

CONSTRAINED PERCOLATION, ISING MODEL AND XOR ISING MODEL ON PLANAR LATTICES

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ABSTRACT. We study site percolation models on planar lattices including the $[m, 4, n, 4]$ lattice and the square tilings on the Euclidean plane (\mathbb{R}^2) or the hyperbolic plane (\mathbb{H}^2), satisfying certain local constraints on degree-4 faces. These models are closely related to Ising models and XOR Ising models (product of two i.i.d Ising models) on regular tilings of \mathbb{R}^2 or \mathbb{H}^2 . In particular, we obtain a description of the numbers of infinite “+” and “-” clusters of the ferromagnetic Ising model on a vertex-transitive triangular tiling of \mathbb{H}^2 for different boundary conditions and coupling constants. Our results show the possibility that such an Ising configuration has infinitely many infinite “+” and “-” clusters, while its random cluster representation has no infinite open clusters. Percolation properties of corresponding XOR Ising models are also discussed.

1. INTRODUCTION

A constrained percolation model is a probability measure on subgraphs of a lattice satisfying certain local constraints. Each subgraph is called a configuration. These models are abstract mathematical models for ubiquitous phenomena in nature, and have been interesting topics in mathematical and scientific research for long. Examples of constrained percolation models include the dimer model (see [28]), the 1-2 model (see [18]), the six-vertex model (or 6V model, see [2, 8, 29]), and general vertex models (see [34, 46, 48]). The study of these models may give deep insights to understand many natural phenomena, such as structure of matter, phase transition, limit shape, and critical behavior.

We are interested in the classical percolation problem in a constrained model: under which probability measure does there exist an infinite connected set (infinite cluster) in which every vertex is present in the random configuration, or equivalently, included in the randomly-chosen subgraph? Such a question has been studied extensively in the unconstrained case - in particular the i.i.d Bernoulli percolation - see, for instance, [3, 15, 16, 23, 24, 31]. The major difference between the constrained percolation and the unconstrained percolation lies in the fact that imposing local constraints usually makes stochastic monotonicity, which is a crucial property when studying the unconstrained model, invalid. Therefore new techniques need to be developed to study constrained percolation models.

Some constrained percolation models, including the 1-2 model, the periodic plane dimer model, certain 6V models, are exactly solvable; see [8, 19, 30, 35–37]. The integrability properties of these models make it possible to compute the correlations. When the parameters associated to the probability measure vary, different behaviors of the local correlations imply a phase transition from a microscopic point of view. If we consider phase transitions

from a macroscopic, or geometric point of view, different approaches may be applied to study the existence of infinite clusters for a large class of constrained percolation models.

In [27], we studied a constrained percolation model on the \mathbb{Z}^2 lattice, and showed that if the underlying probability measure satisfies mild assumptions like symmetry, ergodicity and translation-invariance, then with probability 0 the number of infinite clusters is nonzero and finite. The technique makes use of the planarity and amenability of the 2D square grid \mathbb{Z}^2 . As an application, we obtained percolation properties for the XOR Ising model (a random spin configuration on a graph in which each spin is the product of two spins from two i.i.d Ising models, see [49]) on \mathbb{Z}^2 , with the help of the combinatorial correspondence between the XOR Ising model and the dimer model proved in [9, 13]. In this paper, we further develop the technique to study constrained percolation models on a number of planar lattices, which may be amenable or non-amenable, including the $[m, 4, n, 4]$ lattice and the square tilings of the hyperbolic plane; see [11] for an introduction to hyperbolic geometry.

The XOR Ising model was first introduced in [49] with interesting conformal invariance properties at criticality. For positive integers $m, n \geq 3$, the $[m, 4, n, 4]$ lattice is a vertex-transitive planar graph in which each vertex is incident to 4 faces with degrees $m, 4, n, 4$ in cyclic order. The constrained percolation model on the $[m, 4, n, 4]$ lattice is of special interest because there is a measure-preserving correspondence between its configurations and the XOR Ising configurations on the m -regular lattice or the n -regular lattice. The Euclidean-plane version of such a correspondence was introduced in [9]. When $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, the $[m, 4, n, 4]$ lattice is no longer amenable but can be embedded into the hyperbolic plane. Although phase transitions and conformal invariance for statistical mechanical models in the Euclidean plane have been studied extensively, statistical mechanical models, including the Ising model and the related random cluster model, have been fascinating problems for mathematicians and physicists for a long time, however, a lot of things remain unknown. For example, it is well-known that for statistical mechanical models in the hyperbolic plane, there is an “intermediate” phase between the non-percolation phase and unique-percolation phase, which usually does not exist for statistical mechanical models in the Euclidean plane; a lot of descriptions of the “intermediate” phase seem to be “qualitative” while not “quantitative” - for which values of the parameters does the model have such an “intermediate” phase? Indeed, the general results we obtain in this paper can be used to prove further results concerning percolation properties of the XOR Ising model on the hexagonal and the triangular lattices, as well as on regular tilings of the hyperbolic plane.

The specific geometric properties of non-amenable graphs make it an interesting problem to study percolation models on such graphs; and a set of techniques have been developed in the past few decades; see [3–5, 20–22, 38–40, 42, 44, 45, 50] for an incomplete list. In this paper, we also study the general automorphism-invariant percolation models on transitive planar graphs.

One of the most classical percolation models is the **i.i.d Bernoulli site percolation** on a graph, in which the vertices are open (resp. closed) with probability p (resp. $1 - p$) independently, where $p \in [0, 1]$. The **critical probability** p_c is the supremum of p 's such

that almost surely there are no infinite open clusters. A graph $G = (V, E)$ is a **vertex-transitive graph** if there exists a subgroup $\Gamma \subseteq \mathbf{Aut}(G)$ of the automorphism group G such that for any two vertices $v, w \in V$, there exist $\gamma \in \Gamma$ satisfying $\gamma v = w$. The number of **ends** of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.

Our results may be related to the following two conjectures. More precisely, we prove the following conjectures for some special vertex-transitive planar graphs.

Conjecture 1.1. *(Conjecture 7 of [5]) Suppose that G is a planar, connected graph, and the minimal vertex degree in G is at least 7. In an i.i.d Bernoulli site percolation on G , at every p in the range $(p_c, 1 - p_c)$, there are infinitely many infinite open clusters in the i.i.d Bernoulli site percolation on G . Moreover, we conjecture that $p_c < \frac{1}{2}$, and the above interval is nonempty.*

In Example 2.3, we explain why Conjecture 1.1 is true for the i.i.d Bernoulli site percolation on vertex-transitive triangular tilings of the hyperbolic plane where each vertex has degree $n \geq 7$.

Conjecture 1.2. *(Conjecture 8 of [5]) Let G be a planar, connected graph. Let $p = \frac{1}{2}$ be the probability that a vertex is open and assume that a.s. percolation occurs in the site percolation on G . Then almost surely there are infinitely many infinite clusters.*

Our Proposition 10.8 implies that Conjecture 1.2 is true for automorphism-invariant site percolation (not necessarily independent, or insertion tolerant) on vertex-transitive triangular tilings of the hyperbolic plane where each vertex has degree $n \geq 7$ if the underlying measure is ergodic and invariant under switching state-1 vertices and state-0 vertices.

We then apply our results concerning the general automorphism-invariant percolation models on transitive planar graphs to study the infinite “+”-clusters and “-”-clusters for the Ising model on vertex-transitive triangular tilings of the hyperbolic plane where each vertex has degree $n \geq 7$, and describe the behaviors of such clusters with respect to varying coupling constants under the free boundary condition and the wired boundary condition. A surprising result we obtain is that it is possible that the random cluster representation of the Ising model has no infinite open clusters, while the Ising model has infinitely many infinite “+”-clusters and infinitely many infinite “-”-clusters - in contrast with the Ising percolation and its random cluster representation on the 2d square grid \mathbb{Z}^2 (see [12, 17, 25]) where the Ising model has an infinite “+” or “-”-cluster if and only if its random cluster representation has an infinite open cluster.

The main tools to prove these results are the planar duality of graphs, ergodicity and symmetry of probability measures, as well as properties of amenability and non-amenability. One characteristic of the constrained percolation obtained from a natural correspondence with the XOR Ising model, which is not shared with the unconstrained percolation, is that given such a constraint, there are two sets of “contours” separating clusters of vertices of different states. These two sets of contours lie on two planar graphs dual to each other, and the present edges in these two different sets of contours never cross. As a result, there are four types of infinite components in our constrained percolation model: infinite “0”-cluster, infinite “1”-cluster, infinite planar contour and infinite dual contour. The geometric

configurations of these infinite components, together with the ergodicity and symmetry of the probability measure, lead to interesting properties that are particular and unique to the constrained percolation model.

The organization of the paper is as follows.

In Section 2, we introduce the $[m, 4, n, 4]$ lattice and state the result concerning constrained percolation models on the $[m, 4, n, 4]$ lattice. In Section 3, we state the main results concerning infinite clusters in the Ising model on regular triangular tilings of the hyperbolic plane, and, in particular, provide a description of the numbers of infinite “+” and “-” clusters of the ferromagnetic Ising model with the free boundary condition, the “+” boundary condition or the “-” boundary condition on such a lattice for different values of coupling constants. In Section 4, we state the main results concerning infinite clusters in the XOR Ising model on regular triangular tilings of the hyperbolic plane and its dual graph. In Section 5, we state the result proved in this paper concerning the percolation properties of the XOR Ising model on the hexagonal lattice and the triangular lattice. In Section 6, we introduce the square tilings of the hyperbolic plane, state and prove the main result concerning constrained percolation models on such a lattice.

The remaining sections are devoted to prove the theorems stated in preceding sections. In Section 7, we prove Theorem 2.2. In Section 8, we prove Theorem 2.4. In Section 9, we prove Theorem 2.5. In Section 10, we discuss the applications of the techniques developed in the proof of Theorem 2.2 to prove results concerning unconstrained site percolation on vertex-transitive, triangular tilings of the hyperbolic plane in preparation of proving Theorems 3.3, 4.1 and 4.2. In Section 11, we prove Theorem 3.3. In Section 12, we prove Theorem 4.1 and Theorem 4.2. In Section 13, we prove Theorems 5.1 and 5.2. In Appendix A, we prove combinatorial results concerning contours and clusters in preparation to prove the main theorems.

2. CONSTRAINED PERCOLATION ON THE $[m, 4, n, 4]$ LATTICE

In this section, we state the main result proved in this paper for the constrained percolation models on the $[m, 4, n, 4]$ lattice. We shall start with a formal definition of the $[m, 4, n, 4]$ lattice.

Let m, n be positive integers satisfying

$$(1) \quad m \geq 3, \quad n \geq 3$$

$$(2) \quad \frac{1}{m} + \frac{1}{n} \leq \frac{1}{2}.$$

The $[m, 4, n, 4]$ lattice is a vertex-transitive graph which can be embedded into the Euclidean plane or the hyperbolic plane such that each vertex is incident to 4 faces with degrees $m, 4, n, 4$ in cyclic order. When $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, the graph is amenable and can be embedded into the Euclidean plane. When $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, the graph is non-amenable and can be embedded into the hyperbolic plane ([43]). Note that when $m = n = 4$, the graph is the square grid embedded into the 2D Euclidean plane. See Figure 1 for an illustration of the $[3, 4, 6, 4]$ lattice, Figure 2 for the $[3, 4, 7, 4]$ lattice, and Figure 3 for the $[6, 4, 6, 4]$ lattice.

Let $G = (V, E)$ be an $[m, 4, n, 4]$ lattice. We color all the faces of degree m or n with white and all the other faces with black, such that any two faces sharing an edge have

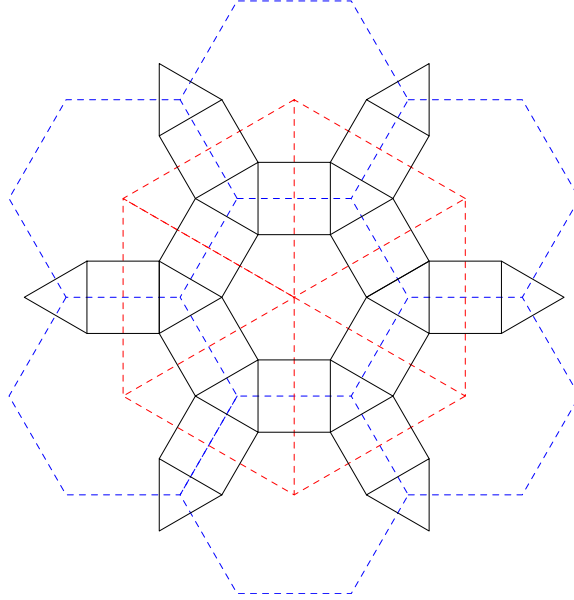


FIGURE 1. The $[3,4,6,4]$ lattice, the auxiliary hexagonal lattice and triangular lattice. Black lines represent the $[3,4,6,4]$ lattice; dashed red lines represent the triangular lattice; dashed blue lines represent the hexagonal lattice.

different colors. We consider the site percolation on V satisfying the following constraint (see Figure 4):

- around each black face, there are six allowed configurations (0000) , (1111) , (0011) , (1100) , (0110) , (1001) , where the digits from the left to the right correspond to vertices in clockwise order around the black face, starting from the lower left corner. See Figure 4.

