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# Free nilpotent minimum algebras

**Manuela Busaniche***

Instituto Argentino de Matemática, CONICET,
Saavedra 15, Piso 3, C1083ACA – Capital Federal, Argentina

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In the present paper we give a description of the free algebra over an arbitrary set of generators in the variety of nilpotent minimum algebras. Such description is given in terms of a weak Boolean product of directly indecomposable algebras over the Boolean space corresponding to the Boolean subalgebra of the free NM-algebra.



## 1 Introduction

The *monoidal t-norm based logic* ($\mathcal{MTL}$) is introduced in [11] to formalize logics in which the conjunction is interpreted by a left continuous t-norm and the implication by its corresponding adjoint. The *nilpotent minimum logic* ($\mathcal{NML}$) is the extension of $\mathcal{MTL}$ that corresponds to an specific t-norm called the *nilpotent minimum t-norm*, which was introduced by Fodor in [12] as an example of a left continuous t-norm which is not continuous. The logic $\mathcal{NML}$ is algebraizable in the sense of Blok and Pigozzi, and the algebraic semantics of this logic is the variety of *nilpotent minimum algebras* (NM-algebras, for short). NM-algebras are bounded residuated lattices that satisfy three extra axioms: prelinearity, involution and the nilpotent minimum axiom, which roughly states that the conjunction of two elements is either their minimum or the bottom element in the lattice. Many researches about NM-algebras have already been done (see for instance [11, 13, 17]).

Since the propositions under equivalence form a free NM-algebra, the description of the free NM-algebra is quite important from the logical point of view. The description of truth functions of $\mathcal{NML}$ given in [19] can be interpreted as a characterization of finitely generated free NM-algebras. In the present paper, we shall give a more structural description of free NM-algebras that also covers the case of an infinite set of generators. Since the variety of NM-algebras is a subvariety of bounded residuated lattices, it is arithmetical. Then (see [1]) any NM-algebra can be represented as a weak Boolean product of directly indecomposable NM-algebras over the spectrum of its Boolean skeleton. It turns out that directly indecomposable objects in the variety of NM-algebras are Girard monoids. This monoids are deeply studied in [15], where the author gives a method, called *rotation*, for constructing the monoids from a substructure of semigroup. Then we can give a description of the free NM-algebra as a weak Boolean product of NM-algebras which are rotations of free objects in a bigger variety of algebras. More precisely, they are connected and disconnected rotations of generalized Gödel algebras.

The key to obtain this description is the characterization of the Boolean skeleton of an NM-algebra. Such characterization relies on the fact that the variety of MV-algebras generated by the three elements Łukasiewicz chain is a proper subvariety of the variety of NM-algebras. Therefore, one can consider the MV-skeleton of an NM-algebra $\boldsymbol{A}$: that is the biggest subalgebra $\mathbf{MV_3}(\boldsymbol{A})$ of $\boldsymbol{A}$ which is an MV-algebra. We prove that there is a retraction term from the NM-algebra onto its MV-skeleton. As a consequence we have that the Boolean skeleton of $\boldsymbol{A}$ coincides with the Boolean skeleton of $\mathbf{MV_3}(\boldsymbol{A})$. Then we are reducing the problem of finding a subalgebra of Boolean elements of an NM-algebra to finding a Boolean skeleton of an MV-algebra in a subvariety generated by a finite chain. This last problem has already been solved in [4], giving us the desired characterization.

* e-mail: manuelabusaniche@yahoo.com.ar

The paper is organized as follows: in the first section we give the preliminaries about NM-algebras and the representation of an NM-algebra as a weak Boolean product of directly indecomposable algebras. Secondly, we define the term function $\varphi$ that allows us to obtain $\mathbf{MV_3}(\boldsymbol{A})$: the greatest subalgebra of $\boldsymbol{A}$ that belongs to the subvariety of MV-algebras generated by the three elements Łukasiewicz chain. We prove that the subalgebra of Boolean elements of $\boldsymbol{A}$ coincides with the subalgebra of Boolean elements of $\mathbf{MV_3}(\boldsymbol{A})$. In the third section, we give a characterization of directly indecomposable algebras as connected or disconnected rotations of the kernel of the term function $\varphi$ defined in the previous section. We also prove that the kernel of $\varphi$ is a generalized Gödel algebra. The fourth section is divided into two parts. The first one is advocated to the study of the Boolean subalgebra of the free NM-algebra and the characterization of its ultrafilters. In second one, we prove that the directly indecomposable quotients of the free algebra are connected or disconnected rotations of free generalized Gödel algebras. Lastly we show how to extend the methods developed previously to describe free objects in any subvariety of NM-algebras.

## 2 Preliminaries

We shall always denote algebras by bold capital letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \dots$ and their corresponding universes by $A, B, C, D, \dots$. Unless otherwise specified, all the notions related with universal algebra used in the paper can be found in [3].

### 2.1 Basic definitions

An *integral residuated lattice* is an algebra $\boldsymbol{A} = (A, \wedge, \vee, *, \to, \top)$ of type $(2, 2, 2, 2, 0)$ such that:

1. $(A, *, \top)$ is a commutative monoid.
2. $(A, \wedge, \vee, \top)$ is a lattice with greatest element $\top$.
3. For any $x, y, z \in A$, the following residuation equation holds:

$$x * y \leq z \quad \text{iff} \quad x \leq y \to z,$$

where $\leq$ is the order given by the lattice structure.
A *bounded residuated lattice* is an integral residuated lattice $\boldsymbol{A}$ with an extra constant $\perp$ such that $\perp \leq x$ for all $x \in A$. In any bounded residuated lattice an additional operation of negation can be defined by:

$$\neg x = x \to \perp.$$

Bounded residuated lattices satisfying the involutive equation $\neg\neg x = x$ are called *involutive residuated lattices*. In any involutive residuated lattice the operations $*$ and $\to$ are related as follows:

(1) $$x * y = \neg(x \to \neg y) \quad \text{and} \quad x \to y = \neg(x * \neg y).$$

Also the De Morgan laws, $x \wedge y = \neg(\neg x \vee \neg y)$ and $x \vee y = \neg(\neg x \wedge \neg y)$, are satisfied. An integral residuated lattice that satisfies the prelinearity equation

(2) $$(x \to y) \vee (y \to x) = \top$$

is called a *generalized MTL-algebra* (GMTL-algebra, for short). A *generalized Gödel algebra* (also known in the literature as *generalized linear Heyting algebra* or as *relative Stone algebra* (see [10])) is a GMTL-algebra that satisfies the equation

(3) $$(x \wedge y) \to (y * x) = \top.$$

Bounded GMTL-algebras are *MTL-algebras*, the algebras corresponding to the monoidal t-norm based logic defined on [11]. *Involutive MTL-algebras* (IMTL-algebras) are MTL-algebras whose underlying residuated lattice is involutive. A *nilpotent minimum algebra* $\boldsymbol{A} = (A, \wedge, \vee, *, \to, \perp, \top)$ is an IMTL-algebra that satisfies the equation

(4) $$(x * y \to \perp) \vee (x \wedge y \to x * y) = \top.$$

 

The class of NM-algebras forms a proper subvariety of the variety of MTL-algebras. We shall refer to this variety as $\mathcal{NM}$. The variety $\mathcal{MV}_3$ of MV-algebras generated by the three element Łukasiewicz chain is a proper subvariety of $\mathcal{NM}$ characterized by the equation

$$(x \wedge y) \to (x * (x \to y)) = \top. \tag{5}$$

Since the variety of Boolean algebras is a proper subvariety of $\mathcal{MV}_3$, it is also a proper subvariety of $\mathcal{NM}$. An NM-algebra $\boldsymbol{A}$ is called *trivial* provided that $A = \{\top\}$.

Some elements of an NM-algebra can be characterized by the order relationship with their own negation. As in [13], we defined *the set of positive elements of* $\boldsymbol{A}$ as the set

$$A^+ = \{x \in A : x > \neg x\}.$$

Similarly

$$A^- = \{x \in A : x < \neg x\}$$

is called *the set of negative elements of* $\boldsymbol{A}$. In [14], it is proved that an NM-algebra can have at most one element $x$ such that $x = \neg x$. Such point is usually called *negation fixpoint* (or simply *fixpoint*) *of the NM-algebra*. If $p$ is the negation fixpoint of an NM-algebra $\boldsymbol{A}$, the set $\{\bot, p, \top\}$ constitutes the universe of a subalgebra of $\boldsymbol{A}$.

If the lattice order of an NM-algebra $\boldsymbol{A}$ is total we shall call the algebra *NM-chain*. Clearly, for every NM-chain $\boldsymbol{A}$ with a fixpoint $p$, we have $A = A^+ \cup A^- \cup \{p\}$. If $\boldsymbol{A}$ has no fixpoint, then $A = A^+ \cup A^-$.

