2(\underline{k}\cdot\underline{r}_A)$ in one and $\cos^2(\underline{k}\cdot\underline{r}_A)$ in the other. These, of course, add to give unity. To evaluate the integral over $d^3\underline{k}$ we notice that $d^3\underline{k} = dk_x dk_y dk_z = k^2 \sin\theta\, d\phi d\theta dk$ in polar coordinates in k space. In addition, if $\underline{\mu}$ is oriented along the z axis, then

$$\underline{\mu} \cdot \underline{e}_{\lambda} = \mu \sin \theta$$

and

$$(\dot{\underline{\mu}} \cdot \underline{e}_{\lambda})_{\nu} = +i\nu\, \mu_{\nu} \sin \theta$$

Putting in all of these changes

$$\Phi(\mu,\mu') = \frac{1}{2\pi\hbar} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{\nu^2 k^2 \sin^3\theta \, d\phi d\theta dk}{2\pi^2} \int_0^{\infty} \left[\frac{\mu'_{\nu}(\mu_{-\nu} - \mu'_{-\nu})}{-\left[(\nu - i\epsilon)^2 - k^2c^2\right]} + \frac{\mu_{-\nu}(\mu_{\nu} - \mu'_{\nu})}{-\left[(\nu + i\epsilon)^2 - k^2c^2\right]} \right] d\nu$$

$$= + \frac{1}{2\pi\hbar} \int_0^{\infty} \frac{4\nu^2\Omega^2}{3\pi c^2} \, d\Omega \int_0^{\infty} \frac{\mu'_{\nu}(\mu_{-\nu} - \mu'_{-\nu})}{-[(\nu - i\epsilon)^2 - \Omega^2]} + \frac{\mu_{-\nu}(\mu_{\nu} - \mu'_{\nu})}{-[(\nu + i\epsilon)^2 - \Omega^2]} \, d\nu \quad . \qquad (A3.7)$$

If instead of a many-particle atom we have a single charged particle harmonically bound to some center of force, then the above becomes

$$\Phi_T(X,X') = \frac{1}{2\pi\hbar} \int_0^{\infty} \left\{ \frac{X'_{\nu}(X_{-\nu} - X'_{-\nu})}{[i\nu Z_T(\nu)]} + \frac{X_{-\nu}(X_{\nu} - X'_{\nu})}{[-i\nu Z_T(-\nu)]} \right\} d\nu \qquad (A3.8)$$

where X is the displacement coordinate of the particle, e is its charge, and

$$\frac{1}{i\nu Z_T(\nu)} = -\int_0^{\infty} \left(\frac{4e^2\nu^2\Omega^2}{3\pi c^3} \right) \frac{d\Omega}{(\nu - i\epsilon)^2 - \Omega^2} \quad .$$

The equivalent distribution of oscillators to which X is coupled is

$$G(\Omega) = \frac{4e^2\nu^2\Omega^2}{3\pi c^3} \qquad (A3.9)$$

b. Spontaneous Emission Probability of an Atom in Free Space

To compute the transition probability for this atom, we use second order perturbation theory developed in Appendix II(a) for a system initially in state $\phi_n(X_\tau)$ and finally in state $\phi_m(X_T)$ when acted on by an influence functional for a linear system at zero temperature. The expression is

$$P_{\substack{n \to m \\ n \neq m}} = \frac{2T|X_{nm}|^2}{\hbar\nu_{nm}} \operatorname{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) \quad , \quad \nu_{nm} > 0 \quad , \tag{A3.10}$$

$$= 0 \quad , \quad \nu_{nm} < 0$$

where $X_{nm} = \int \phi_n^*(X)\, X \phi_m(X) dX$ and $\nu_{nm} = \frac{E_n - E_m}{\hbar}$.

From equation IV.18 and A3.9 we find

$$\operatorname{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) = \frac{\pi G(\nu_{nm})}{2} = \frac{2e^2\nu_{nm}^4}{3c^3} \quad . \tag{A3.11}$$

Using this in A3.10 we find

$$P_{\substack{n \to m \\ n \neq m}} = \frac{4e^2\nu_{nm}^3}{3\hbar c^3} |X_{nm}|^2 \, T \quad , \tag{A3.12}$$

which is the first order spontaneous emission probability for an atom in free space.