Let $\Omega \subset \{0,1\}^V$ be the probability space consisting of all the site configurations on G satisfying the constraint above. To the $[m,4,n,4]$ lattice G , we associate two auxiliary lattices $\mathbb{L}_1 = (V(\mathbb{L}_1), E(\mathbb{L}_1))$ and $\mathbb{L}_2 = (V(\mathbb{L}_2), E(\mathbb{L}_2))$ as follows. Each vertex of \mathbb{L}_1 (resp. \mathbb{L}_2) is located at the center of each degree- m face (resp. degree- n face) of G . Two vertices of \mathbb{L}_1 (resp. \mathbb{L}_2) are joined by an edge of \mathbb{L}_1 (resp. \mathbb{L}_2) if and only if the two corresponding m -faces (resp. n -faces) of G are adjacent to the same square face of G through a pair of opposite edges (edges of a square face that do not share a vertex), respectively.

We say an edge $e \in E(\mathbb{L}_1) \cup E(\mathbb{L}_2)$ **crosses** a square face of the $[m,4,n,4]$ lattice if the edge e crosses a pair of opposite edges of the square face. Note that

- \mathbb{L}_1 (resp. \mathbb{L}_2) is a planar lattice in which each face has degree n (resp. m) and each vertex has degree m (resp. n).
- \mathbb{L}_1 and \mathbb{L}_2 are planar dual to each other.
- Each edge in $E(\mathbb{L}_1) \cup E(\mathbb{L}_2)$ crosses a unique square face of the $[m,4,n,4]$ lattice. When $m \neq 4$ and $n \neq 4$, each square face of the $[m,4,n,4]$ lattice is crossed by a unique edge $e_1 \in E(\mathbb{L}_1)$ and a unique edge $e_2 \in E(\mathbb{L}_2)$; and moreover, e_1 and e_2 are dual to each other.

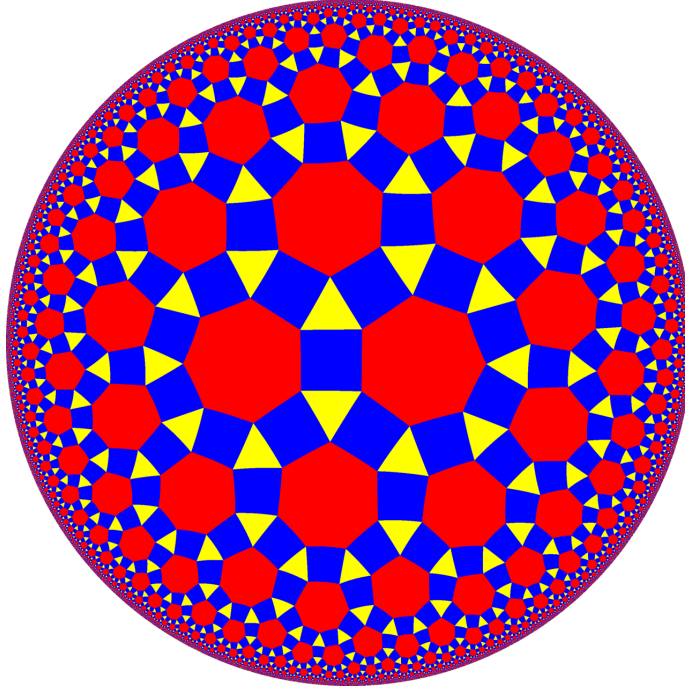


FIGURE 2. The $[3,4,7,4]$ lattice (picture from the wikipedia)

When $m = n$, both \mathbb{L}_1 and \mathbb{L}_2 are lattices in which each face has degree n and each vertex has degree n . When $m = 3$ and $n = 6$, \mathbb{L}_1 is the hexagonal lattice and \mathbb{L}_2 is the triangular lattice; see Figure 1. When $m = 3$ and $n = 7$, \mathbb{L}_2 is a vertex-transitive triangular tiling of the hyperbolic plane, in which each vertex has degree 3; see the left graph of Figure 5; while \mathbb{L}_1 is the $[7,7,7]$ lattice on the hyperbolic plane, see the right graph of Figure 5.

Let $\Phi \subset \{0,1\}^{E(\mathbb{L}_1) \cup E(\mathbb{L}_2)}$ be the set of contour configurations satisfying the condition that each vertex of $V(\mathbb{L}_1)$ and $V(\mathbb{L}_2)$ is incident to an even number of present edges, and present edges in $E(\mathbb{L}_1)$ and $E(\mathbb{L}_2)$ never cross. Any constrained percolation configuration $\omega \in \Omega$ is mapped to a contour configuration $\phi(\omega) \in \Phi$, where an edge e in $E(\mathbb{L}_1)$ or $E(\mathbb{L}_2)$ is present (i.e., have state 1) if and only if the following condition holds

- Let S be the square face of G crossed by e . Let v_1, v_2, v_3, v_4 be the four vertices of S , such that v_1 and v_2 are on one side of e and v_3, v_4 are on the other side of e .

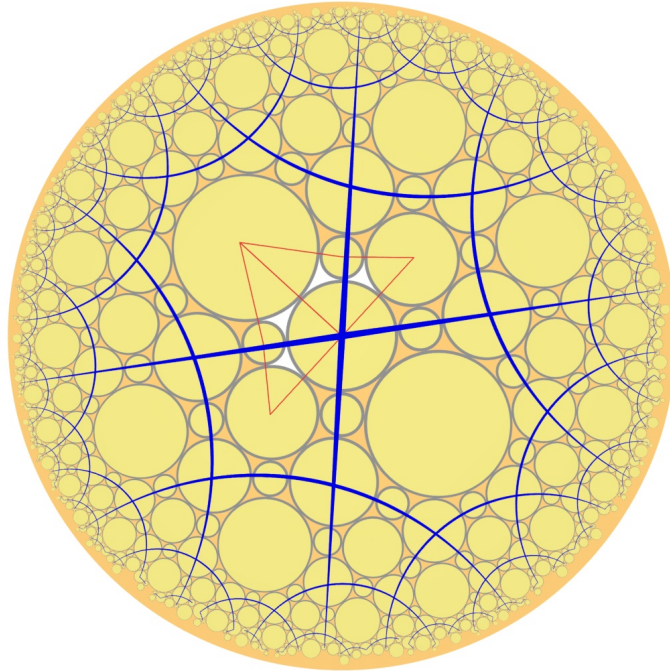


FIGURE 3. The $[6,4,6,4]$ lattice represented by blue lines (picture from <http://epinet.anu.edu.au/>)

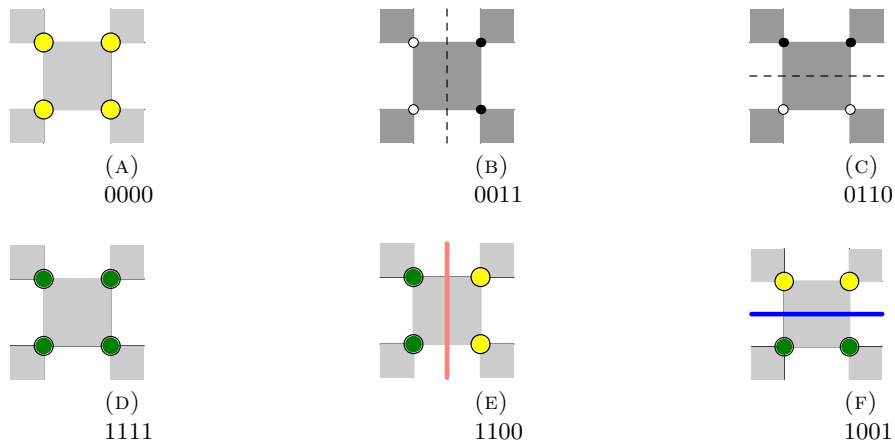


FIGURE 4. Local configurations of the constrained percolation around a black square. Red and blue lines mark contours separating 0's and 1's (in \mathbb{L}_1 and \mathbb{L}_2 respectively). Yellow (resp. green) disks represent 0's (resp. 1's).

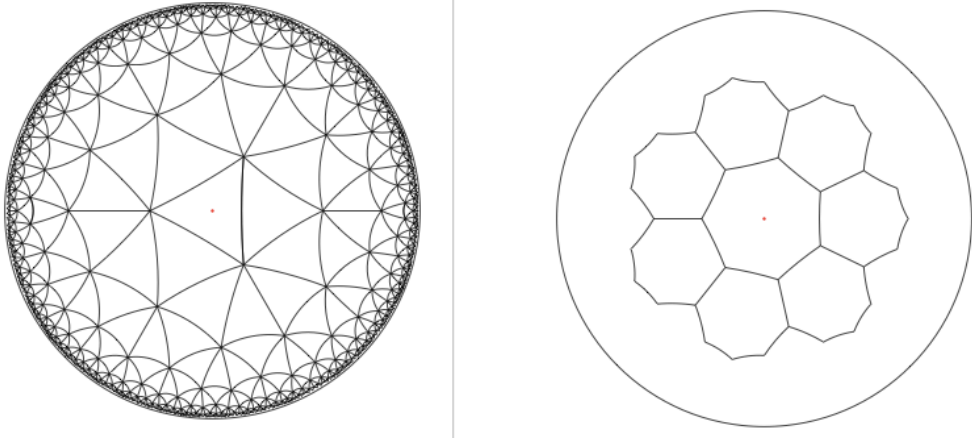


FIGURE 5. The $[3,3,3,3,3,3,3]$ lattice on the left and the $[7,7,7]$ lattice on the right

Then v_1 and v_2 have the same state, v_3 and v_4 have the same state, and v_1 and v_3 have different states.

See Figure 6 for a contour configuration obtained from a constrained percolation configuration on the $[3, 4, 6, 4]$ lattice. Note that the mapping $\phi : \Omega \rightarrow \Phi$ is 2-to-1 since $\phi(\omega) = \phi(1 - \omega)$.

A **contour** is a connected component of present edges in a contour configuration in Φ . A contour may be finite or infinite depending on the number of edges in the contour. Since present edges of a contour configuration in $E(\mathbb{L}_1)$ and in $E(\mathbb{L}_2)$ never cross, either all the edges in a contour are edges of \mathbb{L}_1 , or all the edges in a contour are edges in \mathbb{L}_2 . We call a contour **primal contour** (resp. **dual contour**) if all the edges in the contour are edges of \mathbb{L}_1 (resp. \mathbb{L}_2).

Let Γ be the automorphism group $\text{Aut}(G)$ of the graph G . For $i \in \{1, 2\}$, let Γ_i be the automorphism group $\text{Aut}(\mathbb{L}_i)$ of the graph \mathbb{L}_i . Let μ be a probability measure on Ω . The measure μ naturally induces a measure on Φ under the 2-to-1 mapping ϕ ; with a little abuse of notation, we still use μ to denote this induced measure on contour configurations Φ . We may assume that μ satisfies the following conditions:

- (A1) μ is Γ -invariant;
- (A2) μ is Γ_i -ergodic for $i = 1, 2$; i.e. any Γ_i -invariant event has μ -probability 0 or 1;
- (A3) μ is symmetric: let $\theta : \Omega \rightarrow \Omega$ be the map defined by $\theta(\omega)(v) = 1 - \omega(v)$, for each vertex $v \in V$, then μ is invariant under θ , that is, for any event A , $\mu(A) = \mu(\theta(A))$.

Let Φ_1 (resp. Φ_2) be the set of all contour configurations on \mathbb{L}_1 (resp. \mathbb{L}_2) satisfying the condition that each vertex of \mathbb{L}_1 (resp. \mathbb{L}_2) has an even number of incident present edges. For each contour configuration $\psi \in \Phi$, we have $\psi = \psi_1 \cup \psi_2$, where $\psi_1 \in \Phi_1$ and $\psi_2 \in \Phi_2$; moreover, $\psi_1 \cap \psi_2 = \emptyset$.

Let ν_1 (resp. ν_2) be the marginal distribution of μ on Φ_1 (resp. Φ_2). When $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, the $[m, 4, n, 4]$ lattice is amenable. It is not hard to see that if $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, then $(m, n) \in$

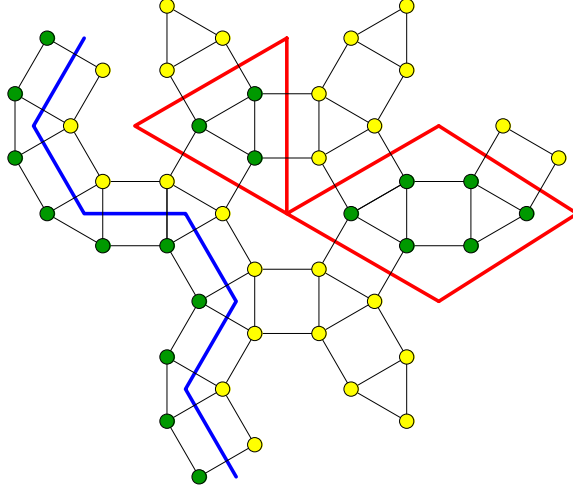


FIGURE 6. A constrained percolation configuration on the $[3,4,6,4]$ lattice. Red lines represent contours on the triangular lattice. Blue lines represent contours on the hexagonal lattice.

$\{(4,4), (3,6), (6,3)\}$. When $m = n = 4$, the $[m,4,n,4]$ lattice is the 2D square grid, on which the constrained percolation models was discussed in [27].