NM-chains play a key role in the study of subvarieties of $\mathcal{NM}$ because of the following subdirect representation theorem that can be derived from [11, Proposition 3]:

**Theorem 2.1** *Every NM-algebra is a subdirect product of NM-chains.*

The most important example of NM-chain is the nilpotent minimum t-norm. Its universe is the real interval $[0,1]$ with the usual order. The operations are given by: $\neg x = 1 - x$, $x \to y = \top$ if $x \le y$, $x \to y = \neg x \vee y$ otherwise. The behaviour of $*$ is given by equation (1). The NM-algebra $[\mathbf{0},\mathbf{1}] = ([0,1], \wedge, \vee, * \to, \neg, 0, 1)$ is known as *the standard NM-algebra*. The point $x = \frac{1}{2}$ is the fixpoint of the algebra. Generalizing the behaviour on $[\mathbf{0},\mathbf{1}]$ (see [13]), it can be proved that in any NM-chain the operations $*$ and $\to$ are related to $\wedge, \vee, \neg, \top, \bot$ in the following way:

$$x * y = \begin{cases} \bot & \text{if } y \le \neg x, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{6}$$

$$x \to y = \begin{cases} \top & \text{if } x \le y, \\ \neg x \vee y & \text{otherwise.} \end{cases} \tag{7}$$

Therefore, up to isomorphism, for each finite $n \in \mathbb{N}$, there is only one nilpotent minimum chain $\boldsymbol{A}_n$ with exactly $n$ elements. The algebra $\boldsymbol{A}_3$ is the Łukasiewicz finite chain with three elements and the algebra $\boldsymbol{A}_2$ is the two elements Boolean chain. We shall refer to the universe of $\boldsymbol{A}_3$ as the set $\{\bot, p, \top\}$, where $p$ denotes the only element of $A_3$ which is not a constant (i. e. the fixpoint). We conclude this section with the following lemma:

**Lemma 2.2** *Let $\boldsymbol{A}$ be an NM-algebra, and let $x \in A^+$ and $z \in A^-$. Then $x > z$. If $p$ is the negation fixpoint, then $x > p > z$.*

P r o o f. Let $\boldsymbol{A}$ be an NM-chain. If we assume conversely that $x \le z$, we have

$$\neg x < x \le z < \neg z.$$

Since the negation operator inverts the order we obtain

$$x = \neg\neg x > \neg x \ge \neg z > \neg\neg z = z$$

leading to an absurdum. If $\boldsymbol{A}$ is an arbitrary NM-algebra, since $\boldsymbol{A}$ is a subdirect product of NM-chains the relation holds coordinatewise and the result follows. In an analogous way one can prove that if $\boldsymbol{A}$ has a fixpoint $p$, then $x > p$ for each $x \in A^+$ and $p > z$ for each $z \in A^-$. $\square$

 

### 2.2 Implicative filters and congruences

**Definition 2.3** An *implicative filter of an NM-algebra $\boldsymbol{A}$* is a subset $F \subseteq A$ satisfying the following conditions:

1. $\top \in F$;
2. for all $x, y \in A$, if $x \in F$ and $x \leq y$, then $y \in F$;
3. if $x, y \in F$, then $x * y \in F$.

Alternatively, a filter $F$ of an NM-algebra $\boldsymbol{A}$ can be defined as a subset $F \subseteq A$ such that $\top \in F$ and if $x \in F$ and $x \to y \in F$, then $y \in F$. Therefore filters of NM-algebras are closed under $*$, $\vee$ and $\to$. Moreover, since in any residuated lattice the equation $x * y \leq x \wedge y$ holds (see [16]), then if $F$ is a filter, $\boldsymbol{F} = (F, \wedge, \vee, *, \to, \top)$ is a residuated lattice that satisfies (2), i. e. $\boldsymbol{F}$ is a GMTL-algebra.

An implicative filter is called *proper* provided that $F \neq A$. If $W$ is a subset of an NM-algebra $\boldsymbol{A}$, the implicative filter generated by $W$ will be denoted by $\langle W \rangle$.

Implicative filters characterize congruences in NM-algebras. Indeed, there is a bijection between the congruences of an NM-algebra $\boldsymbol{A}$ and its implicative filters (see [16, Proposition 1.3]). We shall denote by $\boldsymbol{A}/F$ the quotient of $\boldsymbol{A}$ by the congruence corresponding to $F$.

As usual, if $\psi : \boldsymbol{A} \longrightarrow \boldsymbol{A}'$ is a homomorphism from the NM-algebra $\boldsymbol{A}$ onto the NM-algebra $\boldsymbol{A}'$, then

$$F_\psi = \psi^{-1}(\{\top\}) = \{x \in A \,:\, \psi(x) = \top\}$$

is an implicative filter of $\boldsymbol{A}$.

### 2.3 Representation of NM-algebras as weak Boolean product of directly indecomposable algebras

Letting $\boldsymbol{A}$ be an NM-algebra, one can always consider the set $B(\boldsymbol{A})$ given by

$$B(\boldsymbol{A}) = \{x \in A \,:\, x \vee \neg x = \top \text{ and } x \wedge \neg x = \bot\}.$$

This set is the universe of a subalgebra $\boldsymbol{B}(\boldsymbol{A})$ of $\boldsymbol{A}$ which is a Boolean algebra and it is usually known as *the Boolean skeleton of $\boldsymbol{A}$*. Moreover, if $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A}$ which is a Boolean algebra, then $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{B}(\boldsymbol{A})$. Notice that if $\boldsymbol{A}$ is an NM-chain, then $\boldsymbol{B}(\boldsymbol{A}) \cong \boldsymbol{A}_2$. If $U$ is a filter of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$, then the implicative filter $\langle U \rangle$ is called a *Stone filter of $\boldsymbol{A}$*.

An NM-algebra $\boldsymbol{A}$ is *directly indecomposable* if it can not be decomposed into the direct product of two non-trivial NM-algebras. The following result can be derived from [16].

**Lemma 2.4** *An NM-algebra $\boldsymbol{A}$ is directly indecomposable iff $\boldsymbol{B}(\boldsymbol{A}) \cong \boldsymbol{A}_2$.*

As usual, given a Boolean algebra $\boldsymbol{B}$ one can provide the set of its ultrafilters with the Stone topology to obtain the corresponding Boolean space $\mathrm{Sp}(\boldsymbol{B})$ (see [3]). A *weak Boolean product of a family $(\boldsymbol{A}_y, y \in Y)$ of algebras over a Boolean space $Y$* is a subdirect product $\boldsymbol{A}$ of the given family such that the following conditions hold:

1. If $a, b \in A$, then $[a = b] = \{y \in Y \,:\, a_y = b_y\}$ is open.
2. If $a, b \in A$ and $Z$ is a clopen in $Y$, then $a|_Z \cup b|_{Y \setminus Z} \in A$.

An algebra $\boldsymbol{A}$ is *representable as a weak Boolean product* when there exists a family of algebras $(\boldsymbol{A}_y, y \in Y)$ over a Boolean space $Y$ such that $\boldsymbol{A}$ is isomorphic to a weak Boolean product of the given family. Since the variety of bounded residuated lattices is arithmetical (see [16]), it has the Boolean factor congruence property. By [1], each non-trivial NM-algebra can be represented as a weak Boolean product of directly indecomposable NM-algebras. The explicit representation of each NM-algebra as a weak Boolean product of directly indecomposable NM-algebras is the following:

**Lemma 2.5** *Let $\boldsymbol{A}$ be an NM-algebra and let $\mathrm{Sp}\boldsymbol{B}(\boldsymbol{A})$ be the Boolean space of ultrafilters of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$. Then $\boldsymbol{A}$ is representable as a weak Boolean product of the family*

$$((\boldsymbol{A}/\langle U \rangle) \,:\, U \in \mathrm{Sp}\boldsymbol{B}(\boldsymbol{A}))$$

*over the Boolean space $\mathrm{Sp}\boldsymbol{B}(\boldsymbol{A})$.*

 

The purpose of the present paper is to use this representation to obtain a concrete description of the free algebra in $\mathcal{NM}$ as a weak Boolean product of directly indecomposable NM-algebras. To achieve such aim, we will study the structure of the Boolean skeleton of an NM-algebra and also the structure of directly indecomposable NM-algebras.

## 3 Boolean skeleton of NM-algebras

Let $x^2$ be an abbreviation for $x * x$. We define over any NM-algebra $\boldsymbol{A}$ the following terms:

$$\nabla(x) = \neg(\neg x^2)^2 \quad \text{and} \quad \Delta(x) = (\neg(\neg x)^2)^2.$$

These operators provide a tool to verify whether an NM-chain does or does not have a fixpoint. Precisely, the following result can be found in [13, Theorem 2].

**Lemma 3.1** *An NM-chain $\boldsymbol{A}$ does not have a fixpoint iff*

$$\nabla(x) = \Delta(x)$$

*for all $x \in A$.*

If $\boldsymbol{A} = [\mathbf{0}, \mathbf{1}]$ is the standard NM-algebra we have

$$\nabla(x) = \begin{cases} 1 & \text{if } x > \frac{1}{2}, \\ 0 & \text{if } x \leq \frac{1}{2}, \end{cases}$$

and

$$\Delta(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{2}, \\ 0 & \text{if } x < \frac{1}{2}. \end{cases}$$

Recalling from (6) and (7) the behaviour of $*$ and $\rightarrow$ in NM-chains, the following result can be verified:

**Lemma 3.2** *Let $\boldsymbol{A}$ be an NM-chain. Then we have*

$$\nabla(x) = \begin{cases} \top & \textit{if } x > \neg x, \\ \bot & \textit{if } x \leq \neg x, \end{cases} \tag{8}$$

*and*

$$\Delta(x) = \begin{cases} \top & \textit{if } x \geq \neg x, \\ \bot & \textit{if } x < \neg x. \end{cases} \tag{9}$$

From Theorem 2.1 and Lemma 3.2 we can assert that the images of both operators $\nabla$ and $\Delta$ are included in the Boolean skeleton of the domain. If $\boldsymbol{A}$ is an NM-algebra satisfying the equation $\Delta(x) = \nabla(x)$, the operator $\nabla$ is a retract from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$ (see [9]). Therefore one can always describe the Boolean skeleton of an NM-algebra satisfying $\Delta(x) = \nabla(x)$ as the image of $\nabla$. This is no longer the case for an arbitrary NM-algebra. That is the reason why we have to develop some alternative methods to describe the Boolean skeleton of an arbitrary NM-algebra. We shall do that studying the biggest subalgebra of $\boldsymbol{A}$ that belongs to $\mathcal{MV}_3$.

