Minimum degree conditions for the existence of a sequence of cycles whose lengths differ by one or two

Shuya Chiba* Katsuhiro Ota† Tomoki Yamashita‡

Gao, Huo, Liu and Ma (2019) proved a result on the existence of paths connecting specified two vertices whose lengths differ by one or two. By using this result, they settled two famous conjectures due to Thomassen (1983). In this paper, we improve their result, and obtain a generalization of a result of Bondy and Vince (1998).

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1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively, and $\deg_G(v)$ denotes the degree of a vertex v in G.

In 1983, Thomassen proposed the following two conjectures.

Conjecture A (Thomassen [5]) *For a positive integer k, every graph of minimum degree at least $k+1$ contains cycles of all even lengths modulo k.*

Conjecture B (Thomassen [5]) *For a positive integer k, every 2-connected non-bipartite graph of minimum degree at least $k+1$ contains cycles of all lengths modulo k.*

*Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan; E-mail address: schiba@kumamoto-u.ac.jp; This work was supported by JSPS KAKENHI Grant Number 20K03720

†Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan; E-mail address: ohta@math.keio.ac.jp; This work was supported by JSPS KAKENHI Grant Number 16H03952

‡Department of Science, Kindai University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan; E-mail address: yamashita@math.kindai.ac.jp; This work was supported by JSPS KAKENHI Grant Number 16K05262

The above conjectures originated from the conjecture of Burr and Erdős concerning the extremal problem for the existence of cycles with prescribed lengths modulo k (see [2]). We refer the reader to [4] for more details. In 2018, Liu and Ma proved that Conjectures A and B are true for all even integers k by considering the existence of a sequence of paths whose lengths differ by two in bipartite graphs, see [4, Theorem 1.9].

Recently, Gao, Huo, Liu and Ma [6] announced that they had confirmed Conjectures A and B for all integers k by using the following theorem. Here, we say that a sequence of k paths (or k cycles) $P_1, \dots, P_k$ is *admissible* if $|E(P_1)| \geq 2$, and either $|E(P_{i+1})| - |E(P_i)| = 1$ for $1 \leq i \leq k-1$ or $|E(P_{i+1})| - |E(P_i)| = 2$ for $1 \leq i \leq k-1$.

Theorem C (Gao et al. [6, Theorem 1.2]) *Let k be a positive integer, and let G be a 2-connected graph, and x, y be two distinct vertices of G. If $\deg_G(v) \geq k+1$ for each $v \in V(G) \setminus \{x, y\}$, then G contains k admissible paths from x to y.*

In this paper, we show that the degree condition in Theorem C can be relaxed as follows.

Theorem 1 *Let k be a positive integer, and let G be a 2-connected graph, x, y be two distinct vertices of G, and z be a vertex of G (possibly $z \in \{x, y\}$) such that $V(G) \setminus \{x, y, z\} \neq \emptyset$. If $\deg_G(v) \geq k+1$ for each $v \in V(G) \setminus \{x, y, z\}$, then G contains k admissible paths from x to y.*

This study also originated from the question of whether every graph of minimum degree at least three contains two admissible cycles, which was raised by Erdős (see [1]). In 1998, Bondy and Vince answered this queston by proving the following stronger theorem.

Theorem D (Bondy, Vince [1]) *Every graph of order at least three, having at most two vertices of degree less than three, contains two admissible cycles.*

They also conjectured that every graph of sufficiently large order, having at most m vertices of degree less than three, contains two admissible cycles, and they gave some remarks for the case of small m. In 2020, Gao and Ma [3] settled the conjecture in the affirmative for all m.

We give the following another generalization of Theorem D by using Theorem 1.

Theorem 2 *For an integer $k \geq 2$, every graph of order at least three, having at most two vertices of degree less than $k+1$, contains k admissible cycles.*

Note that a weaker version of Theorem 2 is obtained from Theorem C (see [6, Theorem 1.3]), and the result (and also Theorem 2) settles Conjectures A and B for all integers k.

To show Theorem 1, in the next section, we consider the existence of admissible paths in "rooted graphs" and give a stronger result than Theorem 1 (see Theorem 3 in Section 2). We also extend the concept of "cores" which were used in the argument of [4, 6] in preparation for the proof of Theorem 3. In Section 3, we prove Theorem 3 and also give the proof of Theorem 2 at the end of Section 3.

2 Preliminaries

2.1 Admissible paths in rooted graphs

Let G be a graph. A *cut-vertex* of G is a vertex whose removal increases the number of components of G. A *block* of G is a maximal connected subgraph of G which has no cut-vertex, and a block B of G is called an *end-block* if B has at most one cut-vertex of G. If G itself is connected and has no cut-vertex, then G is a block and is also an end-block. For distinct vertices x and y of G, (G, x, y) is called a *rooted graph*. A rooted graph (G, x, y) is 2*-connected* if

(R1) G is a connected graph of order at least three with at most two end-blocks, and

(R2) every end-block of G contains at least one of x and y as a non-cut-vertex of G.

Note that (G, x, y) is 2-connected if and only if $G + xy$ (i.e., the graph obtained from G by adding the edge xy if $xy \notin E(G)$) is 2-connected. We denote by $(G, x, y; z)$ a rooted graph (G, x, y) with a specified vertex z (this includes the case where $z \in \{x, y\}$ or $z \notin V(G)$). For a rooted graph $(G, x, y; z)$, we define $\delta(G, x, y; z) = \min\{\deg_G(v) : v \in V(G) \setminus \{x, y, z\}\}$ if $V(G) \setminus \{x, y, z\} \neq \emptyset$; otherwise, let $\delta(G, x, y; z) = -\infty$.

In this paper, instead of proving Theorem 1, we prove the following stronger theorem.

Theorem 3 *Let k be a positive integer, and let $(G, x, y; z)$ be a 2-connected rooted graph. If $\delta(G, x, y; z) \geq k + 1$, then G contains k admissible paths from x to y.*

2.2 Terminology and notation

In this subsection, we prepare terminology and notation which will be used in the proof of Theorem 3.

Let G be a graph. We denote by $N_G(v)$ the neighborhood of a vertex v in G. For $S \subseteq V(G)$, we define $N_G(S) = \left(\bigcup_{v \in S} N_G(v)\right) \setminus S$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and let $G - S = G[V(G) \setminus S]$. We denote by $\mathrm{dist}_G(u, v)$ the length of a shortest path from a vertex u to a vertex v in G. For $U, V \subseteq V(G)$ with $U \cap V = \emptyset$, a path in G is a (U, V)*-path* if one end-vertex of the path belongs to U, the other end-vertex belongs to V, and the internal vertices do not belong to $U \cup V$. We write a path P with a given orientation as $\overrightarrow{P}$. For an oriented path $\overrightarrow{P}$ and $u, v \in V(P)$, a path from u to v along $\overrightarrow{P}$ is denoted by $u\overrightarrow{P}v$. For t (≥ 2) pairwise vertex-disjoint sets $V_1, \ldots, V_t$ of vertices, we define the graph $V_1 \vee \cdots \vee V_t$ from the union of $V_1, \ldots, V_t$ by joining every vertex of V_i to every vertex of V_{i+1} for $1 \leq i \leq t - 1$. For convenience, we let $V_1 \vee \cdots \vee V_t \vee \emptyset = V_1 \vee \cdots \vee V_t$.

Let D be a connected graph and v be a vertex. The *v-end-block of D* is an end-block B_v with cut-vertex b_v in D such that $V(B_v) = \{v, b_v\}$. The v-end-block of D, if exists, is unique, and so we always denote it by B_v for a vertex v. We also denote by b_v the unique cut-vertex of D which is contained in B_v. If $\deg_D(b_v) = 2$, then let b'_v denote the unique neighbor of b_v in D which is not v; otherwise, let $b'_v = b_v$. See Figure 1.

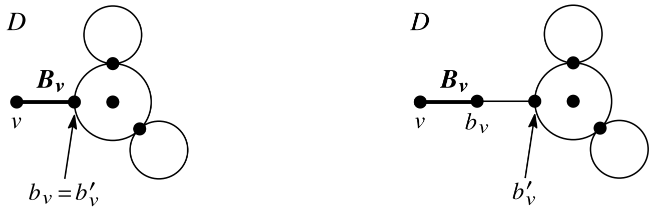

Figure 1: The v-end-block B_v of D

Throughout the rest of this paper, we often denote the singleton set $\{v\}$ by v, and we often identify a subgraph H of G with its vertex set $V(H)$.

2.3 The concept of cores

In this subsection, we extend the concept of cores which were used in the argument of [4, 6].

Let ℓ be an integer. Let x be a vertex of a graph G, and let H be a subgraph of G.

- H is called an *ℓ-core of type 1 with respect to x* if $H = x \vee T \vee S$, where $\ell \geq 1$, $S = \emptyset$ and T is a clique of order exactly $\ell + 1$ in G.
- H is called an *ℓ-core of type 2 with respect to x* if $H = x \vee S \vee T$, where $\ell \geq 2$, S is an independent set of order exactly 2 and T is a clique of order exactly ℓ.
- H is called an *ℓ-core of type 3 with respect to x* if $H = x \vee T \vee S$, where $\ell \geq 0$ and, S and T are independent sets of orders exactly ℓ and at least $\max\{\ell + 1, 2\}$, respectively. (Since $\ell \geq 0$, S may be an empty set.)