Now we consider the case when $(m,n) = (3,6)$. As discussed before, in this case \mathbb{L}_1 is the hexagonal lattice \mathbb{H} , and \mathbb{L}_2 is the triangular lattice \mathbb{T} . We first define the finite energy condition of a random contour configuration on a planar graph.

Definition 2.1. Let $G = (V, E)$ be a vertex-transitive, planar graph. Let Φ be the set of all contour configurations on G , in which each contour configuration is a subset of edges such that each vertex is incident to an even number of present edges. Let ν be a probability measure on Φ . We say ν has **finite energy** if for any face S of G , let $\partial S \subset E$ consist of all the sides of the polygon S . Define ϕ_S to be the configuration obtained by switching the states of each element of ∂S , i.e. $\phi_S(e) = 1 - \phi(e)$ if $e \in \partial S$, and $\phi_S(e) = \phi(e)$ otherwise. Let E be an event, and define

$$(3) \quad E_S = \{\phi_S : \phi \in E\}.$$

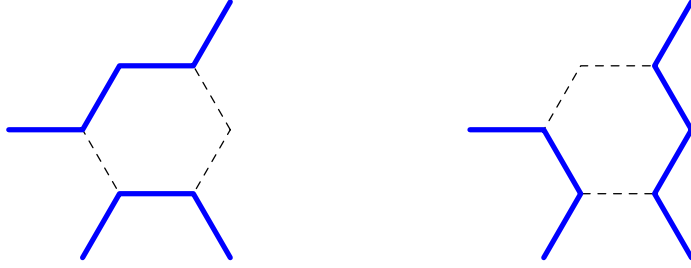
Then $\nu(E_S) > 0$ whenever $\nu(E) > 0$.

We may assume that ν_1 or ν_2 has **finite energy** as follows.

- (A4) ν_1 has finite energy.
- (A5) ν_2 has finite energy.

See Figures 7 and 8 for illustrations of the configuration-changing process on the hexagonal lattice \mathbb{H} and the triangular lattice \mathbb{T} , respectively.

For a random contour configuration $\psi \in \Phi_1$ (resp. $\psi \in \Phi_2$) whose distribution is the marginal distribution ν_1 (resp. ν_2) of μ on Φ_1 (resp. Φ_2), ψ induces a random constrained configuration $\omega \in \phi^{-1}(\psi)$ as follows. Let v_0 be a fixed vertex of G . Assume that $\omega(v_0) = 1$ with probability $\frac{1}{2}$, and $\omega(v_0) = 0$ with probability $\frac{1}{2}$, and is independent of ψ . For two vertices v_1, v_2 of G joined by an edge e , v_1 and v_2 have different states if and only if e

FIGURE 7. Change of contour configurations in $\mathbb{L}_1 = \mathbb{H}$ FIGURE 8. Change of contour configurations in $\mathbb{L}_2 = \mathbb{T}$

crosses a present edge in ψ . Let λ_1 (resp. λ_2) be the distribution of ω . We may further make the following assumptions

- (A6) λ_1 is Γ_1 -ergodic;
- (A7) λ_2 is Γ_2 -ergodic.

Also we may sometimes assume that

- (A8) μ is Γ_1 -invariant.

The main theorems of this section are stated as follows.

Theorem 2.2. *Let G be the $[3, 4, n, 4]$ lattice with $n \geq 7$. Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. 1-clusters). Let t_1 (resp. t_2) be the number of infinite \mathbb{L}_1 -contours (resp. \mathbb{L}_2 -contours).*

- I Let μ be a probability measure on Ω satisfying (A2),(A3),(A7),(A8). Then μ -a.s. $(s_0, s_1, t_1) = (\infty, \infty, \infty)$.
- II Let μ be a probability measure on Ω satisfying (A2),(A3),(A6),(A7),(A8). Then μ -a.s. $(s_0, s_1, t_1, t_2) \in \{(\infty, \infty, \infty, 1), (\infty, \infty, \infty, \infty)\}$.

The case $m = 3, n \geq 7$ is of special interest, because in this case \mathbb{L}_1 is a cubic graph (each vertex has degree 3), and \mathbb{L}_2 is a triangular tiling of the hyperbolic plane. As a result, any infinite contour on \mathbb{L}_1 must be a doubly infinite self-avoiding path. An application of Theorem 2.2 is illustrated in the following example.

Example 2.3. *Consider the i.i.d. Bernoulli site percolation on the regular tiling \mathbb{L}_2 of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Assume that each vertex of \mathbb{L}_2 takes value 1 with probability $\frac{1}{2}$. The corresponding contour configuration on the dual graph \mathbb{L}_1 to the site percolation on \mathbb{L}_2 induces a constrained configuration in the $[3, 4, n, 4]$ lattice satisfying (A8),(A2),(A3),(A7). Then by Theorem 2.2 μ -a.s. $(s_0, s_1) = (\infty, \infty)$.*

Theorem 2.4. *Let G be the $[m, 4, n, 4]$ lattice such that*

$$(4) \quad \begin{aligned} m &\geq 3, & n &\geq 3; \\ \frac{1}{m} + \frac{1}{n} &= \frac{1}{2} \end{aligned}$$

Let μ be a probability measure on Ω . Then

- I if μ satisfies (A1)-(A6), then almost surely there are no infinite contours in \mathbb{L}_2 ;*
- II if μ satisfies (A1)-(A5) and (A7), then almost surely there are no infinite contours in \mathbb{L}_1 ;*
- III if μ satisfies (A1)-(A7), almost surely there are neither infinite contours nor infinite clusters.*

When $m = n$, the two lattice \mathbb{L}_1 and \mathbb{L}_2 are isomorphic to each other, this allows us to use symmetry to obtain the following theorem.

Theorem 2.5. *Let G be the $[m, 4, n, 4]$ lattice satisfying*

$$m = n \geq 5.$$

Let μ be a probability measure on Ω . Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. 1-clusters), and let t_1 (resp. t_2) be the number of infinite \mathbb{L}_1 -contours (resp. \mathbb{L}_2 -contours). If μ satisfies (A1)-(A3); then μ -a.s. $(s_0, s_1, t_1, t_2) = (\infty, \infty, \infty, \infty)$.

Theorem 2.2 is proved in Section 7. Theorem 2.4 is proved in Section 8, and Theorem 2.5 is proved in Section 9.

3. ISING MODEL ON TRANSITIVE, TRIANGULAR TILINGS OF THE HYPERBOLIC PLANE

In this section, we state the main result concerning the percolation properties of the Ising model on transitive, triangular tilings of the hyperbolic plane. These results, as given in Theorem 3.3, will be proved in Section 11.

The random cluster representation of an Ising model on a transitive, triangular tiling of the hyperbolic plane can be defined as in [20]. Here we briefly summarize basic facts about the Fortuin-Kasteleyn random cluster model, which is a probability measure on bond configurations of a graph, and the related Potts model. See [17] for more information.

The **random cluster measure** $RC := RC_{p,q}^{G_0}$ on a finite graph $G_0 = (V_0, E_0)$ with parameters $p \in [0, 1]$ and $q \geq 1$ is the probability measure on $\{0, 1\}^{E_0}$ which to each $\xi \in \{0, 1\}^{E_0}$ assigns probability

$$(5) \quad RC(\xi) :\propto q^{k(\xi)} \prod_{e \in E_0} p^{\xi(e)} (1-p)^{1-\xi(e)},$$

where $k(\xi)$ is the number of connected components in ξ .

Let $G = (V, E)$ be an infinite, locally finite, connected graph. For each $q \in [1, \infty)$ and each $p \in (0, 1)$, let $WRC_{p,q}^G$ be the random cluster measure with the wired boundary condition, and let $FRC_{p,q}^G$ be the random cluster measure with the free boundary condition. More precisely, $WRC_{p,q}^G$ (resp. $FRC_{p,q}^G$) is the weak limit of RC 's defined by (5) on larger and larger finite subgraphs approximating G , where we assume that all the edges outside each finite subgraph are present (resp. absent).

The Gibbs measure μ^+ (resp. μ^-) for the Ising model on G with coupling constant $J \geq 0$ on each edge and “+”-boundary conditions (resp. “-”-boundary conditions) can be obtained by considering a random configuration of present and absent edges according to the law $WRC_{p,2}^G$, $p = 1 - e^{-2J}$, and assigning to all the vertices in each infinite cluster the state “+” (resp. “-”), and to all the vertices in each finite cluster a state from $\{+, -\}$, chosen uniformly at random for each cluster and independently for different clusters.

The Gibbs measure μ^f for the Ising model on G with coupling constant $J \geq 0$ on each edge and free boundary conditions can be obtained by considering a random configuration of present and absent edges according to the law $FRC_{p,2}^G$, $p = 1 - e^{-2J}$, and assigning to all the vertices in each cluster a state from $\{+, -\}$, chosen uniformly at random for each cluster and independently for different clusters.

When there is no confusion, we may write $FRC_{p,q}^G$ and $WRC_{p,q}^G$ as $FRC_{p,q}$ and $WRC_{p,q}$ for simplicity. Assume that G is transitive. Then measures $FRC_{p,q}$ and $WRC_{p,q}$ are $\text{Aut}(G)$ -invariant, and $\text{Aut}(G)$ -ergodic; see the explanations on Page 295 of [45].

Now we introduce the following definitions.

Definition 3.1. A transitive graph $G = (V, E)$ is **unimodular**, if there exists a subgroup $\Gamma \subseteq \text{Aut}(G)$ acting transitively on G , such that for any two vertices $u, v \in V$, we have

$$|\text{Stab}_u(v)| = |\text{Stab}_v(u)|;$$

where $\text{Stab}_u \subseteq \Gamma$ is the stabilizer of u , i.e.

$$(6) \quad \text{Stab}_u = \{\gamma \in \Gamma : \gamma u = u\};$$

and $|\cdot|$ is the cardinality of a set.

Definition 3.2. A graph $G = (V, E)$ is called **amenable**, if its edge isoperimetric constant

$$(7) \quad \iota_E(G) := \inf_{K \subset V, |K| < \infty} \frac{|\partial_E K|}{|K|} = 0.$$

where $\partial_E K$ is the set of edges with exactly one endpoint in K and one endpoint not in K .

If the edge isoperimetric constant is strictly positive, the graph is called **nonamenable**.

If we further assume that G is unimodular, nonamenable and planar, it is known that there exists $p_{c,q}^w, p_{c,q}^f, p_{u,q}^w, p_{u,q}^f \in [0, 1]$, such that $FRC_{p,q}$ -a.s. the number of infinite clusters equals

$$(8) \quad \begin{cases} 0 & \text{for } p \leq p_{c,q}^f \\ \infty & \text{for } p \in (p_{c,q}^f, p_{u,q}^f) \\ 1 & \text{for } p > p_{u,q}^f; \end{cases}$$

and $WRC_{p,q}$ -a.s. the number of infinite clusters equals

$$(9) \quad \begin{cases} 0 & \text{for } p < p_{c,q}^w \\ \infty & \text{for } p \in (p_{c,q}^w, p_{u,q}^w) \\ 1 & \text{for } p \geq p_{u,q}^w. \end{cases};$$

see expressions (17),(18), Theorem 3.1 and Corollary 3.7 of [20].

It is well known that for the i.i.d Bernoulli percolation on a infinite, connected, locally finite transitive graph G , there exist p_c, p_u such that

- (i) $0 < p_c \leq p_u \leq 1$;
- (ii) for $p \in [0, p_c)$ there is no infinite cluster a.s.
- (iii) for $p \in (p_c, p_u)$ there are infinitely many infinite clusters, a.s.
- (iv) for $p \in (p_u, 1]$, there is exactly one infinite cluster, a.s.

The monotonicity in p of the uniqueness of the infinite cluster was proved in [21, 44]. Combining with Theorem 7.5 of [38] (proved first in [41]), we obtain statements (i)-(iv) above. It is proved that $p_c = p_u$ for amenable transitive graphs (see [5]); and conjectured that $p_c < p_u$ for transitive non-amenable graphs. The conjecture $p_c < p_u$ was proved for transitive hyperbolic planar graphs (see [6]) and non-amenable Cayley graphs with small spectral radii (see [42, 45, 47]) or large girths (see [40]).

Theorem 3.3. *Let \mathbb{L}_2 be a triangulation of the hyperbolic plane such that each vertex has degree $n \geq 7$. Consider the Ising model with spins located on vertices of \mathbb{L}_2 and coupling constant $J \in \mathbb{R}$ on each edge. Let p_c, p_u be critical i.i.d Bernoulli site percolation probabilities on \mathbb{L}_2 as defined by (i)-(iv) above.*

I Let $h > 0$ satisfy

$$(10) \quad \frac{e^{-h}}{e^h + e^{-h}} = p_c$$

Let μ^+ (resp. μ^- , μ^f) be the infinite-volume Ising Gibbs measure with “+”-boundary conditions (resp. “-” boundary conditions, free boundary conditions). If

$$(11) \quad n|J| < h,$$

then μ -a.s. there are infinitely many infinite “+”-clusters, infinitely many infinite “-”-clusters and infinitely many infinite contours, where μ is an arbitrary $\text{Aut}(\mathbb{L}_2)$ -invariant Gibbs measure for the Ising model on \mathbb{L}_2 with coupling constant J .