To achieve such aim, let the term $\varphi$ be defined over any NM-algebra by:

$$\varphi(x) = \Delta(x) \wedge (\nabla(x \vee \neg x) \vee x). \tag{10}$$

Over the standard NM-algebra $[\mathbf{0}, \mathbf{1}]$, $\varphi$ has the following behaviour:

$$\varphi(x) = \begin{cases} 1 & \text{if } x > \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = \frac{1}{2}, \\ 0 & \text{if } x < \frac{1}{2}. \end{cases}$$

**Theorem 3.3** *Let $\boldsymbol{A}$ be an NM-algebra. The map $\varphi : A \longrightarrow A$ is a homomorphism.*

P r o o f. To prove that $\varphi$ is a homomorphism, from Theorem 2.1, it is enough to check that in any NM-chain $\boldsymbol{A}$ the following equations are satisfied:

(e1) $\varphi(\top) = \top$ and $\varphi(\bot) = \bot$,
(e2) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$,
(e3) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$,
(e4) $\varphi(\neg x) = \neg\varphi(x)$,
(e5) $\varphi(x * y) = \varphi(x) * \varphi(y)$,
(e6) $\varphi(x \rightarrow y) = \varphi(x) \rightarrow \varphi(y)$.

Equations (e1) are easy to check. In order to prove the remaining equations, we consider the following three possible situations for an element $x$ in an NM-chain $\boldsymbol{A}$:

C a s e 1: $x \in A^+$. Considering equations (8) and (9), we get $\Delta(x) = \top$ and $\nabla(x \vee \neg x) = \top$, thus $\varphi(x) = \top$.
C a s e 2: $x \in A^-$. In this case $\Delta(x) = \bot$ implies $\varphi(x) = \bot$.
C a s e 3: $x = \neg x$. Hence $\Delta(x) = \top$, $\nabla(x \vee \neg x) = \bot$, and $\varphi(x) = x$.
Summing up

$$\varphi(x) = \begin{cases} \top & \text{if } x > \neg x, \\ x & \text{if } x = \neg x, \\ \bot & \text{if } x < \neg x. \end{cases} \tag{11}$$

Checking equations (e2), (e3) and (e4) is left to the reader. To check equation (e5) we consider the following possibilities:

1. $x \in A^+$ and $y \in A^+$. In such case, $\varphi(x) = \varphi(y) = \top$, hence $\varphi(x) * \varphi(y) = \top$. Since either $y \geq x > \neg x$ or $x \geq y > \neg y$, from (6) we know that $x * y = x \wedge y$. From (e2) we obtain

$$\varphi(x * y) = \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) = \top$$

as desired.

2. $x \in A^-$ and $y \in A^-$. This being the case, $\varphi(x) = \varphi(y) = \bot$, thus $\varphi(x) * \varphi(y) = \bot$. Now $x * y = \bot$, and

$$\varphi(x * y) = \varphi(\bot) = \bot.$$

3. $x \in A^-$ and $y \in A^+$. Hence $\varphi(x) = \bot$, $\varphi(y) = \top$, and $\varphi(x) * \varphi(y) = \bot$. From the total order of $A$ one of the following situations must happen: either $x \leq \neg y$ or $\neg x < y$. In the first case $x * y = \bot$. In the second one, from Lemma 2.2, $x * y = x \wedge y = x$. In both cases $\varphi(x * y) = \bot$ as desired.

4. $x = \neg x$ and $y = \neg y$. Since in each NM-algebra there is at most one fixpoint, we have $x = y$. Thus

$$\varphi(x) * \varphi(y) = x * y = x * x = \bot = \varphi(\bot) = \varphi(x * x) = \varphi(x * y).$$

5. $x \in A^+$ and $y = \neg y$. Then $\varphi(x) = \top$ and $\varphi(y) = y$, so $\varphi(x) * \varphi(y) = y$. Lemma 2.2 implies $x > y$. Using again (6) we have $x * y = x \wedge y = y$, thus $\varphi(x * y) = \varphi(y) = y$.

6. $x \in A^-$ and $y = \neg y$. We have $\varphi(x) = \bot$ and $\varphi(y) = y$, hence $\varphi(x) * \varphi(y) = \bot$. In this case $x * y = \bot$, thus $\varphi(x * y) = \varphi(\bot) = \bot$.

Since the operation $*$ is commutative, each of the remaining cases is analogous to one of the previous.

Lastly, according to equation (1) the operator $\rightarrow$ can be defined in terms of $*$ and $\neg$, thus (e6) is a consequence of (e4) and (e5). □

From equation (11) on the chain $\boldsymbol{A}_3 \in \mathcal{MV}_3$, the equation $\varphi(x) = x$ holds. Then we can conclude:

**Lemma 3.4** *If $\boldsymbol{A} \in \mathcal{MV}_3$, then $\varphi(x) = x$ for each $x \in A$, i. e. $\varphi$ is the identity function over $\boldsymbol{A}$.*

For each NM-algebra $\boldsymbol{A}$ we define the set $\mathrm{MV}_3(\boldsymbol{A}) = \{\varphi(a) \,:\, a \in A\}$.

 

**Theorem 3.5** *Let $\boldsymbol{A}$ be an NM-algebra. Then*

$$\mathbf{MV_3}(\boldsymbol{A}) = (\mathrm{MV}_3(\boldsymbol{A}), \wedge, \vee, *, \rightarrow, \neg, \bot, \top)$$

*is a subalgebra of $\boldsymbol{A}$ which is in $\mathcal{MV}_3$. Moreover, if $\boldsymbol{C} \in \mathcal{MV}_3$ is a subalgebra of $\boldsymbol{A}$, then $\boldsymbol{C}$ is a subalgebra of $\mathbf{MV_3}(\boldsymbol{A})$.*

Proof. Since $\varphi$ is a homomorphism, we know that $\mathrm{MV}_3(\boldsymbol{A})$ is closed under the operations of NM-algebras, thus $\mathbf{MV_3}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$. If $\boldsymbol{A}$ is an NM-chain, from equation (11) we know that $\mathbf{MV_3}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}_3$, so $\mathbf{MV_3}(\boldsymbol{A}) \in \mathcal{MV}_3$. The general result follows from Theorem 2.1 and the fact that $\varphi$ is given by a term.

Finally assume that $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{A}$ which is in $\mathcal{MV}_3$. Let $x \in C$. By Lemma 3.4, $x = \varphi(x)$, which means $x \in \mathrm{MV}_3(\boldsymbol{A})$. Therefore $\boldsymbol{C}$ is a subalgebra of $\mathbf{MV_3}(\boldsymbol{A})$. □

Given an NM-algebra $\boldsymbol{A}$, the algebra $\mathbf{MV_3}(\boldsymbol{A})$ shall be called *the MV-skeleton of $\boldsymbol{A}$*. From Lemma 3.4 and Theorem 3.5 we obtain that for each $x \in A$,

$$\varphi(\varphi(x)) = \varphi(x).$$

Therefore we conclude:

**Corollary 3.6** *The homomorphism $\varphi$ is a retract from an NM-algebra $\boldsymbol{A}$ onto $\mathbf{MV_3}(\boldsymbol{A})$.*

**Remark 3.7** Let $\mathcal{L}_3$ denote the three-valued Łukasiewicz propositional logic. As an important consequence of the previous results we obtain a Glivenko like theorem for the nilpotent minimum logic (see [8]), stating that a formula $\alpha$ is deducible in $\mathcal{L}_3$ iff $\varphi(\alpha)$ is deducible in the logic $\mathcal{NML}$. There is a similar result in [18] stating that a formula $\alpha$ is deducible in $\mathcal{L}_3$ iff there is a term depending on $\alpha$ that is deducible in the logic corresponding to a certain subvariety of MTL-algebras that includes the variety $\mathcal{NM}$. It is worth to notice that, although the terms are not the same, they are equivalent over NM-algebras.

**Theorem 3.8** *For any NM-algebra $\boldsymbol{A}$, we have $\boldsymbol{B}(\boldsymbol{A}) = \boldsymbol{B}(\mathbf{MV_3}(\boldsymbol{A}))$.*

Proof. The inclusion $B(\boldsymbol{A}) \supseteq B(\mathbf{MV_3}(\boldsymbol{A}))$ is trivial. For the other inclusion, since $\boldsymbol{A}$ is a subdirect product of NM-chains, if $\boldsymbol{x} \in B(\boldsymbol{A})$, the coordinates $x_i$ of $\boldsymbol{x}$ in the subdirect product are only $\bot$ and $\top$. This being the case, $\varphi(\boldsymbol{x}) = \boldsymbol{x}$, hence $\boldsymbol{x} \in B(\mathbf{MV_3}(\boldsymbol{A}))$. □

Given an NM-algebra $\boldsymbol{A}$, Theorem 3.8 suggests that instead of searching for Boolean elements on the whole universe $A$ we shall seek for them in $\mathrm{MV}_3(\boldsymbol{A})$. Therefore the knowledge of the MV-skeleton provides us of valuable information about the Boolean skeleton.

## 4 Directly indecomposable NM-algebras

As a consequence of Lemma 2.4 we know that in order to check that an NM-algebra $\boldsymbol{A}$ is directly indecomposable we have to check that its only Boolean elements are $\bot$ and $\top$. Due to Theorem 3.8 we can give an alternative way of checking that an NM-algebra is directly indecomposable that depends on the MV-skeleton.