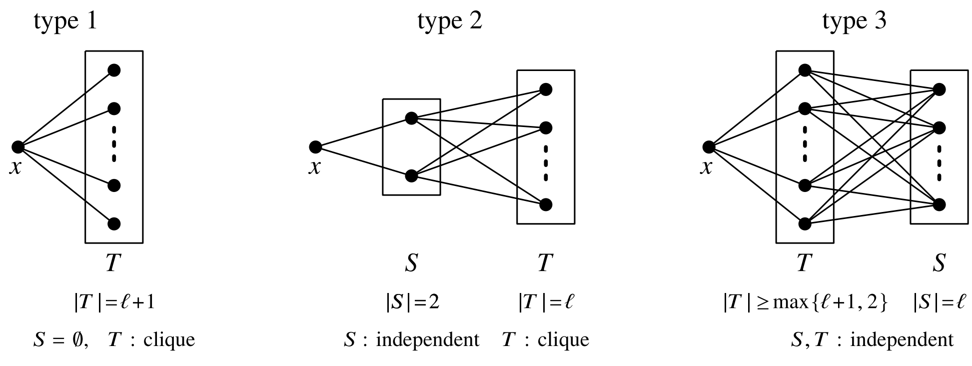

Figure 2: The structures of ℓ-cores of types 1, 2 and 3 with respect to x

See Figure 2. We say that H is an ℓ-core with respect to x when there is no need to specify the type. We also say that H is an *ℓ-core with respect to* (x, y) if H is an ℓ-core with respect to x, and y is a vertex of $V(G) \setminus V(H)$. In what follows, "a core" always means an ℓ-core for some integer ℓ.

Remark 1 *Let G be a graph, and x, y be two distinct vertices of G. If $\deg_G(x) \geq 2$ and $xy \notin E(G)$, then there exists a core of type 1 or type 3 with respect to (x, y) in G.*

We give three facts which will be used frequently in the proof of Theorem 3. Here, an admissible sequence which changed the condition $|E(P_1)| \geq 2$ into $|E(P_1)| \geq 1$, is said to be *semi-admissible*.

Fact 1 (Gao et al. [6, Lemma 3.2]) *Let s, t be positive integers. Let G be a graph, x, y be two distinct vertices and $U \subseteq V(G) \setminus \{x, y\}$. Assume that G contains s semi-admissible (U, y)-paths $P_1, \ldots, P_s$, and let u_i be the unique vertex of $V(P_i) \cap U$ for $1 \leq i \leq s$. Assume further that for each $1 \leq i \leq s$, $G - (V(P_i) \setminus \{u_i\})$ contains t semi-admissible (x, u_i)-paths $Q_{i,1}, \ldots, Q_{i,t}$. If $|V(Q_{1,j})| = |V(Q_{2,j})| = \cdots = |V(Q_{s,j})|$ for $1 \leq j \leq t$, then G contains $s + t - 1$ admissible (x, y)-paths.*

Fact 2 *Let H be an ℓ-core with respect to x. Then the following hold.*

(1) If H is of type 2, then for any $s \in S$, H contains ℓ admissible (x, s)-paths of lengths $3, 4, \ldots, \ell + 2$; if H is of type 3, then for any $s \in S$ and $t \in T$, $H - t$ contains ℓ admissible (x, s)-paths of lengths $2, 4, \ldots, 2\ell$.

(2) For any $t \in T$, H contains $\ell + 1$ semi-admissible (x, t)-paths of lengths $1, 2, \ldots, \ell + 1$ (if H is of type 1) or $2, 3, \ldots, \ell + 2$ (if H is of type 2) or $1, 3, \ldots, 2\ell + 1$ (if H is of type 3). In particular, if H is of type 3 and $|T| \geq \ell + 2$, then for any $t, t' \in T$ with $t \neq t'$, $H - t'$ contains $\ell + 1$ semi-admissible (x, t)-paths of lengths $1, 3, \ldots, 2\ell + 1$.

Fact 2 immediately yields the following.

Fact 3 *Let k be a positive integer and (G, x, y) be a 2-connected rooted graph. Let H be an ℓ-core with respect to (x, y) and C be the component of $G - V(H)$ such that $y \in V(C)$. Assume that G does not contain k admissible (x, y)-paths. Then (1) $\ell \leq k - 1$, and (2) if $N_G(C) \cap T \neq \emptyset$, then $\ell \leq k - 2$.*

3 Proof of Theorem 3

Proof of Theorem 3. We prove it by induction on $|V(G)| + |E(G)|$. Let $(G, x, y; z)$ be a minimum counterexample with respect to $|V(G)| + |E(G)|$.

Claim 3.1 *(1) $k \geq 2$ (and so $\delta(G, x, y; z) \geq 3$), and (2) $|V(G)| \geq 5$.*

Proof. (1) If $k = 1$, then by (R1) and (R2), we can easily see that G contains an (x, y)-path of length at least 2, a contradiction. Thus $k \geq 2$, and so $\delta(G, x, y; z) \geq k + 1 \geq 3$.

(2) Since $\delta(G, x, y; z) \geq k + 1 \geq 3$, we have $|V(G)| \geq 4$. Suppose that $|V(G)| = 4$. Note that, then $\delta(G, x, y; z) = k + 1 = 3$. Let u and v be two distinct vertices of $V(G) \setminus \{x, y\}$ such that $u \neq z$. Since $\deg_G(u) = 3$, we have $N_G(u) = \{x, y, v\}$. By (R2), we also have $N_G(v) \cap \{x, y\} \neq \emptyset$, say $xv \in E(G)$ up to symmetry, and then xuy and $xvuy$ are k $(= 2)$ admissible (x, y)-paths, a contradiction. □

Claim 3.2 *(1) G is 2-connected and (2) $\{x, y, z\}$ is independent.*

Proof. (1) Suppose that G is not 2-connected. Then by (R1), G has a cut vertex c and $G - c$ has exactly two components C_1 and C_2. By (R2), without loss of generality, we may assume that $x \in V(C_1)$ and $y \in V(C_2)$. Since $V(G) \setminus \{x, y, z, c\} \neq \emptyset$ by Claim 3.1 (2), and by the symmetry of x and y, we may assume that $V(C_1) \setminus \{x, z\} \neq \emptyset$. Let $G_i = G[C_i \cup c]$ for $i \in \{1, 2\}$. Then $(G_1, x, c; z)$ is a 2-connected rooted graph such that $\delta(G_1, x, c; z) \geq \delta(G, x, y; z)$. Hence by the induction hypothesis, G_1 contains k admissible (x, c)-paths $\vec{P}_1, \ldots, \vec{P}_k$. Let $\vec{Q}$ be a (c, y)-path in G_2. Then $x\vec{P}_i c\vec{Q}y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction.

(2) Suppose that $xv \in E(G)$ for some $v \in \{y, z\}$, and choose such a vertex v so that $v = y$ if possible. If $G - xv$ (i.e., the graph obtained from G by deleting the edge xv) is 2-connected, then by the induction hypothesis, it follows that $G - xv$ (and also G) contains k admissible (x, y)-paths, a contradiction. Thus $G - xv$ is not 2-connected. Since G is 2-connected by Claim 3.2 (1), this implies that $(G - xv, x, v)$ is a 2-connected rooted graph with exactly two end-blocks.

Let $B_1, \ldots, B_t$ $(t \geq 2)$ be all the blocks of $G - xv$ such that $V(B_i) \cap V(B_{i+1}) \neq \emptyset$ for $1 \leq i \leq t - 1$, say $V(B_i) \cap V(B_{i+1}) = \{b_i\}$ for $1 \leq i \leq t - 1$. Without loss of generality, we may assume that $x \in V(B_1) \setminus \{b_1\}$ and $v \in V(B_t) \setminus \{b_{t-1}\}$. Then $y \in V(B_p) \setminus \{b_{p-1}\}$ for some p with $1 \leq p \leq t$, where we let $b_0 = x$.

Suppose that $p = t$. Then $(G - xv, x, y; z)$ is a 2-connected rooted graph such that $\delta(G - xv, x, y; z) = \delta(G, x, y; z)$. Hence, by the induction hypothesis, $G - xv$ (and also G) contains k admissible (x, y)-paths, a contradiction. Thus $p \leq t - 1$. This implies that $v \neq y$, that is, $v = z$. Then by the choice of v, we have $xy \notin E(G)$.