Now we can form an expression for the intensity of radiation per unit time. The power radiated from this dipole is

$$\hbar\nu_{nm}\, {}_n P_{\to m}/T = 2\,|X_{nm}|^2 \operatorname{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) = \frac{4e^2|X_{nm}|^2\,\nu_{nm}^4}{3c^3} \quad , \tag{A3.13}$$

an expression which is almost the same as that for power radiated from

a classical dipole. The expression becomes exactly the same if we apply the correspondence principle by replacing the matrix element of the time average of the coordinate of the oscillator by its corresponding classical quantity. Thus, if X is the coordinate of the corresponding classical oscillator (X_o is its maximum value) then*

$$2\left|X_{mn}\right|^2 \Rightarrow \left\langle X^2\right\rangle = \frac{1}{2}X_o^2 \quad .$$

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*That this correspondence is true can be seen easily as follows. Consider the dipole, a harmonic oscillator as above, to be in a high quantum state, ϕ_n. Classically the motion of the dipole can be described as $X = X_o \sin \omega t$. We wish to relate the classical value of $\langle X^2\rangle$ to its matrix element. In a quantum mechanical sense,

$$\left\langle X^2\right\rangle = \int \phi_n^*(X)X^2\phi_n(X)dX = \iint \phi_n^*(X)X \sum_k \phi_k(X)\phi_k^*(X')X'\phi_n(X')dX'dX.$$

Since matrix elements exist, in the case of a harmonic oscillation only for $k = n-1$, $k = n+1$, we have

$$\overline{X}^2 = \left|X_{n,n-1}\right|^2 + \left|X_{n,n+1}\right|^2 \quad .$$

For very high quantum numbers these two terms become nearly equal since

$$\left|X_{n,n-1}\right|^2 = \frac{n\cancel{\hbar}}{2m\omega} \ , \quad \left|X_{n,n+1}\right|^2 = \frac{(n+1)\cancel{\hbar}}{2m\omega} \quad .$$

Thus, as $n \to \infty$ $\left\langle X^2\right\rangle = 2\left|X_{n,n-1}\right|^2$. But in the classical case, $\overline{X}^2 = \frac{1}{2}X_o^2$. Therefore, $\left|X_{n,n-1}\right|^2 \Rightarrow \frac{1}{4}X_o^2$.

If $|\underline{X}_{mn}|^2$ is replaced by $X_o^2/4$ then A3.13 becomes the expression for the power radiated from a classical dipole. If it served a useful purpose we could continue and relate the classical resistance of a dipole to the distribution of oscillators $G(\Omega)$ by equating the power radiated by the dipole to $\frac{1}{2} I^2 R$ where I is the current of the dipole.* Our purpose, however, in doing this example, was to show for a specific problem that the effect of a distribution of oscillators interacting on a system is the same as the effect of loss on the system. This has been done by relating the energy lost from the radiating dipole to the distribution. It is not surprising that a sea of oscillators should give this effect. If the dimensions of the box are allowed to be finite then energy emitted from the system under observation is reflected from the walls and eventually finds its way back to be absorbed again. This is equivalent to saying that the number of oscillators comprising the electromagnetic field in the box is infinite with a finite frequency spacing between the modes. Since the oscillators are independent there

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*If the power radiated from the oscillator is related to the classical expression $1/2\ I^2R$ then from equation A3.13 it can be seen that R is proportional to $\mathrm{Re}[1/Z(\nu)]$ which in turn is related to the distribution of oscillators. One might expect $\mathrm{Im}[1/Z(\nu)]$ to be related to the reactance seen by an oscillating dipole, a quantity which is known to be infinite classically. From equations A3.10 and A3.11 we find that

$$\mathrm{Im}\left(\frac{1}{Z(\nu)}\right) = \int_0^\infty \frac{4e^2\nu^3\Omega^2 d\Omega}{3\pi c^3(\nu^2-\Omega^2)} = \left.\frac{4e^2\nu^3}{3\pi c^3}\Omega\right|_{\to\infty}$$

The integral is linearly divergent. This factor is also related to the infinite self energy of a point charge which occurs both classically and in quantum electrodynamics. Here this divergence does not bother us since it never enters into the calculation.

is no coupling between them and energy coupled into one of the oscillators from the test system must eventually return to it. If the dimensions of the box are allowed to get infinitely large, energy emitted from the test system never gets reflected and thus never returns. In oscillator language this means that the frequency spacing between oscillators has become infinitesimal, so close that a little of the energy absorbed by each one gradually leaks into nearby modes and eventually is completely gone.

APPENDIX IV

Spontaneous Emission of an Atom in a Cavity

In this calculation as in the free space calculation the dipole approximation will be used in computing the spontaneous emission probability. The linear coordinates inside the cavity will be represented by the vector $\underline{Q}$ while the time varying coordinates of the single cavity mode being considered will be $X(t)$. The Lagrangian of the system may be written

$$\mathcal{L}(\text{system}) = \mathcal{L}(\dot{Q}_n, Q_n, t) + \sum_n \frac{e_n \dot{\underline{Q}}_n}{c} \cdot \underline{A}(\underline{Q}_p + \underline{Q}_n, t) + \mathcal{L}(\text{cavity}) \qquad (A4.1)$$

where $\underline{Q}_p$ is the atom coordinate, $\underline{Q}_n + \underline{Q}_p$ is the particle coordinate in the atom, and $\underline{A}$ is the vector potential of the cavity field. The interaction term is the one of interest, since from it we find the terms that we wish to solve for classically. This term will be put into more convenient form. Let us write

$$\underline{A}(\underline{Q}, t) = \underline{a}(\underline{Q})\, X(t) \qquad (A4.2)$$

where

$$\int \underline{a}(\underline{Q}) \cdot \underline{a}(\underline{Q}) d^3\underline{Q} = 4\pi c^2 \quad . \qquad (A4.3)$$