II Assume $J \geq 0$. If one of the following conditions

- (a) μ^f is $\text{Aut}(\mathbb{L}_2)$ -ergodic;
 - (b) $\inf_{u,v \in V(\mathbb{L}_2)} \langle \sigma_u \sigma_v \rangle_{\mu^f} = 0$, where σ_u and σ_v are two spins associated to vertices $u, v \in V(\mathbb{L}_2)$ in the Ising model;
 - (c) $0 \leq J < \frac{1}{2} \ln \left(\frac{1}{1-p_{u,2}^f} \right)$, where $p_{u,2}^f$ is the critical probability for the existence of a unique infinite open cluster of the corresponding random cluster representation of the Ising model on \mathbb{L}_2 , with free boundary conditions as given in (8);
 - (d) $0 \leq J < \frac{1}{2} \ln \left(\frac{1}{1-p_{u,1}} \right)$, where $p_{u,1}$ is the critical probability for the existence of a unique infinite open cluster for the i.i.d Bernoulli bond percolation on \mathbb{L}_2 ;
- holds, then μ^f -a.s. there are infinitely many infinite “+”-clusters and infinitely many infinite “-”-clusters. Indeed, we have (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a).

III Assume

$$(12) \quad J \geq \frac{1}{2} \ln \left(\frac{1}{1-p_{u,2}^w} \right).$$

Let \mathcal{A}_+ be the event that there is a unique infinite “+”-cluster, no infinite “-”-clusters and no infinite contours; and let \mathcal{A}_- be the event that there is a unique infinite “-”-cluster, no infinite “+”-clusters and no infinite contours. then

$$(13) \quad \mu^+(\mathcal{A}_+) = 1.$$

$$(14) \quad \mu^-(\mathcal{A}_-) = 1.$$

IV If

$$(15) \quad J > \frac{1}{2} \ln \left(\frac{1}{1 - p_{u,2}^f} \right)$$

then (13) and (14) hold, and moreover,

$$(16) \quad \mu^f(\mathcal{A}_+) = \mu^f(\mathcal{A}_-) = \frac{1}{2}.$$

From Theorem 3.3, we can also obtain the following corollary:

Corollary 3.4. *Let \mathbb{L}_2 be a triangulation of the hyperbolic plane such that each vertex has degree $n \geq 7$. Consider the Ising model with spins located on vertices of \mathbb{L}_2 and coupling constant $J \geq 0$. on each edge. If*

$$(17) \quad J < \frac{1}{2} \ln \left(\frac{1}{1 - p_{c,2}^w} \right),$$

then for any Gibbs measure μ for the Ising model on \mathbb{L}_2 with coupling constant J , μ -a.s. there are infinitely many infinite “+”-clusters, infinitely many infinite “-”-clusters and infinitely many infinite contours. Here $p_{c,2}^w$ is defined as in (9).

We can see that when the conditions of Corollary 3.4 are satisfied, almost surely there are no infinite open clusters in the corresponding random cluster representation of the Ising model, however, the conclusion of the corollary says that there are infinitely many infinite “+”-clusters and infinitely many infinite “-”-clusters in the Ising model.

4. XOR ISING MODEL ON TRANSITIVE, TRIANGULAR TILINGS OF THE HYPERBOLIC PLANE

In this section, we state the main result concerning the percolation properties of the XOR Ising model on transitive, triangular tilings of the hyperbolic plane. These results, as given in Theorems 4.1 and 4.2, will be proved in Section 12 as applications of Theorem 2.2.

Throughout this section, we let \mathbb{L}_1 be the $[n, n, n]$ regular tiling of the hyperbolic plane, such that each face has degree $n \geq 7$, and each vertex has degree 3. Let \mathbb{L}_2 be the planar dual graph of \mathbb{L}_1 . More precisely, \mathbb{L}_2 is the vertex-transitive triangular tiling of the hyperbolic plane such that each vertex has degree $n \geq 7$. An **XOR Ising model** on \mathbb{L}_2 is a probability measure on $\sigma_{XOR} \in \{\pm 1\}^{V(\mathbb{L}_2)}$, such that

$$\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v), \quad \forall v \in V(\mathbb{L}_2),$$

where σ_1, σ_2 are two i.i.d. Ising models with spins located on $V(\mathbb{L}_2)$. A **contour configuration** of the XOR Ising configuration on \mathbb{L}_2 is a subset of edges of \mathbb{L}_1 in which each edge

has a dual edge in $E(\mathbb{L}_2)$ joining two vertices $u, v \in V(\mathbb{L}_2)$ satisfying $\sigma_{XOR}(u) = -\sigma_{XOR}(v)$. A connected component in a contour configuration is called a **contour**. Obviously each vertex of \mathbb{L}_1 has 0 or 2 incident present edges in a contour configuration of an XOR Ising configuration, since \mathbb{L}_1 has vertex degree 3. Each contour of an XOR Ising configuration on \mathbb{L}_2 is either a self-avoiding cycle or a doubly-infinite self-avoiding path.

We can similarly define an XOR Ising model with spins located on vertices of \mathbb{L}_1 , and its contours to be even-degree subgraphs of \mathbb{L}_2 .

Theorem 4.1. *Let σ_1, σ_2 be two i.i.d. Ising models with spins located on vertices of \mathbb{L}_2 , coupling constant $J \in [0, \infty)$ and free boundary conditions. Let μ_1^f (resp. μ_2^f) be the distribution of σ_1 (resp. σ_2). Assume that one of the following cases occurs*

- I If $\mu_1^f \times \mu_2^f$ is $\text{Aut}(\mathbb{L}_2)$ -ergodic; or
- II $\liminf_{|i-j| \rightarrow \infty} \langle \sigma_{1,i} \sigma_{1,j} \rangle_{\mu_{1,f}} = 0$, where $\sigma_{1,i}$ and $\sigma_{1,j}$ are two spins in the Ising model σ_1 with distance $|i-j|$; or
- III J satisfies Condition (c) of Theorem 3.3 II.
- IV J satisfies Condition (d) of Theorem 3.3 II.

then $\mu_1^f \times \mu_2^f$ -a.s. there are infinitely many infinite “+”-clusters and infinitely many infinite “-”-clusters.

Theorem 4.2. *Let σ_1, σ_2 be two i.i.d. Ising models with spins located on vertices of \mathbb{L}_1 , and coupling constant $K \geq 0$. For $i = 1, 2$, let $\mu_{i,+}$ (resp. $\mu_{i,-}$) be the distribution of σ_i with “+”-boundary conditions (resp. “-”-boundary conditions). Let $J \geq 0$ be given by*

$$(18) \quad e^{-2J} = \frac{1 - e^{-2K}}{1 + e^{-2K}},$$

and let t be the number of infinite contours. Let μ_{++} (resp. μ_{--}, μ_{+-}) be the product measure of $\mu_{1,+}$ and $\mu_{2,+}$ (resp. $\mu_{1,-}$ and $\mu_{2,-}, \mu_{1,+}$ and $\mu_{2,-}$). Assume J satisfies the assumption of Theorem 4.1, then we have

$$\mu_{++}(t \in \{0, \infty\}) = \mu_{--}(t \in \{0, \infty\}) = \mu_{+-}(t \in \{0, \infty\}) = 1.$$

5. XOR ISING MODELS ON THE HEXAGONAL AND TRIANGULAR LATTICES

In this section, we define the XOR Ising models on the hexagonal and triangular lattices, and state the main results proved in this paper concerning the percolation properties of these models.

Let σ_1, σ_2 be two i.i.d. ferromagnetic Ising models with spins located on vertices of the hexagonal lattice $\mathbb{H} = (V_{\mathbb{H}}, E_{\mathbb{H}})$. The hexagonal lattice has edges in three different directions. Assume that both σ_1 and σ_2 have nonnegative coupling constants J_a, J_b, J_c on edges of \mathbb{H} with the three different directions, respectively. Assume also that the distributions of both σ_1 and σ_2 are weak limits of Gibbs measures under periodic boundary conditions. Recall that the XOR Ising model $\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v)$, for $v \in V_{\mathbb{H}}$.

A **contour configuration** for an XOR Ising configuration, σ_{XOR} , defined on \mathbb{H} (resp. \mathbb{T}), is a subset of $\{0, 1\}^{E(\mathbb{T})}$ (resp. $\{0, 1\}^{E(\mathbb{H})}$), whose state-1-edges (present edges) are edges of \mathbb{T} (resp. \mathbb{H}) separating neighboring vertices of \mathbb{H} (resp. \mathbb{T}) with different states in σ_{XOR} . (Note that \mathbb{H} and \mathbb{T} are planar duals of each other.) Contour configurations of the XOR

Ising model were first studied in [49], in which the scaling limits of contours of the critical XOR Ising model are conjectured to be level lines of Gaussian free field. It is proved in [9] that the contours of the XOR Ising model on a plane graph correspond to level lines of height functions of the dimer model on a decorated graph, inspired by the correspondence between Ising model and bipartite dimer model in [13]. We will study the percolation properties of the XOR Ising model on \mathbb{H} and \mathbb{T} , as an application of the main theorems proved in this paper for the general constrained percolation process.

Let

$$(19) \quad f(x, y, z) = e^{-2x} + e^{-2y} + e^{-2z} + e^{-2(x+y)} + e^{-2(x+z)} + e^{-2(y+z)} - e^{-2(x+y+z)} - 1.$$

$$(20) \quad g(x, y, z) = e^{2x} + e^{2y} + e^{2z} - e^{2(x+y+z)}.$$

We say the XOR Ising model on \mathbb{H} with coupling constants (J_a, J_b, J_c) is in the **high-temperature state** (resp. **low-temperature state**, **critical state**) if $f(J_a, J_b, J_c) > 0$ (resp. $f(J_a, J_b, J_c) < 0$, $f(J_a, J_b, J_c) = 0$). Note that $f(J_a, J_b, J_c) = 0$ is the well-known condition that an Ising model on the 2D hexagonal lattice \mathbb{H} is critical; see, for example, [35] for a rigorous proof. The XOR Ising model $\sigma_1 \cdot \sigma_2$ on \mathbb{H} is in the high-temperature state (resp. low-temperature state, critical state), if and only if both σ_1 and σ_2 are in the high-temperature state (resp. low-temperature state, critical state).

Let $\mathbb{T} = (V_{\mathbb{T}}, E_{\mathbb{T}})$ be the dual triangular lattice of \mathbb{H} . We also consider the XOR Ising model with spins located on $V_{\mathbb{T}}$. Assume that the coupling constants on edges with 3 different directions are K_a, K_b and K_c , respectively, such that $K_a, K_b, K_c \geq 0$. Also for $i \in \{a, b, c\}$, assume that K_i is the coupling constant on an edge of \mathbb{T} dual to an edge of \mathbb{H} with coupling constant J_i . We say the XOR Ising model on the triangular lattice is in the **low-temperature state** (resp. **high-temperature state**, **critical state**) if $g(K_a, K_b, K_c) < 0$ (resp. $g(K_a, K_b, K_c) > 0$, $g(K_a, K_b, K_c) = 0$). Again these come from the known fact that if $g(K_a, K_b, K_c) < 0$ (resp. $g(K_a, K_b, K_c) > 0$, $g(K_a, K_b, K_c) = 0$), both Ising models, each of which is a factor of the XOR Ising model, are in the low-temperature state (resp. high-temperature state, critical state).

Similar to the square grid case, in the high temperature state, the Ising model on the hexagonal lattice or the triangular lattice has a unique Gibbs measure, and the spontaneous magnetization vanishes; while in the low temperature state, the Gibbs measures are not unique and the spontaneous magnetization is strictly positive under the “+”-boundary condition. See [1, 14, 32, 35].

If

$$(21) \quad e^{-2K_\tau} = \frac{1 - e^{-2J_\tau}}{1 + e^{-2J_\tau}}, \quad \text{for } \tau = a, b, c,$$

then the XOR Ising model on \mathbb{H} with coupling constants (J_a, J_b, J_c) is in the low-temperature state (resp. high-temperature state, critical state) if and only if the XOR Ising model on the triangular lattice with coupling constants (K_a, K_b, K_c) is in the high-temperature state (resp. low-temperature state, critical state).

We define clusters and contours with respect to an XOR Ising configuration on \mathbb{H} or \mathbb{T} in the usual way. Then we have the following theorems.

Theorem 5.1. *Consider the critical XOR Ising model on \mathbb{H} or \mathbb{T} . Then*

- I almost surely there are no infinite clusters;*
- II almost surely there are no infinite contours.*

Theorem 5.2. *In the low-temperature XOR Ising model on \mathbb{H} or on \mathbb{T} , almost surely there are no infinite contours.*

Theorems 5.1 and 5.2 are proved in Section 13.

6. SQUARE TILINGS OF THE HYPERBOLIC PLANE

In this section, we introduce the square tilings of the hyperbolic plane, and then state and prove properties of the constrained percolation models on such graphs. We first discuss known results about percolation on non-amenable graphs that will be used to prove main theorems of the paper.

The following lemma is proved in [4, 6].