**Theorem 4.1** *An NM-algebra $\boldsymbol{A}$ is directly indecomposable iff $\mathbf{MV_3}(\boldsymbol{A}) \cong \boldsymbol{A}_2$ or $\mathbf{MV_3}(\boldsymbol{A}) \cong \boldsymbol{A}_3$. Moreover, if $\boldsymbol{A}$ has no fixpoint, then $\boldsymbol{A}$ is directly indecomposable iff $\mathbf{MV_3}(\boldsymbol{A}) \cong \boldsymbol{A}_2$.*

Proof. Lemma 2.4 together with Theorem 3.8 implies that $\boldsymbol{A}$ is directly indecomposable iff

$$\boldsymbol{B}(\mathbf{MV_3}(\boldsymbol{A})) \cong \boldsymbol{A}_2.$$

But $\boldsymbol{B}(\mathbf{MV_3}(\boldsymbol{A})) \cong \boldsymbol{A}_2$ iff $\mathbf{MV_3}(\boldsymbol{A})$ is directly indecomposable. From [7, Chapter 6] an algebra $\boldsymbol{C} \in \mathcal{MV}_3$ is directly indecomposable iff $\boldsymbol{C} \cong \boldsymbol{A}_2$ or $\boldsymbol{C} \cong \boldsymbol{A}_3$, hence we have proved the first statement of the theorem. To complete the proof, it is enough to observe that since $\mathbf{MV_3}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$ the existence of a fixpoint in $\mathbf{MV_3}(\boldsymbol{A})$ implies the existence of a fixpoint in $\boldsymbol{A}$ and vice versa. □

 

We shall give a method for constructing directly indecomposable algebras from generalized Gödel algebras. For short, we shall refer to these last as GG-algebras.

**Definition 4.2** Let $\boldsymbol{D} = (D, \wedge, \vee, *, \rightarrow, 1)$ be a GG-algebra. We define *the disconnected rotation*

$$\mathbf{DR}(\boldsymbol{D}) = (D \times \{1\} \cup D \times \{0\}, \sqcap, \sqcup, \otimes, \Rightarrow, \neg, \bot, \top)$$

as an algebra of type $(2, 2, 2, 2, 1, 0, 0)$ with the operations given by the following prescriptions:

$$(x,i) \sqcap (y,j) = (y,j) \sqcap (x,i) = \begin{cases} (x \wedge y, 1) & \text{if } i = j = 1, \\ (x \vee y, 0) & \text{if } i = j = 0, \\ (x, 0) & \text{if } i < j, \end{cases}$$

$$(x,i) \sqcup (y,j) = (y,j) \sqcup (x,i) = \begin{cases} (x \vee y, 1) & \text{if } i = j = 1, \\ (x \wedge y, 0) & \text{if } i = j = 0, \\ (y, 1) & \text{if } i < j, \end{cases}$$

$$(x,i) \otimes (y,j) = (y,j) \otimes (x,i) = \begin{cases} (x * y, 1) & \text{if } i = j = 1, \\ (1, 0) & \text{if } i = j = 0, \\ (y \rightarrow x, 0) & \text{if } i < j, \end{cases}$$

$$(x,i) \Rightarrow (y,j) = \begin{cases} (x \rightarrow y, 1) & \text{if } i = j = 1, \\ (y \rightarrow x, 1) & \text{if } i = j = 0, \\ (x * y, 0) & \text{if } i > j, \\ (1, 1) & \text{if } i < j, \end{cases}$$

$$\neg(x,i) = \begin{cases} (x, 1) & \text{if } i = 0, \\ (x, 0) & \text{if } i = 1, \end{cases}$$

$$\top = (1,1), \quad \bot = (1,0).$$

**Theorem 4.3** *Let $\boldsymbol{D}$ be a GG-algebra. The disconnected rotation $\mathbf{DR}(\boldsymbol{D})$ of $\boldsymbol{D}$ is a directly indecomposable NM-algebra without fixpoint.*

P r o o f. In [15], it is proved that $\mathbf{DR}(\boldsymbol{D})$ is an involutive residuated lattice. Clearly, there is no negation fixpoint in this algebra. Then we only need to check that $\mathbf{DR}(\boldsymbol{D})$ satisfies the prelinearity equation (2), the nilpotent minimum equation (4) and that the NM-algebra is directly indecomposable. Since the prelinearity equation is satisfied in $\boldsymbol{D}$ we have

$$((x,i) \Rightarrow (y,j)) \sqcup ((y,j) \Rightarrow (x,i)) = \begin{cases} (x \rightarrow y \vee y \rightarrow x, 1) = \top & \text{if } i = j = 1, \\ (1,1) = \top & \text{if } i \neq j, \\ (y \rightarrow x \vee x \rightarrow y, 1) = \top & \text{if } i = j = 0. \end{cases}$$

Hence $\mathbf{DR}(\boldsymbol{D})$ also satisfies (2). To prove that the equation

(12) $$(((x,i) \otimes (y,j)) \Rightarrow (1,0)) \sqcup (((x,i) \sqcap (y,j)) \Rightarrow ((x,i) \otimes (y,j))) = (1,1)$$

holds we refer to [9, Lemma 5.6]. Finally it is not hard to see that $\boldsymbol{B}(\mathbf{DR}(\boldsymbol{D})) \cong \boldsymbol{A}_2$, since

$$(x,i) \sqcap \neg(x,i) = \bot$$

and

$$(x,i) \sqcup \neg(x,i) = \top$$

only if $x = 1$. □

 

**Definition 4.4** Let $\boldsymbol{D} = (D, \wedge, \vee, *, \rightarrow, 1)$ be a GG-algebra. *The connected rotation*[1)]

$$\mathbf{CR}(\boldsymbol{D}) = (D \times \{1\} \cup (\{\tfrac{1}{2}\} \times \{\tfrac{1}{2}\}) \cup D \times \{0\}, \sqcap, \sqcup, \otimes, \Rightarrow, \neg, \bot, \top)$$

is an algebra of type $(2, 2, 2, 2, 1, 0, 0)$ with the operations $\sqcap, \sqcup, \otimes, \Rightarrow, \neg$ given as in Definition 4.2 over

$$D \times \{1\} \cup D \times \{0\},$$

and extended by:

$$(x, i) \sqcap (\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}) \sqcap (x, i) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } i = 1, \\ (x, i) & \text{otherwise,} \end{cases}$$

$$(x, i) \sqcup (\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}) \sqcup (x, i) = \begin{cases} (x, i) & \text{if } i = 1, \\ (\frac{1}{2}, \frac{1}{2}) & \text{otherwise,} \end{cases}$$

$$(x, i) \otimes (\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}) \otimes (x, i) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } i = 1, \\ (1, 0) & \text{otherwise,} \end{cases}$$

$$(x, i) \Rightarrow (\tfrac{1}{2}, \tfrac{1}{2}) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } i = 1, \\ (1, 1) & \text{otherwise,} \end{cases}$$

$$(\tfrac{1}{2}, \tfrac{1}{2}) \Rightarrow (x, i) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } i = 0, \\ (1, 1) & \text{otherwise,} \end{cases}$$

$$\neg(\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2}), \quad \top = (1, 1), \quad \bot = (1, 0).$$

**Theorem 4.5** *Given a GG-algebra $\boldsymbol{D}$ the connected rotation $\mathbf{CR}(\boldsymbol{D})$ of $\boldsymbol{D}$ is a directly indecomposable NM-algebra with a negation fixpoint.*

P r o o f. The proof that $\mathbf{CR}(\boldsymbol{D})$ is an involutive residuated lattice with a fixpoint can be found in [15]. It is a routine exercise checking that $\mathbf{CR}(\boldsymbol{D})$ satisfies the prelinearity equation and the nilpotent minimum equation. To complete the proof, notice that $\boldsymbol{B}(\mathbf{CR}(\boldsymbol{D})) = \boldsymbol{B}(\mathbf{DR}(\boldsymbol{D})) \cong \boldsymbol{A}_2$, thus the NM-algebra $\mathbf{CR}(\boldsymbol{D})$ is directly indecomposable. □

Theorems 4.3 and 4.5 imply that connected and disconnected rotations of generalized Gödel algebras give us examples of directly indecomposable NM-algebras. The question that arises naturally is: Is every directly indecomposable NM-algebra a connected or disconnected rotation of a generalized Gödel algebra? The next part of the present section is advocated to answer this question.

**Definition 4.6** If $\varphi : \boldsymbol{A} \longrightarrow \mathbf{MV_3}(\boldsymbol{A})$ is the homomorphism given by (10), let $\boldsymbol{P}(\boldsymbol{A})$ be the GMTL-algebra whose universe is $\varphi^{-1}(\{\top\})$.

**Theorem 4.7** *If $\boldsymbol{A}$ is an NM-algebra, then $\boldsymbol{P}(\boldsymbol{A})$ is a GG-algebra.*

To obtain a proof of Theorem 4.7 we will prove first the following result:

**Lemma 4.8** *Let $\boldsymbol{A}$ be a non-trivial NM-algebra and let $x \in A$. If $\varphi(x) = \top$, then $x > \neg x$.*

P r o o f. By equation (11), the quasiequation

$$\varphi(x) = \top \Rightarrow \neg x \rightarrow x = \top$$

holds in any NM-chain. Hence Theorem 2.1 implies that if $\varphi(x) = \top$, then $\neg x \leq x$. Now assume that $x = \neg x$. Then $\top = \varphi(x) = \varphi(\neg x) = \bot$, contradicting the non-triviality of $\boldsymbol{A}$. □

1) Our definition of connected rotation is analogous but not exactly the same as the one given in [15]. The algebra $\mathbf{CR}(\boldsymbol{D})$ that we define is, according to [15], the connected rotation of the semigroup obtained by adding a lower bound to the GG-algebra $\boldsymbol{D}$.