If $\underline{A}$ does not vary much over the atom, then $\underline{A}(\underline{Q}_p) \approx \underline{a}(\underline{Q}_p + \underline{Q}_n)$ and the interaction term is written

$$\sum_n \frac{e_n \dot{\underline{Q}}_n}{c} \cdot \underline{a}(\underline{Q}_p)\, X(t) = \frac{\dot{\mu} \left| \underline{a}(\underline{Q}_p) \right| X(t)}{c} \qquad (A4.4)$$

where $\dot{\mu} = \dfrac{(\sum e_n \dot{\underline{Q}}_n) \cdot \underline{a}(\underline{Q}_p)}{|\underline{a}(\underline{Q}_p)|}$.

Let us now determine the ratio

$$\frac{\dot{\mu}_\nu}{\frac{|\underline{a}(\underline{Q}_p)|\, X_\nu}{c}} \tag{A4.5}$$

classically. The wave equation appropriate for this calculation (high Q)* is

$$\nabla^2 \underline{A} - \frac{1}{c^2}\ddot{\underline{A}} + \frac{\omega}{c^2 Q}\dot{\underline{A}} - \frac{4\pi}{c}\dot{\underline{P}} = 0 \quad . \tag{A4.6}$$

The atom is located at $\underline{Q}_p$ and, since the dipole moment is induced, its direction on the average is the same as that of the field $\underline{A}$ in the cavity. We have then

$$\dot{\underline{P}} = \frac{\underline{a}(\underline{Q}_p)}{|\underline{a}(\underline{Q}_p)|}\,\dot{\mu}\,\delta(\underline{Q} - \underline{Q}_p) \quad . \tag{A4.7}$$

Substituting A4.7 and A4.2 into A4.6 we obtain

$$\left(-\frac{\omega^2}{c^2}X - \frac{\ddot{X}}{c^2} + \frac{\omega}{c^2 Q}\dot{X}\right)\underline{a}(\underline{Q}) - \frac{4\pi\,\underline{a}(\underline{Q}_p)}{c|\underline{a}(\underline{Q}_p)|}\,\dot{\mu}\,\delta(\underline{Q} - \underline{Q}_p) = 0 \tag{A4.8}$$

where ω is the resonant frequency of the cavity. Multiplying by $\underline{a}(Q)\cdot$, integrating over Q, and taking Fourier transforms, A4.8 becomes

$$\left(\nu^2 - \omega^2 + \frac{i\nu\omega}{Q}\right) X_\nu - \frac{|\underline{a}(\underline{Q}_p)|}{c}\,(\dot{\mu})_\nu = 0 \quad . \tag{A4.9}$$

We find for the ratio A4.5

$$\frac{i\nu\mu_\nu}{\frac{|\underline{a}(\underline{Q}_p)|\, X_\nu}{c}} = \left[\nu^2 - \omega^2 + \frac{i\nu\omega}{Q}\right]\frac{c^2}{|\underline{a}(\underline{Q}_p)|^2} = i\nu\, Z_\nu$$

*Q is used here as the dissipation factor of the cavity, $\omega_0 L/R$, while $\underline{Q}$ is a vector representing the linear coordinates inside the cavity.

The influence phase for this is (although it is unnecessary to write it)

$$\Phi(\mu,\mu') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{\nu^2\mu'_\nu(\mu_{-\nu}-\mu'_{-\nu})}{(i\nu\, Z_\nu)} + \frac{\nu^2\mu_{-\nu}(\mu_\nu-\mu'_\nu)}{(-i\nu\, Z_{-\nu})}\right] d\nu$$

From second order perturbation theory we know

$$P_{\substack{n\to m\\ n\neq m}} = \frac{2T\left|\nu_{nm}\mu_{nm}\right|^2}{\hbar\nu_{nm}} \mathrm{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) \quad \text{for} \quad \nu_{nm}>0 \ .$$

Noting that $\mathrm{Re}\left(\frac{1}{Z_{\nu_{nm}}}\right) = \frac{\omega\nu^2/Q}{\left[(\nu^2-\omega^2)^2+\frac{\omega^2\nu_{mn}^2}{Q^2}\right]} \frac{\left|\underline{a}(\underline{Q}_A)\right|^2}{c^2}$

and defining a cavity from factor $f^2 = \frac{V}{4\pi}\frac{\left|\underline{a}(\underline{Q}_A)\right|^2}{c^2}$

$$P_{n\to m} = \left(\frac{8\pi\left|\mu_{nm}\right|^2 f^2}{\hbar V}\right)\left(\frac{\omega\nu_{nm}^3/Q}{(\nu_{nm}^2-\omega^2)^2+\frac{\omega^2\nu_{nm}^2}{Q^2}}\right) \quad T \qquad \text{(A4.10)}$$