Lemma 6.1. *Let G be a quasi-transitive, non-amenable, planar graph with one end, and let ω be an invariant percolation on G . Then a.s. the number of infinite 1-clusters of ω is 0, 1, or ∞ .*

Proof. See Lemma 3.5 of [6]. □

Lemma 6.2. *(Threshold for bond percolation on non-amenable graphs) Let $G = (V, E)$ be a non-amenable graph. Let $\Gamma \subseteq \text{Aut}(G)$ be a closed unimodular quasi-transitive subgroup, and let o_1, \dots, o_L be a complete set of representatives in V of the orbits of G . For $1 \leq i \leq L$, let Stab_{o_i} is defined as in (6) and*

$$\eta_i := |\text{Stab}_{o_i}|.$$

Let \mathbb{P} be a bond percolation on G whose distribution is Γ -invariant. Let D_i be the random degree of o_i in the percolation subgraph, and let d_i be the degree of o_i in G . Write $p_{\infty, v}$ for the probability that $v \in V$ is in an infinite component. Let $p_{\infty, i}$ be the probability that o_i is in an infinite cluster. Then

$$(22) \quad \sum_{i=1}^L \frac{(d_i - \alpha(G))p_{\infty, i}}{\eta_i} \geq \sum_{j=1}^L \frac{\mathbb{E}D_j - \alpha(G)}{\eta_j}$$

where $\alpha(G)$ is a constant depending on the structure of the graph G defined by

$$\begin{aligned} \alpha_K &:= \frac{1}{|K|} \sum_{x \in K} \deg_K(x) \\ \alpha(G) &:= \sup\{\alpha_K : K \subset G \text{ is finite}\} \end{aligned}$$

In particular, if the right-hand side of (22) is positive, then there is an infinite component in the percolation subgraph with positive probability.

Proof. See Theorem 4.1 of [4]. □

Let $G = (V, E)$ be a graph corresponding to a square tiling of the hyperbolic plane. Assume that

- I each face of G has 4 edges; and

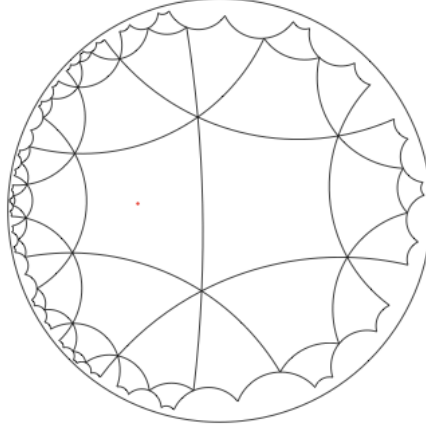


FIGURE 9. The $[4,4,4,4,4,4]$ lattice: each face has degree 4, and each vertex have degree 6

II each vertex of G is incident to $2n$ faces, where $n \geq 3$.

See Figure 9 for an example of such a graph when $n = 3$.

We can color all the faces of G by black and white such that black faces can share edges only with white faces and vice versa. Let $G = (V, E)$ denote the graph embedded into the hyperbolic plane as described above.

We consider the site configurations in $\{0, 1\}^V$. We impose the following constraint on site configurations

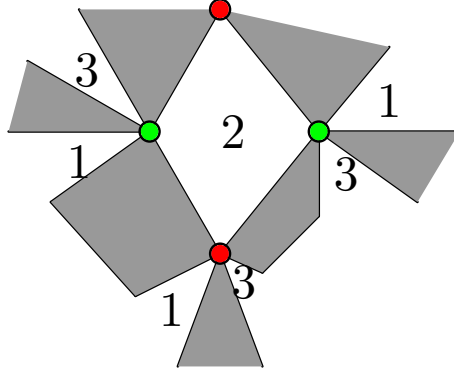
- Around each black face, there are six allowed configurations (0000) , (1111) , (0011) , (1100) , (0110) , (1001) , where the digits from the left to the right correspond to vertices in clockwise order around the black face, starting from the lower left corner. See Figure 4.

Let $\Omega \subset \{0, 1\}^V$ be the set of all configurations satisfying the constraint above. We use Ω to denote the sample space throughout this paper, however, Ω have different meanings in different sections.

Note that G is a vertex-transitive graph. Since each face of G has an even number of edges, G itself is a bipartite graph - we can color the vertices of G by red and green such that red vertices are adjacent only to green vertices and vice versa. We assign an integer in $1, 2, \dots, n$ to each white face of G according to the following rules

- I around each red vertex of G , white faces are assigned integers $1, 2, \dots, n$ clockwise; and
- II around each green vertex of G , white faces are assigned integers $1, 2, \dots, n$ counterclockwise; and
- III any two white faces adjacent to the same black face along two opposite edges have the same assigned integer.

See Figure 10 for an example of assignments of integers $1, 2, 3$ to the white faces of the $[4, 4, 4, 4, 4, 4]$ lattice.


 FIGURE 10. Labels of white faces of the $[4,4,4,4,4]$ lattice

For $1 \leq i \leq n$, we construct a graph \mathbb{L}_i as follows. The vertex set of \mathbb{L}_i consists of all the white faces of G whose assigned integers are i . Two vertices of \mathbb{L}_i are joined by an edge of \mathbb{L}_i if and only if they correspond to two white faces of G adjacent to the same black face along two opposite edges. We have the following proposition regarding the connected components of \mathbb{L}_i

Proposition 6.3. *When $n \geq 3$, each component of \mathbb{L}_i ($1 \leq i \leq n$) is a regular tree of degree 4. For any integer i satisfying $1 \leq i \leq n$, the edges of \mathbb{L}_{i-1} (if $i = 1$, $\mathbb{L}_{i-1} := \mathbb{L}_n$) and \mathbb{L}_i cross; the edges of \mathbb{L}_{i+1} (if $i = n$, $\mathbb{L}_{i+1} := \mathbb{L}_1$) and \mathbb{L}_i cross.*

Proof. We consider a doubly infinite sequence of edges in G consisting of edges $\dots, e_{-1}, e_0, e_1, e_2, \dots$, such that

- For each $k \in \mathbb{Z}$, e_k and e_{k+1} share a vertex v , such that there are exactly $(n-1)$ edges incident to v to the left of e_k and e_{k+1} , and $(n-1)$ edges incident to v to the right of e_k and e_{k+1} .

Then $\dots, e_{-1}, e_0, e_1, \dots$ form a doubly infinite self-avoiding path in G because its left side and right side are symmetric. Indeed, if the path crosses itself, starting from e_0 , we move the path along both the positive direction e_1, e_2, \dots and the negative direction e_{-1}, e_{-2}, \dots , until the first time the movements along the two directions meet, and form a cycle $\mathcal{C}_{ab} := e_{-a}, e_{-a+1}, \dots, e_0, \dots, e_{b-1}, e_b$, where $a, b \in \{0, 1, 2, \dots\}$. Then $G \setminus \mathcal{C}_{ab}$ has a finite component and an infinite component; moving from e_{-a} to e_b along \mathcal{C}_{ab} , the finite component is either on the left or on the right, but this is a contradiction to the fact that on the left and right side of $\dots, e_{-1}, e_0, e_1, e_2, \dots$, G is symmetric. We call the infinite self-avoiding path obtained this way a **central path**.

Assume there is a cycle in \mathbb{L}_i for some $1 \leq i \leq n$, then we can find a face in \mathbb{L}_i . Let (u, v) be an edge of \mathbb{L}_i . Moving from u to v , at v there are 3 other incident edges except the edge (u, v) ; since the graph is embedded in the hyperbolic plane, we may label the three incident edges at v other than (u, v) by the left edge, the middle edge, and the right edge, in such a way that starting from the edge (u, v) and moving around v clockwise along a small circle, one will cross the left edge first, then the middle edge, and finally the right edge. If we can find a face in \mathbb{L}_i , then the face can be found by always moving along the

right edge at each vertex for finitely many times, and the face is on the right of an oriented cycle obtained this way. But when $n \geq 3$, this is not possible since any oriented path in \mathbb{L}_i obtained by always moving along the right edge at each vertex has a central path on its right, which is infinite.

Note that each black face of G has two pairs of opposite edges. There exists $1 \leq i \leq n$, such that along one pair of opposite edges the black face is adjacent to two white faces labeled by i , and along the other pair of opposite edges the black face is adjacent to two white faces labeled by $(i + 1)$ (if $i = n$, then $i + 1 = 1$). Then from the construction of \mathbb{L}_i 's we can see that an edge of \mathbb{L}_i and an edge of \mathbb{L}_{i+1} cross at the black face of G . \square

Any constrained percolation configuration in Ω gives rise to a contour configuration on $\cup_{i=1}^n \mathbb{L}_i$. An edge e in $\cup_{i=1}^n \mathbb{L}_i$ is present in the contour configuration if and only if it crosses a black face b in G , such that the states of the vertices of b on the two sides separated by e in the configuration are different, and any two vertices of b on the same side of e have the same state. This is a contour configuration satisfying the condition that each vertex in $\cup_{i=1}^n \mathbb{L}_i$ has an even number of incident present edges. For any $1 \leq i < j \leq n$, present edges in \mathbb{L}_i and \mathbb{L}_j can never cross.

A **cluster** is a maximal connected set of vertices in G in which every vertex has the same state in a constrained percolation configuration. A **contour** is a maximal connected set of edges in $\cup_{i=1}^n \mathbb{L}_i$ in which every edge is present in the contour configuration. Note that each contour must be a connected subgraph of \mathbb{L}_i , for some $1 \leq i \leq n$. Hence by Proposition 6.3, each contour must be a tree. Since each vertex in a contour has an even number of incident present edges in the contour, each contour must be an infinite tree.

Let μ be a probability measure on Ω . We may assume that μ satisfies the following conditions

- (D1) μ is $\text{Aut}(G)$ -invariant;
- (D2) μ is $\text{Aut}(\mathbb{L}_i)$ -ergodic, for $1 \leq i \leq n$;
- (D3) μ is symmetric, i.e. let $\theta : \Omega \rightarrow \Omega$ be the map defined by $\theta(\omega)(v) = 1 - \omega(v)$, for each $v \in V$, then μ is invariant under θ , that is, for any event A , $\mu(A) = \mu(\theta(A))$.

Note that when $n \geq 3$, the graph G is a non-amenable group. Recall that the number of **ends** of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.

Here is the main theorem concerning the properties of constrained percolations on the square tilings of the hyperbolic plane.

- Theorem 6.4.** (a) *Let μ be a probability measure on Ω satisfying (D1). Let n_0 (resp. n_1) be the number of infinite 0-clusters (resp. 1-clusters). Then μ -a.s. $(n_0, n_1) \in \{(0, 1), (1, 0), (1, \infty), (\infty, 1), (\infty, \infty)\}$.*
- (b) *Let ν be a probability measure on Ω satisfying (D1) - (D3). Then ν -a.s. there are infinitely many infinite 0-clusters and infinitely many infinite 1-clusters.*

In order to prove 6.4, we first prove a few lemmas.

Lemma 6.5. *In a contour configuration in $\cup_{i=1}^n \mathbb{L}_i$ as described above, any contour must be an infinite tree (a tree consisting of infinite many edges of $\cup_{i=1}^n \mathbb{L}_i$) in which each vertex has degree 2 or 4.*

Proof. This lemma is straightforward from the facts that each contour is a connected subgraph of \mathbb{L}_i for some $i \in \{1, \dots, n\}$; each component of \mathbb{L}_i $1 \leq i \leq n$ is a regular tree of degree 4, and each vertex in a contour has an even number of incident present edges. \square

Lemma 6.6 below is proved in [4] and [3] using the mass transport principle.

Lemma 6.6. *Let G be a nonamenable graph whose automorphism group has a closed subgroup acting transitively and unimodularly on G , and let ω be an invariant percolation on G which has a single component a.s. Then $p_c(\omega) < 1$ a.s., where $p_c(\cdot)$ is the critical i.i.d. Bernoulli percolation probability on a graph.*

Proof. See Theorem 1.5 of [4]. \square

Proof of Theorem 6.4 First we show that Part (a) of the theorem together with Assumptions (D2), (D3) implies Part (b). Let ν be a probability measure on Ω satisfying (D1) - (D3). By Assumption (D2) and (D3), there exists a positive integer k (possibly infinite), such that $\nu(n_0 = n_1 = k) = 1$. Then Part (b) follows from Part (a).

Now we prove Part (a). Obviously $(n_0, n_1) \in \{(0, 1), (1, 0)\}$ if there are no contours. Now assume that contours do exist. By Lemma 6.1, $n_0, n_1 \in \{0, 1, \infty\}$. By Lemma A.12, $n_0, n_1 \in \{1, \infty\}$. Let ϕ be the contour configuration. If there are infinitely many contours in ϕ , or there exists a contour of ϕ in which infinitely many vertices have degree 4, then $\mathbb{H}^2 \setminus \phi$ has infinitely many unbounded components. By Lemma A.13, $n_0 + n_1 = \infty$. Therefore $\{n_0, n_1\} \in \{(1, \infty), (\infty, 1), (\infty, \infty)\}$ in this case.

