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FREE BOUNDARY PROBLEMS FOR NONLINEAR WAVE SYSTEMS: MACH STEMS FOR INTERACTING SHOCKS*

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Abstract. We study a family of two-dimensional Riemann problems for compressible flow modeled by the nonlinear wave system. The initial constant states are separated by two jump discontinuities, $x = \pm\kappa_a y$, which develop into two interacting shock waves. We consider shock angles in a range where regular reflection is not possible. The solution is symmetric about the y-axis and on each side of the y-axis consists of an incident shock, a reflected compression wave, and a Mach stem. This has a clear analogy with the problem of shock reflection by a ramp. It is well known that no triple point structure exists in which incident, reflected, and Mach stem shocks meet at a point. In this paper, we model the reflected wave by a continuous function with a singularity in the derivative. This fails to be a weak solution across the sonic line. We show that a solution to the free boundary problem for the Mach stem exists, and we conjecture that the global solution can be completed by the construction of a reflected shock, by a similar free boundary technique.

The point of our paper is the capability to deal analytically with a Mach stem by solving a free boundary problem. The difficulties associated with the analysis of solutions containing Mach stems include (1) loss of obliqueness in the derivative boundary condition corresponding to the jump conditions across the Mach stem, and (2) loss of ellipticity at the formation point of the Mach stem.

We use barrier functions to show that for sufficiently large values of κ_a the subsonic solution is continuous up to the sonic line at the Mach stem.

Key words. two-dimensional conservation laws, degenerate elliptic equations, free boundary problems, self-similar solutions, Riemann problems

AMS subject classifications. Primary: 35L65, 35J70, 35R35; Secondary: 35M10, 35J65, 76J20.

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1. Introduction.

This paper marks another step in our program to solve two-dimensional Riemann problems for hyperbolic conservation laws. Our first results involved a method [6] for solving the free boundary problems which arise in the study of small time-independent perturbations of steady transonic shocks in the small disturbance equation. We extended this technique to analyze quasi-steady transonic shocks that are not necessarily small perturbations of known solutions by focusing on weak shock reflection by a wedge, modeled by the unsteady transonic small disturbance (UTSD) equation. We solved this problem in two stages: first, a case corresponding to strong regular reflection in which the free boundary involved a strictly elliptic subsonic state [3] and, second, the case of weak regular reflection, in which proving existence of the free boundary was complicated by the failure of strict ellipticity in the downstream state [4]. The use of a simplified equation was necessary, as

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our construction relied in an essential way on reducing the self-similar system to a second-order equation with particular structure which changed type from hyperbolic to elliptic across the sonic line. In [3, 4] we obtained an existence result in only a finite neighborhood of the shock reflection point.

More recently, we have outlined a program for extending our results to a larger class of equations, choosing for a model the nonlinear wave system [5]. This system is a slightly more realistic simplification of the compressible Euler equations of gas dynamics and hence is a better test case for the program. It offers the advantage of being linearly well-posed in space and time (which the UTSD equation is not) and of having a nonlinear acoustic-wave dependence similar to the gas dynamics equations. It also has the convenient feature, just as the UTSD equations have, of reducing to a second-order quasi-linear self-similar equation which, at the sonic line, changes type from hyperbolic to elliptic. It has the additional feature, a more realistic prototype for gas dynamics, of being coupled to a transport equation, so that the change of type takes one from a hyperbolic to a mixed type system. The feature that makes the system more tractable than gas dynamics is that the coupling is very weak: it comes into play only at the point of reconstructing the solution in primitive variables.

As indicated in [5], a number of obstacles must be overcome before a theory for the general Riemann solution, even for this simplified model, can be given. The present result looks at a prototype for a Mach stem. We consider a problem characterized by symmetry and otherwise simplified data. Our eventual goal is to cover all situations which arise with general sectorially constant data. The innovations in this paper are twofold:

1. We are able handle the entire Mach stem without cutoff functions.
2. We overcome the technical difficulty posed by the fact that at the foot of a Mach stem the static boundary condition on the free boundary is no longer a uniformly oblique derivative condition.

We prove existence of a solution in a case where the equation is sonic at the formation point of the Mach stem. However, the correct modeling of the shock interaction is limited to the Mach stem and interaction point itself; we have not attempted to construct the reflected shock. Rather, we have replaced the reflected shock by a weak shock at the sonic boundary, which does not give a weak solution in the neighborhood of the sonic line. Although we have not completely solved the problem, we feel that our result is a significant advance and that this approach will help in solving the full problem. We explain this in section 5.

The analysis applies to the nonlinear wave system (NLWS), a reduction of the inviscid system for compressible isentropic gas dynamics, obtained by neglecting the inertial terms. The system is

$$
\begin{aligned}
\rho_t + m_x + n_y &= 0, \\
m_t + p(\rho)_x &= 0, \\
n_t + p(\rho)_y &= 0.
\end{aligned} \tag{1.1}
$$

We consider (1.1) with sectorially constant Riemann data consisting of two states separated by discontinuities at $x = \pm\kappa_a y$ for $y \geq 0$ and with the states chosen so that the one-dimensional Riemann problems at each discontinuity are solved by upward-moving shocks and linear waves only. These determine the solution in the far field. One expects to see a shock interaction consisting either of regular reflection or of Mach reflection, depending on whether the angle between the incident shocks is small or large (see [16]). The scenario we study here for Mach reflection places the formation

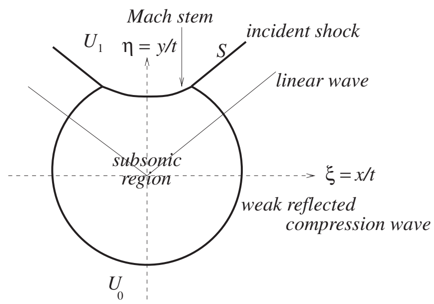

FIG. 1.1. *Sketch of global solution structure.*

point of the Mach stem exactly at the sonic circle, and hence, since this system does not admit triple points, the reflected wave has strength zero at this formation point. This scenario thus requires that the angle between the incident shocks be large enough that the shocks intersect the sonic circle before their extensions intersect each other. Numerical simulations in [17] suggest that such a formation does indeed occur and suggest, further, that the reflected shock is weak.

In this paper, we match a piecewise constant solution outside the sonic circle with a solution of the self-similar equation inside the sonic circle, demanding continuity at the circle. See Figure 1.1 for a sketch of such a solution. Our main result is the existence of a solution to the subsonic problem. The composite function is not a weak solution across the sonic circle. This leaves open the question of what is the actual solution; it differs from the construction here and from the simulations. One possibility is that the reflected wave is a weak, nearly circular shock, which has strength zero at the formation point. Based on the successful construction of the Mach stem in this paper, it may be possible to solve the complete problem by finding this reflected shock as the solution of another free boundary problem. Another possibility is a cascade of supersonic patches, as reported by Tesdall and Hunter for the UTSD equation [25]. We leave this for a future paper.

The techniques we use in this paper to prove global existence of a solution rely on an application of the Schauder fixed point theorem, developed in [6, 3, 4]. A similar approach was used by Chen and Feldman to prove stability of steady transonic shocks for the full potential equation [8, 9]. Chen and Feldman use the potential formulation of the equation to obtain a second-order operator. Both approaches prove existence of a fixed point which solves the underlying free boundary problem. The main difference lies in the compactness arguments used. Owing to the presence of the gradient of the potential in the principal coefficient of the full potential operator, the mapping in [8, 9] is not compact, but it is shown to operate on a compact space. Steady transonic shock perturbation analyses, both in [6] and in [8, 9], examine small perturbations of a uniform solution. A perturbation analysis of steady transonic shocks is also given by Chen, Geng, and Li in [12]. Using partial hodograph transformations which map the free boundary (shock) into a fixed boundary, combined with classical elliptic techniques, [12] obtains stability results for perturbations of conical shocks attached to the tip of a perturbed cone. Chen has used this same partial hodograph technique in a quasisteady problem [11] and has also found an analytical solution for a linearized problem corresponding to quasisteady regular reflection in the gas dynamics equations, [10].

The compressible Euler equations cannot, in general, be written in potential form and self-similar reduction of the compressible Euler equations (see [5, 24, 26]) leads to a system related in structure to the model studied in the present paper. In this connection, we mention also recent work by Zheng on diverging shocks in the pressure gradient system, a type of nonlinear wave system, [27].

In section 2, we derive the second-order operator and derivative boundary condition at the shock for the nonlinear wave system, (1.1); give the technical statement of our result, Theorem 2.3; set up the mapping to find the free boundary; and establish some preliminary estimates. In section 3, using a regularized differential operator, with $\varepsilon\Delta$ added, we prove the existence of a fixed point corresponding to the free boundary for the uniformly elliptic problem. The main point here is to deal with loss of obliqueness in the derivative boundary condition. In section 4, we proceed to the limit $\varepsilon \to 0$. The novelty here is that a uniform upper barrier at the intersection of the Mach stem with the sonic line cannot be found by standard barrier estimates. In section 5, we explain the significance of the result in providing a first step in the construction of Mach stems and other configurations where oblique derivative boundary conditions can become degenerate and where shocks cross the sonic line.

2. Background on the nonlinear wave system. Our point of departure is the compressible Euler system for isentropic flow in two space dimensions,

$$
\begin{aligned}
\rho_t + (u\rho)_x + (v\rho)_y &= 0,\\
(u\rho)_t + (u^2\rho + p)_x + (uv\rho)_y &= 0,\\
(v\rho)_t + (uv\rho)_x + (v^2\rho + p)_y &= 0,
\end{aligned}
\tag{2.1}
$$

where ρ, u, and v are the density and the components of velocity, respectively, and $p = p(\rho)$ is the pressure. While we have in mind a power-law relation $p(\rho) = A\rho^\gamma$, where $\gamma > 1$ is the ratio of specific heats, all that we require in this paper is $p' > 0$ and $p'' > 0$. We recall that the local speed of sound is c and that $c^2 = dp/d\rho$. The nonlinear wave system is a reduction of (2.1) obtained by neglecting the quadratic terms in u and v. (We do not know if any physical situation is represented by this assumption. However, it underlies the scaling for Stokes flow and was used by Pironneau [23] in a case study of the shallow-water equations, which are modeled by (2.1) with $\gamma = 2$.) In the resulting nonlinear wave system, (1.1), we work with the conserved momentum variables $(m, n) = (\rho u, \rho v)$. The NLWS (1.1) can be written as a second-order nonlinear wave equation for the density and a transport equation for the specific vorticity $\omega = n_x - m_y$:

$$
\begin{aligned}
\rho_{tt} &= \nabla(c^2(\rho)\nabla\rho),\\
\omega_t &= 0.
\end{aligned}
\tag{2.2}
$$

Since ω is stationary in this simplification of (2.1), then in any regions where the initial data satisfy the irrotationality condition $n_x = m_y$, the solutions, classical or weak, satisfy the same condition.

Introducing self-similar coordinates $\xi = x/t, \eta = y/t$, we can write the system (1.1) as

$$-\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta = 0, \tag{2.3}$$

$$-\xi m_\xi - \eta m_\eta + c^2(\rho)\rho_\xi = 0, \tag{2.4}$$

$$-\xi n_\xi - \eta n_\eta + c^2(\rho)\rho_\eta = 0. \tag{2.5}$$

In self-similar coordinates the nonlinear wave equation in (2.2), with its principal part in divergence form, is

$$Q(\rho) \equiv \left((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta\right)_\xi + \left((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi\right)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0. \tag{2.6}$$

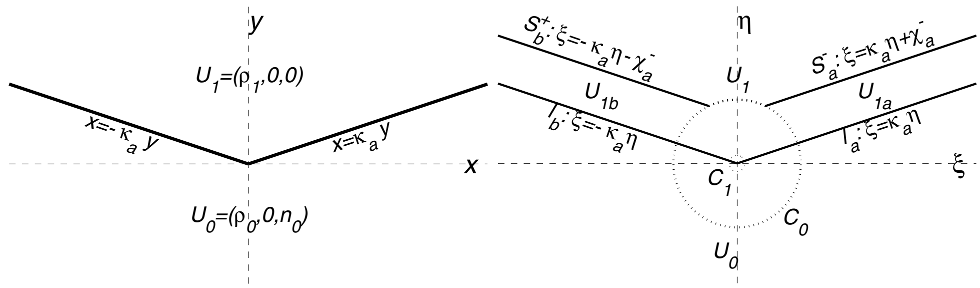

FIG. 2.1. *Riemann data and far-field solution.*

The equation is hyperbolic when $c^2(\rho) < \xi^2 + \eta^2$, elliptic when $c^2(\rho) > \xi^2 + \eta^2$, and degenerate on the sonic circle $c^2(\rho) = \xi^2 + \eta^2$.

It is because we can formulate the problem in terms of ρ that we can apply our fixed point method to this equation.

2.1. Setting up the problem. We consider two-dimensional Riemann data which are constant in sectors. Specifically, in this paper we look at data which correspond to two symmetric converging shocks. This may alternatively be regarded as the reflection of an oblique shock at a vertical wall. The data are constant in two sectors bounded by $\{x = \pm\kappa_a y, y \geq 0\}$ and symmetric with respect to $x = 0$, as shown in Figure 2.1. Let U denote the vector of conserved quantities, $U = (\rho, m, n)$. The Riemann data are

$$U(x,y,0) = \begin{cases} U_1 \equiv (\rho_1, 0, 0), & -\kappa_a y < x < \kappa_a y, \quad y > 0, \\ U_0 \equiv (\rho_0, 0, n_0) & \text{otherwise.} \end{cases} \tag{2.7}$$

To obtain converging shocks in the far field, we choose $\rho_0 > \rho_1$ and determine n_0, depending on ρ_1, ρ_0, and κ_a, so that the one-dimensional wave between U_0 and U_1 at angle κ_a consists of a backward shock, S_a^-, and a linear wave, l_a, with a state U_{1a} between them:

$$S_a^- : \{\xi = \kappa_a \eta + \chi_a^-\}, \quad l_a : \{\xi = \kappa_a \eta\}, \quad U_{1a} = (\rho_0, m_{1a}, n_{1a}). \tag{2.8}$$

Using the formula (6.1) in [5] these values are

$$\begin{aligned} \chi_a^- &= -\sqrt{1+\kappa_a^2}\sqrt{\frac{p(\rho_0) - p(\rho_1)}{\rho_0 - \rho_1}}; \\ m_{1a} &= -\sqrt{\frac{(p(\rho_0) - p(\rho_1))(\rho_0 - \rho_1)}{1+\kappa_a^2}}; \quad n_{1a} = -\kappa_a m_{1a}; \\ n_0 &= \frac{1}{\kappa_a}\sqrt{(1+\kappa_a^2)(p(\rho_0) - p(\rho_1))(\rho_0 - \rho_1)}. \end{aligned} \tag{2.9}$$

By symmetry, the resolution of the discontinuity at $x = -\kappa_a y$ is

$$S_b^+ : \{\xi = -\kappa_a \eta - \chi_a^-\}, \quad l_b : \{\xi = -\kappa_a \eta\}, \quad U_{1b} = (\rho_0, -m_{1a}, n_{1a}).$$

For the Riemann data (2.7), the sonic circle is important:

$$C_0 \equiv \{(\xi,\eta) : \xi^2 + \eta^2 = c_0^2 \equiv c^2(\rho_0)\}. \tag{2.10}$$

We also define $C_1 \equiv \{(\xi,\eta); \xi^2 + \eta^2 = c_1^2 \equiv c^2(\rho_1)\}$.