At resonance this expression reduces to

$$P_{n\to m} = \frac{8\pi\left|\mu_{mn}\right|^2 f^2}{\hbar V}\ \frac{Q\nu_{nm}}{\omega}\ T \ . \qquad \text{(A4.11)}$$

The quantity usually computed is the ratio of the transition probability in the cavity at resonance to that in free space. This ratio is

$$\frac{P_{nm}\ \text{cavity}}{P_{nm}\ \text{free space}} = \frac{\left[8\pi\left|\mu_{nm}\right|^2\ \nu_{nm}Qf^2/V\hbar\omega\right]T}{\left[4\left|\mu_{mn}\right|^2\ \nu_{nm}^3/3\hbar c^3\right]\ T} = \frac{6\pi c^3 Q f^2}{V\ \nu_{nm}^3} \ .$$

At resonance, the ratio increases with respect to Q as one might expect and decreases with respect to the cavity volume V and ν_{nm}^3 .

This expression agrees with the one given by E. M. Purcell (22) although the form factor in his calculation was left out. This does not matter since for a particle located near the maximum field point in a cavity the magnitude of f is of the order of unity.

APPENDIX V

a. Influence Functional for the Effect of a Damped Harmonic Oscillator

The problem was formulated in Section IV.4. From equation IV.24 the Lagrangian for the total system is

$$\mathcal{L}(\text{system}) = \mathcal{L}_o(\dot{Q},Q,t) + \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} + QX + \int_0^\infty G(\Omega)\left[\frac{\dot{y}^2}{2} - \frac{\Omega^2 y^2}{2} + XY\right] d\Omega.$$

The influence functional is then from equation IV.25,

$$\mathcal{F}(Q,Q') = \int \delta(X_T - X'_T)\, \mathcal{F}(X,X') \exp\left\{\frac{i}{\hbar}\left[S(X) - S(X') + \int_\tau^T (QX - Q'X')dt\right]\right.$$
$$\left. - \frac{m\omega}{2\hbar}(X^2 + X'^2)\right\} dX_\tau \cdots \mathcal{D}X'(t) \qquad \text{(IV.25)}$$

where $\mathcal{F}(X,X')$ is the effect of the distribution on X. From equation IV.8, the explicit form for $\mathcal{F}(X,X')$ is

$$\mathcal{F}(X,X') = \exp\left\{-\frac{1}{2\hbar}\int_\tau^T\int_\tau^t (Q_t - Q'_t)\left[Q_s\left(\int_0^\infty \frac{G(\Omega)e^{-i\Omega(t-s)}}{\Omega} d\Omega\right)\right.\right.$$
$$\left.\left. - Q'_s\left(\int_0^\infty \frac{G(\Omega)e^{i\Omega(t-s)}}{\Omega} d\Omega\right)\right] dsdt\right\}$$

Substituting this in equation IV.25 we have

$$\mathcal{F}(Q,Q') = \int \delta(X_T - X'_T)\exp\left\{\frac{i}{\hbar}\int_\tau^T\left[\frac{m}{2}(\dot{X}^2 - \dot{X}'^2) - \frac{m\omega^2}{2}(X^2 - X'^2) + (QX - Q'X') + \right.\right.$$
$$\left.\left.\frac{i}{2}\int_\tau^t (X_t - X'_t)(X_s f(t-s) - X'_s f^*(t-s))ds\right] dt - \frac{m\omega}{2\hbar}(X^2 + X'^2)\right\} dX_\tau \cdots \mathcal{D}X'(t) \qquad \text{(A5.1)}$$

where $$f(t-s) = \int_0^\infty \frac{G(\Omega)}{\Omega} e^{-i\Omega(t-s)} d\Omega \qquad \text{(A5.2)}$$

If a change of variables is made from X,X',Q,Q' to x,x',q,q' as defined in equation A1.2,

$$\mathcal{F}(Q,Q')=\int \delta(x'_T)\exp\left\{\frac{i}{\hbar}\int_\tau^T\left[\frac{m}{2}\dot{x}\dot{x}'-\frac{m\omega^2}{2}xx'+\frac{qx'}{2}+\frac{q'x}{2}+\frac{i}{4}\int_\tau^t x'_t x'_s\, h(t-s)ds\right.\right.$$

$$\left.\left.+\frac{i}{4}\int_\tau^t x'_t x_s\, j(t-s)ds\right]dt-\frac{m\omega}{4\hbar}(x_\tau^2+x_\tau'^2)\right\}dx_\tau\cdots\mathcal{D}x'(t) \qquad (A5.3)$$

where $h(t-s)=f(t-s)+f^*(t-s)=\int_0^\infty \frac{2G(\Omega)}{\Omega}\cos\Omega(t-s)d\Omega$

and $j(t-s)=f(t-s)-f^*(t-s)=-\int_0^\infty \frac{2iG(\Omega)}{\Omega}\sin\Omega(t-s)d\Omega$.