Now consider the case that the number of contours is finite and nonzero, and on each contour only finitely many vertices have degree 4. Fix an i satisfying $1 \leq i \leq n$, and conditional on the event that the number of contours on \mathbb{L}_i is finite and nonzero. Choose a contour τ on \mathbb{L}_i uniformly at random; then τ forms an invariant bond percolation on \mathbb{L}_i which has a single component. By Lemma 6.6, almost surely τ has infinitely many vertices with degree 4 - since otherwise $p_c(\tau) = 1$. Therefore this case does not occur a.s. \square

7. PROOF OF THEOREM 2.2

In this section, we prove Theorem 2.2. The idea of the proof is to consider all the possible values of (s_0, s_1, t_1, t_2) and exclude those with probability 0 to occur using the symmetry and ergodicity of the probability measure. In Lemma 7.1, we exclude the case $t_1 = t_2 = 1$; the proof is based on constructing a superimposition \hat{G} of the lattice \mathbb{L}_1 and its dual lattice \mathbb{L}_2 ; and the union of contour configurations on \mathbb{L}_1 and \mathbb{L}_2 form an invariant bond percolation on \hat{G} , in which the number of infinite clusters can only be 0, 1, ∞ by Lemma 6.1 a.s.; however, if $t_1 = t_2 = 1$, since the contour configurations on \mathbb{L}_1 and \mathbb{L}_2 do not cross each other, the number of infinite clusters in the union would be 2. Proposition 7.2 excludes the case when $(s_0, s_1, t_1, t_2) = (0, 0, 0, 0)$, the proof applies planarity to obtain an infinite sequence of contours, one surrounding another, and then obtain a contradiction with non-amenability. Lemma 7.3 excludes the case $(t_1, t_2) = (0, k)$ and $(t_1, t_2) = (k, 0)$ for $1 \leq k \leq \infty$ by ergodicity, symmetry and planarity. Lemma 7.4 excludes the case that $(s_0, s_1) = (1, 1)$ again by constructing an invariant bound percolation on G with 2 infinite clusters and obtaining a contradiction to Lemma 6.1. In the proof

of Theorem 2.2, we use symmetry, ergodicity, Lemma 6.1 and Lemma 7.4 to obtain that a.s. $(s_0, s_1) \in \{(0, 0), (\infty, \infty)\}$; to rule out the case $(s_0, s_1) = (0, 0)$, we apply Lemma 6.1 again to show that if $(s_0, s_1) = (0, 0)$, then $(s_0, s_1) \in \{(0, 0, 0), (0, 0, 1), (0, 0, \infty)\}$, we then show that each of the cases has probability 0 to occur by applying Lemmas 7.1 and 7.3 and Proposition 7.2.

We start with Lemma 7.1.

Lemma 7.1. *Let G be the $[m, 4, n, 4]$ lattice satisfying (1) $m \geq 3, n \geq 3$ and*

$$(23) \quad \frac{1}{m} + \frac{1}{n} < \frac{1}{2}.$$

Let μ be a probability measure on Ω satisfying (A1). Let t_1 (resp. t_2) be the number of infinite \mathbb{L}_1 -contours (resp. \mathbb{L}_2 -contours). Then

$$\mu((t_1, t_2) = (1, 1)) = 0.$$

Proof. The proof is inspired by the proof of Corollary 3.6 of [6].

We embed \mathbb{L}_1 and \mathbb{L}_2 in the hyperbolic plane in such a way that every edge e intersects its dual edge e^* at one point v_e , and there are no other intersections of \mathbb{L}_1 and \mathbb{L}_2 . We define a new graph $\hat{G} = (\hat{V}, \hat{E})$, where $\hat{V} = V(\mathbb{L}_1) \cup V(\mathbb{L}_2) \cup \{v_e, e \in E(\mathbb{L}_1)\}$, and an edge in \hat{E} is either a half-edge of $E(\mathbb{L}_1)$ joining a vertex in $V(\mathbb{L}_1)$ and a vertex in $\cup\{v_e, e \in E(\mathbb{L}_1)\}$, or a half-edge of $E(\mathbb{L}_2)$ joining a vertex in $V(\mathbb{L}_2)$ and a vertex in $\cup\{v_e, e \in E(\mathbb{L}_1)\}$.

For $i \in \{1, 2\}$, let $\phi_i \in \Phi_i$ be the random contour configuration restricted on \mathbb{L}_i . Let

$$\hat{\phi} := \{[v, v_e] \in \hat{E} : v \in V(\mathbb{L}_1), e \in \phi_1\} \cup \{[v_*, v_e] \in \hat{E} : v_* \in V(\mathbb{L}_2), e_* \in \phi_2\}$$

We say $\hat{\phi}$ is a **contour configuration** on \hat{G} , and each connected component of $\hat{\phi}$ is called a **contour**. Then $\hat{\phi}$ is an invariant bond percolation on the quasi-transitive, non-amenable, planar, one-ended graph \hat{G} . Note that the number of infinite components of $\hat{\phi}$ is the number of infinite contours of ϕ_1 plus the number of infinite contours of ϕ_2 . If there is a positive probability that $(t_0, t_1) = (1, 1)$, then the number of infinite components in $\hat{\phi}$ is 2. This contradicts Lemma 6.1, which says that the number of infinite components in the invariant percolation $\hat{\phi}$ on the quasi-transitive, one-ended, nonamenable, planar graph \hat{G} can only be 0, 1 or ∞ . \square

Proposition 7.2. *Let G be the $[m, 4, n, 4]$ lattice with m, n satisfying (1) $m \geq 3, n \geq 3$ and (23) $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$. Let $\omega \in \Omega$ be a Γ -invariant, Γ_1 -ergodic constrained percolation on G . Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. 1-clusters) in ω , and let t_1 (resp. t_2) be the number of infinite \mathbb{L}_1 -contours (resp. infinite \mathbb{L}_2 -contours) in ω . Then almost surely $(s_0, s_1, t_1, t_2) \neq (0, 0, 0, 0)$.*

Proof. The proof is inspired by Lemma 3.3 of [6]. Let $\hat{G} = (\hat{V}, \hat{E})$, $\hat{\phi}$ be defined as in the proof of Lemma 7.1. Note that when m, n satisfy (1) and (23), \hat{G} is a quasi-transitive, non-amenable, planar and one-ended graph; and that the $[m, 4, n, 4]$ lattice is exactly the dual graph of \hat{G} . It is also known that quasi-transitive planar graphs with one end are unimodular; see [38].

Define a **generalized contour** in a contour configuration $\hat{\phi}$ of \hat{G} to be either a single vertex in \hat{V} which has no incident present edges in $\hat{\phi}$, or a contour in $\hat{\phi}$. This way each vertex $v \in \hat{V}$ has a unique generalized contour in $\hat{\phi}$ passing through the vertex v .

Suppose that $(s_0, s_1, t_1, t_2) = (0, 0, 0, 0)$ a.s. Then a.s. given a generalized contour C of $\hat{\phi}$, there is a cluster C' of ω surrounding it. Similarly, for every cluster C in ω , there is a contour C' in $\hat{\phi}$ that surrounds it. Let \mathcal{C}_0 denote the set of all generalized contours of $\hat{\phi}$. We set

$$\mathcal{C}_{j+1} := \{C'' : C \in \mathcal{C}_j\};$$

in which C is a generalized contour, C' is a cluster, and C'' is a contour. For $C \in \mathcal{C}_0$ and $v \in \hat{V}$, let $r(C) := \sup\{j : C \in \mathcal{C}_j\}$, and define $r(v) := r(C)$ if C is the generalized contour of v in $\hat{\phi}$. Intuitively, we may consider $r(C)$ as the maximal length of sequences of nesting contours, in which C is the outermost contour.

Then there exist $i \in \{1, 2\}$ and a sequence of finite contours $C_1, C_2, \dots, C_n, \dots$ in \mathbb{L}_i , such that C_{n+1} surrounds C_n , and

$$\lim_{n \rightarrow \infty} r(C_n) = \infty.$$

For each r let ω^r be the set of edges in $E(\mathbb{L}_i)$ whose both endpoints $u, v \in V(\mathbb{L}_i)$ satisfy $r(v) \leq r$ and $r(u) \leq r$. Then ω^r is an invariant bond percolation and for any $v \in V(\mathbb{L}_i)$,

$$\deg_{\mathbb{L}_i} v = \mathbf{E} \lim_{r \rightarrow \infty} [\deg_{\omega^r} v] \leq \liminf_{r \rightarrow \infty} \mathbf{E}[\deg_{\omega^r} v] \leq \limsup_{r \rightarrow \infty} \mathbf{E}[\deg_{\omega^r} v] \deg_{\mathbb{L}_i} v.$$

Note that \mathbb{L}_i is a transitive, non-amenable graph. We have

$$\alpha(\mathbb{L}_i) = \deg_{\mathbb{L}_i} v - \iota_E(\mathbb{L}_i) < \deg_{\mathbb{L}_i} v$$

where $\iota_E(\mathbb{L}_i)$ is the edge isoperimetric constant defined as in (7), and $\alpha(\mathbb{L}_i)$ is defined in Lemma 6.2. By Lemma 6.2, the right hand side of (22) is strictly positive for sufficiently large r ; we deduce that ω^r has infinite components with positive probability for all sufficiently large r .

However, since $(s_0, s_1, t_1, t_2) = (0, 0, 0, 0)$, by the arguments above each vertex in \mathbb{L}_i is surrounded by infinitely many finite contours in \mathbb{L}_i . This implies that for any $r \in \mathbb{N}$, for any vertex $v \in V(\mathbb{L}_i)$, there exists a finite contour C surrounding v , such that $r(C) > r$, and therefore $C \cap \omega^r = \emptyset$. As a result, the components in ω^r including v is finite. Then the proposition follows from the contradiction. \square

Lemma 7.3. *Let G be the $[m, 4, n, 4]$ lattice with m, n satisfying (1), (23). Let (s_0, s_1, t_1, t_2) be given as in Theorem 2.2.*

I Let μ be a probability measure on Ω satisfying (A2)(A7). Then

$$\mu((t_1, t_2) = (0, k)) = 0.$$

for any integer $1 \leq k \leq \infty$.

II Let μ be a probability measure on Ω satisfying (A2)(A6). Then

$$\mu((t_1, t_2) = (k, 0)) = 0.$$

for any integer $1 \leq k \leq \infty$.

Proof. We prove Part I here; Part II can be proved using exactly the same technique. By (A2) μ is Γ_i ergodic, either $\mu((t_1, t_2) = (0, k)) = 0$ or $\mu((t_1, t_2) = (0, k)) = 1$. Assume that $\mu((t_1, t_2) = (0, k)) = 1$; we shall obtain a contradiction. Since there exists an infinite

\mathbb{L}_2 -contour; hence there exists an infinite cluster in λ_2 containing the infinite \mathbb{L}_2 -contour. By (A7) λ_2 is Γ_2 -ergodic, and the symmetry of λ_2 , there exist an infinite 0-cluster and an infinite 1-cluster in λ_2 a.s.. Note that the configuration in $\lambda_2 \in \{0, 1\}^{V(\mathbb{L}_2)}$ naturally induces a configuration $\omega \in \Omega$ by the condition that the contour configurations corresponding to λ_2 and ω are the same. We can see that if in λ_2 there exist both an infinite 0-cluster and an infinite 1-cluster, then in the induced constrained configuration $\omega \in \Omega$, there is both an infinite 0-cluster and an infinite 1-cluster. By Lemma A.3, there exist an infinite \mathbb{L}_1 -contour. But this is a contradiction to the fact that $t_1 = 0$. \square

Lemma 7.4. *Let $G = (V, E)$ be the $[m, 4, n, 4]$ lattice with m, n satisfying (1) $m \geq 3, n \geq 3$ and (2) $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$. Let μ be a probability measure on Ω satisfying (A2), (A8). Let (s_0, s_1, t_1, t_2) be given as in Theorem 2.2. Then*

$$\mu((s_0, s_1) = (1, 1)) = 0.$$

Proof. By (A2) μ is Γ_i -ergodic, either $\mu((s_0, s_1) = (1, 1)) = 0$ or $\mu((s_0, s_1) = (1, 1)) = 1$. Assume that $\mu((s_0, s_1) = (1, 1)) = 1$; we shall obtain a contradiction.

Let $\omega \in \Omega$. We first construct a bond configuration $\omega_b \in \{0, 1\}^E$ by letting an edge $e \in E$ to be present if and only if it joins two edges in ω with the same state; i.e. either both its endpoints have state 0; or both its endpoints have state 1. It is easy to check that the (0 or 1) clusters in ω are exactly the components in ω_b . Then ω_b forms a Γ_1 -invariant percolation on G . If $(s_0, s_1) = (1, 1)$, then ω_b has exactly two infinite components. But this is a contradiction to Lemma 6.1. \square

Proof of Theorem 2.2 I. Assume that μ is a probability measure on Ω satisfying (A2),(A3),(A7),(A8).

Let (s_0, s_1, t_1, t_2) be given as in the theorem. By Lemma 6.1, we have μ -a.s. $s_0 \in \{0, 1, \infty\}$, $s_1 \in \{0, 1, \infty\}$ and $t_1 \in \{0, 1, \infty\}$. By (A2) μ is Γ_i -ergodic and (A3) μ is symmetric with respect to interchanging state “0” and state “1”, we have μ -a.s. $(s_0, s_1) \in \{(0, 0), (1, 1), (\infty, \infty)\}$. Hence we need to rule out the case that $(s_0, s_1) = (1, 1)$ and the case that $(s_0, s_1) = (0, 0)$. Almost surely we have $(s_0, s_1) \neq (1, 1)$ by Lemma 7.4. Now we show that almost surely $(s_0, s_1) \neq (0, 0)$.