Let $G' = G[\bigcup_{1 \leq i \leq p} V(B_i)]$, and let $z' = b_p$. Note that $z \notin V(G')$. Note also that if $p = 1$, then since $xy \notin E(G)$, $V(G') \setminus \{x, y, z'\} = V(B_1) \setminus \{x, y, b_1\} \neq \emptyset$ holds; if $p \geq 2$, $V(G') \setminus \{x, y, z'\} \neq \emptyset$ clearly holds. Then $(G', x, y; z')$ is a 2-connected rooted graph such that $\delta(G', x, y; z') \geq \delta(G, x, y; z)$, and so the the induction hypothesis yields that G' (and also G) contains k admissible (x, y)-paths, a contradiction. Thus $xv \notin E(G)$ for each $v \in \{y, z\}$. By the symmetry of x and y, we also have $yz \notin E(G)$. □

By Remark 1 and Claim 3.2, there exist cores with respect to (x, y) and (y, x), respectively, in G. By the symmetry of x and y, we can rename the vertices x and y so that

(XY1) there exists a core H with respect to (x, y) so that the number of type of H is as small as possible,

(XY2) $\deg_G(x) \le \deg_G(y)$, subject to (XY1), and

(XY3) $\mathrm{dist}_G(x, z) \le \mathrm{dist}_G(y, z)$, subject to (XY1) and (XY2).

Let H be an ℓ-core with respect to (x, y) in G for some integer ℓ, and let C be the component of $G - V(H)$ such that $y \in V(C)$. Choose H so that

(H1) the number of type of H is as small as possible, and

(H2) subject to (H1),

- (H2-1) if H is of type 1 or type 2, then $|T|$ is maximum;
- (H2-2) if H is of type 3, then (i) $|S|$ is maximum, (ii) $|T|$ is maximum, subject to (i).
- (H2-3) If H and C satisfy the following condition (T):
  - (T) H is of type 3, $|T| \ge 3$, $V(C) = \{y\}$, $N_G(x) = N_G(y) = T$, and there exists a component D_0 of $G - V(H)$ such that $D_0 \ne C$, $V(D_0) \setminus \{z\} \ne \emptyset$ and $N_G(D_0) \cap T \ne \emptyset$,

  then let $t_0 \in N_G(D_0) \cap T$ ($\cap N_G(y)$), and we modify H (and C depending on H) by resetting ℓ, S and T as follows:

  - (M1) if $|T| = |S| + 1$[1], then let $s_0 \in S$, and we reset $\ell := \ell - 1$, $S := S \setminus \{s_0\}$ and $T := T \setminus \{t_0\}$;
  - (M2) if $|T| \ge |S| + 2$, then we reset $\ell := \ell$, $S := S$ and $T := T \setminus \{t_0\}$.

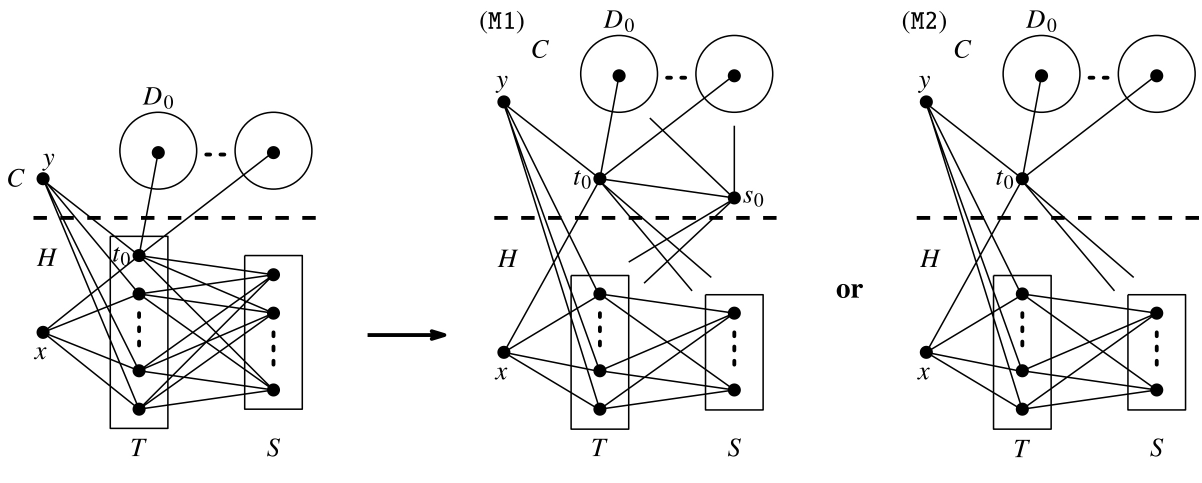

Figure 3: The modifications of H and C

[1]Note that, in this case, $\ell \ge 2$.

Note that if we modify H as in (H2-3), then the new graph H is also an ℓ-core of type 3 with respect to (x, y) (see also Figure 3). However, to make the difference clear, we sometimes say that H is of type $3^\flat$ if H is modified as above; otherwise H is of type $3^\natural$.

Claim 3.3 *If v is a vertex of $V(G) \setminus (V(H) \cup \{y, t_0\})$, then $|N_G(v) \cap V(H)| \leq \ell + 1$, where $t_0 = y$ for the case where H is not of type $3^\flat$. In particular, if the equality holds, then $N_G(v) \cap T \neq \emptyset$.*

Proof. Let v be a vertex of $V(G) \setminus (V(H) \cup \{y, t_0\})$. We show the claim as follows.

Assume first that H is of type 1. If $|N_G(v) \cap V(H)| \geq \ell + 2$, then we have $N_G(v) \cap V(H) = x \cup T$, which contradicts the maximality of T (see (H2-1)). Thus $|N_G(v) \cap V(H)| \leq \ell + 1$. If the equality holds, we clearly have $N_G(v) \cap T \neq \emptyset$, since $\ell + 1 \geq 2$.

Assume next that H is of type 2, and suppose that $|N_G(v) \cap V(H)| \geq \ell + 2$. Since there exists no core of type 1 with respect to (x, y) by (H1), we have $x \notin N_G(v)$ or $N_G(v) \cap S = \emptyset$. Since $|N_G(v) \cap V(H)| \geq \ell + 2$, $|S| = 2$ and $|T| = \ell$, this yields that $N_G(v) \cap V(H) = S \cup T$, which contradicts the maximality of T (see (H2-1)). Thus $|N_G(v) \cap V(H)| \leq \ell + 1$. If the equality holds, then since $x \notin N_G(v)$ or $N_G(v) \cap S = \emptyset$ holds, it follows that $|N_G(v) \cap T| = (\ell + 1) - |N_G(v) \cap (x \cup S)| \geq (\ell + 1) - 2 = \ell - 1 \geq 1$. Thus we have $N_G(v) \cap T \neq \emptyset$.

Assume finally that H is of type 3. Suppose that $|N_G(v) \cap V(H)| \geq \ell + 1$. Since there exists no core of type 1 with respect to (x, y) by (H1), we have $x \notin N_G(v)$ or $N_G(v) \cap T = \emptyset$. Since there also exists no core of type 2 with respect to (x, y) by (H1), we have $|N_G(v) \cap T| \leq 1$ or $N_G(v) \cap S = \emptyset$. If H is of either type $3^\natural$ or type $3^\flat$ in (M2), then $|N_G(v) \cap T| \leq \ell + 1$ (by (H2-2)-(i)); if H is of type $3^\flat$ in (M1), then we have $|N_G(v) \cap T| \leq |T| = \ell + 1$. If H is of type $3^\natural$, then $x \cup S \nsubseteq N_G(v)$ (by (H2-2)-(ii)); if H is of type $3^\flat$, then since $N_G(x) = T \cup \{t_0\}$ and $v \neq t_0$, we have $x \cup S \nsubseteq N_G(v)$. Since $|S| = \ell$, combining these facts yields that $|N_G(v) \cap V(H)| = \ell + 1$, and $N_G(v) \cap T \neq \emptyset$. □

By Claim 3.2 (2) and (T), we can easily obtain the following.

Claim 3.4 *If H is of type $3^\flat$, then $|V(C) \setminus \{y, z\}| \geq 2$.*

Proof. Since $yz \notin E(G)$ by Claim 3.2 (2) and $V(D_0) \setminus \{z\} \neq \emptyset$ by (T), we have $|V(C) \setminus \{y, z\}| \geq |V(D_0) \setminus \{z\}| + |\{t_0\}| \geq 2$. □

We now divide the proof into two cases according to $V(C) = \{y\}$ or $V(C) \neq \{y\}$.

Case 1. $V(C) = \{y\}$.

Note that by Claim 3.4, H is not of type $3^\flat$.

Claim 3.5 *If H is of type 1 or type 3, then $N_G(x) = N_G(y) = T$. If H is of type 2, then $N_G(x) = N_G(y) = S$.*

Proof. Note that by Claim 3.2, $|N_G(y) \cap V(H - x)| = \deg_G(y) \geq 2$.

Assume first that H is of type 1. Then $|N_G(y) \cap T| \geq 2$, and so there exists a core of type 1 with respect to (y, x). Since $N_G(y) \subseteq T \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = T$. Thus the claim follows.

Assume next that H is of type 2. Since $|N_G(y) \cap V(H - x)| \geq 2$, and since there exists no core of type 1 with respect to (y, x) by (XY1), we have $N_G(y) = S$. This in particular implies that $y \vee S \vee T$ is an ℓ-core of type 2 with respect to (y, x). Since $N_G(y) = S \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = S$. Thus the claim follows.