It is convenient to change the order of integration in the last integral in the exponent

$$\int_\tau^T\int_t^t x'_t x_s\, j(t-s)dsdt=\int_\tau^T\int_t^T x_t x'_s\, j(s-t)\,dsdt \quad . \qquad (A5.4)$$

In equation A5.3 we integrate $\int_\tau^T \dot{x}\dot{x}'dt$ by parts, substitute A5.4, and obtain

$$\mathcal{F}(Q,Q')=\int \delta(x'_T)\exp\left\{-\frac{m\omega}{4\hbar}(x^2+x'^2)+\frac{im}{2\hbar}(x_T\dot{x}'_T-x_\tau\dot{x}'_\tau)+\frac{i}{\hbar}\int_\tau^T\left[x_t g(x')\right.\right.$$

$$\left.\left.+\frac{i}{4}x'_t\int_\tau^t x'_s\, h(t-s)ds+\frac{qx'}{2}\right]dt\right\}dx_\tau\cdots\mathcal{D}x' \qquad (A5.5)$$

where $g(x')=-\frac{m}{2}\ddot{x}'-\frac{m\omega^2}{2}x'+\frac{q'}{2}+\frac{i}{4}\int_t^T x'_s\, j(s-t)\,ds$. $\qquad (A5.6)$

The procedure here is similar to that of Appendix I. The path integral

on $\mathcal{D}x(t)$ yields a delta-functional of the path $g(x')$, i.e., $\delta[g(x')]$. Then the integral on dx_T yields $\delta(\dot{x}'_T)$ and the dx_τ integration is Gaussian. Doing all this, equation A5.5 becomes

$$\mathcal{F}(Q,Q') = \int \delta(x'_T)\delta(\dot{x}'_T)\delta[g(x')] \exp\left\{-\frac{m\omega}{4\hbar}(x'^2 + \frac{\dot{x}'^2}{\omega^2}) + \frac{i}{\hbar}\int_\tau^T \left[\frac{q_t x'_t}{2} + \frac{i}{4}\int_\tau^t x'_t x'_s\, h(t-s)ds\right] dt\right\} dx'_\tau dx'_T \mathcal{D}x' \quad . \tag{A5.7}$$

Again we have the situation that integration on $\mathcal{D}x'\, dx'_\tau dx'_T$ means substituting the classical path $g(x') = 0$ subject to the conditions $x'_T = \dot{x}'_T = 0$. Rather than doing this in the time domain, we will go directly to Fourier transform notation, extend the time interval ($\tau \to -\infty$ to $T = +\infty$), and simply substitute the Fourier transform of the classical path in A5.7. In addition, it will be shown later in part b that as $\tau \to -\infty$, $x'^2 + (\dot{x}'^2_\tau/\omega^2) = 0$; therefore, it will be left out of the calculation now. All that is left of A5.7 now may be written

$$i\bar{\Phi}(q;q') = \frac{i}{\hbar}\int_{-\infty}^{\infty}\left\{\frac{q_t x'_t}{2} + \frac{i}{4}\int_{-\infty}^{t} x'_t x'_s\, h(t-s)ds\right\}dt \tag{A5.8}$$

where x'_τ satisfies $g[x'_t] = 0$, or in transform notation,

$$\bar{\Phi}(q,q') = \frac{1}{4\pi\hbar}\int_{-\infty}^{\infty}\left[q_{-\nu}x'_\nu + \frac{i}{2}\, x'_{-\nu}x'_\nu h_\nu\right]d\nu \tag{A5.9}$$

where $h_\nu = \int_{-\infty}^{\infty} 1(t)h(t)e^{-i\nu t}\, dt$. Similarly, equation A5.6 may be written

$$g(x'_\nu) = \frac{m}{2}(\nu^2 - \omega^2)x'_\nu + \frac{q'_\nu}{2} + \frac{i}{4}\, x'_\nu\, j_{-\nu} \tag{A5.10}$$

where $j_\nu = \int_{-\infty}^{\infty} 1(t)\, j(t)\, e^{-i\nu t}\, dt$

Setting $g(x'_\nu) = 0$ we find

$$x'_\nu = -\frac{q'_\nu}{\left[m(\nu^2 - \omega^2) + \frac{i}{2} j_{-\nu}\right]} = \frac{q'_\nu}{(-i\nu Z_{-\nu})} \qquad (A5.11)$$

Putting this in A5.9 and changing the limits of integration we have

$$\Phi(q,q') = \frac{1}{4\pi\hbar}\int_0^{\infty}\left[\frac{q_{-\nu}q'_{+\nu}}{-\nu Z_{-\nu}} + \frac{q_\nu q'_{-\nu}}{i\nu Z_\nu} + \frac{i}{2}\,\frac{q'_\nu q'_{-\nu}(h_\nu + h_{-\nu})}{(i\nu Z_\nu)(-i\nu Z_{-\nu})}\right] d\nu \,. \qquad (A5.12)$$