We claim that μ -a.s. $t_1 \in \{0, \infty\}$. Assume that μ -a.s. $t_1 = 1$, we shall obtain a contradiction. Let τ be the unique infinite \mathbb{L}_1 -contour. Then τ forms an invariant bond percolation on \mathbb{L}_1 which has a single component a.s.. By Lemma 6.6, $p_c(\tau) < 1$ a.s. However, τ is an even-degree subgraph of \mathbb{L}_1 and \mathbb{L}_1 has vertex-degree 3; as a result, τ must be a doubly-infinite self-avoiding path. This is a contradiction to the fact that $p_c(\tau) < 1$. Therefore we have either μ -a.s. $t_1 = 0$ or μ -a.s. $t_1 = \infty$.

If μ -a.s. $t_1 = \infty$, let ϕ be the contour configuration on $\mathbb{L}_1 \cup \mathbb{L}_2$ corresponding to the constrained percolation configuration. Since each infinite contour in ϕ is a doubly-infinite self-avoiding path, if there are infinitely many infinite contours, then $\mathbb{H}^2 \setminus \phi$ has infinitely many unbounded components. Note also that there exists an infinite cluster in each infinite component of $\mathbb{H}^2 \setminus \phi$; hence μ -a.s. $(s_0, s_1, t_1) = (\infty, \infty, \infty)$ in this case.

Now consider the case that μ -a.s. $t_1 = 0$.

We assume that μ -a.s. $(s_0, s_1, t_1) = (0, 0, 0)$ and shall again obtain a contradiction. By Proposition 7.2, a.s. $(s_0, s_1, t_1, t_2) \neq (0, 0, 0, 0)$. Moreover, it is impossible to have

$(s_0, s_1, t_1, t_2) = (0, 0, 0, \infty)$ since if $t_2 = \infty$, then there are infinitely many infinite clusters. By Lemma 7.3, a.s. $(s_0, s_1, t_1, t_2) \neq (0, 0, 0, 1)$. Therefore $\mu((s_0, s_1, t_1) = (0, 0, 0)) = 0$.

We next assume that μ -a.s. $(s_0, s_1, t_1) = (\infty, \infty, 0)$. By Lemma 7.3, μ -a.s. $(s_0, s_1, t_1, t_2) = (\infty, \infty, 0, 0)$. Since there exists an infinite 0-cluster and an infinite 1-cluster, by Lemma A.3, there exists an infinite contour, and $s_0 + s_1 > 0$. The contradiction implies that $\mu((s_0, s_1, t_1) = (\infty, \infty, 0)) = 0$. This completes the proof of Part I of Theorem 2.2. \square

Proof of Theorem 2.2 II. Assume that μ is a probability measure on Ω satisfying (A2),(A3),(A6),(A7),(A8). By Theorem 2.2 I, μ -a.s. $(s_0, s_1, t_1) = (\infty, \infty, \infty)$. Part II of Theorem 2.2 then follows from Lemma 7.3. \square

8. PROOF OF THEOREM 2.4

In this section, we prove Theorem 2.4.

We first prove that Parts (a) and (b) implies Part (c). If μ satisfies (A1)-(A7), then by (a) and (b), μ -a.s. there are neither infinite primal contours nor infinite dual contours. Therefore μ -a.s. there are no infinite contours.

Let \mathcal{E}_0 (resp. \mathcal{E}_1) be the event that there exists an infinite 0-cluster (resp. infinite 1-cluster). Assume that $\mu(\mathcal{E}_0 \cup \mathcal{E}_1) > 0$. Then by (A2) μ is Γ_i ergodic,

$$(24) \quad \mu(\mathcal{E}_0 \cup \mathcal{E}_1) = 1.$$

By (A3) μ is symmetric with respect to exchanging state “0” and state “1”, $\mu(\mathcal{E}_0) = \mu(\mathcal{E}_1)$. By (A2), either $\mu(\mathcal{E}_0) = \mu(\mathcal{E}_1) = 1$ or $\mu(\mathcal{E}_0) = \mu(\mathcal{E}_1) = 0$. By (24), we have $\mu(\mathcal{E}_0) = \mu(\mathcal{E}_1) = 1$. By Lemma A.3, μ -a.s. there exists an infinite contour. But this is a contradiction to the fact that μ -a.s. there are no infinite contours. Therefore μ -a.s. there are no infinite clusters.

Next we prove (a) and (b). Note that the $[m, 4, n, 4]$ lattice G is amenable if and only if

$$(25) \quad \frac{1}{m} + \frac{1}{n} = \frac{1}{2}.$$

When m, n are positive integers greater than or equal to 3, the only pairs of (m, n) satisfying (25) are $(m, n) = (4, 4)$, $(m, n) = (3, 6)$ and $(m, n) = (6, 3)$. When $(m, n) = (4, 4)$, G is the square grid embedded into \mathbb{R}^2 . In this case (a) and (b) were proved in [27]. Then cases $(m, n) = (3, 6)$ and $(m, n) = (6, 3)$ can be proved in the same way. We write down the proof of the case when $(m, n) = (3, 6)$ here.

When $(m, n) = (3, 6)$, \mathbb{L}_1 is the hexagonal lattice $\mathbb{H} = (V(\mathbb{H}), E(\mathbb{H}))$ and \mathbb{L}_2 is the triangular lattice $\mathbb{T} = (V(\mathbb{T}), E(\mathbb{T}))$.

Lemma 8.1. *Assume that $(m, n) = (3, 6)$. When μ satisfies (A1), (A2) and (A5), almost surely there exists at most one infinite contour in \mathbb{T} .*

Proof. Let \mathbb{N} be the set of all nonnegative integers. Let \mathcal{N} be the number of infinite contours in \mathbb{T} . By (A2), there exists $k_0 \in \mathbb{N} \cup \{\infty\}$, s.t. $\mu(\mathcal{N} = k_0) = 1$.

By [4] (see also Exercise 7.24 of [38]), (A1) and the fact that the triangular lattice \mathbb{T} is transitive and amenable, μ -a.s. no infinite contours has more than 2 ends.

The triangular lattice \mathbb{T} can be obtained from a square grid \mathbb{S} by adding a diagonal in each square face of \mathbb{S} .

Let B_n be an $n \times n$ box of \mathbb{S} . Let \tilde{B}_n be the corresponding box in \mathbb{T} , i.e. \tilde{B}_n can be obtained from B_n by adding a diagonal edge on each square face of B_n .

Let ϕ (resp. $\tilde{\phi}$) be a contour configuration on \mathbb{S} (resp. \mathbb{T}), such that ϕ and $\tilde{\phi}$ satisfy the following conditions (note that the vertices in ∂B_n and $\partial \tilde{B}_n$ are in 1-1 correspondence)

- for each vertex $v \in \partial B_n$, no edges incident to v outside \tilde{B}_n are present in $\tilde{\phi}$ if and only if no edges incident to v outside B_n are present in ϕ ;
- for each vertex $v \in \partial B_n$, if there are incident present edges of v in $\tilde{\phi}$ outside \tilde{B}_n , then the parity of the number of incident present edges of v outside \tilde{B}_n in $\tilde{\phi}$ is the same as the parity of the number of incident present edges of v outside B_n in ϕ ; i.e. either both numbers are even or both are odd.

Let $n \geq 2$. Given ϕ , we can find a configuration ξ in B_n , such that $[\phi \setminus B_n] \cup \xi$ is a contour configuration on \mathbb{S} (i.e. each vertex of \mathbb{S} has an even number of incident present edges in $[\phi \setminus B_n] \cup \xi$), and all the incident present edges of ∂B_n outside B_n are in the same contour; see Lemma 4.2 of [27]. If $\tilde{\phi}$ and ϕ satisfy the conditions described above, then $[\tilde{\phi} \setminus \tilde{B}_n] \cup \xi$ is a contour configuration on \mathbb{T} , and all the incident present edges of $\partial \tilde{B}_n$ outside \tilde{B}_n are in the same contour.

Note that ξ can be obtained from $\tilde{\phi} \cap \tilde{B}_n$ by changing configurations on finitely many triangles in \tilde{B} as described in (A6). That is because any contour configuration on \mathbb{T} naturally induces two site configurations $\omega, 1 - \omega$, in $\{0, 1\}^{V(\mathbb{H})}$, such that two adjacent vertices in \mathbb{H} have different states if and only if the edge in \mathbb{T} separating the two vertices are present in the contour configuration. Any two site configurations in $\{0, 1\}^{V(\mathbb{H})}$ differ only in \tilde{B}_n can be obtained from each other by changing states on finitely many vertices in $V(\mathbb{H}) \cap \tilde{B}_n$. Changing the state at a vertex in $V(\mathbb{H})$ corresponds to changing the states on all the edges of the dual triangle face including the vertex in the contour configuration of \mathbb{T} .

We claim that $k_0 \in \{0, 1\}$. Indeed, if $2 \leq k_0 < \infty$, we can find a box \tilde{B}_n in \mathbb{T} , such that \tilde{B}_n intersects all the k_0 infinite contours. Then we can change configurations on finitely many triangles in \tilde{B}_n , such that after the configuration change, there is exactly one infinite contour. By the finite energy assumption (A5), with positive probability, there exists exactly one infinite contour, but this is a contradiction to $\mu(\mathcal{N} = k_0) = 1$, where $2 \leq k_0 < \infty$.

If $k_0 = \infty$, we can find a box \tilde{B}_m in \mathbb{T} , such that \tilde{B}_m intersects at least 3 infinite contours. Then we can change configurations on finitely many triangles in \tilde{B}_m , such that after the configuration change, all the infinite contours intersecting \tilde{B}_m merge into one infinite contour, which has at least 3 ends. By (A5), with positive probability there exists an infinite contour with more than 2 ends. But this is a contradiction to the fact that almost surely no infinite contours have more than two ends. \square

Lemma 8.2. *Assume that $(m, n) = (3, 6)$. When μ satisfies (A1) and (A4), almost surely there exists at most one infinite contour in \mathbb{H} .*

Proof. Recall that a contour is a connected set of edges in which each vertex has an even number of incident present edges. Since the hexagonal lattice \mathbb{H} is a cubic graph, i.e. each vertex has 3 incident edges; each vertex in a contour of \mathbb{H} has 2 incident present edges. As a result, each contour in \mathbb{H} is either a self-avoiding cycle or a doubly-infinite self-avoiding path. In particular, each infinite contour in \mathbb{H} is a doubly-infinite self-avoiding path.

Each contour configuration in \mathbb{H} naturally induces two site configurations $\omega, 1 - \omega$ in $\{0, 1\}^{V(\mathbb{T})}$, in which two adjacent vertices of \mathbb{T} have the same state if and only if the dual edge in \mathbb{H} separating the two vertices is absent in the contour configuration. The finite energy assumption (A4) implies the finite energy in the induced site configuration in $\{0, 1\}^{V(\mathbb{T})}$; see [10] for a definition. When μ satisfies (A1) (A4), by the result in [10], almost surely there exists at most one infinite 1-cluster and at most one infinite 0-cluster. In particular, there exist at most two infinite clusters. However, if in \mathbb{H} there are more than one infinite contour, then there are at least two doubly-infinite self-avoiding paths in \mathbb{H} . As a result, the number of infinite clusters in the induced site configuration on \mathbb{T} is at least 3. The contradiction implies the lemma. \square

Lemma 8.3. *Let $\omega \in \Omega$ be a constrained percolation configuration on the $[3, 4, 6, 4]$ lattice G . Let $\psi = \phi(\omega) \in \Phi$ be the corresponding contour configuration in $E(\mathbb{H}) \cup E(\mathbb{T})$. Assume that $\psi = \psi_1 \cup \psi_2$, where ψ_1 (resp. ψ_2) is the contour configuration in \mathbb{H} (resp. \mathbb{T}). If there is an infinite contour in ψ_1 (resp. ψ_2), then there is an infinite cluster in $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$).*

Proof. Assume that there is an infinite contour C in \mathbb{H} (resp. \mathbb{T}). Let V_C be the set consisting of all the vertices of G such that

- $v \in V_C$ if and only if v is a vertex of a face of G crossed by an edge present in the contour C .

Let F be a square face of G crossed by C ; then all the vertices in F are in the same cluster of $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$). That is because $\psi_1 \cap \psi_2 = \emptyset$, if F is crossed by $C \subseteq \psi_1$ (resp. $C \subseteq \psi_2$), then $F \cap \psi_2 = \emptyset$ (resp. $F \cap \psi_1 = \emptyset$).

Let F' be a triangle (resp. hexagon) face of G crossed by C ; then all the vertices in F' are also in the same cluster of $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$). That is because the boundary edges of F' cannot be crossed by edges of \mathbb{T} (resp. \mathbb{H}) at all.

We claim that all the vertices in V_C are in the same cluster in $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$). Indeed, for any two vertices $u, v \in V_C$, we can find a sequence of faces F_0, F_1, \dots, F_k , such that

- $u \in F_0$ and $v \in F_k$; and
- for $0 \leq i \leq k$, F_i is crossed by C ;
- for $1 \leq j \leq k$, F_j and F_{j-1} share a vertex.

Then u and v are in the same cluster in $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$) since all the vertices in $\cup_{i=0}^k F_i$ are in the same cluster in $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$). Moreover, $|V_C| = \infty$ since C is an infinite contour. Therefore, $\phi^{-1}(\psi_2)$ (resp. $\phi^{-1}(\psi_1)$) has an infinite cluster. \square

Parts (a) and (b) can be proved using similar techniques; we write down the proof of (a) here.