Several types of shock interaction seem possible in this model, depending on the relative positions of the incident shock and the sonic circle. They are described in more detail in [17]. For small κ_a, the shocks intersect at a point $\Xi_c \equiv (0, \eta_c) = S_b^+ \cap S_a^- = (0, -\chi_a^-/\kappa_a)$ on the η axis, and two symmetric downward-moving shocks leave Ξ_c. For values of κ_a less than a critical value κ_R which depends on ρ_0 and ρ_1 one expects two solutions of this form, corresponding to "weak" and "strong" regular reflection. For $\kappa_a > \kappa_R$, no solutions of this form exist. On the other hand, for κ_a greater than a value κ_A (with $\kappa_A > \kappa_R$), one finds that $\eta_c < c_0$, so Ξ_c is inside the sonic circle C_0, and the farfield shocks intersect C_0 before reaching the symmetry axis. In this case, it is appealing to believe that a solution like that shown in Figure 1.1 is possible: the subsonic flow interacts with the shocks, which bend to form a single discontinuity; and the flow is continuous at C_0 below the shock. This phenomenon can be thought of as a perturbation of the uniform case $\kappa_a = \infty$.

In this paper, we prove the existence of a solution to the subsonic problem which contains a Mach stem and is continuous up to the sonic line, for sufficiently large values of κ_a; that is, $\kappa_a > \kappa_* > \kappa_A$. In the remainder of the paper, we assume $\kappa_a > \kappa_A$. The paper [17] gives a more detailed discussion of the regions. There, we also give scenarios (without proof) for solutions in the intermediate range of κ where neither regular reflection nor a solution with a weak reflected wave exists.

2.2. The shock evolution equation. At a shock, the Rankine–Hugoniot jump conditions are satisfied across the line of discontinuity. A key element of our solution method has been to rewrite the equations as a problem for a single variable—in this case, ρ. With this goal, we reformulate the Rankine–Hugoniot conditions to obtain two equations: an evolution equation for the shock curve—that is, a relation between the slope of the curve, $\eta' = d\eta/d\xi$, and the variable ρ which appears in (2.6)—and an oblique derivative boundary condition for ρ—that is, an equation linear in the gradient of ρ with coefficients depending on (ξ, η), ρ, and η'. The second equation then becomes a boundary condition for the differential equation (2.6), and we play these two conditions against each other to obtain a mapping on approximate shock positions.

We proceed to derive the jump conditions and formulate the shock evolution equation using the Rankine–Hugoniot conditions.

Writing $U \equiv (\rho, m, n)$ and $\Xi = (\xi, \eta)$, system (2.3)–(2.5) can be put in conservation form:

$$\partial_\xi F(U, \Xi) + \partial_\eta G(U, \xi) = -2U \tag{2.11}$$

with

$$F \equiv \begin{pmatrix} m - \xi\rho \\ p(\rho) - \xi m \\ -\xi n \end{pmatrix} \quad \text{and} \quad G \equiv \begin{pmatrix} n - \eta\rho \\ -\eta m \\ p(\rho) - \eta n \end{pmatrix}.$$

Inside the sonic circle $C_0 = \{\xi^2 + \eta^2 = c^2(\rho_0)\}$, the incident shock need no longer be rectilinear. The state ahead of the shock, U_1, is constant, but the state on the other side, U, is subsonic and is not uniform. The Rankine–Hugoniot conditions along the line of discontinuity $\eta = \eta(\xi)$ are, from (2.11),

$$\frac{d\eta}{d\xi} = \frac{-\eta[\rho] + [n]}{-\xi[\rho] + [m]}, \tag{2.12}$$

$$\frac{d\eta}{d\xi} = \frac{-\eta[m]}{[p] - \xi[m]}, \tag{2.13}$$

$$\frac{d\eta}{d\xi} = \frac{[p] - \eta[n]}{-\xi[n]}, \tag{2.14}$$

where $[f] = f - f_1$ denotes a jump in the state f across the shock $\eta(\xi)$. There are three families of discontinuities; two are genuinely nonlinear, and one is linear (see [5]). For nonlinear waves, $[\rho] \neq 0$. Solving for $[m]$ in (2.13) and for $[n]$ in (2.14) yields

$$[m] = \frac{-[p]\eta'}{-\eta'\xi + \eta}, \quad [n] = \frac{[p]}{-\eta'\xi + \eta}. \tag{2.15}$$

A simple consequence of (2.15) is

$$[m] = -\eta'[n]. \tag{2.16}$$

Using (2.16) in (2.13) we obtain

$$\eta = \frac{[p] - \xi[m]}{[n]}, \tag{2.17}$$

while equating the right sides of (2.12) and (2.13) and using (2.17) gives a relation

$$[p][\rho] = [m]^2 + [n]^2 \tag{2.18}$$

valid for states across a shock.

Using equations (2.15) in (2.12) we get an equation for η' involving only the state variable ρ:

$$([p] - \xi^2[\rho])(\eta')^2 + 2\xi\eta[\rho]\eta' + [p] - \eta^2[\rho] = 0. \tag{2.19}$$

To streamline the discussion, we define a function

$$s(a,b) \equiv \sqrt{\frac{(p(a) - p(b))}{(a-b)}}; \tag{2.20}$$

s is the speed of a one-dimensional shock between states with densities a and b.

PROPOSITION 2.1. *If p is a convex function of ρ, then s^2 is an increasing function of a for fixed b; $s(b,b) \equiv \lim_{a\to b} s(a,b) = c(b)$; and $s(a,b) < c(a)$ for $a > b$.*

Proof. We have

$$\frac{d}{da}s^2 = \frac{p'(a)}{a-b} - \frac{p(a)-p(b)}{(a-b)^2} = \frac{p'(a)(a-b) - (p(a)-p(b))}{(a-b)^2}.$$

Expanding $p(b) = p(a) + p'(a)(b-a) + p''(\beta)(b-a)^2/2$ for some $\beta \in (a,b)$, we obtain $ds^2/da = p''(\beta)/2 > 0$ if p is convex. As $a \to b$, $s^2 \to p'(b) = c^2(b)$ and if $a > b$,

$$c^2(a) - s^2(a,b) = \frac{p'(a)(a-b) - (p(a)-p(b))}{a-b} > 0. \qquad \square$$

For fixed b, we can write

$$a = s_b^{-1}(\eta) \quad \text{when} \quad s(a,b) = \eta. \tag{2.21}$$

Now, solving (2.19) for η' in terms of ρ and writing s^2 for $[p]/[\rho]$ yields

$$\frac{d\eta}{d\xi} = \frac{-\xi\eta \pm \sqrt{s^2(\xi^2+\eta^2-s^2)}}{s^2-\xi^2}. \tag{2.22}$$

Since the subsonic region is symmetric with respect to $\xi = 0$, we solve the problem in the half of the domain in the right half-plane, $\xi \geq 0$, and impose a zero Neumann boundary condition on $\xi = 0$. We may now specify the plus sign in (2.22) for the shock curve Σ in the first quadrant, as we anticipate (and will prove) that the shock slope is nonnegative. This gives the shock evolution equation

$$\frac{d\eta}{d\xi} = \frac{-\xi\eta + \sqrt{s^2(\xi^2+\eta^2-s^2)}}{s^2-\xi^2} = \frac{\eta^2 - s^2}{\xi\eta + \sqrt{s^2(\xi^2+\eta^2-s^2)}}. \tag{2.23}$$

The second expression is equivalent to the first, and so both are well defined provided

$$s^2 \leq \xi^2 + \eta^2. \tag{2.24}$$

We will establish this condition in Proposition 2.5. We define $\Xi_s \equiv (0, \eta_s) \equiv (0, \eta(0))$, the point at the foot of the shock, and observe that we want $\eta'(0) = \sqrt{\eta^2 - s^2}/s$ to equal zero, by symmetry, and so $\eta^2 = s^2$ at Ξ_s. Thus we require

$$\eta_s = \eta(0) = s(\rho, \rho_1) = \sqrt{\frac{p(\rho) - p(\rho_1)}{\rho - \rho_1}}. \tag{2.25}$$

This can be interpreted as a condition which determines $\rho(\Xi_s)$ in the subsonic region at the base of the shock (the symmetry boundary).

We also define $\Xi_0 \equiv (\xi_0, \eta_0) = S_a^- \cap C_0$, the point where the incident shock S_a^- and the sonic circle C_0 meet. Using (2.8) for S_a^- and (2.10) for C_0 we determine Ξ_0:

$$\xi_0 = \frac{\kappa_a\sqrt{c_0^2 - s_0^2} - s_0}{\sqrt{1+\kappa_a^2}}, \quad \eta_0 = \frac{\kappa_a s_0 + \sqrt{c_0^2 - s_0^2}}{\sqrt{1+\kappa_a^2}}, \tag{2.26}$$

where $s_0^2 = (p(\rho_0) - p(\rho_1))/(\rho_0 - \rho_1)$. The initial condition for the shock position is $\eta(\xi_0) = \eta_0$.

2.3. The oblique derivative boundary condition. We next use the Rankine–Hugoniot conditions to formulate a boundary condition along the shock $\Sigma = \{(\xi, \eta(\xi))\}$.

Since vorticity is confined to the lines of discontinuity of the Riemann data (see (2.2) and [5]), and these lie below the shock (see Figure 2.1), the vorticity is zero along the shock:

$$m_\eta - n_\xi = 0. \tag{2.27}$$

Using this equation and (2.3)–(2.5), we express all the partial derivatives of m and n in terms of the derivatives of ρ:

$$n_\xi = m_\eta = \frac{1}{\xi^2+\eta^2}\Big(\eta(c^2-\xi^2)\rho_\xi + \xi(c^2-\eta^2)\rho_\eta\Big), \tag{2.28}$$

$$m_\xi = \frac{1}{\xi^2+\eta^2}\Big(\xi(c^2+\eta^2)\rho_\xi - \eta(c^2-\eta^2)\rho_\eta\Big), \tag{2.29}$$

$$n_\eta = \frac{1}{\xi^2+\eta^2}\Big(\xi(-c^2+\xi^2)\rho_\xi + \eta(c^2+\xi^2)\rho_\eta\Big). \tag{2.30}$$

Differentiating (2.18) along Σ ($' = d/d\xi = \partial_\xi + \eta'\partial_\eta$) we get

$$\begin{aligned}(c^2(\rho)[\rho] + [p])(\rho_\xi + \eta'\rho_\eta) &= 2[m]m' + 2[n]n' \\ &= 2[n](-\eta' m' + n') = 2[n](-\eta' m_\xi + (1-(\eta')^2)m_\eta + \eta' n_\eta),\end{aligned}$$

where $[m] = -\eta'[n]$ (equation (2.16)) is used in the second equality and $m_\eta = n_\xi$ (equation (2.27)) in the last equality. We simplify the last expression, replacing derivatives Dm and Dn by $D\rho$ using (2.28), (2.29), (2.30), and

$$[n] = \frac{[p]}{-\eta'\xi + \eta} \tag{2.31}$$

from (2.15), and finally we get

$$\beta \cdot \nabla\rho \equiv \beta_1\rho_\xi + \beta_2\rho_\eta = 0, \tag{2.32}$$

where β is a function of Ξ, ρ, and η' with components

$$\begin{aligned}\beta_1 = {} & (\xi^2+\eta^2)(-\eta'\xi+\eta)(c^2(\rho)+s^2(\rho,\rho_1)) \\ & - 2s^2\left\{-\eta'\xi(c^2+\eta^2) + (1-(\eta')^2)\eta(c^2-\xi^2) + \eta'\xi(-c^2+\xi^2)\right\}\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\beta_2 = {} & \eta'(\xi^2+\eta^2)(-\eta'\xi+\eta)(c^2(\rho)+s^2(\rho,\rho_1)) \\ & - 2s^2\left\{\eta'\eta(c^2-\eta^2) + (1-(\eta')^2)\xi(c^2-\eta^2) + \eta'\eta(c^2+\xi^2)\right\}.\end{aligned} \tag{2.34}$$

We now examine the obliqueness condition by comparing β with the inward normal to Ω at Σ, $\nu = (\eta', -1)$. It turns out that the operator $\beta \cdot \nabla$ in (2.32) is oblique at all points on the shock except the symmetry point. In fact, obliqueness holds along any monotonic curve which satisfies the shock equation (2.23) at (ξ_0, η_0), that is, $\eta(\xi_0) = \eta_0$ and $\eta'(\xi_0) = 1/\kappa_a$, and for any subsonic function ρ. We prove the following result.

Proposition 2.2. *Let $\Sigma = \{(\xi, \eta(\xi))\}$ be any curve which has positive slope on $(0, \xi_0]$, lies inside the sonic circle C_0, and at $\xi = \xi_0$ satisfies (2.23) and $\eta = \eta_0$; let ν be its inward normal. Then for any function $\rho(\xi,\eta)$ with $c^2(\rho) > \xi^2+\eta^2$, we have $\beta\cdot\nu > 0$ on Σ for $\xi \in (0, \xi_0]$.*

Proof. We calculate

$$\begin{aligned}\beta\cdot\nu &= \beta_1\eta' - \beta_2 \\ &= -2s^2\left\{-(\eta')^2\xi(c^2+\eta^2) + \eta'(1-(\eta')^2)\eta(c^2-\xi^2) + (\eta')^2\xi(-c^2+\xi^2)\right. \\ &\qquad \left. -\eta'\eta(c^2-\eta^2) - (1-(\eta')^2)\xi(c^2-\eta^2) - \eta'\eta(c^2+\xi^2)\right\} \\ &= 2s^2(\eta'\eta+\xi)\left\{(c^2-\xi^2)(\eta')^2 + 2\xi\eta\eta' + c^2 - \eta^2\right\}.\end{aligned}$$

Now $s^2 = [p]/[\rho] \neq 0$, since $c^2(\rho) > \xi^2+\eta^2 > c^2(\rho_1)$. Also, if $\eta' > 0$ and $\xi > 0$ we have $\eta'\eta + \xi > 0$; so to get obliqueness we need only verify that

$$(c^2-\xi^2)(\eta')^2 + 2\xi\eta\eta' + c^2 - \eta^2 > 0. \tag{2.35}$$

We first note that (2.35) holds at $\xi = \xi_0$, since $c^2(\rho_0) > s^2(\rho_0, \rho_1)$ and $(s^2-\xi^2)(\eta')^2 + 2\xi\eta\eta' + s^2 - \eta^2 = 0$ (equation (2.19)) holds at (ξ_0, η_0).