Adding and subtracting the term $q'_{-\nu}q'_\nu\left(\frac{1}{-i\nu Z_{-\nu}} - \frac{1}{i\nu Z_\nu}\right)$ this becomes

$$\Phi(q,q') = \frac{1}{4\pi\hbar}\int_0^{\infty}\left[\frac{q'_{-\nu}(q_\nu - q'_\nu)}{i\nu Z_\nu} + \frac{q'_\nu(q_{-\nu} + q'_{-\nu})}{-i\nu Z_{-\nu}}\right] d\nu$$

$$+ \frac{1}{8\pi\hbar}\int_0^{\infty} q'_\nu q'_{-\nu}\left[\frac{ih_\nu + ih_{-\nu} - (i\nu Z_\nu) + (-i\nu Z_{-\nu})}{(i\nu Z_\nu)(-i\nu Z_{-\nu})}\right] d\nu \,. \qquad (A5.13)$$

From equation A5.11, $(-i\nu Z_\nu) + (-i\nu Z_{-\nu}) = +\frac{i}{2}(j_\nu - j_{-\nu})$ so that the last integral in A5.13 is

$$\frac{i}{8\pi\hbar}\int_0^{\infty}\frac{q'_\nu q'_{-\nu}}{\nu^2|Z_\nu|^2}(h_\nu + ij_\nu + h_{-\nu} - ij_{-\nu})\,d\nu \,. \qquad (A5.14)$$

But,

$$h_\nu + ij_\nu + h_{-\nu} - ij_{-\nu} = \int_0^{\infty}\frac{2G(\Omega)}{\Omega}\,d\Omega\int_0^{\infty}\Big[(\cos\Omega t - i\sin\Omega t)e^{-i\nu t} + (\cos\Omega t + i\sin\Omega t)e^{i\nu t}\Big]\,dt$$

$$= \int_0^{\infty}\frac{2G(\Omega)}{\Omega}\,d\Omega\int_{-\infty}^{\infty} e^{-i(\nu+\Omega)t}\,dt = \int_0^{\infty}\frac{4\pi G(\Omega)\,\delta(\nu+\Omega)}{\Omega}\,d\Omega = 0.$$

Therefore, equation A5.13 becomes in terms of Q,Q' :

$$\tilde{\Phi}(Q,Q') = \frac{1}{2\pi\hbar}\int_0^\infty \left[\frac{Q'_\nu(Q_{-\nu}-Q'_{-\nu})}{i\nu Z_\nu} + \frac{Q_{-\nu}(Q_\nu-Q'_\nu)}{-i\nu Z_{-\nu}}\right] d\nu \qquad (A5.15)$$

where

$$i\nu Z_\nu = -m(\nu^2-\omega^2) - \frac{i}{2}\, j_\nu \quad .$$

Since $-\frac{i}{2}\, j_\nu = \frac{1}{2i}\int_0^\infty \frac{G(\Omega)}{\Omega}\, d\Omega \int_0^\infty (e^{-i\Omega t}-e^{i\Omega t})e^{-i\nu t}\, dt$

$$= -\int_0^\infty \frac{G(\Omega)}{\Omega}\, d\Omega \int_0^\infty \sin\Omega t\, e^{-i\nu t} dt = \int_0^\infty \frac{G(\Omega)\, d\Omega}{(\nu-i\epsilon)^2-\Omega^2}$$

then

$$i\nu Z_\nu = -m(\nu^2-\omega^2) + \int_0^\infty \frac{G(\Omega)\, d\Omega}{(\nu-i\epsilon)^2-\Omega^2} \qquad (A5.16)$$

and

$$Z_\nu = \frac{\pi G(\nu)}{\nu^2} + \frac{im}{\nu}(\nu^2-\omega^2) - \frac{i}{\nu}\int_0^\infty \frac{G(\Omega)d\Omega}{\nu^2-\Omega^2} \quad . \qquad (A5.17)$$

b. Demonstration that $x'^2_\tau + \dot{x}'^2_\tau/\omega^2 = 0$, $\tau \to -\infty$

Before showing that the above assertion is true, a physical argument can be made which makes it seem very plausible. As we already know the sea of oscillators $G(\Omega)$ has the effect of adding a finite amount of loss to the system being considered. If the initial time τ is considered to be infinitely far in the past, then all the transient effects will be gone at any finite time t . Thus, it really does not matter what initial state was used, the influence functional is independent of it.