 

P r o o f o f T h e o r e m 4.7. Consider $x, y \in P(\boldsymbol{A}) \subseteq A$. Since every NM-algebra satisfies equation (4), we have

(13) $$(x * y \to \bot) \vee (x \wedge y \to x * y) = \top.$$

From the hypothesis $x, y \in P(\boldsymbol{A})$ we conclude that $\varphi(x * y) = \varphi(x) * \varphi(y) = \top$. Thus Lemma 4.8 implies

$$\neg(x * y) < x * y.$$

Since $x*y \leq (x \wedge y) \to (x*y)$ the left-hand side of (13) becomes $x \wedge y \to x*y$. Hence $\boldsymbol{P}(\boldsymbol{A})$ is a GMTL-algebra that satisfies equation (3), i. e. $\boldsymbol{P}(\boldsymbol{A})$ is a GG-algebra. □

Therefore, given an NM-algebra $\boldsymbol{A}$, both the connected and the disconnected rotations of $\boldsymbol{P}(\boldsymbol{A})$ are well defined.

**Theorem 4.9** *Let $\boldsymbol{A}$ be an NM-algebra without negation fixpoint. We define $\alpha : \mathbf{DR}(\boldsymbol{P}(\boldsymbol{A})) \longrightarrow \boldsymbol{A}$ by*

$$\alpha(x, i) = \begin{cases} x & \text{if } i = 1, \\ \neg x & \text{if } i = 0. \end{cases}$$

*Then $\alpha$ is an injective homomorphism from $\mathbf{DR}(\boldsymbol{P}(\boldsymbol{A}))$ into $\boldsymbol{A}$. Moreover, $\alpha$ is onto iff $\boldsymbol{A}$ is directly indecomposable.*

P r o o f. To check that $\alpha$ is a homomorphism notice that if $x, y \in P(\boldsymbol{A})$, by Lemma 4.8, we have $x > \neg x$ and $y > \neg y$. Then Lemma 2.2 yields

(14) $$x > \neg y.$$

Obviously $\alpha(\top, 1) = \top$, $\alpha(\top, 0) = \neg\top = \bot$. That $\alpha((x, i) \wedge (y, j)) = \alpha(x, i) \wedge \alpha(y, j)$ follows easily if at least one of $i$ or $j$ equals 1, and follows from the De Morgan laws if $i = j = 0$. In a similar way one can prove that $\alpha((x, i) \vee (y, j)) = \alpha(x, i) \vee \alpha(y, j)$. It is trivial that $\alpha(\neg(x, i)) = \neg\alpha(x, i)$. From (1) we can define $\to$ in terms of $*$ and $\neg$, therefore it only remains to check that $\alpha((x, i) * (y, j)) = \alpha(x, i) * \alpha(y, j)$. Since

$$\alpha((x, i) * (y, j)) = \begin{cases} x * y & \text{if } i = j = 1, \\ \neg(y \to x) & \text{if } i < j, \\ \bot & \text{if } i = j = 0, \end{cases}$$

and

$$\alpha(x, i) * \alpha(y, j) = \begin{cases} x * y & \text{if } i = j = 1, \\ \neg x * y & \text{if } i < j, \\ \neg x * \neg y & \text{if } i = j = 0, \end{cases}$$

we only need to check that $\neg x * y = \neg(y \to x)$ and that $\neg x * \neg y = \bot$ if $\varphi(x) = \varphi(y) = \top$. The first equality follows from equation (1). For the second one, notice that equation (6) implies that the quasiequation

$$\neg y \to x = \top \Rightarrow \neg x * \neg y = \bot$$

holds in any NM-chain. Then, it holds in any NM-algebra and because of (14) the result follows.

To check injectivity, let $\alpha(x, i) = \alpha(y, j)$, for some $x, y \in P(\boldsymbol{A})$ and $i, j \in \{0, 1\}$. If $i = j$, then $x = y$. Otherwise, without loss of generality we may assume that $i < j$. Then $\alpha(x, i) = \alpha(y, j)$ implies that $\neg x = y$. Since $y, x \in P(\boldsymbol{A})$, $\varphi(x) = \varphi(y) = \top$. But $\top = \varphi(y) = \varphi(\neg x) = \neg\varphi(x) = \bot$. The absurdum indicates that the case $\alpha(x, i) = \alpha(y, j)$ with $i \neq j$ cannot occur.

Since $\mathbf{DR}(\boldsymbol{P}(\boldsymbol{A}))$ is directly indecomposable, if $\alpha$ is onto, then $\boldsymbol{A}$ is directly indecomposable. Conversely, assume that $\boldsymbol{A}$ is directly indecomposable. From Theorem 4.1 we know that $\mathbf{MV_3}(\boldsymbol{A}) \cong \boldsymbol{A}_2$. Then

$$A = \varphi^{-1}(\{\top\}) \cup \varphi^{-1}(\{\bot\}) = \varphi^{-1}(\{\top\}) \cup \varphi^{-1}(\{\neg\top\}).$$

 

Let $\neg\varphi^{-1}(\{\top\}) = \{\neg y \in A : \varphi(y) = \top\}$. Hence $\varphi^{-1}(\{\neg\top\}) = \neg\varphi^{-1}(\{\top\})$, and

$$A = P(\boldsymbol{A}) \cup \neg P(\boldsymbol{A}) = \alpha(\mathrm{DR}(\boldsymbol{P}(\boldsymbol{A}))).$$

Therefore $\alpha$ is onto. □

**Theorem 4.10** *Let $\boldsymbol{A}$ be an NM-algebra with a negation fixpoint $p$. We define*

$$\beta : \mathbf{CR}(\boldsymbol{P}(\boldsymbol{A})) \to \boldsymbol{A}$$

*by $\beta(\frac{1}{2}, \frac{1}{2}) = p$ and if $x \in P(\boldsymbol{A})$, let*

$$\beta(x, i) = \begin{cases} x & \text{if } i = 1, \\ \neg x & \text{if } i = 0. \end{cases}$$

*Then $\beta$ is an injective homomorphism from $\mathbf{CR}(\boldsymbol{P}(\boldsymbol{A}))$ into $\boldsymbol{A}$. Moreover, $\beta$ is onto iff $\boldsymbol{A}$ is directly indecomposable.*

Proof. Note that $\varphi(x) = p$ iff $x = p$. Indeed, we know that $\varphi(p) = \varphi(\neg p) = \neg\varphi(p)$. On the other hand, if $\boldsymbol{A}$ is a chain and $\varphi(x) = p$, we know from (11) that $x = p$. If $\boldsymbol{A}$ is a subdirect product of chains, and $\varphi(x) = p$, then each coordinate $\varphi(x)_i$ on the product is a fixpoint. Since $\varphi$ is given by a term, we know that $\varphi(x)_i = \varphi(x_i)$. Thus $x_i$ is the fixpoint of the corresponding chain and $x$ is the fixpoint $p$. Taking this into account, one can prove that $\beta$ is a injective homomorphism as in Theorem 4.9.

Obviously $\beta$ onto implies that $\boldsymbol{A}$ is directly indecomposable. To prove the other implication, assume that $\boldsymbol{A}$ is directly indecomposable. By Theorem 4.1 we have $\mathbf{MV_3}(\boldsymbol{A}) \cong \boldsymbol{A}_3$. Thus

$$A = \varphi^{-1}(\{\top\}) \cup \varphi^{-1}(\{\bot\}) \cup \varphi^{-1}(\{p\}) = \varphi^{-1}(\{\top\}) \cup \neg\varphi^{-1}(\{\top\}) \cup \{p\}.$$

Then

$$A = P(\boldsymbol{A}) \cup \neg P(\boldsymbol{A}) \cup \{p\} = \alpha(\mathrm{CR}(\boldsymbol{P}(\boldsymbol{A})))$$

implies $\beta$ is onto. □

Theorem 4.9 and Theorem 4.10 answer our original question. As an immediate consequence of these two theorems and of the definitions of connected and disconnected rotations we have:

**Corollary 4.11** *If $\boldsymbol{A}$ is a directly indecomposable NM-algebra without a fixpoint, then $A = A^+ \cup A^-$. If $\boldsymbol{A}$ is a directly indecomposable NM-algebra with a fixpoint $p$, then $A = A^+ \cup \{p\} \cup A^-$. Moreover, in both cases $A^+ = P(\boldsymbol{A})$ and $A^- = \neg P(\boldsymbol{A})$.*

In the terminology of [17] and [18], the previous corollary is asserting that directly indecomposable NM-algebras without a fixpoint are perfect IMTL-algebras and directly indecomposable NM-algebras with fixpoint are perfect IMTL-algebras plus fixpoint.

If $\boldsymbol{A}$ is directly indecomposable, another useful consequence of Theorems 4.9 and 4.10, because of the definitions of the operations $*$ and $\neg$ in the connected and disconnected rotations, is the next lemma:

**Lemma 4.12** *Let $\boldsymbol{A}$ be a directly indecomposable NM-algebra. Then the operators $\nabla, \Delta$ and $\varphi$ have the following behaviour:*

$$\nabla(x) = \begin{cases} \top & \text{if } x > \neg x, \\ \bot & \text{if } x \leq \neg x, \end{cases} \tag{15}$$

$$\Delta(x) = \begin{cases} \top & \text{if } x \geq \neg x, \\ \bot & \text{if } x < \neg x, \end{cases} \tag{16}$$

$$\varphi(x) = \begin{cases} \top & \text{if } x > \neg x \\ x & \text{if } x = \neg x, \\ \bot & \text{if } x < \neg x. \end{cases} \tag{17}$$

 

As an immediate consequence of this result, if $\boldsymbol{C}$ be a GG-algebra, we have

$$P(\mathbf{CR}(\boldsymbol{C})) = P(\mathbf{DR}(\boldsymbol{C})) = \{(x,1) : x \in C\}.$$

Therefore we obtain:

**Theorem 4.13** *If $\boldsymbol{C}$ is a GG-algebra, the applications $\delta : \boldsymbol{P}(\mathbf{CR}(\boldsymbol{C})) \longrightarrow \boldsymbol{C}$ and $\gamma : \boldsymbol{P}(\mathbf{DR}(\boldsymbol{C})) \longrightarrow \boldsymbol{C}$ defined by*

$$\delta((x,1)) = x \quad \textit{and} \quad \gamma((x,1)) = x$$

*are isomorphisms from their respective domains onto $\boldsymbol{C}$.*

### 4.1 Generators of the GG-algebra $\boldsymbol{P}(\boldsymbol{A})$

In the present section we shall see how to obtain a set of generators for the GG-algebra $\boldsymbol{P}(\boldsymbol{A})$ from a set of generators of the whole NM-algebra $\boldsymbol{A}$.