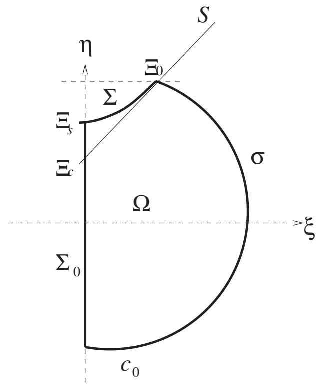

FIG. 2.2. *Sketch of the domain.*

Now, the left-hand side of (2.35) is a quadratic polynomial, $P(\eta')$, where $P(Y) = (c^2 - \xi^2)Y^2 + 2\eta\xi Y + (c^2 - \eta^2)$, with coefficients depending smoothly on ξ, η, and ρ. For any (ξ, η, ρ) with $\xi^2 + \eta^2 < c^2(\rho)$, $P(Y)$ has a fixed sign for all Y since $\mathrm{disc}(P) = c^2\big(\xi^2 + \eta^2 - c^2(\rho)\big) < 0$. Thus, P has a fixed sign inside C_0. Since $P(\eta') > 0$ at (ξ_0, η_0), then $P > 0$ on $\{(\xi, \eta(\xi)) \mid \xi \in [0, \xi_0]\}$.

Thus obliqueness holds for $\xi > 0$. However, obliqueness fails at $\xi = 0$, where the factor $\eta'\eta + \xi$ vanishes because we impose the condition $\eta' = 0$. □

2.4. The free boundary problem. We can now give a technical statement of the main result in this paper. The subsonic domain is bounded by the part of the circle $\xi^2 + \eta^2 = c^2(\rho_0)$ which lies below the shock and by the a priori unknown curved transonic shock itself. Taking advantage of the symmetry, we solve the problem in the right half of this domain, which we will call Ω in the remainder of the paper. We define σ to be the closed segment of C_0 bounding Ω and Σ_0 to be the relatively open segment of the η axis which forms the symmetry boundary. See Figure 2.2. The use of a half-domain results in a technical issue at the bottom corner, where Σ_0 meets σ, which is easily dealt with by standard continuity arguments. In addition, the fact that the upper boundary Σ is free means that Σ_0 is also not defined a priori. This matter of nomenclature we shall also ignore in the interest of simplicity.

We define Q to be the governing second-order quasi-linear operator in the domain Ω, given in (2.6) (repeated indices are summed):

$$Q\rho = \big((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta$$

$$\equiv D_i(a_{ij}(\Xi, \rho)D_j\rho) + b_i(\Xi)D_i\rho = 0. \tag{2.36}$$

In principle, we should modify Q so that it is elliptic in Ω for any value of ρ. However, in Proposition 2.4, we immediately obtain a priori bounds which enable us to use the original operator. We denote by M the quasi-linear oblique derivative boundary operator on $\Sigma = \{(\xi, \eta(\xi)) | \ \xi \in (0, \xi_0)\}$:

$$M\rho \equiv \beta(\Xi, \rho, \eta') \cdot \nabla\rho = 0. \tag{2.37}$$

Here β is the vectorfield defined by (2.33) and (2.34). The second condition on the free boundary is the shock evolution equation (2.23) for Σ:

$$\frac{d\eta}{d\xi} = f(\Xi, \rho) \equiv \frac{-\xi\eta + \sqrt{s^2(\xi^2 + \eta^2 - s^2)}}{s^2 - \xi^2} \quad \text{with} \quad \eta(\xi_0) = \eta_0. \tag{2.38}$$

Here $s = s(\rho(\xi, \eta(\xi)), \rho_1)$ is the function given by (2.20). On the fixed segments of the boundary, Σ_0 and σ, we impose Neumann and Dirichlet conditions, respectively:

$$\rho_\xi = 0 \quad \text{on} \quad \Sigma_0 \subset \{\xi = 0\}; \quad \rho = \rho_0 \quad \text{on} \quad \sigma \subset \{\xi^2 + \eta^2 = c^2(\rho_0)\}. \tag{2.39}$$

At the Dirichlet boundary, the equation is degenerate elliptic, in a manner described in our previous work, [1, 2, 7]. In particular, we expect that the solution will have an algebraic singularity along this boundary segment.

Now, it is easy to see that the trivial solution $\rho(\xi, \eta) \equiv \rho_0$ solves this problem, with Σ simply the straight-line extension of the incoming shock S_a^-, except at the point where the shock meets the symmetry boundary. Thus, we must in addition impose a one-point condition at this a priori unknown point, which we label Ξ_s. We impose the condition that the curved shock is smooth for the full domain problem, and hence that $\eta'(0) = 0$. As shown in section 2.2, this is equivalent to (2.25). We may alternatively express this as a one-point Dirichlet condition at the corner Ξ_s by solving $\eta_s = s(\rho(0, \eta_s), \rho_1)$, for $\rho(\Xi_s)$, or, using the notation of equation (2.21),

$$\rho(\Xi_s) = s_{\rho_1}^{-1}(\eta_s). \tag{2.40}$$

We establish the following existence theorem.

THEOREM 2.3. *There is a value κ_* such that for any Riemann data* (2.7) *with $\kappa_a > \kappa_*$, the free boundary problem consisting of* (2.36), (2.37), (2.38), (2.39), *and* (2.40) *has a classical solution $\rho \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ which is twice continuously differentiable up to Σ and Σ_0 except at Ξ_s and Ξ_0. The free boundary is of Hölder class $H_{2+\alpha}$ for some α which is determined by the Riemann data of the problem.*

We prove this theorem using the fixed point argument we developed in our earlier papers and in work with Lieberman [3, 4, 6] for the slightly simpler small disturbance equations. The main technical difficulty which is new in this case is that the boundary condition on the free boundary is no longer uniformly oblique. To be precise, obliqueness fails at the point Ξ_s. On the other hand, because it is the nature of the Mach stem to strengthen as it approaches the wall, we find that we can control the quantity under the square root sign in (2.38). Thus our result is not restricted to being local, as in [3] and [4], or perturbative, as in [6].

We formulate the fixed point argument in terms of the position of the free boundary. We work with a regularized, uniformly elliptic, operator $Q^\varepsilon = Q + \varepsilon\Delta$ and then, as in [4], send the regularizing parameter, ε, to zero. The mapping on the free boundary is obtained by solving a fixed boundary problem using the oblique derivative condition on the shock boundary and then integrating the shock evolution equation to update the position of the shock. However, unlike our problem in [4], obliqueness fails at the corner Ξ_s representing the foot of the Mach stem. Following ideas outlined by Lieberman [20, 21], we establish local Schauder estimates at Ξ_s which are independent of the obliqueness ratio (Theorem 3.5). In section 3.2 we apply these results to the nonlinear regularized fixed boundary problem. The regularized free boundary problem is solved in section 3.3, and results for the limit $\varepsilon \to 0$ are obtained in section 4.

Before beginning the analysis, we establish that the equations above are well-defined for the approximations we use. The following monotonicity result is used throughout.

PROPOSITION 2.4. *For a given monotonic function $\eta(\xi)$ forming the boundary Σ, suppose that $\rho \in C^1(\Omega \cup \Sigma \cup \Sigma_0) \cap C(\overline{\Omega})$ is a solution of the boundary value problem* (2.36), (2.37), (2.39), *and* (2.40) *with $\rho \geq \rho_0$. Then $\rho(0, \eta_s) = \rho_{\max}$ is the maximum value of ρ in $\overline{\Omega}$ and ρ is monotonic on Σ.*

Proof. Since the operator Q in (2.36) has no undifferentiated terms, the classical and Hopf maximum principles apply. That is, the local and absolute extrema of ρ occur on the boundary $\partial\Omega$ (classical); and at any point on $\partial\Omega$ where ρ has a local extremum, the normal derivative is nonzero (Hopf [15, p. 34]). On the Neumann and oblique derivative boundaries, Σ_0 and Σ, if ρ has an extremum along the boundary then two linearly independent directional derivatives of ρ are zero, and so $\nabla\rho$ is zero there, which is impossible, by the Hopf maximum principle. Thus there are no local extrema in the interior of Σ_0 or of Σ. There cannot be absolute extrema, either, and hence $\rho_{\max} = \rho(0, \eta_s)$ is the absolute maximum of ρ in Ω, and we obtain the bounds $\rho_0 < \rho < \rho_{\max}$ in Ω from the classical maximum principle. And since in Ω we have $\xi^2 + \eta^2 < c^2(\rho_0) < c^2(\rho)$, it follows that the solution is strictly subsonic in Ω.

To prove monotonicity, we argue by contradiction. Let us first examine the C^1 function ρ restricted to Σ. This is now a function of a single variable, say, the first component of a point $\Xi = (\xi, \eta)$ on Σ. Without confusion, we can label this component by the name of the point, we can order the points along Σ by this component, and we can refer to intervals along Σ by the labels. Then lack of monotonicity means there exist points Z_1 and Z_2 on Σ with $\Xi_s < Z_1 < Z_2 < \Xi_0$ at which $\rho(Z_1) < \rho(Z_2)$. We immediately deduce that

1.

2. $$\text{in} \quad (\Xi_s, Z_2) \quad \exists \quad \widetilde{C} \quad \text{with} \quad \rho(\widetilde{C}) = \min_{[\Xi_s, Z_2]} \rho;$$

$$\text{in} \quad (\widetilde{C}, \Xi_0) \quad \exists \quad D \quad \text{with} \quad \rho(D) = \max_{[\widetilde{C}, \Xi_0]} \rho.$$

We want to identify points C and D, $C < D$, on Σ such that the following three properties hold:

(i) $\rho(\Xi_s) \geq \rho \geq \rho(C)$ on $[\Xi_s, C]$;
(ii) $\rho(C) \leq \rho \leq \rho(D)$ on $[C, D]$;
(iii) $\rho(D) \geq \rho \geq \rho(\Xi_0)$ on $[D, \Xi_0]$.

Now, property (ii) may not hold with $C = \widetilde{C}$ because $\rho(\widetilde{C})$ is the minimum value of ρ only on the interval $[\Xi_s, Z_2]$, and we may have $D > Z_2$. So, if there is a point in (Z_2, D) at which $\rho < \rho(\widetilde{C})$, then we let C be a point at which ρ has its minimum value in this interval; if there is no such point, then let $C = \widetilde{C}$. Then all three properties hold.

Now we look at the function ρ in the domain Ω. The idea is to partition Ω into subdomains by two curves Γ_C and Γ_D from C and D, respectively, to points A and B, respectively, on Σ_0, in such a way that $\rho(A) < \rho(B)$ and so that we can deduce that there is a point m on Σ_0 at which ρ reaches a minimum on either the domain Ω_A or the domain Ω_B, thus violating the Hopf maximum principle, as stated in the first paragraph of this proof. See Figure 2.3. It is of course sufficient to show that $\rho(m)$ is the minimum value of ρ on the boundary of Ω_A or Ω_B.

It would be simplest to find curves on which ρ is monotonic, but it is not clear that such curves exist, or what properties they would have. Instead, we construct Lipschitz curves on which ρ is monotonic on average. To be precise, we construct curves on which, for a certain number μ,

$$\rho(A) \leq \rho \leq \rho(C) + \mu \quad \text{on } \Gamma_C \quad \text{and } \rho(A) < \rho(C); \tag{2.41}$$

$$\rho(B) \geq \rho \geq \rho(D) - \mu \quad \text{on } \Gamma_D \quad \text{and } \rho(B) > \rho(D). \tag{2.42}$$

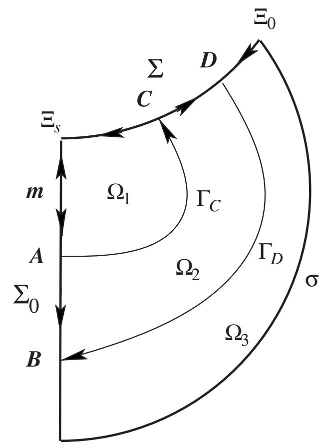

FIG. 2.3. *Illustration of the proof of Proposition* 2.4.

We begin by identifying some useful constants. Let

$$\mu = \frac{1}{4}\min\{\rho(D) - \rho(C), \rho(\Xi_s) - \rho(D), \rho(C) - \rho(\Xi_0)\}.$$

Since $\rho \in C(\overline{\Omega})$, then ρ is uniformly continuous, and there is an $\epsilon > 0$ such that $\rho(\Xi) \le \rho_0 + \mu$ if $\mathrm{dist}(\Xi, \sigma) < \epsilon$. Let $\Omega_\epsilon = \{\Xi \in \Omega \mid \mathrm{dist}(\Xi, \sigma) > \epsilon\}$, and let $\sigma_\epsilon = \{\Xi \in \Omega \mid \mathrm{dist}(\Xi, \sigma) = \epsilon\}$. As we shall see, we can restrict our attention to $\overline{\Omega_\epsilon}$. The purpose of constructing this domain is to be able to bound $|\nabla\rho|$. Since $\rho \in C^1(\overline{\Omega_\epsilon})$, we have $|\nabla\rho| \le M$ there, say. (We could estimate M from Schauder theory, but this is not important here.)

Now, on any ball of radius r, the oscillation of ρ is bounded by $2Mr$, and we now choose a radius, $R = \mu/(2M)$, so that

$$\operatorname*{osc}_{B_R \cap \Omega_\epsilon} \rho \le \mu.$$

Now we construct Γ_D as follows. Consider a ball $B_R(D)$ centered at D. In $B_R(D) \cap \Omega_\epsilon$, $\rho(D)$ cannot be the maximum value of ρ (because D is not a point of local maximum in Ω); hence there are points of $\partial B_R(D) \cap \Omega_\epsilon$ where $\rho > \rho(D)$. Let X_1 be a point at which ρ attains its maximum value in $\overline{B_R(D)}$. The first segment of Γ_D is a straight line from D to X_1. We have $\rho(X_1) > \rho(D)$, and on the segment, $\rho(X) \ge \rho(D) - \mu$ and $\rho(X) < \rho(X_1)$.

Now we continue inductively, forming a sequence of line segments with corners at $\{X_i\}$ (take $D = X_0$), along which $\rho \ge \rho(D) - \mu$ and such that $\rho(X_1) < \rho(X_2) < \cdots$. To show that we can do this, let

$$\Omega_j = \Omega_\epsilon \backslash \overline{\{\cup_0^{j-1} B_R(X_i)\}};$$

we have $X_j \in \partial\Omega_j$, and we consider $B_R(X_j)$. We note that $\rho(X_j)$ is the largest value of ρ on the part of $B_R(X_j)$ inside the complement of Ω_j. However, $\rho(X_j)$ is less than the maximum value of ρ on $B_R(X_j)$, by the mean value property. Hence there is a point $X_{j+1} \in \partial B_R(X_j) \cap \Omega_j$ at which ρ attains its maximum value in $\overline{B_R(X_j)}$. Again, along the straight line from X_j to X_{j+1} we have $\rho \ge \rho(X_j) - \mu > \rho(D) - \mu$.