Mathematically, we can proceed by starting with the equation $g[x'(t)] = 0$. This is from equation A5.6

$$\ddot{x}' + \omega^2 x' + q'/m + \frac{i}{2m}\int_t^T x'_s \, j(s-t)ds = 0 \quad . \tag{A5.18}$$

Multiplying each term by $\dot{x}'$ and integrating from $t = \tau$ to $t = T$ we can write the above as

$$-\left(\frac{\dot{x}'^2}{2} + \frac{\omega^2 x'^2}{2}\right)\Bigg|_\tau^T = \int_\tau^T \frac{\dot{x}' q'}{m}\, dt + \frac{i}{2m}\int_\tau^T\int_t^T \dot{x}'_t x'_s \, j(s-t)dsdt \; . \tag{A5.19}$$

From the boundary conditions indicated by equation A5.7, $x'_T = 0$ and $\dot{x}'_T = 0$, therefore equation A5.19 can be rewritten

$$\frac{\omega^2}{2}\left(x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2}\right) = \int_\tau^T \frac{\dot{x}'_t q'_t}{m}\, dt + \frac{i}{2m}\int_\tau^T\int_t^T \dot{x}'_t x'_s \, j(s-t)dsdt \quad . \tag{A5.20}$$

Extending the time interval to be infinite and changing to Fourier transforms, the right hand side of equation A5.20 becomes

$$\frac{\omega^2}{2}\left(x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2}\right) = \int_{-\infty}^{\infty} \left(\frac{-i\nu\, x'_\nu q'_{-\nu}}{2\pi m} + \frac{1}{4\pi m}\, \nu x'_\nu x'_{-\nu} j_\nu\right) d\nu$$

$$= \frac{i}{2\pi m}\int_{-\infty}^{\infty} \nu x'_\nu \left[-q'_{-\nu} - \frac{i}{2}\, x'_{-\nu}\, j'_\nu\right] d\nu \quad . \tag{A5.21}$$

But from $g(x'_\nu) = 0$ we find

$$m(\nu^2 - \omega^2)x'_{-\nu} = -q'_{-\nu} - \frac{i}{2}\, x'_{-\nu} j_\nu \quad . \tag{A5.22}$$

Substituting this in A5.21,

$$\frac{\omega^2}{2}\left(x'^2_\tau + \frac{\dot{x}'^2_\tau}{\omega^2}\right) = \frac{i}{2\pi m}\int_{-\infty}^{\infty} m(\nu^2-\omega^2)\nu x'_\nu x'_{-\nu}\,d\nu = 0$$

since the $d\nu$ integral is antisymmetric. Thus, the initial transients disappear as was expected.

APPENDIX VI

Equation VII.20 which is to be putinto transform notation is as follows:

$$i\Phi(E,E') = -\frac{1}{2\hbar}\int_{T}^{T}\int_{T}^{t} (E_t^{\alpha}-E_t'^{\alpha})\,[E_s^{\beta}F^{*}_{\alpha\beta}(t-s)- E_s'^{\beta}F_{\alpha\beta}(t-s)]dsdt\ . \qquad (VII.20)$$

If we use the notation as before that $F_{\alpha\beta}(t) = A_{\alpha\beta}(t)+ iB_{\alpha\beta}(t)$ and change to the variables

$$e = E + E'$$
$$e' = E - E'$$

the above equation reads

$$i\Phi(e,e') = \frac{1}{2\hbar}\int_{T}^{T}\int_{T}^{t} [ie_t'^{\alpha}e_s^{\beta}B_{\alpha\beta}(t-s) - e_t'^{\alpha}e_s'^{\beta}A_{\alpha\beta}(t-s)]\,dsdt\ . \qquad (A6.1)$$

Converting this to transform notation:

$$i\Phi(e,e')= \frac{1}{4\pi\hbar}\int_{0}^{\infty} d\nu[ie_{-\nu}'^{\alpha}e_{\nu}^{\beta}B_{\alpha\beta}(\nu)+ie_{\nu}'^{\alpha}e_{-\nu}^{\beta}B_{\alpha\beta}(-\nu)-e_{-\nu}'^{\alpha}e_{\nu}'^{\beta}A_{\alpha\beta}(\nu)-e_{\nu}'^{\alpha}e_{-\nu}'^{\beta}A_{\alpha\beta}(-\nu)] \qquad (A6.2)$$

Adding and subtracting $-ie_{-\nu}'^{\alpha}\,e_{\nu}'^{\beta}\,B_{\alpha\beta}(\nu)+ ie_{\nu}'^{\alpha}\,e_{-\nu}'^{\beta}\,B_{\alpha\beta}(-\nu)$, equation A6.2 becomes

$$i\Phi(e,e')= \frac{1}{4\pi\hbar}\int_{0}^{\infty} d\nu[ie_{-\nu}'^{\alpha}(e_{\nu}^{\beta}-e_{\nu}'^{\beta})B_{\alpha\beta}(\nu)+ie_{\nu}'^{\alpha}(e_{-\nu}^{\beta}+e_{-\nu}'^{\beta})B_{\alpha\beta}(-\nu)]$$
$$- \frac{1}{4\pi\hbar}\int_{0}^{\infty} d\nu[e_{-\nu}'^{\alpha}\,e_{\nu}'^{\beta}\big(A_{\alpha\beta}(\nu)-iB_{\alpha\beta}(\nu)\big)+e_{\nu}'^{\alpha}\,e_{-\nu}'^{\beta}\big(A_{\alpha\beta}(-\nu)+iB_{\alpha\beta}(-\nu)\big)]\ . \qquad (A6.3)$$