Assume finally that H is of type 3. By (XY1) and (H1), there exist no cores of type 1 or type 2 with respect to (y, x), and so any core with respect to (y, x) is of type 3 (by Remark 1, Claim 3.2). This also implies that $N_G(y) \subseteq T$ or $N_G(y) \subseteq S$. Since $|T| \geq \max\{\ell + 1, 2\} > \ell = |S|$ and $T \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = T$. Thus the claim follows. □

Claim 3.6 *Assume that H is of either type 1 or type 3. Let D be a component of $G - V(H)$ such that $D \neq C$ and $V(D) \setminus \{z\} \neq \emptyset$. Then $N_G(D) \cap S \neq \emptyset$. (This in particular implies that, if H is of type 1, then $G - V(H)$ does not have a component D such that $D \neq C$ and $V(D) \setminus \{z\} \neq \emptyset$.)*

Proof. Suppose that $N_G(D) \subseteq T$. By Claim 3.5, there exists a vertex $t_{cd} \in N_G(y) \cap N_G(D) \cap T$. Let D^* be the graph obtained from $G[D \cup N_G(D)]$ by contracting $N_G(D) \setminus \{t_{cd}\}$ into a new vertex t^*. Since $N_G(D) \subseteq T$ and G is 2-connected, it follows that $(D^*, t^*, t_{cd}; z)$ is a 2-connected rooted graph. Since $V(D) \setminus \{z\} \neq \emptyset$, we also have $\emptyset \neq V(D^*) \setminus \{t^*, t_{cd}, z\} \subseteq V(G) \setminus \{x, y, z\}$. Let

$$\epsilon = \begin{cases} 1 & \text{if } |T| = \ell + 1, \\ 0 & \text{if } |T| \geq \ell + 2. \end{cases}$$

If H is of type 3, then $|T| \geq \max\{\ell + 1, 2\}$, and so $\ell \geq 1$ holds for the case of $|T| = \ell + 1$, which implies that $\ell - \epsilon \geq 0$; if H is of type 1, then since $|T| = \ell + 1 \geq 2$, we clearly have $\ell - \epsilon \geq 0$. In either case, the inequality $\ell - \epsilon \geq 0$ holds. Then, for a vertex v of $V(D^*) \setminus \{t^*, t_{cd}, z\}$, the following hold:

- If $|N_G(v) \cap T| = 0$, then $\deg_{D^*}(v) = \deg_G(v) \geq \deg_G(v) - \ell + \epsilon$.
- If $1 \leq |N_G(v) \cap T| \leq \ell$, then $\deg_{D^*}(v) \geq \deg_G(v) - \ell + 1 \geq \deg_G(v) - \ell + \epsilon$.
- If $|N_G(v) \cap T| = \ell + 1$[2], then $\deg_{D^*}(v) \geq \deg_G(v) - (\ell + 1) + (1 + \epsilon) = \deg_G(v) - \ell + \epsilon$.

Thus the definition of D^* and Claim 3.3 yield that

$$\delta(D^*, t^*, t_{cd}; z) \geq \delta(G, x, y; z) - \ell + \epsilon \geq (k - \ell + \epsilon) + 1.$$

By the induction hypothesis, D^* contains $k - \ell + \epsilon$ admissible (t^*, t_{cd})-paths. This implies that $G[T \cup D]$ contains $k - \ell + \epsilon$ admissible $(T \setminus \{t_{cd}\}, t_{cd})$-paths $\vec{P}_1, \ldots, \vec{P}_{k-\ell+\epsilon}$. Let t_i be the unique

[2] In this case, if $\epsilon = 1$ then v is adjacent to all the vertices of T.

vertex of $V(P_i) \cap (T \setminus \{t_{cd}\})$ for $1 \le i \le k - \ell + \epsilon$. Then $t_i \overrightarrow{P_i} t_{cd} y$ $(1 \le i \le k - \ell + \epsilon)$ are $k - \ell + \epsilon$ admissible $(T \setminus \{t_{cd}\}, y)$-paths in $G[T \cup D \cup C]$. On the other hand, it follows from Fact 2 (2) that for each $1 \le i \le k - \ell + \epsilon$, $H - t_{cd}$ contains $\ell - \epsilon + 1$ (x, t_i)-paths of lengths $1, 2, \ldots, \ell - \epsilon + 1$ (if H is of type 1) or $1, 3, \ldots, 2(\ell - \epsilon) + 1$ (if H is of type 3). Hence by Fact 1, we obtain $k\ \big(= (k - \ell + \epsilon) + (\ell - \epsilon + 1) - 1 \big)$ admissible (x, y)-paths in G, a contradiction. Thus $N_G(D) \not\subseteq T$. Combining this with Claim 3.5, we have $N_G(D) \cap S = N_G(D) \setminus (T \cup x) = N_G(D) \setminus T \neq \emptyset$. □

Case 1.1. H is of type 1.

By Claim 3.5, $N_G(x) = N_G(y) = T$. By Claim 3.6, we also have $V(G) = T \cup \{x, y, z\}$. Since $|T| = \ell + 1 \le k - 1$ by Fact 3 (2), and since $T \setminus \{z\} \neq \emptyset$, the degree condition yields that $(\ell + 1 =)\ |T| = k - 1$, $z \notin T \cup \{x, y\}$, and $N_G(v) = (T \setminus \{v\}) \cup \{x, y, z\}$ for all $v \in T$. This implies that G contains (x, y)-paths of lengths $2, 3, \ldots, k + 1$. Thus G contains k admissible (x, y)-paths, a contradiction.

Case 1.2. H is of type 2.

By Claim 3.5, we have $N_G(x) = N_G(y) = S$, say $N_G(x) = N_G(y) = \{s_1, s_2\}$. Let $G' = G - \{x, y\}$. Since G and $H - x$ are 2-connected, respectively, and $|V(H - x)| \ge 4$, it follows that $(G', s_1, s_2; z)$ is a 2-connected rooted graph such that $\delta(G', s_1, s_2; z) \ge \delta(G, x, y; z)$. Therefore, by the induction hypothesis, we obtain k admissible (s_1, s_2)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k$ in G'. Then $x s_1 \overrightarrow{P_i} s_2 y$ $(1 \le i \le k)$ are k admissible (x, y)-paths in G, a contradiction.

Case 1.3. H is of type 3.

Claim 3.7 *There exists a component D of $G - V(H)$ such that $D \neq C$, $V(D) \setminus \{z\} \neq \emptyset$ and $N_G(D) \cap T \neq \emptyset$.*

Proof. If there exists $t \in T$ such that $N_G(t) \setminus (V(H) \cup \{y, z\}) \neq \emptyset$, then the assertion clearly holds. Thus, we may assume that $N_G(t) \setminus (V(H) \cup \{y, z\}) = \emptyset$ for all $t \in T$. Since $N_G(y) \cap T \neq \emptyset$ by Claim 3.5, Fact 3 (2) yields $|S| = \ell \le k - 2$. Then for a vertex $t \in T \setminus \{z\}$ $(\neq \emptyset)$, we have

$$0 = |N_G(t) \setminus (V(H) \cup \{y, z\})| \ge (k + 1) - (|S| + |\{x, y, z\}|) \ge (k + 1) - (k + 1) = 0.$$

Thus the equality holds, which implies that $|S| = \ell = k - 2$, $z \notin V(H) \cup \{y\}$ and $tz \in E(G)$ for all $t \in T$. By Claim 3.5 and since there exists no core of type 2 with respect to (x, y) by (H1), we also have $N_G(x) = N_G(y) = T = N_G(z) \cap V(H)$.

If $N_G(z) \setminus V(H) \neq \emptyset$, then since $T \subseteq N_G(z)$, the claim follows, and so we may assume that $N_G(z) \setminus V(H) = \emptyset$, that is, $N_G(z) = T$. If $|T| \ge \ell + 2$, then $x \vee T \vee (S \cup z)$ is an $(\ell + 1)$-core of type 3, contradicting to (H2-2)-(i). Thus we have $|T| = \ell + 1 = k - 1$, which also implies that $|S| = \ell \ge 1$. Since $N_G(x) = N_G(y) = N_G(z) = T$, a vertex $s \in S$ satisfies

$$|N_G(s) \setminus (V(H) \cup \{y, z\})| = |N_G(s) \setminus T| \ge (k + 1) - |T| = (k + 1) - (k - 1) > 0.$$