Now,

$$\operatorname{dist}(X_{i-1}, \Omega_i) = R$$

and

$$\Omega_j \subset \Omega_{j-1} \subset \cdots \subset \Omega_1,$$

so

$$\operatorname{dist}(X_i, \Omega_k) \geq R$$

for $k \geq i+1$; and since $X_k \in \partial\Omega_k$, the estimate

$$\operatorname{dist}(X_{j+1}, X_i) \geq R \quad \forall \quad i \leq j$$

follows.

Hence $\operatorname{dist}(X_i, X_j) \geq R$ for $i \neq j$ for all points in the sequence. But only a finite number of balls with radius R and separated centers will fit in Ω_ϵ, so this process must terminate after a finite number of steps when we reach a point $X_L = B \in \partial\Omega_\epsilon$. By construction, Γ_D has the properties indicated in (2.42).

Similarly, we construct Γ_C, with termination point $A \in \partial\Omega_\epsilon$.

Next we show that the points A and B lie on Σ_0. First, the curves cannot cross each other, because at every point on Γ_D, $\rho \geq \rho(D) - \mu > \rho(C) + \mu$, while at every point on Γ_C we have $\rho < \rho(C) + \mu$. Also, Γ_D cannot terminate at σ_ϵ where $\rho \leq \rho_0 + \mu < \rho(D)$. For the same reason, B cannot lie on Σ in the segments $[D, \Xi_0]$ or $[C, D]$ where $\rho \leq \rho(D)$. Finally, B cannot lie in the segment $[\Xi_s, C]$ of Σ because this would trap Γ_C in a region where $\rho \geq \rho(C)$ (or, more simply, this would contradict the fact that C is not a local minimum in Ω). Hence $B \in \Sigma_0$.

Similarly, A cannot lie on Σ, where $\rho \geq \rho(C)$ in the interval $[\Xi_s, D]$, and must lie on Σ_0, between B and Ξ_s.

Now we find our final contradiction. Since there is a point, A, in the interval $[\Xi_s, B]$ of Σ_0 where ρ is smaller than its value at either endpoint, then there must be a point m where ρ, restricted to the interval $[\Xi_s, B]$ of Σ_0 attains its minimum. We recall that m cannot be a local minimum in Ω, and so it cannot be a minimum in Ω_1 or in Ω_2. The relevant domain is Ω_1 if $m \in [\Xi_s, A]$; otherwise it is Ω_2. In particular, there would have to be points on the boundary of the relevant domain at which $\rho < \rho(m)$. But the construction we have performed prevents this. To verify this, suppose first that $m \in [\Xi_s, A]$. Then $\rho \geq \rho(m)$ on $[\Xi_s, A]$. In particular, $\rho(m) \leq \rho(A) \leq \rho(X)$ for $X \in \Gamma_C$, and $\rho(m) \leq \rho(A) < \rho(C)$, by (2.41). In addition, $\rho \geq \rho(C)$ on the top boundary, $[\Xi_s, C]$ in Σ, of Ω_1. Thus, we have a contradiction to the maximum principle if $m \in [\Xi_s, A]$.

But if $m \in [A, B]$, then again there are no points on the interval $[A, B]$ of Σ_0 at which $\rho < \rho(m)$, and again $\rho \geq \rho(A) \geq \rho(m)$ along Γ_C. As before, $\rho(C) > \rho(A) \geq \rho(m)$. Now, $\rho \geq \rho(C)$ on the interval $[C, D]$ of the top boundary, Σ, of Ω_2, and by (2.42) we have $\rho \geq \rho(D) - \mu > \rho(C)$ along Γ_D. Thus in this case also, $\rho(m)$ is the smallest value of ρ along the entire boundary of Ω_2. This again contradicts the maximum principle, as stated in the first paragraph of the proof.

We conclude that C and D do not exist, and hence that Z_1 and Z_2 do not exist, and ρ is monotonic on Σ. $\square$

As a second basic result, we prove that the shock evolution equation can always be integrated, defining the mapping whose fixed point is the free boundary. Beginning with a given curve $\eta(\xi)$, assume we have solved the fixed boundary value problem (2.36), (2.37), (2.39), and (2.40). We then produce a new approximate shock position $\tilde{\eta}(\xi)$ by integrating (2.38):

$$\tilde{\eta}(\xi) = \eta_0 + \int_{\xi_0}^{\xi} f(x, \eta(x), \rho(x, \eta(x)))\, dx, \tag{2.43}$$

where f is defined in (2.38). Note that on the right side of (2.43) we evaluate all quantities along the old shock position, $\eta(\xi)$. We have the following proposition.

PROPOSITION 2.5. *Suppose that η is a monotone function and that ρ satisfies the boundary value problem* (2.36), (2.37), (2.39), *and* (2.40). *Then $\eta^2 > s^2$ and $\eta^2 + \xi^2 > s^2$ for all $\xi \in (0, \xi_0)$ so the new curve $\tilde{\eta}$ is defined for all $\xi \in [0, \xi_0]$ and is monotonic. Furthermore, $\tilde{\eta}'(0) = 0$.*

Proof. Because ρ satisfies (2.40), we see that at $\xi = 0$ the quantity under the square root sign in (2.38) is zero. Since η is monotonic, the quantity $\eta^2(\xi)$ is an increasing function of ξ. We use Proposition 2.4 to conclude that s^2 along Σ is a decreasing function of ξ (since ρ decreases and s is a monotonic function of ρ). Hence $\eta^2 - s^2$ is strictly positive when $\xi > 0$. In addition, this implies that $\xi^2 + \eta^2 - s^2$ is positive, and so the right-hand side of (2.43) is well defined (see the equivalent form in (2.23)). In addition, (2.23) also shows that $d\tilde{\eta}/d\xi$ is positive as long as $\eta^2 - s^2 > 0$. Finally, this derivative is zero at $\xi = 0$, where the right side of (2.38) vanishes. ∎

We now define $\mathcal{K} = \mathcal{K}^{\varepsilon}$, a closed, convex subset of a Hölder space $H_{1+\alpha_1}([0, \xi_0])$; the value of $\alpha_1 \in (0, 1)$ depends on the regularizing parameter ε and will be specified later. The functions in $\mathcal{K}$ satisfy

(K1) $\eta(\xi_0) = \eta_0$, and $\eta'(\xi_0) = 1/\kappa_a$, where ξ_0 and η_0 are defined in (2.26);
(K2) $\eta'(0) = 0$;
(K3) $\eta_c \le \eta(\xi) \le \eta_0$; recall that $\eta_c = \sqrt{1 + \kappa_a^2} s_0/\kappa_a < \eta_0 < c_0$ if $\kappa_a > \kappa_A$;
(K4) $0 \le \eta' \le \sqrt{c_0^2/s_0^2 - 1}$.

Then (2.43) defines a mapping on $\mathcal{K}$:

$$J : \eta \mapsto \tilde{\eta}. \tag{2.44}$$

The upper bound in **(K4)** is justified by the following proposition.

PROPOSITION 2.6. *If $\eta(\xi)$ is a monotonic function with $\eta(\xi_0) = \eta_0$ and ρ a solution to* (2.36), (2.37), (2.39), *and* (2.40), *then the function f given by* (2.38) *is bounded above by $\sqrt{c_0^2 - s_0^2}/s_0 \equiv 1/\kappa_A$.*

Proof. By Proposition 2.4, $s(\xi, \eta)$ is a decreasing function on $\eta(\xi)$ with $s^2(0, \eta(0)) = \eta^2(0)$, and by Proposition 2.5, $\eta \ge s$ on $\eta(\xi)$. For the function f defined by (2.38), a calculation shows

$$\begin{aligned}
\frac{\partial f}{\partial \xi} &= -\frac{(\eta^2 - s^2)\big(s\xi + \eta\sqrt{\xi^2 + \eta^2 - s^2}\big)}{\sqrt{\xi^2 + \eta^2 - s^2}\big(\xi\eta + \sqrt{s^2(\xi^2 + \eta^2 - s^2)}\big)^2} < 0,\\
\frac{\partial f}{\partial \eta} &= \frac{\xi^2 + \eta^2}{\sqrt{\xi^2 + \eta^2 - s^2}\big(s\eta + \xi\sqrt{\xi^2 + \eta^2 - s^2}\big)} > 0,\\
\frac{\partial f}{\partial s^2} &= -\frac{\frac{1}{2}\eta^2(\xi^2 + \eta^2 - s^2) + \frac{1}{2}s^2\xi^2 + \xi\eta s\sqrt{\xi^2 + \eta^2 - s^2}}{\big(\xi\eta + \sqrt{s^2(\xi^2 + \eta^2 - s^2)}\big)^2} < 0,
\end{aligned}$$

Hence, f is largest when η has its maximum value η_0, and ξ and s their minimum values, 0 and s_0, respectively. This gives the stated upper bound, which is the reciprocal of the limiting value κ_A, as calculated in [17]. ∎

We also note the upper bound for the solution ρ of (2.36), (2.37), (2.39), and (2.40) when $\eta \in \mathcal{K}$. Since $\eta_s \leq \eta_0$ and s^2 is monotonic, for given Riemann data $(\rho_0, \rho_1, \kappa_a)$, the value of $\rho_{\max}$ in Proposition 2.4 is bounded above by ρ_M, where, from (2.40),

$$\rho_M = s_{\rho_1}^{-1}(\eta_0). \tag{2.45}$$

We will use this upper bound in the proofs.

We prove Theorem 2.3 in two stages. First, in section 3 we solve the regularized free boundary value problem for $Q^\varepsilon = Q + \varepsilon\Delta$. In section 4, we consider the limit $\varepsilon \to 0$ and show that this limit yields a solution of (2.36)–(2.40).

3. The regularized problem. For a fixed $\varepsilon \in (0, 1)$ we solve the free boundary problem defined at the beginning of section 2.4, but with Q replaced by the regularized operator Q^ε. The equation for ρ in the subsonic region is now

$$Q^\varepsilon \rho = Q\rho + \varepsilon\Delta\rho = 0; \tag{3.1}$$

the shock evolution equation remains the same,

$$\eta' = f(\xi, \eta, \rho), \quad \eta(\xi_0) = \eta_0; \tag{3.2}$$

and the boundary conditions are, as before,

$$M\rho = \beta \cdot \nabla\rho = 0 \quad \text{on} \quad \Sigma \equiv \{(\xi, \eta(\xi)); 0 < \xi < \xi_0\}, \tag{3.3}$$

$$\rho = \rho_0 \quad \text{on} \quad \sigma; \quad \rho_\xi = 0 \quad \text{on} \quad \Sigma_0, \tag{3.4}$$

and

$$\rho(\Xi_s) = \rho_s \equiv s_{\rho_1}^{-1}(\eta_s). \tag{3.5}$$

The theorem we prove in this section is as follows. (See (3.7) for the spaces.)

THEOREM 3.1. *For each $\varepsilon \in (0, 1)$, there exists a solution $(\rho^\varepsilon, \eta^\varepsilon) \in H_{1+\alpha}^{(-\gamma)}(\Omega^\varepsilon) \times H_{1+\alpha}([0, \xi_0])$ to the regularized free boundary problem* (3.1), (3.2), (3.3), (3.4), *and* (3.5) *such that*

$$\rho_0 < \rho^\varepsilon \leq \rho_s \leq \rho_M \quad \text{and} \quad c^2(\rho^\varepsilon) > \xi^2 + \eta^2 \quad \text{in} \quad \overline{\Omega^\varepsilon} \setminus \sigma. \tag{3.6}$$

Here, $\alpha, \gamma \in (0, 1)$ both depend on ε and on the Riemann data κ_a, ρ_0, and ρ_1. The curve $\eta^\varepsilon(\xi)$, defining the position of the free boundary Σ^ε, is in $\mathcal{K}^\varepsilon$; Ω^ε is bounded by σ, Σ_0, and Σ^ε.

We prove Theorem 3.1 in the following steps (which take up the three subsections of this section).

Step 1. First we show the existence of a solution to a linear problem with fixed boundary Σ defined by $\eta(\xi) \in \mathcal{K}$ and establish Hölder and Schauder estimates at Σ. For this, it is convenient to define a weighted Hölder space; see [15] for the general definition of weighted Hölder spaces. Let $V = \{\Xi_0\}$ denote the corner point at which Σ meets the degenerate boundary σ. Set $\Omega' = \Omega \cup \sigma \cup \Sigma_0 \setminus V$. We anticipate loss of regularity at V, because of the mixed boundary condition and the degeneracy of the operator Q at σ. At Ξ_s, we also find loss of regularity because of loss of obliqueness of the operator M. The third corner, between Σ_0 and σ, is an artifact of our decision to work in a half-domain. Since it does not contribute to any loss of regularity, we ignore it in the discussion. We define the corner region near Ξ_0:

$$\Omega_V(\delta) = \{x \in \Omega : \text{dist}(V, x) \leq \delta\}.$$

In [3, 4, 6], in which the derivative condition was uniformly oblique, the only loss of regularity came from the corners. In the present problem, we overcome the loss of obliqueness at a single point on Σ, but at a cost: the Schauder estimates are no longer independent of the gradients of the coefficients, and hence we do not get a compact mapping in the same spaces. In this paper, we therefore modify the weighted Hölder spaces, as follows. We define a region which is close to Σ but does not contain the corner Ξ_0 by taking a covering of Σ with balls of radius δ centered at points on Σ which are bounded away from Ξ_0. Define $\Sigma''(\delta) = \{\Xi \in \Sigma \mid \text{dist}(\Xi, \Xi_0) > \delta\}$ and

$$\Sigma(\delta) = \left\{ x \in \Omega \cap \bigcup_{\Xi \in \Sigma''(\delta)} B_\delta(\Xi) \right\},$$

where $B_\delta(\Xi)$ is a ball of radius δ centered at Ξ. We then define

$$H_a^{(b)} \equiv \left\{ \|u\|_a^{(b)} \equiv \sup_{\delta > 0} \delta^{a+b} |u|_{a, \overline{\Omega} \setminus \{\Sigma(\delta) \cup \Omega_V(\delta)\}} < \infty \right\}. \tag{3.7}$$

For the linear problem, we establish a priori Schauder and Hölder bounds at Σ, in particular near the point where the data lose obliqueness; we use Hölder estimates near V, and $C_{2+\alpha}$ estimates locally in the rest of the domain. We prove existence of a solution by regularizing the oblique boundary condition to be uniformly oblique, then passing to the limit using the a priori bounds.