Changing back to the variables E,E' this becomes

$$i\bar{\Phi}(E,E') = \frac{i}{2\pi\hbar}\int_0^\infty d\nu[E'^{\beta}_{\nu}(E^{\alpha}_{-\nu}-E'^{\alpha}_{-\nu})B_{\alpha\beta}(\nu)+ E^{\beta}_{-\nu}(E^{\alpha}_{\nu}-E'^{\alpha}_{\nu})B_{\alpha\beta}(-\nu)]$$

$$- \frac{1}{\pi\hbar^2}\int_0^\infty d\nu(E^{\alpha}_{\nu}-E'^{\alpha}_{\nu})(E^{\beta}_{-\nu}-E'^{\beta}_{-\nu}) \left(\frac{\hbar}{4}\right)[A_{\alpha\beta}(\nu)-iB_{\alpha\beta}(\nu)+A_{\beta\alpha}(-\nu)+iB_{\beta\alpha}(-\nu)] \quad . \tag{A6.4}$$

The first integral of A6.4 is the same as the first integral of equation VII.24 when it is recognized that

$$B_{\alpha\beta}(\nu) = \frac{1}{i\nu Z_{\alpha\beta}(\nu)} \quad .$$

The second integral indicates the power spectrum is given by

$$\phi_{\alpha\beta}(\nu) = \left(\frac{\hbar}{4}\right)[A_{\alpha\beta}(\nu)-iB_{\alpha\beta}(\nu) + A_{\beta\alpha}(-\nu)+iB_{\beta\alpha}(-\nu)] \quad . \tag{A6.5}$$

To simplify this we notice that

$$A_{\alpha\beta}(\nu) + iB_{\alpha\beta}(\nu) = F_{\alpha\beta}(\nu)$$

and

$$A_{\beta\alpha}(-\nu)-iB_{\beta\alpha}(-\nu) = F^*_{\beta\alpha}(-\nu) \quad .$$

Substituting the expressions for $F_{\alpha\beta}$ from equation VII.21, and taking the transforms

$$\phi_{\alpha\beta}(\nu) = \sum_{a,b} \frac{e^2\rho_{aa}(X_\alpha)_{ab}(X_\beta)_{ba}}{2}\int_0^\infty \left[\exp\left\{-i(\nu+\omega_{ba})t\right\} + \exp\left\{i(\nu+\omega_{ba})t\right\}\right]dt$$

$$= \sum_{a,b} \pi\, e^2\rho_{aa}(X_\alpha)_{ab}(X_\beta)_{ba}\,\delta(\nu+\omega_{ba})$$

$$= \sum_{a,b} \pi\, e^2(X_\alpha)_{ab}(X_\beta)_{ba}\left(\frac{\rho_{bb}-\rho_{aa}}{\frac{\rho_{bb}}{\rho_{aa}}-1}\right)\delta(\nu+\omega_{ba}) \quad . \tag{A6.6}$$

If a state of temperature equilibrium exists initially, ρ_{bb}/ρ_{aa} can be written $e^{\beta\hbar\omega_{ab}}$ so that

$$\phi_{\alpha\beta}(\nu) = \sum_{a,b} \pi e^2 (X_\alpha)_{ab}(X_\beta)_{ba} \left(\frac{\rho_{bb}-\rho_{aa}}{e^{\beta\hbar\nu}-1}\right)\delta(\nu+\omega_{ba}) \quad . \tag{A6.7}$$

To relate this to the impedance we take equation VII.23 and form the expression

$$\frac{1}{(Z_\nu)_{\alpha\beta}} + \frac{1}{(Z_\nu)^*_{\beta\alpha}} = \sum_{a,b} \frac{e^2(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar}(\rho_{aa}-\rho_{bb})\left(\frac{i\nu}{\nu-i\epsilon+\omega_{ba}} - \frac{i\nu}{\nu+i\epsilon+\omega_{ba}}\right) \tag{A6.8}$$

$$= \sum_{a,b} \frac{\nu e^2(X_\alpha)_{ab}(X_\beta)_{ba}}{\hbar} (\rho_{aa}-\rho_{bb})(-2\pi)\delta(\nu+\omega_{ba}) \quad .$$

Substitution of this in equation A6.7 yields

$$\phi_{\alpha\beta}(\nu) = \frac{\hbar\left[\frac{1}{(Z_\nu)_{\alpha\beta}} + \frac{1}{(Z_\nu)^*_{\beta\alpha}}\right]}{2\nu\left[e^{\beta\hbar\nu} - 1\right]} \quad . \tag{A6.9}$$

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