Hence there exists a component D of $G - V(H)$ such that $D \neq C$, $V(D) \setminus \{z\} \neq \emptyset$, and $N_G(D) \subseteq S$. Let $s_0 \in N_G(D)$ and D^* be the graph obtained from $G[D \cup N_G(D)]$ by contracting $N_G(D) \setminus \{s_0\}$

into a new vertex s^*. Since $N_G(D) \subseteq S$ and G is 2-connected, it follows that $(D^*, s^*, s_0; z)$ is a 2-connected rooted graph. Since $V(D) \setminus \{z\} \neq \emptyset$ and $|S| = \ell = k-2$, we also have $\delta(D^*, s^*, s_0; z) \geq \delta(G, x, y; z) - (k-2) + 1 \geq 3 + 1$. Therefore, by the induction hypothesis, D^* contains three admissible (s^*, s_0)-paths. This implies that $G[S \cup D]$ contains three admissible $(S \setminus \{s_0\}, s_0)$-paths $\overrightarrow{P}_1, \overrightarrow{P}_2$ and $\overrightarrow{P}_3$. Let s_i be the unique vertex of $V(P_i) \cap (S \setminus \{s_0\})$ for $1 \leq i \leq 3$, and let $t_0 \in N_G(y) \cap T$. Then $s_i \overrightarrow{P}_i s_0 t_0 y$ $(1 \leq i \leq 3)$ are three admissible $(S \setminus \{s_0\}, y)$-paths in $G[t_0 \cup S \cup D \cup C]$. On the other hand, since $|T \setminus \{t_0\}| = |(S \setminus \{s_0\}) \cup \{z\}| = \ell = k-2$ and $N_G(z) = T$, it follows that for each $1 \leq i \leq 3$, $G[\big(V(H) \setminus \{t_0, s_0\}\big) \cup \{z\}]$ contains $k-2$ (x, s_i)-paths of lengths $2, 4, \ldots, 2(k-2)$. Hence by Fact 1, we obtain k $\big(= 3 + (k-2) - 1\big)$ admissible (x, y)-paths in G, a contradiction. □

Let D be a component of $G - V(H)$ as in Claim 3.7. Since (T) does not hold, it follows from Claims 3.5 and 3.7 that $|T| = 2$, say $T = \{t_1, t_2\}$. Since $N_G(D) \cap S \neq \emptyset$ by Claim 3.6 and since $|T| \geq |S| + 1$, we also have $|S| = 1$, say $S = \{s\}$. Let $G' = G - \{x, y\}$. Since G is 2-connected and $N_G(x) = N_G(y) = \{t_1, t_2\}$ by Claim 3.5, it is easy to check that $(G', t_1, t_2; z)$ is a 2-connected rooted graph. Since $\emptyset \neq V(D) \setminus \{z\} \subseteq V(G')$, we also have $\delta(G', t_1, t_2; z) \geq \delta(G, x, y; z)$. Therefore, by the induction hypothesis, we obtain k admissible (t_1, t_2)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k$ in G'. Then $x t_1 \overrightarrow{P}_i t_2 y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction.

This completes the proof of Case 1.

Case 2. $V(C) \neq \{y\}$.

Claim 3.8 *Assume that H is of type 3. If $|S| = 1$, then $N_G(C) \cap T \neq \emptyset$.*

Proof. Suppose that $|S| = 1$, say $S = \{s\}$, and $N_G(C) \cap T = \emptyset$. Let $G' = G - V(C)$. Since G and H are 2-connected, $y \notin V(G')$ and $|V(H)| \geq |\{x\}| + |T| + |S| \geq 1 + 2 + 1 \geq 4$, it follows that $(G', x, s; z)$ is a 2-connected rooted graph such that $\delta(G', x, s; z) \geq \delta(G, x, y; z)$. By the induction hypothesis, G' contains k admissible (x, s)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k$. Since G is 2-connected and $N_G(C) \cap T = \emptyset$, we have $s \in N_G(C)$, and so there exists an (s, y)-path $\overrightarrow{Q}$ in $G[C \cup s]$. Then $x \overrightarrow{P}_i s \overrightarrow{Q} y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction. □

In this case, we will apply the induction hypothesis for new graphs obtained from H and blocks with at most two cut-vertices of C. However, the z-end-block of C will not help us to find admissible paths in the argument. So, in the following two claims, we study the structure for the case where C contains the z-end-block. In particular, we show that C is not a (y, z)-path of order exactly 3 at this stage. (See Subsection 2.2 for the definitions of the z-end-block B_z and the vertices b_z, b'_z.)

Claim 3.9 *Assume that there exists the z-end-block B_z with cut-vertex b_z in C such that $y \notin \{z, b_z\}$. Assume further that $\deg_C(b_z) = 2$. Then the following hold.*

(1) $\ell = k - 2$.

(2) $|N_G(b_z) \cap V(H)| = \ell + 1$.

(3) $\big(N_G(z) \cup N_G(b'_z)\big) \cap T = \emptyset$.

(4) If $b'_z \neq y$, *then* $\deg_C(b'_z) \geq 3$.

Proof. By our assumption, $\deg_G(b_z) \geq k+1$ and there exists a (b_z, y)-path $\overrightarrow{R}$ in $C - z$. If H is of type $3^\flat$, then since $y \neq z$, $N_C(y) = \{t_0\}$ and by Claim 3.4, note that $b_z \neq t_0$.

(1),(2) To show (1) and (2), we first prove that

$$\ell \leq k-2. \tag{3.1}$$

Since $\ell \leq k-1$ by Fact 3 (1), it suffices to show that $\ell \neq k-1$. Suppose to the contrary that $\ell = k-1$. Then it follows from Fact 3 (2) that $N_G(C) \cap T = \emptyset$. Combining this with Claim 3.2, we have $N_G(z) \cap S \neq \emptyset$, say $s_z \in N_G(z) \cap S$. This in particular implies that H is of type 2 or type 3.

Suppose that $N_G(b_z) \cap S \neq \emptyset$, say $s_b \in N_G(b_z) \cap S$. Then $s_b b_z \overrightarrow{R} y$ and $s_z z b_z \overrightarrow{R} y$ are two admissible $(\{s_b, s_z\}, y)$-paths in $G[S \cup C]$. On the other hand, it follows from Fact 2 (1) that for each $s \in \{s_b, s_z\}$, H contains $k-1$ $(=\ell)$ admissible (x, s)-paths. Hence by Fact 1, we obtain $k\ \big(=2+(k-1)-1\big)$ admissible (x, y)-paths in G, a contradiction. Thus $N_G(b_z) \cap S = \emptyset$, that is, $N_G(b_z) \cap (S \cup T) = \emptyset$. Then $1 \leq k-1 \leq \deg_G(b_z) - \deg_C(b_z) = |N_G(b_z) \cap V(H)| \leq |\{x\}| = 1$. Thus the equality holds, which implies that $\ell = k-1 = 1$ and $N_G(b_z) \cap V(H) = \{x\}$. If H is of type 2, then $x b_z \overrightarrow{R} y$ and $x s_z z b_z \overrightarrow{R} y$ are k $(=2)$ admissible (x, y)-paths in G, a contradiction; if H is of type 3, then since $|S| = \ell = 1$ and $N_G(C) \cap T = \emptyset$, this contradicts Claim 3.8. Thus (3.1) is proved.

Now, by Claim 3.3 and (3.1), we have

$$k-1 \leq \deg_G(b_z) - \deg_C(b_z) = |N_G(b_z) \cap V(H)| \leq \ell + 1 \leq k-1.$$

Thus the equality holds, which implies that $\ell = k-2$ and $|N_G(b_z) \cap V(H)| = \ell + 1$.

(3) Note that by Claims 3.3 and 3.9 (2), $N_G(b_z) \cap T \neq \emptyset$, say $t_b \in N_G(b_z) \cap T$. To show (3), suppose that $N_G(v) \cap T \neq \emptyset$ for some $v \in \{z, b'_z\}$, and let $t_v \in N_G(v) \cap T$.

Since $N_C(b_z) = \{z, b'_z\}$, it follows that $G[\{z, b_z, b'_z, t_v\}]$ contains a (t_v, b'_z)-path $\overrightarrow{P}$ of length 1 or 3. Hence P and $t_b b_z b'_z$ are $(\{t_v, t_b\}, b'_z)$-paths of lengths $1, 2$ or $3, 2$. By adding $b'_z \overrightarrow{R} y$ to each of the two paths, we obtain two semi-admissible $(\{t_v, t_b\}, y)$-paths in $G[C \cup \{t_v, t_b\}]$. On the other hand, it follows from Fact 2 (2) and Claim 3.9 (1) that for each $t \in \{t_v, t_b\}$, H contains $k-1$ $(=\ell+1)$ semi-admissible (x, t)-paths. Hence by Fact 1, G contains $k\ \big(=2+(k-1)-1\big)$ admissible (x, y)-paths, a contradiction.

(4) Assume that $b'_z \neq y$ and $\deg_C(b'_z) \leq 2$. We first claim that $b'_z \neq t_0$ if H is of type $3^\flat$. Suppose to the contrary that H is of type $3^\flat$ and $b'_z = t_0$. (See Figure 3.) If H is of type $3^\flat$ in (M1), then $\deg_C(b'_z) = \deg_C(t_0) \geq |N_G(t_0) \cap V(D_0)| + |\{y, s_0\}| \geq 1 + 2 = 3$, a contradiction. Thus H is of type $3^\flat$ in (M2). Recall that $(b_z b'_z =)\ b_z t_0 \in E(G)$ and $x \cup S \subseteq N_G(t_0)$. Since there exist no cores of type 1 or type 2 with respect to (x, y) by (H1), this together with Claim 3.9 (2) implies that $|N_G(b_z) \cap T| = |N_G(b_z) \cap V(H)| = \ell + 1$. Hence $H' := x \vee \big((N_G(b_z) \cap T) \cup t_0\big) \vee (S \cup b_z)$ is an $(\ell+1)$-core of type 3, which contradicts (H2-2)-(i). Thus $b'_z \neq t_0$ if H is of type $3^\flat$.