Step 2. Using the Hölder gradient bounds to the linear problem, we establish existence results for the nonlinear fixed boundary problem, via the Schauder fixed point theorem.

Step 3. We apply the Schauder fixed point theorem again to prove existence of a solution to the nonlinear free boundary problem.

3.1. The regularized linear fixed boundary problem. Replace ρ in the coefficients a_{ij} of (2.36) and β_i of (2.33), (2.34) by a function w in a set $\mathcal{W}$ defined with respect to a given boundary component Σ, and depending on given values Ξ_s and ρ_s (see (3.5)), as follows.

DEFINITION 3.2. *The elements of* $\mathcal{W} \subset H_2^{(-\gamma_1)}$ *satisfy*

(W1) $\rho_0 \le w \le \rho_M$, $w = \rho_0$ *on* σ, $w(\Xi_s) = \rho_s$, $w_\xi = 0$ *on* Σ_0*;*

(W2) $\|w\|_2^{(-\gamma_1)} \le K$*;*

(W3) $|w|_{\alpha_0, \Omega'_{loc}} \le K_0$.

The weighted Sobolev space is defined by (3.7); the values of $\gamma_1, \alpha_0 \in (0, 1)$ will be specified following (3.29), as will the values of K and K_0. The set $\mathcal{W}$ is clearly closed, bounded, and convex.

The quasilinear equations (3.1) and (3.3) are now replaced by linear partial differential and boundary equations (repeated indices are summed)

$$\begin{aligned} L^\varepsilon u &= D_i(a_{ij}(\Xi, w) D_j u) + \varepsilon \Delta u + b_i(\Xi) D_i u = 0 \quad \text{in} \quad \Omega, \\ N u &= \beta_i D_i u = \beta_i(\Xi, w) D_i u = 0 \quad \text{on} \quad \Sigma = \{\eta = \eta(\xi)\}, \end{aligned} \tag{3.8}$$

with a given $\eta \in \mathcal{K} \subset H_{1+\alpha_1}$ and $w \in \mathcal{W}$. Because of the bound (W1), L^ε is uniformly elliptic in Ω with ellipticity ratio depending on the Riemann data and on ε. In this section, we demonstrate the key point that for a given function $w \in \mathcal{W}$, the solution u to the linear equations (3.8) with the remaining boundary conditions

$$u = \rho_0 \text{ on } \sigma, \; u_\xi = 0 \text{ on } \Sigma_0 \text{ and } u(\Xi_s) = \rho_s, \tag{3.9}$$

satisfies Hölder and Schauder estimates in Ω' and a uniform $H_{1+p,\Sigma(d_0)}$ bound near Σ for any $p < \min\{\gamma_1, \alpha_1\}$. This bound gives rise to enough compactness to establish the existence of a solution to the quasilinear problem by applying the Schauder fixed point theorem.

We first note L^∞ a priori bounds for the solution u to the linear problem.

PROPOSITION 3.3. *The solution u to the linear problem* (3.8), (3.9) *satisfies*

$$\rho_0 < u \le \rho_s \le \rho_M \quad in \quad \Omega \cup \Sigma \cup \Sigma_0, \tag{3.10}$$

where $\rho_s = \rho(0, \eta_s)$ is defined in (3.5) *and ρ_M, defined in* (2.45), *is independent of ε. Moreover,*

$$c^2(u) > c^2(\rho_0) > \xi^2 + \eta^2 \quad in \quad \Omega \cup \Sigma \cup \Sigma_0. \tag{3.11}$$

Proof. The linear problem is uniformly elliptic for $\varepsilon > 0$ and $w \in \mathcal{W}$, so the classical maximum principle applies, as well as the boundary considerations used in the proof of Proposition 2.4. ◻

Next, we state the Schauder estimates including the Dirichlet and fixed Neumann boundaries, σ and Σ_0, and the Hölder estimates at the corner, Ξ_0.

THEOREM 3.4. *Assume that Σ is given by $\{(\xi, \eta(\xi))\}$ with $\eta \in \mathcal{K}$ for some $\alpha_1 \in (0,1)$ and that w is in $\mathcal{W}$ for given K, K_0, α_0, and γ_1. Then there exist $\gamma_V, \alpha_\Omega \in (0,1)$ such that any solution $u \in H_{2+\alpha_\Omega,\Omega'} \cap H_{\gamma_V,\Omega_V(d_0)}$ to the linear problem* (3.8), (3.9) *satisfies*

$$|u|_{\gamma,\Omega_V(d_0)} \le C_1 |u|_0 \tag{3.12}$$

for any $\gamma \le \gamma_V$ and

$$|u|_{2+\alpha,\Omega'_{loc}} \le C_2 |u|_0 \tag{3.13}$$

for any $\alpha \le \alpha_\Omega$. The exponent γ_V depends on the Riemann data, and both α_Ω and γ_V depend on ε but are independent of α_1 and γ_1. The constant C_1 is independent of the bounds K and K_0. The constant C_2 is independent of K but depends on K_0.

Proof. The proof is immediate. We refer to Theorem 1 of Lieberman [22] for the corner estimate. Here γ_V depends on the angle between Σ and σ at V, a fixed value that depends only on the Riemann data, and on the obliqueness ratio at V, which is also fixed, as well as on the ellipticity ratio ε, but not on γ_1, α_1, K, or K_0.

Standard interior and boundary Schauder estimates, for example, [15, p. 98], give the local estimate (3.13). The constant C_2 depends on ε, on the H_α norm of the coefficients a_{ij}, and on the domain. ◻

Because interior Schauder estimates can be applied once more, a solution in $H_{2+\alpha,\Omega'}$ is actually in $C^3(\Omega)$.

Finally, we establish Hölder gradient estimates at Σ. It is at this point that we need to derive basic estimates at the point Ξ_s where the boundary operator N is not oblique. To avoid discussing the Neumann boundary separately at each step of this proof, we reflect Ω across the ξ axis, without further comment; Ω includes Σ_0 and we let Σ stand for the full $H_{1+\alpha_1}$ boundary in Theorem 3.5. The remaining assumptions are the same as in Theorem 3.4.

THEOREM 3.5. *Assume that Σ is given by $\{(\xi, \eta(\xi))\}$ with $\eta \in \mathcal{K}$ for some $\alpha_1 \in (0,1)$ and that w is in $\mathcal{W}$ for given K, K_0, α_0, and γ_1. Then, there exists a* *positive constant d_0 such that for every $d \le d_0$, any solution $u \in C^1(\Omega \cup \Sigma) \cup C^3(\Omega)$ to the linear problem* (3.8), (3.9) *satisfies*

$$|u|_{1+p,\Sigma(d)} \le C(\varepsilon, \alpha_1, \gamma_1, K, d_0)|u|_0 \tag{3.14}$$

for any $p < \min\{\gamma_1, \alpha_1\}$.

Proof. Away from a neighborhood $B_{d_0}(\Xi_s)$ of Ξ_s the boundary operator N in (3.8) is oblique and thus we can apply known regularity theory, for example, [15, Theorem 6.30], to get (3.14) in $\Sigma(d_0) \setminus B_{d_0}(\Xi_s)$, with a constant C which depends on ε, α_1, Ω, d_0, and K_0. Hence we consider only estimates near Ξ_s in the remainder of the proof.

For a given solution u to (3.8) and (3.9) we define

$$v = \frac{u}{1 + |Du|_0} \quad \text{and} \quad z = Nv = \beta_i(\Xi)D_i v. \tag{3.15}$$

We construct a barrier function f for z on $B \equiv B_{d_0}(\Xi_s) \cap \overline{\Omega}$ to get a Hölder estimate for the gradient of the solution of (3.8), (3.9). Let $\psi = z + f(\zeta)$, where ζ is the regularized distance function (from the boundary component Σ); see [18]. A smooth approximation to $d(\Xi) = \operatorname{dist}(\Sigma, \Xi)$ is necessary since Σ has minimal regularity. The regularized distance function has the properties $1 \le \zeta/d \le 2$, $0 < \zeta_0 \le |D\zeta| \le \zeta_D$ and $|D^2\zeta| \le \zeta_D d^{\alpha_1 - 1}$. We let $f(0) = 0$ and we first construct the lower barrier, $-f$, by finding a suitable positive, increasing function f such that $\psi > 0$. Note that, with f positive, we get $\psi \ge z$ on ∂B. Where no confusion is likely, we let subscripts denote partial derivatives and calculate

$$D_i\psi = \beta_j D_{ij} v + D_i\beta_j D_j v + f'\zeta_i, \tag{3.16}$$

whence

$$\beta_j D_{ij} v = D_i\psi - (D_i\beta_j D_j v + f'\zeta_i). \tag{3.17}$$

We also have

$$D_{ij}\psi = \beta_k D_{ijk} v + D_j\beta_k D_{ik} v + D_i\beta_k D_{jk} v + D_{ij}\beta_k D_k v + f'\zeta_{ij} + f''\zeta_i\zeta_j. \tag{3.18}$$

In addition, since w satisfies (W2) with a given constant K, we get estimates on the derivatives of a_{ij}. Using the definition of the weighted norms, we have (noting $|Dw| \le |w|_1$ and so on)

$$\begin{aligned} |D(a_{ij})| &\le |a_{ij,x}| + |a_{ij,u}||Dw| \le |a_{ij,x}| + |a_{ij,u}|\|w\|_1^{(-\gamma_1)} d^{\gamma_1 - 1} \le m d^{\gamma_1 - 1}, \\ |D^2(a_{ij})| &\le |a_{ij,x,x}| + 2|a_{ij,x,u}||Dw| + |a_{ij,u,u}||Dw|^2 + |a_{ij,u}||D^2 w| \\ &\le |a_{ij,x,x}| + 2|a_{ij,x,u}|\|w\|_1^{(-\gamma_1)} d^{\gamma_1 - 1} + |a_{ij,u,u}|(\|w\|_1^{(-\gamma_1)} d^{\gamma_1 - 1})^2 \\ &\quad + |a_{ij,u}|\|w\|_2^{(-\gamma_1)} d^{\gamma_1 - 2} \\ &\le m(d^{\gamma_1 - 2} + d^{2\gamma_1 - 2}). \end{aligned} \tag{3.19}$$

Here subscripts denote derivatives of a_{ij} with respect to the variables in Ξ $(, x)$ and with respect to w $(, u)$. The symbol $m = m(K)$ denotes a quantity which depends on the structure of the derivatives of a_{ij} and the bound K on w. We absorb terms which are less singular as $d \to 0$. We also get estimates on the derivatives of β_i. Let $\gamma_2 = \min\{\gamma_1, \alpha_1\}$. Then

$$|D\beta_i| \le md^{\gamma_2 - 1}, \qquad |D^2\beta_i| \le m(d^{\gamma_2-2} + d^{2\gamma_2-2}), \tag{3.20}$$

where $m = m(K) > 0$ depends on the structure of the derivatives of β. In deriving this estimate, we use the fact that η', η'', and η''' are bounded by d^{α_1}, d^{α_1-1}, and d^{α_1-2}, respectively, as we can apply Lemma 2.8 of [14] to $\eta(\xi) - \eta$.

Since $\beta_2(\Xi_s, w) = 0$ and $\beta_1(\Xi_s, w) \neq 0$, we can take $0 < d_1 \le 1$ small enough so that for all $0 < d_0 \le d_1$ and all $w \in \mathcal{W}$ we have $\beta_1(\Xi, w) \neq 0$ in B_{d_0}. Now we solve the two equations in (3.17) along with $Lv = 0$, that is,

$$a_{ij}D_{ij}v = -(D_j a_{ij} D_i v + b_i D_i v), \tag{3.21}$$

as a linear system for the three derivatives $D_{ij}v$. The assumption that β_1 is bounded away from zero, coupled with the ellipticity of L, gives a uniform bound $c_1(\Lambda, \lambda, |\beta|_0)$ on the inverse of the coefficient matrix of the linear system. Here we may let Λ and λ be the eigenvalues of (a_{ij}) restricted to B. These are order one constants which depend only on the Riemann data. Furthermore, we can estimate the right-hand sides of (3.17) and (3.21) using (3.19) and (3.20). We get

$$|D^2v| \le c_1(\Lambda, \lambda, |\beta|_0)\big(|D\psi| + (md^{\gamma_2-1} + |b|_0)|Dv| + f'\zeta_D\big). \tag{3.22}$$

This bounds the second derivatives of v in terms of $|D\psi|$. Now we proceed to obtain bounds for ψ. The idea is to find an elliptic operator for which ψ is a subsolution in B and simultaneously to force $\psi > 0$ on ∂B, by choice of the function f. A second-order operator for ψ involves third derivatives of v, so we estimate these. By using $Lv = 0$, (3.22), (3.19), and (3.20) (recall that $|Dv| \le 1$), we get

$$\begin{aligned} a_{ij}D_{ijk}v &= -\big(D_k a_{ij} D_{ij} v + D_j a_{ij} D_{ik} v + b_i D_{ik} v + D_{jk} a_{ij} D_i v + D_k b_i D_i v\big) \\ &\le (md^{\gamma_2-1} + |b|_0)|D^2v| + (md^{\gamma_2-2} + md^{2\gamma_2-2} + |b|_1)|Dv| \\ &\le c_1(md^{\gamma_2-1} + |b|_0)|D\psi| + c_1(md^{\gamma_2-1} + |b|_0)^2 \\ &\quad + c_1(md^{\gamma_2-1} + |b|_0)f'\zeta_D + md^{\gamma_2-2} + md^{2\gamma_2-2} + |b|_1 \\ &\le c_2\big\{(md^{\gamma_2-1} + 1)|D\psi| + (md^{\gamma_2-1} + 1)^2 \\ &\quad + (md^{\gamma_2-1} + 1)f' + md^{\gamma_2-2} + md^{2\gamma_2-2} + 1\big\}, \end{aligned}$$

where $c_2 = c_2(\Lambda, \lambda, \rho_M, |\beta|_0, |b|_0, |b|_1, \zeta_D)$. Thus, using (3.18) and making the estimates indicated, we have

$$\begin{aligned} a_{ij}D_{ij}\psi &\le c_2|\beta|_0\big\{(md^{\gamma_2-1} + 1)|D\psi| + (md^{\gamma_2-1} + 1)^2 + (md^{\gamma_2-1} + 1)f' \\ &\qquad + md^{\gamma_2-2} + md^{2\gamma_2-2} + 1\big\} \\ &\qquad + 2\Lambda md^{\gamma_2-1}c_1\big\{|D\psi| + md^{\gamma_2-1} + |b|_0 + f'\zeta_D\big\} \\ &\qquad + \Lambda m(d^{\gamma_2-2} + d^{2\gamma_2-2}) + \Lambda f'|\zeta_{ij}| + f'' a_{ij}\zeta_i\zeta_j \\ &\le c_3\big\{(md^{\gamma_2-1} + 1)|D\psi| + md^{\gamma_2-2} + (m^2 + m)d^{2\gamma_2-2} \\ &\qquad + md^{\gamma_2-1}(1 + f')\big\} + \Lambda f'|\zeta_{ij}| + f'' a_{ij}\zeta_i\zeta_j. \end{aligned}$$

Here c_3 is a constant depending on the same parameters as c_1 and c_2, and terms which are bounded as $d \to 0$ have again been omitted. Now we define

$$L_1\psi \equiv a_{ij}D_{ij}\psi - c_3(md^{\gamma_2-1} + 1)|D\psi|$$

and we calculate

(3.23)
$$L_1\psi \le c_3\big\{md^{\gamma_2-2} + (m^2+m)d^{2\gamma_2-2} + md^{\gamma_2-1}(1+f')\big\} + \Lambda\zeta_D f' d^{\alpha_1-1} + \lambda f''\zeta_0^2.$$

To obtain this estimate, we have assumed $f'' < 0$ and estimated

$$f'' a_{ij}\zeta_i\zeta_j \le f'' \min a_{ij}\zeta_i\zeta_j = f''\lambda|D\zeta|^2 \le f''\lambda\zeta_0^2.$$

We have also used the property of regularized distance: $|\zeta_{ij}| \le \zeta_D d^{\alpha_1-1}$. We now specify $f(\zeta) = f_0\zeta^p$ for any $p < \gamma_2$, so that

$$f'' = f_0 p(p-1)\zeta^{p-2} \le f_0 p(p-1)d^{p-2} < 0,$$

and $f' d^{\alpha_1-1} \le 2^{p-1} f_0 p d^{p+\alpha_1-2}$. Finally, we choose f_0 big enough and $d_2 \in (0,1)$ small enough to get $L_1\psi < 0$ in B_{d_0} for every $d_0 \le d_2$. We now define $d_0 \equiv \min\{d_1, d_2\}$.