**Theorem 4.14** *Let $\boldsymbol{A}$ be a directly indecomposable NM-algebra and let $X$ be a set of generators of $\boldsymbol{A}$. If $\boldsymbol{A}$ has no fixpoint, then the set*

$$H = \{x \in X : \varphi(x) = \top\} \cup \{\neg x : x \in X \textit{ and } \varphi(x) = \bot\}$$

*also generates $\boldsymbol{A}$. If $\boldsymbol{A}$ has a fixpoint p, the set $H' = H \cup \{p\}$ generates the algebra.*

P r o o f. Assume that $\boldsymbol{A}$ has no fixpoint. Call $\langle \boldsymbol{H} \rangle$ the subalgebra of $\boldsymbol{A}$ generated by $H$. It is enough to check that $X \subseteq \langle H \rangle$. Let $x \in X$. If $\varphi(x) = \top$, then $x \in H$. Otherwise, $\neg x \in H$. Hence $\neg\neg x = x \in \langle H \rangle$ as required.

Now suppose that $p$ is the fixpoint of the directly indecomposable $\boldsymbol{A} = \mathbf{CR}(\boldsymbol{P}(\boldsymbol{A}))$. Then

$$A \setminus \{p\} = \mathrm{DR}(\boldsymbol{P}(\boldsymbol{A})),$$

and clearly $\langle \boldsymbol{H} \rangle = \mathbf{DR}(\boldsymbol{P}(\boldsymbol{A}))$. Thus $\langle H' \rangle = \mathrm{DR}(\boldsymbol{P}(\boldsymbol{A})) \cup \{p\} = A$. $\square$

**Theorem 4.15** *Let $\boldsymbol{A}$ and $H$ be as in Theorem* 4.14*. Then $H$ generates $\boldsymbol{P}(\boldsymbol{A})$ as a GG-algebra.*

P r o o f. Let $\boldsymbol{G}(H)$ be the GG-algebra generated by $H$. Since $H \subseteq P(\boldsymbol{A})$, we have $G(H) \subseteq P(\boldsymbol{A})$. Assume that $G(H) \subset P(\boldsymbol{A})$. By construction $\mathrm{DR}(\boldsymbol{G}(H)) \subset A = \mathrm{DR}(\boldsymbol{P}(\boldsymbol{A}))$. But $H \subseteq \mathrm{DR}(\boldsymbol{G}(H))$ and $H$ generates $\boldsymbol{A}$. Then we conclude that $H$ generates $\boldsymbol{P}(\boldsymbol{A})$ as a GG-algebra. $\square$

## 5 Free NM-algebras

Recall that an algebra $\boldsymbol{A}$ in a variety $\mathcal{K}$ is said to be *free over a set* $Y$ iff for every algebra $\boldsymbol{C}$ in $\mathcal{K}$ and every function $f : Y \longrightarrow \boldsymbol{C}$, $f$ can be uniquely extended to a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{C}$. Given a variety $\mathcal{K}$ of algebras, we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in $\mathcal{K}$ over $X$. As a consequence of Lemma 2.5 we have:

**Theorem 5.1** *The free NM-algebra $\mathbf{Free}_{\mathcal{NM}}(X)$ can be represented as a weak Boolean product of the family*

$$((\mathbf{Free}_{\mathcal{NM}}(X)/\langle U \rangle) : U \in \mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X)))$$

*over the Boolean space $\mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$.*

In the present section we shall give an explicit description of the Boolean skeleton $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ and of the directly indecomposable NM-algebras $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U \rangle$ for each $U \in \mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$.

### 5.1 Boolean skeletons of free algebras

**Theorem 5.2** *Let $X$ be a set of free generators of the free NM-algebra $\mathbf{Free}_{\mathcal{NM}}(X)$ and let*

$$Z = \{\varphi(x) : x \in X\}.$$

*Then*

$$\mathbf{MV_3}(\mathbf{Free}_{\mathcal{NM}}(X)) = \mathbf{Free}_{\mathcal{MV}_3}(Z).$$

P r o o f. Let $\boldsymbol{C} \in \mathcal{MV}_3$ and $f : Z \longrightarrow \boldsymbol{C}$. We define $f' : X \longrightarrow \boldsymbol{C}$ by

$$f'(x) = f(\varphi(x)).$$

Since $\boldsymbol{C} \in \mathcal{NM}$, $f'$ can be extended to a homomorphism $h : \mathbf{Free}_{\mathcal{NM}}(X) \longrightarrow \boldsymbol{C}$ such that

$$h(x) = f'(x) = f(\varphi(x)).$$

Let $h'$ be the restriction of $h$ to the subalgebra $\mathbf{MV_3}(\mathbf{Free}_{\mathcal{NM}}(X))$. From its definition, $\mathbf{MV_3}(\mathbf{Free}_{\mathcal{NM}}(X))$ is the subalgebra of $\mathbf{Free}_{\mathcal{NM}}(X)$ generated by $Z$. Since $\varphi(x)$ is a term, we have

$$h'(\varphi(x)) = h(\varphi(x)) = \varphi(h(x)) = \varphi(f'(x)) = \varphi(f(\varphi(x)).$$

Since $f((\varphi(x))$ is an element of an algebra $\boldsymbol{C} \in \mathcal{MV}_3$, from Lemma 3.4 we obtain

$$\varphi(f(\varphi(x)) = f(\varphi(x)).$$

Then $h'(\varphi(x)) = f(\varphi(x))$ and the result follows. □

In [4] a characterization of the Boolean skeleton of a free MV-algebra in the subvariety $\mathcal{MV}_3$ is given in terms of the Moisil operators $\sigma_1, \sigma_2$ defined on each algebra in $\mathcal{MV}_3$. Information about these Moisil operators can be obtained in [2, 5, 6]. One can check that the operator $\sigma_1$ used in [4] coincides with $\nabla$ and $\sigma_2$ with $\Delta$ on the chain $\boldsymbol{A}_3$. Since $\boldsymbol{A}_3$ generates the variety $\mathcal{MV}_3$ and all these operators are given by terms, we have

$$\sigma_1(x) = \nabla(x) \quad \text{and} \quad \sigma_2(x) = \Delta(x)$$

for all $x$ in an algebra $\boldsymbol{A} \in \mathcal{MV}_3$. Replacing $\nabla$ and $\Delta$ for $\sigma_1$ and $\sigma_2$, the result in [4, Theorem 3.12] asserts:

**Theorem 5.3** $\boldsymbol{B}(\mathbf{Free}_{\mathcal{MV}_3}(Z))$ *is the free Boolean algebra over the poset* $Z' = \{\nabla(z), \Delta(z) : z \in Z\}$.

Because of equation (11) and Theorem 2.1, for any element $x$ in an NM-algebra $\boldsymbol{A}$, $\nabla(\varphi(x)) = \nabla(x)$ and $\Delta(\varphi(x)) = \Delta(x)$. Then applying Theorems 3.8, 5.2 and 5.3 we conclude:

**Theorem 5.4** $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ *is the free Boolean algebra over the poset* $Z = \{\nabla(x), \Delta(x) : x \in X\}$.

We know that the ultrafilters of a Boolean algebra are in bijective correspondence with the homomorphisms from the algebra into the two elements Boolean algebra, $\boldsymbol{A}_2$. Since every upward closed subset of the poset

$$Z = \{\nabla(x), \Delta(x) : x \in X\}$$

is in correspondence with an increasing function from $Z$ onto $\boldsymbol{A}_2$, and every increasing function from $Z$ can be extended to a homomorphism from $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ onto $\boldsymbol{A}_2$, the ultrafilters of $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ are in correspondence with the upward closed subsets of $Z$. This is summarized in the following lemma:

**Lemma 5.5** *Consider the poset* $Z = \{\nabla(x), \Delta(x) : x \in X\}$. *For each upward closed subset* $S \subseteq Z$ *consider the set* $G_S$ *given by the joint of the following four sets:*

$$\{\nabla(x) : \nabla(x) \in S\}, \quad \{\neg\nabla(x) : \nabla(x) \notin S\}, \quad \{\Delta(x) : \Delta(x) \in S\}, \quad \{\neg\Delta(x) : \Delta(x) \notin S\}.$$

*Then the correspondence that assigns to each upward closed subset* $S \subseteq Z$ *the Boolean filter* $U_S$ *generated by* $G_S$ *defines a bijection from the set of upward closed subsets of* $Z$ *onto the ultrafilters of* $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$.

Taking this fact into account, we shall refer to each ultrafilter of the Boolean skeleton $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ by $U_S$ making explicit reference to the upward closed subset $S$ that corresponds to it.

### 5.2 Directly indecomposable quotients of free algebras

**Theorem 5.6** *The directly indecomposable NM-algebra* $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ *has a fixpoint iff there is* $x \in X$ *such that* $\nabla(x) \notin S$ *and* $\Delta(x) \in S$.