By Claim 3.9 (3), $N_G(b'_z) \cap T = \emptyset$, and so Claims 3.3 and 3.9 (1) yield that $|N_G(b'_z) \cap V(H)| \leq \ell = k - 2$. Then we obtain

$$\deg_G(b'_z) \leq \deg_C(b'_z) + |N_G(b'_z) \cap V(H)| \leq 2 + \ell = k,$$

a contradiction.

This completes the proof of Claim 3.9. □

Claim 3.10 *C is not a (y, z)-path of order exactly* 3.

Proof. Suppose that C is a (y, z)-path of order exactly 3. By Claims 3.2 and 3.9 (3), we have $N_G(z) \cap S \neq \emptyset$, say $s_z \in N_G(z) \cap S$. This in particular implies that H is not of type 1.

Suppose that H is of type 2. Since $N_G(b_z) \cap T \neq \emptyset$ by Claims 3.3 and 3.9 (2), it follows from Fact 2 (2) and Claim 3.9 (1) that $G[H \cup b_z]$ contains $k - 1$ $(= \ell + 1)$ admissible (x, b_z)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_{k-1}$ of lengths $3, 4, \ldots, \ell + 3$. On the other hand, by Fact 2 (1), H contains an (x, s_z)-path $\overrightarrow{Q}$ of length $\ell + 2$, and so $\overrightarrow{P}_k := x\overrightarrow{Q}s_z z b_z$ is an (x, b_z)-path of length $\ell + 4$. Then $x\overrightarrow{P}_i b_z y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction. Thus H is not of type 2, that is, H is of type 3.

Since $N_G(y) \cap T$ $(= N_G(b'_z) \cap T) = \emptyset$ by Claim 3.9 (3), and since $xy \notin E(G)$ by Claim 3.2 (2), it follows that $N_G(y) \subseteq S \cup b_z$. By (XY1) and (H1), there exist no cores of type 1 or type 2 with respect to (y, x), and so any core with respect to (y, x) is of type 3 (by Remark 1, Claim 3.2). Then we can use the inequality in (XY2). Note that $\ell \geq 1$, since $S \neq \emptyset$. Hence

$$\ell + 1 = \max\{\ell + 1, 2\} \leq |T| \leq \deg_G(x) \leq \deg_G(y) \leq |S \cup b_z| = \ell + 1.$$

Thus the equality holds, which implies that $\deg_G(x) = \deg_G(y)$ and $N_G(x) = T$. Then it follows from the first equality and (XY3) that $\text{dist}_G(x, z) \leq \text{dist}_G(y, z) = 2$ holds. On the other hand, since $xz \notin E(G)$ by Claim 3.2 (2), and since $N_G(z) \cap N_G(x) = N_G(z) \cap T = \emptyset$ by Claim 3.9 (3), it follows that $\text{dist}_G(x, z) \geq 3$. This is a contradiction. □

Let V_c be the set of cut-vertices of C. A block B of C is said to be *feasible* if B satisfies the following condition (F).

(F) $|V(B) \cap (V_c \cup \{y, z\})| \leq 2$ and $V(B) \setminus (V_c \cup \{y, z\}) \neq \emptyset$.

Note that by the assumption of Case 2 and Claim 3.2 (2), if C itself is a block, then C is feasible.

Claim 3.11 *There exists a feasible block of C.*

Proof. Suppose that there exists no feasible block of C. Then the condition (F) yields the following: C is not a block; its block-tree is a path; one of the two end-blocks of C is the y-end-block and the other is the z-end-block. By the definition of b_z and b'_z, if $\deg_C(b_z) \geq 3$, then $b_z = b'_z$ and so $\deg_C(b'_z) \geq 3$; if $\deg_C(b_z) = 2$, then it follows from Claims 3.9 (4) and 3.10

that $\deg_C(b'_z) \geq 3$. In either case, $\deg_C(b'_z) \geq 3$ holds. Hence there exists a block B of C which is not an end-block and satisfies (F). $\square$

In the rest of the proof, B, b and z' denote any one of the following (B1), (B2)-(i), (B2)-(ii) and (B3) (note that by Claim 3.11 and (F), such a tuple (B, b, z') exists, see also Figure 4):

(B1) B is a feasible block of C such that $|V(B) \cap V_c| = 0$ (i.e., C itself is a block and $B = C$) and, $b := y$ and $z' := z$.

(B2) B is a feasible block of C such that $|V(B) \cap V_c| = 1$, say $V(B) \cap V_c = \{b'\}$, and

- (i) if $y \in V(B) \setminus \{b'\}$, then $b := y$ and $z' := b'$;
- (ii) if $y \notin V(B) \setminus \{b'\}$, then $b := b'$ and $z' := z$.

(B3) B is a feasible block of C such that $|V(B) \cap V_c| = 2$, and b is the unique vertex of $V(B) \cap V_c$ such that $C - (V(B) \setminus V_c)$ contains a (b, y)-path (possibly $b = y$) and $\{z'\} := (V(B) \cap V_c) \setminus \{b\}$.

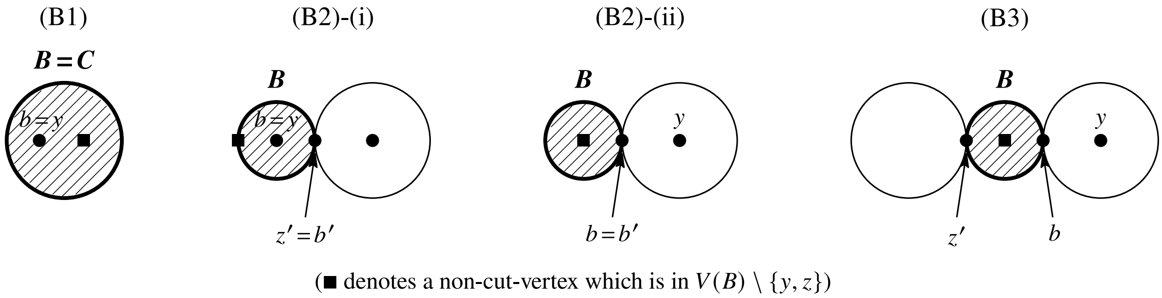

Figure 4: The definitions of B, b and z'

Note that $\emptyset \neq V(B) \setminus \{b, z'\} \subseteq V(G) \setminus \{x, y, z\}$ and $N_G(v) \subseteq V(B) \cup V(H)$ for $v \in V(B) \setminus \{b, z'\}$. Note also that if H is of type $3^\flat$, then since t_0 is a cut-vertex of C, $t_0 \notin V(B) \setminus \{b, z'\}$. Let $\overrightarrow{R}$ be a (b, y)-path in C such that $V(R) \cap V(B - b) = \emptyset$.

Claim 3.12 *(1) $N_G(B - b) \cap V(H) \subseteq x \cup S$ and (2) $|N_G(B - b) \cap (x \cup S)| \geq 2$.*

Proof. (1) Suppose that $N_G(B-b) \cap T \neq \emptyset$. Let B^* be the graph obtained from $G[B \cup (N_G(B-b) \cap T)]$ by contracting $N_G(B-b) \cap T$ into a new vertex t^*. Since $\emptyset \neq V(B) \setminus \{b, z'\} \subseteq V(G) \setminus \{x, y, z\}$, $(B^*, t^*, b; z')$ is a 2-connected rooted graph such that $\emptyset \neq V(B^*) \setminus \{t^*, b, z'\} \subseteq V(G) \setminus \{x, y, z\}$. Then, it follows from Claim 3.3 that for a vertex v of $V(B^*) \setminus \{t^*, b, z'\}$, the following hold:

- If $|N_G(v) \cap T| = 0$, then $\deg_{B^*}(v) \geq \deg_G(v) - \ell$.
- If $|N_G(v) \cap T| \geq 1$, then $\deg_{B^*}(v) \geq \deg_G(v) - (\ell + 1) + 1 = \deg_G(v) - \ell$.

Thus the definition of B^* yields that

$$\delta(B^*, t^*, b; z') \geq \delta(G, x, y; z) - \ell \geq (k - \ell) + 1.$$

By the induction hypothesis, B^* contains $k - \ell$ admissible (t^*, b)-paths. Thus G contains $k - \ell$ admissible (T, b)-paths $\overrightarrow{P}_1, \dots, \overrightarrow{P}_{k-\ell}$. Let $t_i \in V(P_i) \cap T$ for $1 \leq i \leq k - \ell$. Then $t_i \overrightarrow{P}_i b \overrightarrow{R} y$ is a (t_i, y)-path in $G[C \cup t_i]$ for $1 \leq i \leq k - \ell$. On the other hand, by Fact 2 (2), H contains $\ell + 1$ semi-admissible (x, t_i)-paths for $1 \leq i \leq k - \ell$. Hence by Fact 1, G contains $k \ \big(= (k - \ell) + (\ell + 1) - 1 \big)$ admissible (x, y)-paths, a contradiction. Thus (1) holds.