Additionally, since (3.14) holds near Σ, away from Ξ_s, and hence is valid on ∂B, we can choose f_0 larger if necessary so that $\psi > 0$ on ∂B. Therefore, by the maximum principle, $\psi > 0$ in B. Thus, $z > -f$ in B.

Similarly, f is an upper barrier for z. We now have an estimate for z. In addition we have, since $\psi = z + f$,

$$|\psi| \le c_4(m^2+1)d^p \quad \text{for} \quad d \le d_0.$$

Since $\psi = 0$ on Σ, we can use Schauder estimates, applying [15, Lemma 6.20] or [14, Lemma 7.1, Theorem 7.2], using the fact that ψ and $-\psi$ are upper and lower solutions of an operator L_1 with a Dirichlet boundary condition and estimating the right side of (3.23), to obtain

$$\|\psi\|^{(-p)}_{2+\gamma_2} \le C_1\big(\sup d^{-p}|\psi| + |\psi|_0 + |\psi|_{p,\partial B}\big) \le c_4(m^2+1) + c(m) = C(m).$$

The constant C_1 depends only on λ and Λ (the ellipticity constants in B) and on γ_2. To obtain the second inequality in this expression, we have used the fact that $|\psi|_{p,\partial B}$ is bounded, with a bound which depends only on $|\psi|_0$ and on Λ/λ. This follows from $\psi = 0$ on Σ and from interior Schauder estimates for v, a solution to a linear problem, on $\partial B \cap \Omega$. Finally, this leads to

$$|D\psi| \le \|D\psi\|^{(1-p)}_{\gamma_2+1} d^{p-1} \le C(m)d^{p-1} \quad \text{for} \quad d < d_0. \tag{3.24}$$

We now use (3.24) in (3.22) and drop lower-order terms to get

$$|D^2 v| \le c_1(|D\psi| + md^{\gamma_2-1} + f') \le c_1(C(m)d^{p-1} + md^{\gamma_2-1} + f_0 p d^{p-1}) \le Cd^{p-1}.$$

Now Hölder estimates on Dv follow by integrating the last inequality. More precisely, $|D^2v| \le Cd^{p-1}$ implies that $\|Dv\|_1^{(-p)} \le C$, and by [14, Lemma 2.1] we have

$$|Dv|_p = \|Dv\|_p^{(-p)} \le C(p)\|Dv\|_1^{(-p)},$$

and therefore we get

$$|v|_{1+p} \le C. \tag{3.25}$$

Finally, using the definition of v in (3.15), we apply the interpolation inequality, [15, Lemma 6.32], with a small $\delta > 0$ to get

$$|u|_{1+p} \leq C(1 + |Du|_0) \leq C(1 + \delta|u|_{1+p} + C_\delta|u|_0) \tag{3.26}$$

and thus (3.14) holds. Therefore we get Hölder gradient estimates at Σ for the solution u of (3.8). $\square$

Now we can establish existence of a solution to (3.8) and (3.9).

THEOREM 3.6. *Assume that* Σ *is given by* $\{(\xi, \eta(\xi))\}$ *with* $\eta \in \mathcal{K}$ *for some* $\alpha_1 \in (0,1)$ *and that* w *is in* $\mathcal{W}$ *for given* K, K_0, α_0, *and* γ_1. *Then there exist* $\gamma_V, \alpha_\Omega \in (0,1)$, *and* $d_0 > 0$, *where* γ_V, α_Ω, *and* d_0 *are independent of* γ_1 *and* α_1, *such that a solution* $u \in H_{1+p,\Sigma(d)} \cap H_{2+\alpha,\Omega'} \cap H_{\gamma,\Omega_V(d_0)}$ *for the linear problem* (3.8) *and* (3.9) *exists for any* $\alpha \leq \alpha_\Omega$, $p < \min\{\gamma_1, \alpha_1\}$, $\gamma \leq \gamma_V$, *and* $d \leq d_0$ *and satisfies* (3.12), (3.13), *and* (3.14).

Proof. To show the existence of a solution u to (3.8) and (3.9), we approximate the oblique derivative boundary condition on Σ. To be precise, noting that the unit inward normal to Σ at Ξ_s is $(0, -1)$, for $0 < \delta < 1$ we let $\beta_\delta = \beta + (0, -\delta)$ so that $\beta_\delta \cdot \nu = \beta \cdot \nu + \delta \geq \delta > 0$ at Ξ_s. Then, for sufficiently small δ, β_δ is uniformly oblique. The boundary condition is now discontinuous at the corner Ξ_s, where Σ and Σ_0 meet. Results from [21] and [19] imply that there exists a solution u^δ to $Lu^\delta = 0$ in Ω, $\beta_\delta \cdot \nabla u^\delta = 0$ on Σ, and (3.9). Now we apply Theorems 3.4 and 3.5, which are independent of δ, to see that the sequence u^δ is uniformly bounded in $H_{1+p,\Sigma(d_0)} \cap H_{2+\alpha_\Omega,\Omega'} \cap H_{\gamma_V,\Omega_V(d_0)}$ for any $p < \min\{\gamma_1, \alpha_1\}$. Thus by the Arzela–Ascoli theorem, there exists a subsequence converging uniformly to a function u. Using the uniform bounds (3.12), (3.13), and (3.14), we conclude that the limiting function solves the problem (3.8), (3.9). $\square$

3.2. The regularized nonlinear fixed boundary problem. This subsection is devoted to proving the existence of solutions to the nonlinear problem (3.1) with a fixed boundary. We again assume that an approximate shock boundary Σ is given by a function $\eta = \eta(\xi) \in \mathcal{K}$. We also are given the value $\rho_s = s_{\rho_1}^{-1}(\eta(0))$. We prove the following theorem.

THEOREM 3.7. *For each* $\varepsilon \in (0,1)$, *and for given* $\eta \in \mathcal{K} \subset H_{1+\alpha_1}$, *there exists a solution* $\rho^\varepsilon \in H_{2+\alpha}^{(-\gamma)}(\Omega^\varepsilon)$ *to* (3.1), (3.3), (3.4), *and* (3.5) *such that*

$$\rho_0 < \rho^\varepsilon \leq \rho_s \leq \rho_M, \quad \text{and} \quad c^2(\rho^\varepsilon) > \xi^2 + \eta^2 \quad \text{in} \quad \overline{\Omega}^\varepsilon \setminus \sigma \tag{3.27}$$

for some $\alpha(\varepsilon), \gamma(\varepsilon) \in (0,1)$. *Moreover, for some* $d_0 > 0$ *the solution* ρ^ε *satisfies*

$$|\rho^\varepsilon|_{\gamma,\Sigma(d_0)\cup\Omega_V(d_0)} \leq K_1, \tag{3.28}$$

where γ *and* K_1 *depend on* ε, γ_V *and* K *but both are independent of* α_1.

Proof. We suppress the dependence on ε to simplify the notation.

Recall that $\mathcal{K} \subset H_{1+\alpha_1}([0, \xi_0])$ is a closed convex set of functions satisfying the additional conditions (K1) to (K4) given in section 2.4. For any function w in $\mathcal{W}$ we define a mapping $T : \mathcal{W} \subset H_2^{(-\gamma_1)} \to H_2^{(-\gamma_1)}$ by letting $\rho = Tw$ be the solution to the linear regularized fixed boundary problem, (3.8), (3.9) solved in Theorem 3.6. Because w satisfies (W1), L^ε is strictly elliptic, with ellipticity ratio depending on ε. By Theorem 3.6, T maps $\mathcal{W} \subset H_2^{(-\gamma_1)}$ to a bounded set in $H_{2+\alpha}^{(-\gamma_V)}$, where γ_V is the value given by Theorem 3.6. Since γ_V is independent of γ_1, we may take $\gamma_1 = \gamma_V/2$ and then $T(\mathcal{W})$ is precompact in $H_2^{(-\gamma_1)}$.

To show T maps $\mathcal{W}$ into itself, we need to show that Tw satisfies (W1), (W2), and (W3). Now, (W1) is immediate by Proposition 3.3 and the boundary conditions.

By applying interior and boundary Hölder estimates (see [15, Theorems 8.22 and 8.27]), we get the local estimate

$$|\rho|_{\alpha*,\Omega_1'} \leq C_0, \tag{3.29}$$

where $0 < \alpha* < 1$ and C_0 depend only on ε (the ellipticity ratio), the Riemann data, and on $d' = \operatorname{dist}(\Omega_1', \partial\Omega')$ with $\Omega_1' \subset \Omega'$. Notice that, as in the remark following Theorem 8.24 in [15, p. 202], the constant C_0 is nondecreasing and the constant $\alpha*$ nonincreasing with respect to d'. Since $\Omega' \subset \Omega$ is bounded, we can find an upper bound for C_0 and a lower bound for $\alpha*$ depending only on the size of Ω and the ellipticity ratio. Thus, if we define $\mathcal{W}$ with $K_0 = C_0$ and $\alpha_0 = \alpha*$, with C_0 the upper bound and $\alpha*$ the lower bound, then $\rho = Tw$ satisfies (W3). Note that K_0 and $\alpha*$ are independent of α_1 and γ_V.

To verify (W2), we need to find a value K such that

$$\sup_{\delta>0} \delta^{2-\gamma_1} |\rho|_{2,\overline{\Omega}\setminus\{\Sigma(\delta)\cup\Omega_V(\delta)\}} < K, \tag{3.30}$$

assuming $\|w\|_2^{-\gamma_1} \leq K$. We start by noting that Theorem 3.5 implies the existence of a positive constant $d_0 > 0$ such that for every $d \leq d_0$, any solution $u \in C^1(\Omega\cup\Sigma)\cup C^3(\Omega)$ to the linear problem (3.8), (3.9) satisfies the Hölder gradient estimate (3.14), where the constant C depends on K but is uniform in $d \leq d_0$. Based on this estimate, we get a local bound for the weighted norm of ρ on $\Sigma(d_0)$ of the form

$$d^{2-\gamma_1}|\rho|_2 \leq d^{1-\gamma_1+p}C \tag{3.31}$$

which holds for all $d < d_0$. Here C depends on K, α_1, and γ_1. To show (3.30) we estimate the supremum by considering separately domains $\overline{\Omega} \setminus \{\Sigma(\delta) \cup \Omega_V(\delta)\}$ for which $\delta > \tilde{d}$, where $\tilde{d} \leq d_0$ will be specified later, and domains for which $\delta \leq \tilde{d}$.

In domains of the first kind, $\overline{\Omega}\setminus\{\Sigma(\delta)\cup\Omega_V(\delta)\}$ with $\delta > \tilde{d}$, the solution is smooth and its C^2-norm bound is independent of K. More precisely, we can use the uniform Hölder estimate (3.29) and bootstrap iteratively (see [15, Theorem 6.6]) to get the local Schauder estimate

$$|\rho|_{2+\alpha_\Omega,\Omega'} \leq C(K_0). \tag{3.32}$$

Notice that since the Hölder estimate (3.29) is independent of the distance between Ω_1' and the boundary Σ, so is the Schauder estimate (3.32). The interpolation inequality [15, Lemma 6.32] gives

$$|\rho|_{2,\Omega'} \leq c|\rho|_0 + \mu|\rho|_{2+\alpha,\Omega'} \leq c\rho_M + \mu C(K_0) \tag{3.33}$$

for any $\mu > 0$ and $c = c(\mu)$. We fix $\mu = 1$ and get

$$\sup_{\delta>\tilde{d}} \delta^{2-\gamma_1} |\rho|_{2,\overline{\Omega}\setminus\{\Sigma(\delta)\cup\Omega_V(\delta)\}} \leq K', \tag{3.34}$$

where K' depends on the size of the domain Ω, on $C(K_0)$, and on ρ_M but is independent of the distance to Σ.

Next we study $\delta^{2-\gamma_1}|\rho|_{2,\overline{\Omega}\setminus\{\Sigma(\delta)\cup\Omega_V(\delta)\}}$ when $\delta \leq \tilde{d}$. We divide the subdomain $\overline{\Omega} \setminus \{\Sigma(\delta) \cup \Omega_V(\delta)\}$ into two: the part for which $\delta > \tilde{d}$ and the complement. Then the upper bound over the subdomain $\overline{\Omega} \setminus \{\Sigma(\delta) \cup \Omega_V(\delta)\}$ is equal to the larger of the suprema over the two subdomains. The supremum over the subdomain for which $\delta > \tilde{d}$ has been calculated above. The supremum over the complement is calculated using the estimates for the behavior of the solution near Σ, namely, estimate (3.31) and the corner estimate (3.12). In (3.12), the constants C_1 and γ_V are independent of K, K_0, and α_1, while $|\rho|_0$ is bounded by ρ_M from Proposition 3.10. By the interpolation inequality [14, Lemma 2.1], since $\gamma_1 = \gamma_V/2$ we have

$$|\rho|_{\gamma_1,\Omega_V(d_V)} \leq C_V|\rho|_{\gamma_V,\Omega_V(d_V)} \leq C_V C_1 \rho_M, \tag{3.35}$$

where $C_V = C_V(\gamma_1, \gamma_V, \Omega_V(d_V))$, for some $d_V > 0$. From here we get that

$$d^{2-\gamma_1}|\rho|_2 \leq K_V \quad \forall d < d_V,$$

where K_V is independent of K. Hence we can take $K \equiv \max\{K_V, K'\}$, using the bound (3.34), and now K is independent of α_1 and of $\tilde{d}$. Since K_V and K' are independent of $\tilde{d}$ we can change $\tilde{d}$ without affecting K. Therefore, we can choose $\tilde{d} \leq \min\{d_0, d_V\}/2$ in (3.31) small enough that $\tilde{d}^{1-\gamma_1+p}C < K$. Therefore, (3.30) is satisfied and we have chosen parameters K, K_0, and α_0 defining $\mathcal{W}$ so that T maps $\mathcal{W}$ into itself.