P r o o f. Recall from Lemma 4.12 that the directly indecomposable algebra $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ has a fixpoint iff there is $y \in \mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ such that $\nabla(y) \neq \Delta(y)$. Assume that there is $x \in X$ such that $\nabla(x) \notin S$ and $\Delta(x) \in S$. Then $\nabla(x/\langle U_S\rangle) \neq \top$ and $\Delta(x/\langle U_S\rangle) = \top$, meaning that $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ has a fixpoint.

 

For the converse implication, notice that since the equation $\nabla(x) \to \Delta(x) = \top$ holds in any NM-chain (see Lemma 3.2), we have that the inequality $\nabla(x) \leq \Delta(x)$ holds in any NM-algebra. This means that if $\nabla(x) \in S$, then $\Delta(x) \in S$, because $S$ is upward closed. Assume that for every $x \in X$,

$$\nabla(x) \in S \quad \text{iff} \quad \Delta(x) \in S.$$

If $\nabla(x) \in S$, then $\nabla(x/\langle U_S\rangle) = \Delta(x/\langle U_S\rangle) = \top$ and if $\Delta(x) \notin S$, then $\neg\nabla(x/\langle U_S\rangle) = \neg\Delta(x/\langle U_S\rangle) = \top$. Since $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ is generated by $X_S = \{x/\langle U_S\rangle \,:\, x \in X\}$ and $\nabla$ and $\Delta$ are given by terms, we conclude that $\nabla(y) = \Delta(y)$ for all $y \in \mathrm{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$. Then $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ has no fixpoint. ☐

Applying the results of Theorem 4.9 and Theorem 4.10 we conclude:

**Theorem 5.7** *Let $X$ be a set of generators of the NM-algebra $\mathbf{Free}_{\mathcal{NM}}(X)$. Let $S$ be an increasing subset of the poset $Z = \{\nabla(x), \Delta(x) \,:\, x \in X\}$ and let $U_S$ be the ultrafilter of $\boldsymbol{B}(\mathbf{Free}_{\mathcal{NM}}(X))$ corresponding to $S$ according to Lemma* 5.5.

1. *If for all $x \in X$, $\nabla(x) \in S$ iff $\Delta(x) \in S$, then*

$$\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle \cong \mathbf{DR}(\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)).$$

2. *Otherwise*

$$\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle \cong \mathbf{CR}(\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)).$$

To obtain a precise description of the directly indecomposable NM-algebras $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ we shall study the structure of the GG-algebras $\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)$.

**Theorem 5.8** *Under the notation of Theorem* 5.7*, let the set $X_S \subseteq X/\langle U_S\rangle$ be given by*

$$X_S = \{x/\langle U_S\rangle \,:\, \varphi(x/\langle U_S\rangle) = \top\} \cup \{\neg x/\langle U_S\rangle \,:\, \varphi(x/\langle U_S\rangle) = \bot\}.$$

*Then $\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)$ is the free GG-algebra generated by $X_S$.*

P r o o f. Let $\boldsymbol{C}$ be a GG-algebra and consider an arbitrary function $f : X_S \longrightarrow C$. We define

$$f' : X \longrightarrow \mathrm{CR}(\boldsymbol{C})$$

by:

$$f'(x) = \begin{cases} (f(x/\langle U_S\rangle), 1) & \text{if } \varphi(x/\langle U_S\rangle) = \top, \\ (\frac{1}{2}, \frac{1}{2}) & \text{if } \varphi(x/\langle U_S\rangle) = p, \\ (f(\neg x/\langle U_S\rangle), 0) & \text{if } \varphi(x/\langle U_S\rangle) = \bot, \end{cases}$$

where $p$ denotes the fixpoint of $\boldsymbol{A}_3$. By the definition of free algebra, there exists a homomorphism

$$g' : \mathbf{Free}_{\mathcal{NM}}(X) \longrightarrow \mathbf{CR}(\boldsymbol{C})$$

such that $g'(x) = f'(x)$ for all $x \in X$.

We claim that $g'(\langle U_S\rangle) \subseteq \{\top\}$. To prove this, we shall see that $g'(y) = \top$ for each $y$ in the set $G_S$ given in Lemma 5.5. By Lemma 4.12, $\nabla(x/\langle U_S\rangle) = \top$ iff $\varphi(x/\langle U_S\rangle) = \top$ and $\Delta(x/\langle U_S\rangle) = \top$ iff $\varphi(x/\langle U_S\rangle) \neq \bot$.

If $\nabla(x) \in S$, then $\nabla(x/\langle U_S\rangle) = \top$, thus $\varphi(x/\langle U_S\rangle) = \top$. Then

$$g'(\nabla(x)) = \nabla(g'(x)) = \nabla(f'(x)) = \nabla((f(x/\langle U_S\rangle), 1)) = \top,$$

where the last equality follows also from Lemma 4.12.

If $\Delta(x) \in S$, then $\Delta(x/\langle U_S\rangle) = \top$. Hence $\varphi(x/\langle U_S\rangle) \neq \bot$. Therefore

$$g'(\Delta(x)) = \Delta(g'(x)) = \Delta(f'(x))$$

and either $f'(x) = (\frac{1}{2}, \frac{1}{2})$ or $f'(x) = (f(x/\langle U_S\rangle), 1)$. In both cases $\Delta(f'(x)) = \top$, as desired.

 

If $\nabla(x) \notin S$, then $\neg\nabla(x/\langle U_S\rangle) = \top$. Since $\nabla(x/\langle U_S\rangle) = \bot$ implies $\varphi(x/\langle U_S\rangle) \neq \top$, we have

$$g'(\neg\nabla(x)) = \neg\nabla(g'(x)) = \neg\nabla(f'(x))$$

and either $f'(x) = (\frac{1}{2}, \frac{1}{2})$ or $f'(x) = (f(\neg x/\langle U_S\rangle), 0)$. Hence $\neg\nabla(f'(x)) = \top$.

Finally if $\Delta(x) \notin S$, then $\neg\Delta(x/\langle U_S\rangle) = \top$. This means $\Delta(x/\langle U_S\rangle) = \bot$ and $\varphi(x/\langle U_S\rangle) = \bot$. In this case

$$g'(\neg\Delta(x)) = \neg\Delta(g'(x)) = \neg\Delta(f'(x)) = \neg\Delta(f(\neg x/\langle U_S\rangle), 0) = \top.$$

Therefore there exists a unique homomorphism

$$g : \mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle \longrightarrow \mathbf{CR}(\boldsymbol{C})$$

such that $g(y/\langle U_S\rangle) = g'(y)$ for all $y \in \mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$.

Let $h$ be the restriction of $g$ to the GG-subalgebra of $\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle$ generated by $X_S$. By Theorem 4.15 this algebra is $\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)$. Clearly the image of $h$ is contained in $\boldsymbol{P}(\mathbf{CR}(\boldsymbol{C})) \cong \boldsymbol{C}$. Consider the composition of $h$ with the function $\gamma$ given in Theorem 4.13, that is, $\gamma \circ h : P(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle) \longrightarrow C$. If $x/\langle U_S\rangle \in X_S$, then $\varphi(x/\langle U_S\rangle) = \top$, thus

$$\gamma \circ h(x/\langle U_S\rangle) = \gamma(g'(x)) = \gamma(f'(x)) = \gamma((f(x/\langle U_S\rangle), 1)) = f(x/\langle U_S\rangle).$$

If $\neg(x/\langle U_S\rangle) \in X_S$, then $\varphi(x/\langle U_S\rangle) = \bot$. Hence

$$\begin{aligned}
\gamma \circ h(\neg(x/\langle U_S\rangle)) &= \gamma(g'(\neg(x))) \\
&= \gamma(\neg(g'(x))) \\
&= \gamma(\neg(f'(x))) \\
&= \gamma(\neg(f(\neg x/\langle U_S\rangle), 0)) \\
&= f(\neg x/\langle U_S\rangle).
\end{aligned}$$

Then for an arbitrary GG-algebra $\boldsymbol{C}$ we have found a homomorphism $\gamma \circ h$ from $\boldsymbol{P}(\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle)$ into $\boldsymbol{C}$ that extends the map $f : X_S \longrightarrow C$. This implies the desired result. □

These results imply that the directly indecomposable algebras in the representation of $\mathbf{Free}_{\mathcal{NM}}(X)$ are either connected or disconnected rotations of free generalized Gödel algebras. More precisely, if we denote by $\mathcal{GG}$ the variety of generalized Gödel algebras we have proved:

**Theorem 5.9** *Let $X$ be a set of free generators of the free NM-algebra* $\mathbf{Free}_{\mathcal{NM}}(X)$. *Let*

$$Z = \{\nabla(x), \Delta(x) \,:\, x \in X\}$$

*and consider an increasing subset $S$ of $Z$. Finally, consider the ultrafilter $U_S$ related to $S$ by Lemma* 5.5 *and let $X_S = \{x/\langle U_S\rangle \,:\, \varphi(x/\langle U_S\rangle) = \top\} \cup \{\neg(x/\langle U_S\rangle) \,:\, \varphi(x/\langle U_S\rangle) = \bot\}$.*

1. *If for all $x \in X$, $\nabla(x) \in S$ iff $\Delta(x) \in S$, then*

$$\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle \cong \mathbf{DR}(\mathbf{Free}_{\mathcal{GG}}(X_S)).$$

2. *Otherwise*

$$\mathbf{Free}_{\mathcal{NM}}(X)/\langle U_S\rangle \cong \mathbf{CR}(\mathbf{Free}_{\mathcal{GG}}(X_S)).$$

Since finitely generated free generalized Gödel algebras were completely described in [10], we can have an explicit description of $\mathbf{Free}_{\mathcal{NM}}(X)$ when $X$ is a finite set.