(2) Suppose that either (i) $|N_G(B - b) \cap (x \cup S)| = 1$ or (ii) $N_G(B - b) \cap (x \cup S) = \emptyset$ holds. Note that if B satisfies (B1) or (B2)-(ii), then the 2-connectivity of G implies that (i) holds; that is to say, if (ii) holds, then B satisfies (B2)-(i) or (B3). If (i) holds, say $N_G(B - b) \cap (x \cup S) = \{v\}$, then let $B' = G[B \cup v]$; if (ii) holds, then let $v = z'$ and $B' = B$. Since $\emptyset \neq V(B) \setminus \{b, z'\} \subseteq V(G) \setminus \{x, y, z\}$, $(B', v, b; z')$ is a 2-connected rooted graph such that $\delta(B', v, b; z') \geq \delta(G, x, y; z)$. By the induction hypothesis, B' contains k admissible (v, b)-paths $\overrightarrow{P}_1, \dots, \overrightarrow{P}_k$. If (i) holds, then let $\overrightarrow{Q}$ be an (x, v)-path in H; if (ii) holds, then by the 2-connectivity of G, there exists an (x, v)-path $\overrightarrow{Q}$ in $G[H \cup \big(V(C) \setminus (V(B' - v) \cup V(R))\big)]$. Then $x \overrightarrow{Q} v \overrightarrow{P}_i b \overrightarrow{R} y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction. □

Case 2.1. H is of type 1.

By Claim 3.12 (2), we have $N_G(B - b) \cap S \neq \emptyset$, which contradicts $S = \emptyset$.

Case 2.2. H is of type 2.

By Claim 3.12 (2), $N_G(B - b) \cap S \neq \emptyset$. Let B^* be the graph obtained from $G[B \cup (N_G(B - b) \cap S)]$ by contracting $N_G(B - b) \cap S$ into a new vertex s^*. Then $(B^*, s^*, b; z')$ is a 2-connected rooted graph such that $V(B^*) \setminus \{s^*, b, z'\} = V(B) \setminus \{b, z'\} \neq \emptyset$. Since there exists no core of type 1 with respect to (x, y) by (H1), it follows that $x \notin N_G(v)$ or $N_G(v) \cap S = \emptyset$ for $v \in V(B) \setminus \{b\}$. This together with the definition of B^* and Claim 3.12 (1) implies that $\delta(B^*, s^*, b; z') \geq \delta(G, x, y; z) - 1 \geq (k - 1) + 1$. Hence, by the induction hypothesis, B^* contains $k - 1$ admissible (s^*, b)-paths, and so $G[S \cup B]$ contains $k - 1$ admissible (S, b)-paths $\overrightarrow{P}_1, \dots, \overrightarrow{P}_{k-1}$. Let $s_i \in V(P_i) \cap S$ for $1 \leq i \leq k - 1$. Then $s_i \overrightarrow{P}_i b \overrightarrow{R} y$ is an (s_i, y)-path in $G[C \cup s_i]$ for $1 \leq i \leq k - 1$. On the other hand, by Fact 2 (1), H contains two admissible (x, s_i)-paths for $1 \leq i \leq k - 1$. By Fact 1, G contains $k \ \big(= (k - 1) + 2 - 1 \big)$ admissible (x, y)-paths, a contradiction.

Case 2.3. H is of type 3.

Note that, by Claim 3.12 (2), $\ell = |S| \geq 1$. Let

$$V_{nc} = V(C) \setminus (V_c \cup \{y, z\}).$$

We divide H into three cases as follows:

- *H is of type I* if $|N_G(v_0) \cap T| = \ell + 1$ for some $v_0 \in V_{nc}$.
- *H is of type II* if $\ell = 1$ and $|N_G(v_0) \cap S| = |N_G(v_0) \cap T| = 1$ for some $v_0 \in V_{nc}$.

- *H is of type III* if H is of neither type I nor type II.

If H is of type I or type II, then let v_0 be a vertex as described above, and let $S^\sharp = S \cup v_0$; if H is of type III, then let $S^\sharp = S$. We then let

$$H^\sharp = G[x \cup T \cup S^\sharp] \quad \text{and} \quad \ell^\sharp = |S^\sharp|.$$

Then the following (i) and (ii) hold: (i) $\ell^\sharp \geq \ell \geq 1$, and if H is of type I or type II, then $\ell^\sharp \geq 2$; (ii) if H is of type I or type II, then by the definitions of V_{nc} and the types, and by Claim 3.12 (1), $v_0 \notin V(B)$ and v_0 does not separate B and y in C. In particular, by (ii), B is still a block of a component of $G - V(H^\sharp)$, there exists a (b, y)-path internally disjoint from B in $G - V(H^\sharp)$, and $N_G(v) \subseteq V(B) \cup (x \cup S)$ for $v \in V(B) \setminus \{b, z'\}$ (by Claim 3.12 (1)).

Claim 3.13 $\ell^\sharp = 1$.

Proof. Suppose that $\ell^\sharp \geq 2$. Let B^* be the graph obtained from $G[B \cup (N_G(B - b) \cap S)]$ by contracting $N_G(B - b) \cap S$ into a new vertex s^*. By Claim 3.12 (2), $(B^*, s^*, b; z')$ is a 2-connected rooted graph such that $V(B^*) \setminus \{s^*, b, z'\} \neq \emptyset$. Recall that $t_0 \notin V(B) \setminus \{b, z'\}$ for the case where H is of type 3^b. For a vertex $v \in V(B^*) \setminus \{s^*, b, z'\}$, it follows from Claims 3.3, 3.12 (1) and $\ell^\sharp \geq 2$ that

$$\deg_{B^*}(v) \geq \begin{cases} \deg_G(v) - |\{x\}| \geq k \geq (k - \ell^\sharp + 1) + 1 & \text{if } N_G(v) \cap S = \emptyset, \\ \deg_G(v) - \ell + 1 \geq \deg_G(v) - \ell^\sharp + 1 \geq (k - \ell^\sharp + 1) + 1 & \text{otherwise,} \end{cases}$$

and thus $\delta(B^*, s^*, b; z') \geq (k - \ell^\sharp + 1) + 1$. By the induction hypothesis, B^* contains $k - \ell^\sharp + 1$ admissible (s^*, b)-paths. Therefore $G[B \cup (N_G(B - b) \cap S)]$ contains $k - \ell^\sharp + 1$ admissible (S, b)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_{k-\ell^\sharp+1}$. Let $s_i \in V(P_i) \cap S$ for $1 \leq i \leq k - \ell^\sharp + 1$, and let $\overrightarrow{R'}$ be a (b, y)-path internally disjoint from B in $G - V(H^\sharp)$. Then $s_i \overrightarrow{P}_i b \overrightarrow{R'} y$ is an (s_i, y)-path in $G[(V(G) \setminus V(H^\sharp)) \cup \{s_i\}]$ for $1 \leq i \leq k - \ell^\sharp + 1$. On the other hand, it follows from Fact 2 (1) and the definition of type II that for each s_i, $H^\sharp$ contains $\ell^\sharp$ admissible (x, s_i)-paths of lengths $2, \ldots, 2\ell^\sharp$ (if H is of type I or type III) or $2, 3$ (if H is of type II). Hence by Fact 1, G contains k $\big(= (k - \ell^\sharp + 1) + \ell^\sharp - 1 \big)$ admissible (x, y)-paths, a contradiction. □

Since $S \neq \emptyset$, it follows from Claim 3.13 that

$$S^\sharp = S, \text{ that is, } H \text{ is of type III}, \;\; \ell^\sharp = \ell = |S| = 1, \text{ say } S = \{s_1\}, \;\; H^\sharp = H.$$

Then the following hold (note that $t_0 \notin V_{nc}$, since t_0 is a cut-vertex of C):

$$N_G(B - b) \cap V(H) = \{x, s_1\} \quad \text{(by Claim 3.12 (2))}, \tag{3.2}$$

$$|N_G(v) \cap V(H)| \leq 1 \text{ for each } v \in V_{nc} \quad \text{(by Claim 3.3)} \tag{3.3}$$

Claim 3.14 *(1) $k \geq 3$, and (2) if $z \in V(C)$, then $N_G(T) \cap V(C - y) \neq \emptyset$.*

Proof. By Claim 3.8, we have $N_G(T) \cap V(C) \neq \emptyset$. Therefore, it follows from Fact 2 (2) that $k \geq 3$. Thus (1) holds. To show (2), suppose that $z \in V(C)$ and $N_G(T) \cap V(C - y) = \emptyset$. Since $N_G(T) \cap V(C) \neq \emptyset$, we have $N_G(T) \cap V(C) = \{y\}$. Let $G' = G - V(C - y)$. Since G and H are 2-connected, $z \in V(C - y)$ and $|V(H)| \geq 4$, it follows that $(G', x, y; s_1)$ is a 2-connected rooted graph such that $\delta(G', x, y; s_1) \geq \delta(G, x, y; z)$. Therefore, by the induction hypothesis, G' (and also G) contains k admissible (x, y)-paths in G, a contradiction. Thus (2) also holds. □