Now, by the Schauder fixed point theorem, there exists a fixed point ρ such that $T\rho = \rho \in H_2^{(-\gamma_1)}$. Thus, ρ solves (3.1), (3.3), (3.4), and (3.5). By a bootstrap argument we get $\rho \in H_{2+\alpha}^{(-\gamma_1)}$ for any $\alpha \leq \alpha_\Omega$, the value given in Theorem 3.6. For reference, we note that we have chosen $\gamma_1 = \gamma_V/2$; the exponent $\gamma_V \in (0,1)$ depends on the corner angle at Ξ_0 and α_Ω and γ_V depend on ε. The bounds on ρ in Proposition 3.3 give the first estimate in (3.27), and the second follows.

Finally, since $T(\mathcal{W}) \subset \mathcal{W}$ is a bounded set in $H_2^{(-\gamma_1)}$, then by (W2) and by the interpolation inequality [14, Lemma 2.1], any fixed point ρ satisfies (3.28) for any $\gamma \leq \gamma_1 = \gamma_V/2$. Note that K_1 and γ_1 are independent of α_1. □

3.3. The regularized nonlinear free boundary problem. We now prove existence of a solution to the regularized free boundary problem.

Proof of Theorem 3.1. Again, we suppress the ε dependence.

For each $\eta \in \mathcal{K} \subset H_{1+\alpha_1}$, using the solution ρ of the nonlinear fixed boundary problem (3.1), (3.3), (3.4), and (3.5) given by Theorem 3.7, we define the map J on $\mathcal{K}$, $\tilde{\eta} = J\eta$ as in (2.44), by integrating (2.43):

$$\tilde{\eta}(\xi) = \eta_0 + \int_{\xi_0}^{\xi} f(x, \eta(x), \rho(x, \eta(x)))\, dx. \tag{3.36}$$

First, we check that J maps $\mathcal{K}$ into itself. Property (K1) follows from (3.36). By Proposition 2.5, property (K2) holds, while the upper and lower bounds in (K4) hold by Proposition 2.6 and in turn imply (K3).

The Hölder class of ρ at Σ is given by the estimate (3.28), along with a bound on the Hölder γ-norm, and from estimate (3.28) in the proof of Theorem 3.7 we saw that we could choose $\gamma = \gamma_V/2$. Evaluating $f(\Xi, \rho(\Xi))$, we get a bound $|f|_{\gamma_V/2} \leq C(K_1)$, and thus $|\tilde{\eta}|_{1+\gamma_V/2} \leq C(K_1)$. The constants here are simple functions of the Riemann data and the structure of the pressure function. The important feature of the mapping is that γ_V is independent of α_1, the Hölder exponent of the space $\mathcal{K}$. Thus, we have $J(\mathcal{K}) \subset H_{1+\gamma_V/2}$; since properties (K1)–(K4) hold, we then have $J(\mathcal{K}) \subset \mathcal{K}$ if

$\alpha_1 \le \gamma_V/2$. Furthermore, J is compact if $\alpha_1 < \gamma_V/2$. We now take $\alpha_1 = \gamma_V/3$. By standard arguments, the map J is continuous.

Therefore, J has a fixed point $\eta^\varepsilon \in H_{1+\gamma_V/3}([0,\xi_0])$ by the Schauder fixed point theorem. This gives a curve Σ^ε on which (3.2) holds. Together with the corresponding solution ρ^ε from Theorem 3.7, this establishes the existence of a solution $(\rho^\varepsilon, \eta^\varepsilon) \in H_{2+\alpha}^{(-\gamma)} \times H_{1+\alpha}$ of the regularized free boundary problem (3.1), (3.2), (3.3), (3.4), and (3.5) for sufficiently small $\gamma(\varepsilon)$ and $\alpha(\varepsilon)$.

This completes the proof of Theorem 3.1. $\square$

4. The limiting solution. In this section we study the limiting solution, as the elliptic regularization parameter ε tends to zero. We start with the regularized solutions of (3.1), (3.2), (3.3), (3.4), and (3.5), whose existence is guaranteed by Theorem 3.1. Denote by ρ^ε a sequence of regularized solutions of the partial differential equation.

PROPOSITION 4.1. *For each ε the constant function ρ_0 is a lower barrier for ρ^ε and $c^2(\rho_0) > \xi^2 + \eta^2$ in $\overline{\Omega^\varepsilon} \setminus \sigma$.*

Proof. For each ε we have $\rho^\varepsilon > \rho_0$, and by the monotonicity of c^2 we get $c^2(\rho^\varepsilon) > c^2(\rho_0) > \xi^2 + \eta^2$ in $\Omega^\varepsilon \cup \Sigma_0$. The same inequality holds on Σ^ε since $(\xi, \eta^\varepsilon(\xi))$ lies inside C_0. Thus $c^2(\rho_0) > \xi^2 + \eta(\xi)^2$ for $\xi \in [0, \xi_0)$ and ρ_0 is a uniform lower barrier. $\square$

The existence of a uniform lower bound ρ_0 in ε allows us to apply standard local compactness arguments (see, for example, [3, Lemma 4.2]) to get a limit ρ, locally, in the interior of the domain. Here, the issue is ensuring ellipticity uniformly in ε in compact subsets of Ω. We first show that the sequence of domains Ω^ε converges to a domain Ω, as $\varepsilon \to 0$.

LEMMA 4.2. *The sequence η^ε has a convergent subsequence, whose limit η belongs to $C^\gamma([0,\xi_0])$ for all $\gamma \in (0,1)$. The limiting curve η is convex.*

Proof. Theorem 3.1 gives the existence of a sequence $(\rho^\varepsilon, \eta^\varepsilon)$ of solutions of the regularized free boundary problems for which η^ε belongs to the set $\mathcal{K}^\varepsilon$ for each ε. Now, $\rho_0 < \rho^\varepsilon \le \rho_s^\varepsilon \le \rho_M$, where ρ_M is independent of ε, and the property (K4) of $\mathcal{K}^\varepsilon$, specified in section 2.4, immediately gives a C^1 bound on η^ε, uniformly in ε. Thus by the Arzela–Ascoli theorem, η^ε has a convergent subsequence, and the limit $\eta \in C^\gamma([0,\xi_0])$ for all $\gamma \in (0,1)$.

To see that η is convex we first show that η^ε is convex for each $\varepsilon > 0$. Recall that $\eta' = f(\xi, \eta(\xi), \rho(\xi, \eta(\xi)))$ and calculate $\eta'' = f_\xi + f_\eta \eta' + f_\rho \rho'$. By observing that if ρ were constant the shock would be a straight line, we get $f_\xi + f_\eta \eta' = 0$. Therefore, the sign of η'' is determined entirely by the sign of f_ρ and ρ'. Since ρ is decreasing by Proposition 2.4, this implies $\rho' \le 0$. Furthermore, by Lemma 2.1 we have $d(s^2)/d\rho \ge 0$ and by the proof of Proposition 2.6 we have $f_{s^2} < 0$, so $f_\rho = f_{s^2}(s^2)' \le 0$. This shows that η^ε is convex for each $\varepsilon > 0$, and so the limiting function is convex. $\square$

The limit value $\eta(0) = \lim \eta^\varepsilon(0)$ is also established, and the corresponding subsequence of domains Ω^ε also has a limit, Ω.

In the remaining lemmas, without further comment, we carry out the limiting argument using the convergent subsequence of η^ε, which we again call η^ε.

LEMMA 4.3. *The sequence ρ^ε has a limit $\rho \in C^{2+\alpha'}(\Omega)$ for some $\alpha' > 0$. The limit ρ satisfies the quasi-linear degenerate elliptic equation* (2.36). *In addition, $\rho_0 < \rho < \rho_M$ in Ω.*

Proof. The proof is based on local compactness arguments and on uniform L^∞ bounds for ρ^ε: $\rho_0 < \rho^\varepsilon < \rho_s^\varepsilon \le \rho_M$, where ρ_M is independent of ε. The main ideas follow those used in [13, Theorem 1] and the proof is almost identical to the proof of [4, Lemma 4.2]. We omit the details. ◻

In the next lemma, we show that the limiting functions ρ and η satisfy both the shock evolution equation (2.38) and the oblique derivative boundary condition (2.37), $M\rho = 0$, on Σ.

LEMMA 4.4. *The limits η and ρ satisfy*

$$\eta' = f(\xi, \eta, \rho) \quad \text{and} \quad M\rho = \beta(\eta(\xi), \rho) \cdot \nabla\rho = 0 \quad \text{on} \quad \Sigma. \tag{4.1}$$

Furthermore, $\eta \in C^{2+\alpha'}(0, \xi_0) \cap C^1([0, \xi_0))$ and $\rho \in C^{2+\alpha'}(\Omega \cup \Sigma \cup \Sigma_0 \setminus \Xi_s) \cap C(\Omega \cup \Sigma \cup \Sigma_0)$ for some $\alpha' > 0$. In addition, ρ satisfies $\rho = \rho_s$ at $\Xi_s = (0, \eta(0))$, where $\rho_s = s_{\rho_1}^{-1}(\eta(0))$.

Proof. The proof is similar to that of [4, Lemma 4.3] except for the loss of uniform obliqueness at Ξ_s. We omit the local arguments away from Ξ_s and concentrate on dealing with the behavior of the solution near Ξ_s.

The arguments presented in the proof of [4, Lemma 4.3] imply $\eta^\varepsilon(\xi) \to \eta(\xi)$ in $C^{2+\alpha'}_{loc}$ for $\xi \neq 0$, and since the subsequence ρ^ε converges to ρ in $C^{1+\alpha'}_{loc}$, we get

$$(\eta^\varepsilon)' = f(\xi, \eta^\varepsilon, \rho^\varepsilon) \to f(\xi, \eta, \rho) \quad \forall \xi \neq 0,$$

thus $\eta' = f(\xi, \eta, \rho)$ for $\xi \neq 0$. Furthermore, by continuity of β and ρ we have

$$0 = \beta(\eta^\varepsilon, \rho^\varepsilon) \cdot \nabla\rho^\varepsilon(\xi, \eta^\varepsilon(\xi)) \to \beta(\eta, \rho) \cdot \nabla\rho(\xi, \eta(\xi)) \quad \forall \xi \neq 0,$$

and thus $\beta(\eta, \rho) \cdot \nabla\rho = 0$ on $\Sigma \setminus \{(0, \eta(0))\}$.

We now focus on the behavior of the solution at Ξ_s. By Lemma 4.2 we have $\eta^\varepsilon \to \eta$ in $C^\gamma([0, \xi_0])$ for any $0 < \gamma < 1$. Furthermore, by construction, for each $\varepsilon > 0$,

$$s^2(\rho_s^\varepsilon, \rho_1) = (\eta^\varepsilon(0))^2.$$

Therefore, as $\varepsilon \to 0$, the right-hand side converges to $\eta^2(0)$; hence $s^2(\rho_s^\varepsilon, \rho_1) \to \eta^2(0)$. By continuity and monotonicity of s^2 this implies that the sequence of numbers ρ_s^ε also has a limit, R. Moreover, $s^2(R, \rho_1) = \eta^2(0)$. But, this equation defines ρ_s; therefore $R = \rho_s$ and we have shown that the sequence of traces of the functions ρ^ε evaluated at $(0, \eta^\varepsilon(0))$ converges to ρ_s. We still have to show that ρ is continuous at Ξ_s, that is, that $\lim_{\xi\to 0} \rho(\xi, \eta(\xi)) = \rho_s$.

Since η'_ε has a limit $\eta' = f(\xi, \eta(\xi), \rho(\xi, \eta(\xi)))$ in $C^{1+\alpha}$ for $\xi \neq 0$, and since for each $\varepsilon > 0$ we have $\eta'_\varepsilon(0) = 0$, then for any $\delta > 0$ there exists an $h_0 \neq 0$ such that

$$|\eta'(h)| \leq |\eta'(h) - \eta'_\varepsilon(h)| + |\eta'_\varepsilon(h)| \leq \delta$$

for $0 < h < h_0$, which implies continuity of η' at $\xi = 0$ and $\eta'(0) = 0$. Thus

$$f(h, \eta(h), \rho(h, \eta(h))) = \eta'(h) \to \eta'(0) = 0 = f(0, \eta(0), \rho_s) \text{ as } h \to 0.$$

This implies, among other things, that $\rho(h, \eta(h)) \to \rho_s$ and so ρ is continuous at Ξ_s, $\rho(\Xi_s) = \rho_s$ and the boundary condition (2.40) is satisfied. ◻

The final task is to prove continuity of ρ up to the degenerate boundary σ. It is here that we need an additional condition on the Riemann data.

LEMMA 4.5. *For Riemann data satisfying a bound $\kappa_a > \kappa_*(\rho_1, \rho_0)$, the limit ρ satisfies $\rho = \rho_0$ on σ and $\rho \in C(\overline{\Omega})$.*

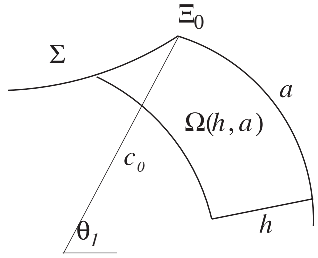

FIG. 4.1. *A sketch of the corner barrier domain.*

Proof. Continuity of solutions of $Q\rho = 0$ up to a degenerate boundary was proved as Corollary 3.3 in [7], at points where the degenerate boundary σ is convex, when the problem satisfies a Dirichlet condition on the entire boundary, and the entire boundary is degenerate. In [7], a pointwise upper barrier function ψ was constructed, uniformly in ε, with $\psi > \rho^\varepsilon$ in Ω and $\psi = \rho^\varepsilon$ at $\Xi \in \sigma$. This proof can easily be adapted to give a local barrier at every interior point of σ in our problem. Thus, to show continuity everywhere on σ we need only to show continuity at Ξ_0. We construct an upper barrier ψ with $\psi(\Xi_0) = \rho_0$ so that $\psi \geq \rho^\varepsilon$ in a fixed set $\Omega(h, a)$ (see Figure 4.1) for all $\varepsilon > 0$. Since ρ_0 is a lower barrier, we then have continuity at Ξ_0.