## 6 Free algebras in subvarieties of $\mathcal{NM}$

In this last section we shall characterize free algebras in subvarieties of NM-algebras. Some of these free algebras are very well known, as it is the case of the free Boolean algebra and the free algebra in the variety of three valued Łukasiewicz algebras (see [7]). Also the free algebra in any subvariety of NM-algebras satisfying the equation $\nabla(x) = \Delta(x)$ is described in [9].

 

Given any subvariety $\mathcal{V}$ of $\mathcal{NM}$ and following Lemma 2.5, we can give a description of the free algebra $\mathbf{Free}_{\mathcal{V}}(X)$ as a weak Boolean product of directly indecomposable algebras of the form

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$$

for each $U \in \mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X))$. Therefore, as done for the free NM-algebra, one must first describe the Boolean algebra $\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X))$ and then the directly indecomposable algebras in the product. The following result is easy to verify.

**Lemma 6.1** *Let $\mathcal{V}$ be a subvariety of $\mathcal{NM}$ and let $\mathcal{V}'$ be the subvariety of $\mathcal{V}$ characterized by equation* (5). *Then*

$$\mathcal{V}' = \{\boldsymbol{B} \in \mathcal{MV}_3 \,:\, \boldsymbol{B} = \mathbf{MV_3}(\boldsymbol{A}) \textit{ for some } \boldsymbol{A} \in \mathcal{V}\}.$$

The only non-trivial subvarieties of $\mathcal{MV}_3$ are the whole variety $\mathcal{MV}_3$ and the variety of Boolean algebras $\mathcal{B}$.

**Lemma 6.2** $\mathcal{V}' = \mathcal{B}$ *iff no member of $\mathcal{V}$ has a fixpoint.*

Proof. Assume that an algebra $\boldsymbol{A} \in \mathcal{V}$ has a fixpoint $p$. Then the set $\{\bot, p, \top\}$ is the universe of a subalgebra of $\boldsymbol{A}$ which is isomorphic to $\boldsymbol{A}_3$. This implies $\boldsymbol{A}_3 \in \mathcal{V}'$, hence $\mathcal{V}' = \mathcal{MV}_3$. For the other implication, assume no member of $\mathcal{V}'$ has a fixpoint. Then $\boldsymbol{A}_3 \notin \mathcal{V}'$, thus $\mathcal{V}' = \mathcal{B}$. □

The variety $\mathcal{V}'$ plays a role in the description of $\mathbf{Free}_{\mathcal{V}}(X)$ analogous to that of $\mathcal{MV}_3$ in the description of $\mathbf{Free}_{\mathcal{NM}}(X)$. In a similar way of that of Theorem 5.2 it can be proved that

$$\mathbf{Free}_{\mathcal{V}'}(Y) = \mathbf{MV_3}(\mathbf{Free}_{\mathcal{V}}(X)),$$

where $Y = \{\varphi(x) \,:\, x \in X\}$. Notice also that if $\mathcal{V}' = \mathcal{B}$, then the equation $\varphi(y) = \nabla(y) = \Delta(y)$ holds in $\mathcal{V}$. Then the characterization of the Boolean subalgebras follows from Theorem 3.8. Precisely we have:

**Theorem 6.3** *Let $\mathcal{V}$ be a subvariety of $\mathcal{NM}$ and let $\mathcal{V}'$ be defined as in Lemma* 6.1.

(a) *If $\mathcal{V}' = \mathcal{B}$, then $\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X)) = \mathbf{Free}_{\mathcal{B}}(Z)$, with $Z = \{\nabla(x) \,:\, x \in X\}$.*

(b) *If $\mathcal{V}' = \mathcal{MV}_3$, then $\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra over the poset $Z = \{\nabla(x), \Delta(x) \,:\, x \in X\}$.*

For the first case in the previous theorem, for each subset $S \subseteq Z = \{\nabla(x) \,:\, x \in X\}$ there is an ultrafilter $U_S \in \mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X)$ generated by the sets $\{\Delta(x) \,:\, \Delta(x) \in S\}$ and $\{\neg\Delta(x) \,:\, \Delta(x) \notin S\}$. These being the case, the directly indecomposable algebras are disconnected rotations, i. e.

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle = \mathbf{DR}(\boldsymbol{P}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)).$$

For the second case, for each increasing subset

$$S \subseteq Z = \{\nabla(x), \Delta(x) : x \in X\},$$

the ultrafilter $U_S \in \mathrm{Sp}\boldsymbol{B}(\mathbf{Free}_{\mathcal{V}}(X)$ that corresponds to $S$ is the one generated by the set $G_S$ defined in Lemma 5.5. Then we have

1. If for every $x \in X$, $\nabla(x) \in S$ iff $\Delta(x) \in S$, then

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle \cong \mathbf{DR}(\boldsymbol{P}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)).$$

2. Otherwise

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle \cong \mathbf{CR}(\boldsymbol{P}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)).$$

There only remains to describe $\boldsymbol{P}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ for each of these cases. To achieve such aim, we define

$$\mathcal{V}_{\mathcal{G}} = \{\boldsymbol{C} \in \mathcal{GG} \,:\, \boldsymbol{C} = \boldsymbol{P}(\boldsymbol{A}) \text{ for some } \boldsymbol{A} \in \mathcal{V}\}.$$

As in [9, Theorem 3.9] we have:

**Lemma 6.4** *For each variety $\mathcal{V} \subseteq \mathcal{NM}$, $\mathcal{V}_{\mathcal{G}}$ is a variety of GG-algebras.*

For each $U_S \in \mathrm{Sp}\boldsymbol{B}(\mathrm{Free}_{\mathcal{V}}(X))$, let $X_S \subseteq X/\langle U_S\rangle$ be given by

$$X_S = \{x/\langle U_S\rangle \,:\, \varphi(x/\langle U_S\rangle) = \top\} \cup \{\neg x/\langle U_S\rangle \,:\, \varphi(x/\langle U_S\rangle) = \bot\}. \tag{18}$$

Let $\boldsymbol{C} \in \mathcal{V}_{\mathcal{G}}$ and let $f : X_S \longrightarrow C$ be an arbitrary map. If $\mathcal{V}$ has no algebra with fixpoint, then we can define $f' : X \longrightarrow \mathrm{DR}(\boldsymbol{C})$ by

$$f'(x) = \begin{cases} (f(x/\langle U_S\rangle), 1) & \text{if } \varphi(x/\langle U_S\rangle) = \top, \\ (f(\neg x/\langle U_S\rangle), 0) & \text{if } \varphi(x/\langle U_S\rangle) = \bot. \end{cases}$$

If $\mathcal{V}$ has an algebra with fixpoint, we can define $f' : X \to \mathbf{CR}(\boldsymbol{C})$ by

$$f'(x) = \begin{cases} (f(x/\langle U_S\rangle), 1) & \text{if } \varphi(x/\langle U_S\rangle) = \top, \\ (\frac{1}{2}, \frac{1}{2}) & \text{if } \varphi(x/\langle U_S\rangle) = p, \\ (f(\neg x/\langle U_S\rangle), 0) & \text{if } \varphi(x/\langle U_S\rangle) = \bot, \end{cases}$$

where $p$ denotes the fixpoint of $\boldsymbol{A}_3$. From the definition of $\mathcal{V}_{\mathcal{G}}$, in the first case, $\mathbf{DR}(\boldsymbol{C}) \in \mathcal{V}$ while in the second one $\mathbf{CR}(\boldsymbol{C}) \in \mathcal{V}$. In both cases, an argument analogous to the proof of Theorem 5.8 give us the following result:

**Lemma 6.5** $\boldsymbol{P}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) = \mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X_S)$.

As a summary of all the results in this section we have: let $\mathcal{V}$ be a subvariety of $\mathcal{NM}$ and let $\mathcal{V}'$ and $\mathcal{V}_{\mathcal{G}}$ be given as in Lemma 6.1 and Lemma 6.4, respectively.

1. If $\mathcal{V}' = \mathcal{B}$, then $\mathbf{Free}_{\mathcal{V}}(X)$ is a weak Boolean product of algebras of the form

$$\mathbf{DR}(\mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X_S))$$

over the Boolean space $\mathrm{Sp}\mathbf{Free}_{\mathcal{B}}(Z)$, where $Z = \{\nabla(x) \,:\, x \in X\}$ and for each $U_S \in \mathrm{Sp}\mathbf{Free}_{\mathcal{B}}(Z)$, $X_S$ is given by (18).

2. If $\mathcal{V}' = \mathcal{MV}_3$, then $\mathbf{Free}_{\mathcal{V}}(X)$ is a weak Boolean product of algebras of the form

$$\mathbf{CR}(\mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X_S)) \quad \text{or} \quad \mathbf{DR}(\mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X_S))$$

over the Boolean space corresponding to the free Boolean algebra over the poset $Z = \{\nabla(x), \Delta(x) \,:\, x \in X\}$ and $X_S$ is given by (18).

As we mentioned before, free NM-algebras in subvarieties satisfying the equation $\Delta(x) = \nabla(x)$ were described in [9]. These subvarieties only contain algebras without fixpoint. Our description coincides with the one obtained in that paper. Notice also that if $\mathcal{V} = \mathcal{MV}_3$, then the only algebra of $\mathcal{V}_{\mathcal{G}}$ is the trivial algebra, thus $\mathrm{Free}_{\mathcal{V}_{\mathcal{G}}}(X) = \{\top\}$. So we have $\mathbf{DR}(\mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X)) \cong \boldsymbol{A}_2$ and $\mathbf{CR}(\mathbf{Free}_{\mathcal{V}_{\mathcal{G}}}(X)) \cong \boldsymbol{A}_3$. Therefore our description of the free algebra in $\mathcal{MV}_3$ coincides with the one given in [7].

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