Claim 3.15 *(1) $V(B - b) \cap \{y, z'\} \neq \emptyset$, and (2) C is not a block.*

Proof. Suppose that $y, z' \notin V(B - b)$. (Note that then B satisfies (B1) or (B2)-(ii).) Recall that (3.2) holds. If $|N_G(s_1) \cap V(B)| \geq 2$, then let $B' = G[B \cup \{x, s_1\}]$ and $z_B = s_1$; if $|N_G(s_1) \cap V(B)| = 1$, say $N_G(s_1) \cap V(B) = \{v\}$, then let $B' = G[B \cup x]$ and $z_B = v$. Note that $|V(B)| \geq 3$, since $V(B) \setminus \{b, z'\} \neq \emptyset$ and $\delta(G, x, y; z) \geq k + 1 \geq 4$ by Claim 3.14 (1). Then $(B', x, b; z_B)$ is a 2-connected rooted graph and $\delta(B', x, b; z_B) \geq \delta(G, x, y; z)$. Hence by the induction hypothesis, B' contains k admissible (x, b)-paths $\overrightarrow{P}_1, \ldots, \overrightarrow{P}_k$. Then $x\overrightarrow{P}_i b\overrightarrow{R}y$ $(1 \leq i \leq k)$ are k admissible (x, y)-paths in G, a contradiction. Thus (1) holds.

Suppose next that C is a block. Then by (B1), note that $B = C$, $b = y$ and $z' = z$. In particular, Claim 3.15 (1) implies that $z = z' \in V(C)$. Then by Claim 3.14 (2), $N_G(T) \cap V(B - b) = N_G(T) \cap V(C - y) \neq \emptyset$. But, this contradicts Claim 3.12 (1). Thus (2) also holds. □

Recall that (B, b, z') denotes any one of (B1), (B2)-(i), (B2)-(ii) and (B3). By Claim 3.15, C has exactly two end-blocks, and each end-block of C contains exactly one of z and y as a non-cut-vertex of C (otherwise, there is a feasible end-block of C which satisfies no Claim 3.15 (1)). In particular, (C, z, y) is a 2-connected rooted graph.

Let $B_1, \ldots, B_t$ $(t \geq 2)$ be all the blocks of C such that $V(B_i) \cap V(B_{i+1}) \neq \emptyset$ for $1 \leq i \leq t - 1$, say $V(B_i) \cap V(B_{i+1}) = \{b_i\}$ for $1 \leq i \leq t - 1$. Without loss of generality, we may assume that $z \in V(B_1) \setminus \{b_1\}$ and $y \in V(B_t) \setminus \{b_{t-1}\}$, and let $b_0 = z$ and $b_t = y$. Then $B = B_p$ for some p with $1 \leq p \leq t$. Note that B_p, b_{p-1}, b_p satisfy (B2)-(i) (if $p = t$) or (B2)-(ii) (if $1 = p < t$) or (B3) (if $2 \leq p < t$) as $(B, b, z') = (B_p, b_p, b_{p-1})$, and so it follows from Claim 3.12 (1) that

$$N_G(T) \cap V(B_p - b_p) = \emptyset. \tag{3.4}$$

Claim 3.16 $N_G(T) \cap \left(\bigcup_{1 \leq i \leq p-1} V(B_i)\right) = \emptyset$.

Proof. Suppose that $N_G(T) \cap \left(\bigcup_{1 \leq i \leq p-1} V(B_i)\right) \neq \emptyset$. Then it follows from Fact 2 (2) that $G\left[H \cup \left(\bigcup_{1 \leq i \leq p-1} V(B_i)\right)\right]$ contains two admissible (x, b_{p-1})-paths. On the other hand, since $v \in V_{nc}$ for $v \in V(B_p) \setminus \{b_{p-1}, b_p\}$, it follows from (3.3) that $(B_p, b_{p-1}, b_p; z)$ is a 2-connected rooted graph such that $\delta(B_p, b_{p-1}, b_p; z) \geq \delta(G, x, y; z) - 1 \geq (k - 1) + 1$, and hence the induction hypothesis yields that B_p contains $k - 1$ admissible (b_{p-1}, b_p)-paths. Let $\overrightarrow{R'}$ be a (b_p, y)-path in $G[\bigcup_{p \leq i \leq t} V(B_i)]$. Then $b_{p-1}\overrightarrow{P}_i b_p\overrightarrow{R'}y$ $(1 \leq i \leq k - 1)$ are $k - 1$ admissible (b_{p-1}, y)-paths.

Therefore, by Fact 1, G contains $k\ \big(= (k-1)+2-1\big)$ admissible (x,y)-paths, a contradiction. □

Choose $B = B_p$ so that p is as large as possible. If $p = t$, then by (3.4) and Claim 3.16, we have $N_G(T) \cap V(C-y) = \emptyset$; since $z \in V(B_1) \setminus \{b_1\}$, this contradicts Claim 3.14 (2). Thus $p < t$ and the choice of $B = B_p$ implies that $|V(B_t)| = 2$, i.e., $V(B_t) = \{b_{t-1}, y\}$ $(= \{b_{t-1}, b_t\})$.

Recall that any core with respect to (y,x) is of type 3 (by $(\mathtt{XY}1)$, $(\mathtt{H}1)$, Remark 1, Claim 3.2). By (3.2), $\deg_G(x) \geq |T| + |N_G(x) \cap (B_p - b_p)| \geq |T| + 1$, and so $(\mathtt{XY}2)$ yields that $\deg_G(y) \geq |T| + 1$. Since $N_G(y) \subseteq H \cup b_{t-1}$, we obtain $|N_G(y) \cap V(H)| \geq |T| \geq 2$. This implies that $N_G(y) \cap V(H) = T$ and $N_G(b_{t-1}) \cap T = \emptyset$ (otherwise, $xy \in E(G)$ or there exists a core of type 1 with respect to (y,x), a contradiction). If $p = t-1$, then by the same argument as the case $p = t$, we get a contradiction to Claim 3.14 (2). Thus $p < t-1$ and the choice of $B = B_p$ implies that $|V(B_{t-1})| = 2$, i.e., $V(B_{t-1}) = \{b_{t-2}, b_{t-1}\}$. Since $\deg_G(y) = |T| + 1$, we have $N_G(x) = T \cup \big(N_G(x) \cap (B_p - b_p)\big)$, and so $x \notin N_G(b_{t-1})$ because $b_{t-1} \notin V(B_p)$. Therefore $N_G(b_{t-1}) \subseteq \{y, b_{t-2}, s_1\}$. Since $\deg_G(b_{t-1}) \geq k+1$, we obtain $k \leq 2$, contradicting to Claim 3.14 (1).

This completes the proof of Theorem 3. □

We finally give the proof of Theorem 2.

Proof of Theorem 2. It suffices to show the case where a given graph is connected. Let $k \geq 2$ be an integer, and let G be a connected graph of order at least three having at most two vertices of degree less than $k+1$. Let x and z be two vertices of degree less than $k+1$ if exist; otherwise, let x and z be arbitrary two vertices. Suppose now that G is a counterexample.

We first consider the case where G is 2-connected. Choose arbitrary edge xy in G (possibly $y = z$). Since $|V(G)| \geq 3$ and $\deg_G(v) \geq k+1 \geq 3$ for $v \in V(G) \setminus \{x,z\}$, we have $V(G) \setminus \{x,y,z\} \neq \emptyset$ and $\deg_G(v) \geq k+1$ for $v \in V(G) \setminus \{x,y,z\}$. Hence Theorem 1 yields that G contains k admissible (x,y)-paths. By adding xy to each of the k paths, we obtain k admissible cycles, a contradiction. Thus G is not 2-connected.

Suppose that there exists an end-block B with cut-vertex b such that $|V(B)| \geq 3$ and $|V(B-b) \cap \{x,z\}| \leq 1$. Let $x' \in V(B-b) \cap \{x,z\}$ if exists; otherwise, $x' \in V(B-b)$. Then the same argument as in the case where G is 2-connected can work with $(G,x,z) = (B,x',b)$, and so we obtain k admissible cycles in B, a contradiction. This, together with the degree condition, implies that the block-tree of G is a path, and the two end-blocks of G are the x-end-block and the z-end-block, respectively. Since $|V(G)| \geq 3$ and $\deg_G(v) \geq k+1 \geq 3$ for $v \in V(G) \setminus \{x,z\}$, there exists a block B with exactly two cut-vertices b_1, b_2 such that $|V(B)| \geq 3$. Then by replacing (G,x,z) and (B,b_1,b_2) in the above argument for the case where G is 2-connected, we obtain k admissible cycles in B, a contradiction again. □

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