It is convenient to work in polar coordinates $(\xi, \eta) = (r\cos\theta, r\sin\theta)$. In this coordinate system, the operator Q^ε becomes

$$Q^\varepsilon \rho = \left(c^2(\rho) - r^2 + \varepsilon\right)\rho_{rr} + \frac{c^2}{r^2}\rho_{\theta\theta} + p''(\rho)\left(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2\right) + \left(\frac{c^2}{r} - 2r\right)\rho_r.$$

To compare ψ and ρ^ε we introduce an operator $Q_1^\varepsilon(\rho^\varepsilon)$ which is partially linearized:

$$Q_1^\varepsilon(\rho^\varepsilon)u = \left(c^2(u) - r^2 + \varepsilon\right)u_{rr} + \frac{c^2(\rho^\varepsilon)}{r^2}u_{\theta\theta} + p''(\rho^\varepsilon)\left(u_r^2 + \frac{1}{r^2}u_\theta^2\right) + \left(\frac{c^2(\rho^\varepsilon)}{r} - 2r\right)u_r.$$

The barrier function has the form

$$\psi(r, \theta) = \rho_0 + A(c_0 - r)^b + B(\theta_1 - \theta)^2. \tag{4.2}$$

Here θ_1 is the angle subtended by Ξ_0; A and B are constants to be determined and the exponent b is a value, also to be determined, in $(0, 1)$. The barrier is constructed on a curvilinear quadrilateral, $c_0 \geq r \geq c_0 - h$, $\theta_1 - a \leq \theta \leq \theta^\varepsilon(r)$, where $\theta^\varepsilon(r)$ is the boundary Σ^ε in polar coordinates and h and a are small numbers to be determined. The use of a barrier function with a singular derivative is motivated by [7], following [13]. In fact, we conjecture that the solution to the equation, in this case, does have a square root singularity at C_0 and that our value of b, which can be refined a posteriori to be any number less than 1/2, is optimal.

Before evaluating $Q_1^\varepsilon\psi$, we write $c^2 = p'$ and expand $c^2 - r^2$ as $c^2(\psi) - r^2 = c^2(\psi) - c_0^2 + c_0^2 - r^2 = (\psi - \rho_0)p''(\overline{\rho}) + c_0^2 - r^2$, where $\overline{\rho}$ is a value in the range of ρ^ε. By assumption, p'' is bounded above and below by positive numbers for $\rho \in [\rho_0, \rho_M]$.

We have

$$\begin{aligned}
&Q_1^\varepsilon(\rho^\varepsilon)\psi \\
&= \Big(p''(\overline{\rho})\big\{A(c_0-r)^b + B(\theta_1-\theta)^2\big\} + (c_0+r)(c_0-r)+\varepsilon\Big) b(b-1)A(c_0-r)^{b-2} \\
&\qquad + \frac{c^2(\rho^\varepsilon)}{r^2}(2B) + p''(\rho^\varepsilon)\left\{\big(Ab(c_0-r)^{b-1}\big)^2 + \frac{1}{r^2}\big(2B(\theta_1-\theta)\big)^2\right\} \\
&\qquad + \left(\frac{c^2(\rho^\varepsilon)}{r} - 2r\right)Ab(c_0-r)^{b-1}.
\end{aligned}$$

The coefficient of $(c_0-r)^{b-2}$, the most singular term as $r\to c_0$, is

$$p''(\overline{\rho})(B(\theta_1-\theta)^2+\varepsilon)b(b-1)A<0.$$

The next most singular power is $(c_0-r)^{2b-2}$, and its coefficient is

$$A^2 b\Big(p''(\overline{\rho})(b-1)+p''(\rho)b\Big) \le A^2 k_0 < 0 \qquad \text{if} \qquad b < \frac{\min p''}{2\max p''}, \tag{4.3}$$

which we now assume. The next power is $(c_0-r)^{b-1}$, whose coefficient is a bounded multiple of A; the remaining terms are bounded and involve only powers of B. Once we have fixed the lower limit, c_0-h, for r, and have chosen B, we can then choose A, which appears quadratically in (4.3) with a negative coefficient, large enough to make the entire expression negative. This is sufficient to guarantee that $\psi-\rho^\varepsilon$ does not have a negative minimum in the interior of $\Omega(h,a)$ provided that $\psi-\rho^\varepsilon$ is nonnegative on the boundary of $\Omega(h,a)$. For at a negative interior minimum, $\nabla\psi=\nabla\rho^\varepsilon$, and

$$\begin{aligned}
0 &\ge Q_1^\varepsilon(\rho^\varepsilon)\psi - Q^\varepsilon(\rho^\varepsilon) \\
&> \big(c^2(\psi)-c^2(\rho^\varepsilon)\big)\psi_{rr} + \big(c^2(\rho^\varepsilon)-r^2+\varepsilon\big)(\psi-\rho)_{rr} + \frac{c^2(\rho^\varepsilon)}{r^2}(\psi-\rho)_{\theta\theta}.
\end{aligned} \tag{4.4}$$

However, $\psi<\rho^\varepsilon$ implies $c^2(\psi)-c^2(\rho)<0$, while $\psi_{rr}<0$ by the concavity of ψ in r; in addition $(\psi-\rho)_{rr}$ and $(\psi-\rho)_{\theta\theta}$ are nonnegative at the minimum, so the sum of the three terms is positive. This contradiction establishes the conclusion that if $\psi\ge\rho^\varepsilon$ on the boundary of $\Omega(h,a)$, then ψ is an upper barrier for each ρ^ε.

We now turn to establishing bounds for ψ on the sides of the quadrilateral. First, on σ: $\rho^\varepsilon=\rho_0<\psi$. We fix an angular interval by choosing some $a>0$; then we can choose B large enough that $Ba^2>\rho_M$. This gives $\psi>\rho^\varepsilon$ on the boundary $\theta=\theta_1-a$ of $\Omega(h,a)$.

The appropriate condition on the oblique derivative boundary is more delicate. We linearize the boundary condition, obtain an estimate of the form $N_1(\rho^\varepsilon)\psi\le 0$, and use the Hopf maximum principle to show that $\psi-\rho^\varepsilon$ is positive on Σ^ε. Getting the estimate $N_1(\rho^\varepsilon)\psi\le0$ is rendered difficult by the fact that the part of $\nabla\psi$ which becomes singular near Ξ_0 is not the normal derivative (over which we have some control because the problem is oblique near Ξ_0) but the derivative in the direction r.

We can obtain the bound we need, at least as long as κ_a is large enough. To see this, we compute the derivative of ψ along Σ^ε, using the linearized operator $N_1(\rho^\varepsilon)=\beta(\rho^\varepsilon)\cdot\nabla$. To focus on the singular part, we write $\beta(\rho^\varepsilon)\cdot\nabla$ in terms of its radial and angular components,

$$N_1\psi = \beta^r\psi_r + \beta^\theta\psi_\theta,$$

where

$$\beta^r = \beta \cdot (\cos\theta, \sin\theta) = \frac{1}{r}(\beta_1\xi + \beta_2\eta),$$

and we have an analogous expression for β^θ. Now a calculation gives

$$\beta_1\xi + \beta_2\eta = (\eta - \eta'\xi)(\xi + \eta'\eta)\big(r^2(c^2 + 3s^2) - 4c^2s^2\big). \tag{4.5}$$

The first two factors are uniformly positive near Ξ_0, and if ρ_M is sufficiently close to ρ_0, then we claim there exists an interval $[c_0 - h, c_0]$ in which $\beta_1\xi + \beta_2\eta$ has a positive lower bound, for there will be a value $h > 0$ such that the expression in (4.5) is positive for $r > c_0 - h$ as long as $4c^2s^2/(c^2 + 3s^2) < c_0^2$ for all values in the range of ρ. Since the left side of this expression is monotone increasing in ρ, it is sufficient to impose the restriction on $c^2(\rho_M)$ and $s^2(\rho_M)$. The condition obviously holds for $\rho = \rho_0$, and so it certainly holds for ρ_M sufficiently close to ρ_0. Furthermore, for large κ_a, $\rho_M = \rho_0 + \mathcal{O}(1/\kappa_a)$, by a calculation given in [17]. That is, for κ_a large enough we have $\beta_1\xi + \beta_2\eta \geq C > 0$ in (4.5). Estimates on κ_* are given in [17].

We now complete the calculation of

$$N_1(\rho^\varepsilon)\psi = -\beta^r A(c_0 - r)^{b-1} - 2\beta^\theta B(\theta_1 - \theta)$$

by choosing A large enough that $N_1\psi \leq 0$ on Σ^ε. We also ensure $\psi - \rho^\varepsilon > 0$ at $r = c_0 - h$, by increasing A again if necessary, so that $Ah^b > \rho_M$.

Finally, we confirm that the inequality $N_1(\rho^\varepsilon)\psi < 0$ precludes negative values of $\psi - \rho^\varepsilon$ on Σ^ε. If there are negative values, then there is a negative minimum, at which the tangential derivative of $\psi - \rho^\varepsilon$ vanishes, so we have

$$0 \geq N_1(\psi - \rho^\varepsilon) = \beta^t(\psi - \rho^\varepsilon)_t + \beta^n(\psi - \rho^\varepsilon)_n = \beta^n(\psi - \rho^\varepsilon)_n,$$

where the superscripts mark the tangential and (inward) normal components of β, and the subscripts the derivatives of $\psi - \rho^\varepsilon$. Since $\beta^n > 0$, this implies that $(\psi - \rho^\varepsilon)_n \leq 0$. However, we can write $\overline{L}(\psi - \rho^\varepsilon) \leq 0$ at such a point for a suitable linear operator $\overline{L}$, and thus the Hopf maximum principle requires that $(\psi - \rho^\varepsilon)_n > 0$, a contradiction. Thus we conclude that $\psi - \rho^\varepsilon \geq 0$ on the entire boundary of $\Omega(h, a)$. By the argument following the inequality (4.4), this establishes ψ as an upper barrier. We note that this construction depends on ε only through the location of the curve $\Sigma = \Sigma^\varepsilon$ and that A, B, b, h, and a are independent of ε. Thus, since the domains Ω^ε converge, it follows that ψ is a barrier for all ρ^ε for sufficiently small ε.

Thus the solution ρ is continuous up to the degenerate boundary. ◻

Continuity of ρ at Ξ_0 allows a strengthening of Lemma 4.4, as follows.

COROLLARY 4.6. *The free boundary η is smooth up to the degenerate boundary, namely, $\eta \in C^1[0, \xi_0]$.*

Proof of Theorem 2.3. Lemmas 4.2, 4.3, 4.4, and 4.5 show that there exists a solution pair $(\rho, \eta) \in C^{2+\alpha'}(\overline{\Omega} \setminus \{\sigma \cup \Xi_s\}) \cap C(\overline{\Omega}) \times C^{2+\alpha'}(0, \xi_0)$ satisfying (2.36), (2.37), (2.38), (2.39), and (2.40). This completes the proof of Theorem 2.3. ◻

5. Conclusions. Theorem 2.3 has constructed a solution ρ of the differential equation (2.36) in Ω; combining this function with the piecewise constant solution far from the origin, we obtain a function which is piecewise constant in the supersonic region, continuous across the degenerate boundary σ, and consistent with the derived form of the Rankine–Hugoniot conditions across the Mach stem. To recover the momentum components, m and n, we could in principle integrate equations (2.4) and (2.5), which can be written as transport equations in the radial variable r,

$$\frac{\partial m}{\partial r} = \frac{1}{r}c^2(\rho)\rho_\xi, \qquad \frac{\partial n}{\partial r} = \frac{1}{r}c^2(\rho)\rho_\eta, \tag{5.1}$$

and integrated from the boundary of the subsonic region toward the origin. We note that the sonic boundary can be written $r = r_0(\theta)$ and the boundary conditions for m and n are of the form $m(r_0(\theta), \theta) = m_0(\theta)$, where m_0 is piecewise continuous on σ and is determined from the Rankine–Hugoniot relation (2.15) on Σ; the component n is treated exactly the same way.

At σ and at Ξ_s, where we have proved only that ρ is continuous, equations (5.1), may not be meaningful. Elsewhere, m and n have the same regularity as ρ, except that discontinuities in m and n on the line $\xi = \kappa_a \eta$ may persist all the way in to the origin. In addition, the behavior of $c^2\rho_\eta/r$ in (5.1) at the origin causes a logarithmic singularity in n (but not in m: $c^2\rho_\xi/r$ remains bounded since $\rho_\xi(0,0) = 0$).

Remark. There is some evidence of the unbounded behavior near the origin in the numerical simulations in [17]. This may presage difficulties in extending these results to the gas dynamics equations.

We argue heuristically that there is a difficulty at σ. Because of the construction, (ρ, m, n) is a weak solution of the system (1.1), or equivalently of the self-similar form (2.3)–(2.5), except possibly at the sonic boundary. It can be checked that the system (1.1) and the second-order equation (2.6), $Q(\rho) = 0$, are equivalent for weak solutions (that is, they conserve the same quantities). We can write (2.6) in the form $\mathrm{div} A = S$, with

$$A = (p_\xi - \xi^2\rho_\xi - \xi\eta\rho_\eta + \xi\rho, p_\eta - \eta^2\rho_\eta - \xi\eta\rho_\xi + \eta\rho),$$

and $S = -2\rho$. The usual multiplication by a smooth test function ϕ supported on a compact set D containing a segment of the degenerate boundary σ, followed by integration by parts, gives the weak form of the equation which must be satisfied for any weak solution in which $\nabla\rho$ is integrable (as is the case for our constructed function). Integrating by parts in the opposite sense on each side of $\Gamma \equiv \sigma \cap D$ yields the condition

$$\int_\Gamma \phi[A \cdot \nu]\, ds = 0,$$

where [ ] denotes the jump in the quantity and ν is the normal to σ. Since this must hold for all choices of D and ϕ, it holds pointwise. Furthermore, since the normal direction is the radial direction at σ, this means we need the function ρ inside Ω to satisfy

$$\lim_{r \to r_0} r\big(c^2(\rho) - r_0^2\big)\rho_r = 0. \tag{5.2}$$

We observe that for a linear wave equation, c^2 is constant and $r_0 = c$, and so this equation holds. However, for the function we constructed in Theorem 2.3 we have only the estimate $\rho - \rho_0 < A(r_0 - r)^\beta$ with $\beta < 1/2$ (see Lemma 4.5 and [7]) and this is not strong enough to give the limit (5.2). In fact, we have calculated, in [2] and [5], that the the behavior of solutions near a degenerate boundary like σ is exactly a square root singularity ($\beta = 1/2$), and so the function we have constructed fails to give a weak solution in the neighborhood of σ.

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