Let μ be a probability measure on Ω satisfying (A1)-(A6). Assume that with strictly positive probability, there exist infinite contours in \mathbb{T} . Then by (A2), μ -a.s. there exist infinite contours in \mathbb{T} . By Lemma 8.1, μ -a.s. there exists exactly one infinite contour C_1 in \mathbb{T} . By Lemma 8.3, μ -a.s. there exist infinite clusters in $\phi^{-1}(\psi_1)$. Let \mathcal{F}_0 (resp. \mathcal{F}_1) be the event that there exists an infinite 0-cluster (resp. infinite 1-cluster) in $\phi^{-1}(\psi_1)$, then

$$(26) \quad \lambda_1(\mathcal{F}_0 \cup \mathcal{F}_1) = 1,$$

where the probability measure λ_1 is defined before (A6). By (A6), and the symmetry of λ_1 , either $\lambda_1(\mathcal{F}_0) = \lambda_1(\mathcal{F}_1) = 0$, or $\lambda_1(\mathcal{F}_0) = \lambda_1(\mathcal{F}_1) = 1$. By (26), we have $\lambda_1(\mathcal{F}_0 \cap \mathcal{F}_1) = 1$. By Lemma A.3, μ -a.s. there are infinite contours in \mathbb{H} . By Lemma 8.2, μ -a.s. there is exactly one infinite contour C_2 in \mathbb{H} .

Hence there is exactly one infinite contour C_2 in \mathbb{H} and exactly one infinite contour C_1 in \mathbb{T} . By Lemma A.7, there exists an infinite cluster incident to both C_1 and C_2 .

Let \mathcal{H}_0 (resp. \mathcal{H}_1) be the event that there exists an infinite 0-cluster (resp. infinite 1-cluster) in ω incident to both the infinite contour in \mathbb{H} and the infinite contour in \mathbb{T} . Then

$$(27) \quad \mu(\mathcal{H}_0 \cup \mathcal{H}_1) = 1.$$

By (A3), $\mu(\mathcal{H}_0) = \mu(\mathcal{H}_1)$. By (A2), either $\mu(\mathcal{H}_0) = \mu(\mathcal{H}_1) = 0$, or $\mu(\mathcal{H}_0) = \mu(\mathcal{H}_1) = 1$. By (27), $\mu(\mathcal{H}_0) = \mu(\mathcal{H}_1) = 1$, therefore $\mu(\mathcal{H}_0 \cap \mathcal{H}_1) = 1$, i.e. there exist an infinite 0-cluster ξ_0 and an infinite 1-cluster ξ_1 , such that ξ_0 is incident to both C_1 and C_2 , and ξ_1 is incident to both C_1 and C_2 . But this is a contradiction to Lemma A.10. Therefore we conclude that μ -a.s. there are no infinite contours in \mathbb{T} ; this completes the proof of Part (a).

9. PROOF OF THEOREM 2.5

In this section, we prove Theorem 2.5.

Lemma 9.1. *Let G be an $[m, 4, m, 4]$ lattice with $m \geq 5$. Let μ be a probability measure on Ω satisfying (A1)-(A3). Then the distribution of infinite clusters can only be one of the following 2 cases.*

I There are no infinite clusters μ -a.s.

II There are infinitely many infinite 1-clusters and infinitely many infinite 0-clusters μ -a.s.

Lemma 9.1 can be obtained from Lemma 7.4; it may also be proved using different arguments below.

Proof. Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. 1-clusters). By Lemma 6.1 and (A2) (A3), there exist $k \in \{0, 1, \infty\}$, such that $\mu((s_0, s_1) = (k, k)) = 1$. It suffices to show that $k \neq 1$.

Let \mathcal{A} be the event that $(s_0, s_1) = (1, 1)$. Assume that $\mu(\mathcal{A}) = 1$, we will obtain a contradiction.

As explained before the constrained site configurations on G correspond to contour configurations in $\mathbb{L}_1 \cup \mathbb{L}_2$.

Since μ -a.s. there exists exactly one infinite 0-cluster and exactly one infinite 1-cluster simultaneously, then by Lemma A.8, μ -a.s. there exists an infinite primal or dual contour incident to both the infinite 0-cluster and the infinite 1-cluster. Let \mathcal{D}_1 (resp. \mathcal{D}_2) be the event that there exists an infinite primal (resp. dual) contour in \mathbb{L}_1 (resp. \mathbb{L}_2), incident to both the infinite 0-cluster and the infinite 1-cluster. So we have

$$(28) \quad \mu(\mathcal{D}_1 \cup \mathcal{D}_2) = 1.$$

By (A1) μ is Γ_i invariant, we have

$$(29) \quad \mu(\mathcal{D}_1) = \mu(\mathcal{D}_2).$$

Moreover, by (A2), we have either

$$(30) \quad \mu(\mathcal{D}_1) = 0 \text{ or } \mu(\mathcal{D}_1) = 1.$$

Combining (28), (29) and (30), we have

$$(31) \quad \mu(\mathcal{D}_1 \cap \mathcal{D}_2) = 1.$$

Thus, by (31), we have exactly one infinite 1-cluster on G , denoted by ξ_1 and exactly one infinite 0-cluster on G , denoted by ξ_0 . There is an infinite primal contour incident to both ξ_0 and ξ_1 , denoted by C_1 ; as well as an infinite dual contour incident to both ξ_0 and ξ_1 , denoted by C_2 . But this is impossible by Lemma A.10. The contradiction implies the lemma. \square

Lemma 9.2. *Let μ be a probability measure on Ω satisfying (A1)-(A3). Then the distribution of infinite contours can only be one of the following 2 cases.*

- I There are neither infinite primal contours nor infinite dual contours μ -a.s..*
- II There are infinitely many infinite primal contours and infinitely many infinite dual contours μ -a.s..*

Proof. The primal (resp. dual) contours form an invariant bond percolation on \mathbb{L}_1 (resp. \mathbb{L}_2) under μ . Let t_1 (resp. t_2) be the number of infinite primal (resp. dual) contours. By Lemma 6.1 and (A1)-(A3), only 3 cases may occur:

- i. μ -a.s. $(t_1, t_2) = (0, 0)$;
- ii. μ -a.s. $(t_1, t_2) = (\infty, \infty)$;
- iii. μ -a.s. $(t_1, t_2) = (1, 1)$.

It remains to exclude in Case iii.. Assume that Case iii. occurs. Let C_1 (resp. C_2) be the unique infinite primal (resp. dual) contour. By Lemma A.7, there exists an infinite cluster incident to both C_1 and C_2 . Moreover, by (A2)-(A3), μ -a.s. there exists an infinite 0-cluster incident to both C_1 and C_2 , as well as an infinite 1-cluster incident to both C_1 and C_2 . But this is impossible by Lemma A.10. \square

Proof of Theorem 2.5. By Lemmas 9.1 and 9.2 and proposition 7.2, it suffices to show that there exists an infinite cluster μ -a.s. if and only if there exists an infinite contour μ -a.s. if μ satisfies (A1)-(A3).

First assume that there exists an infinite cluster μ -a.s. By (A2)-(A3), μ -a.s. there exist both an infinite 0-cluster and an infinite 1-cluster. By Lemma A.3, μ -a.s. there exists an infinite contour.

Now assume that there exists an infinite contour μ -a.s. By (A1)-(A2), μ -a.s. there exist both an infinite primal contour and an infinite dual contour. By Lemma A.2, μ -a.s. there exists an infinite cluster. \square

10. PERCOLATION ON TRANSITIVE, TRIANGULAR TILINGS OF HYPERBOLIC PLANE

In this section, we discuss the applications of the techniques developed in the proof of Theorem 2.2 to prove results concerning unconstrained site percolation on vertex-transitive, triangular tilings of the hyperbolic plane. We also discuss results about Bernoulli percolation on such graphs. These results will be used to prove Theorems 3.3, 4.1 and 4.2.

Lemma 10.1. *Let \mathbb{L}_2 a vertex-transitive, regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider an $\text{Aut}(\mathbb{L}_2)$ -invariant site percolation ω on \mathbb{L}_2 with distribution μ . Assume that μ is $\text{Aut}(\mathbb{L}_2)$ -ergodic. Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. infinite 1-clusters) in the percolation. Then*

$$\mu((s_0, s_1) = (0, 0)) = 0.$$

Proof. Since the event $\{(s_0, s_1) = (0, 0)\}$ is $\text{Aut}(\mathbb{L}_2)$ -invariant, and μ is $\text{Aut}(\mathbb{L}_2)$ -ergodic, either $\mu((s_0, s_1) = (0, 0)) = 0$ or $\mu((s_0, s_1) = (0, 0)) = 1$. Assume that $\mu((s_0, s_1) = (0, 0)) = 1$; we shall obtain a contradiction.

Note that the dual graph \mathbb{L}_1 of \mathbb{L}_2 is a vertex-transitive, non-amenable, planar graph in which each vertex has degree 3. A contour configuration $\phi(\omega) \subset E(\mathbb{L}_1)$ is a subset of edges of \mathbb{L}_1 in which each present edge has a dual edge in $E(\mathbb{L}_2)$ joining exactly one vertex with state 0 and one vertex with state 1 in ω . As usual, a contour is a maximal connected component of present edges in a contour configuration. Each contour configuration, by definition, must be an even-degree subgraph of \mathbb{L}_1 . Given that \mathbb{L}_1 has vertex-degree 3, each vertex in \mathbb{L}_1 is incident to zero or two present edges in a contour configuration. Let t be the number of infinite contours. Each infinite contour on \mathbb{L}_1 must be a doubly infinite self-avoiding path.

If $t \geq 1$, let C_∞ be an infinite contour. Then $\mathbb{H}^2 \setminus C_\infty$ has exactly two unbounded components, since C_∞ is a doubly-infinite self-avoiding path. Let V_∞ be the set of all vertices of \mathbb{L}_2 lying on a face crossed by C_∞ . Given C_∞ a fixed orientation. Let V_∞^+ (resp. V_∞^-) be the subset of V_∞ consisting of all the vertices in V_∞ on the left hand side (resp. right hand side) of C_∞ when traversing C_∞ along the given orientation. Then exactly one of V_∞^+ and V_∞^- is part of an infinite 1-cluster, and the other is part of an infinite 0-cluster. Therefore we have $s_0 \geq 1$ and $s_1 \geq 1$ if $t \geq 1$.

The rest of the proof is an adaptation of the proof of Proposition 7.2 to different graphs. Define a **generalized contour** in a contour configuration $\phi \subset E(\mathbb{L}_1)$ to be either a single vertex in $V(\mathbb{L}_1)$ which has no incident present edges in ϕ , or a contour in ϕ . This way for each vertex $v \in V(\mathbb{L}_1)$, and each contour configuration $\phi \subset E(\mathbb{L}_1)$, there is a unique generalized contour passing through v . By the arguments in the last paragraph, if $(s_0, s_1) = (0, 0)$, then $t = 0$.

Now consider the case when $(s_0, s_1, t) = (0, 0, 0)$. Given a cluster C in ω , there is a unique contour C' of $\phi(\omega)$ surrounding ξ . Similarly, for every generalized contour C' , there is a cluster C'' that surrounds C' . Let \mathcal{C}_0 denote the set of all generalized contours of ω . We set

$$\mathcal{C}_{j+1} := \{C'' : K \in \mathcal{C}_j\}.$$

For $C \in \mathcal{C}_0$ and $v \in V(\mathbb{L}_1)$, let $r(C) := \sup\{j : C \in \mathcal{C}_j\}$, and define $r(v) := r(C)$ if C is the generalized contour of v in $\phi(\omega)$. For each r let ω^r be the set of edges in $E(\mathbb{L}_1)$ whose both end-vertices $u, v \in V(\mathbb{L}_1)$ satisfy $r(v) \leq r$ and $r(u) \leq r$. Then ω^r is an invariant bond percolation and

$$\lim_{r \rightarrow \infty} \mathbf{E}[\deg_{\omega^r} v] = 3.$$

By Lemma 6.2, we deduce that ω^r has infinite components with positive probability for all sufficiently large r .

However, since $(s_0, s_1, t) = (0, 0, 0)$, by the arguments above each vertex in $v \in V(\mathbb{L}_1)$ is surrounded by infinitely many contours. This implies that for any $r \in \mathbb{N}$, for any vertex $v \in V(\mathbb{L}_1)$, there exists a contour C surrounding v , such that $r(C) > r$, and therefore $C \cap \omega^r = \emptyset$. As a result, the components in ω^r including v is finite. The contradiction implies the lemma. \square

Lemma 10.2. *Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider an $\text{Aut}(\mathbb{L}_2)$ -invariant site percolation ω on \mathbb{L}_2 with distribution μ . Let t be the number of infinite contours. Then μ -a.s. $t \in \{0, \infty\}$*

Proof. By Lemma 6.1, μ -a.s. $t \in \{0, 1, \infty\}$. Let \mathcal{A} be the event that $t = 1$. Assume $\mu(\mathcal{A}) > 0$, we shall obtain a contradiction. Conditional on the event \mathcal{A} , let τ be the unique infinite contour. Since τ is a infinite, connected, even-degree subgraph of \mathbb{L}_1 , and each vertex of \mathbb{L}_1 has degree 3, τ must be a doubly infinite self-avoiding path. Then $p_c(\tau) = 1$. But this is a contradiction to Lemma 6.6. Then the lemma follows. \square

Lemma 10.3. *Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider an $\text{Aut}(\mathbb{L}_2)$ -invariant site percolation ω on \mathbb{L}_2 with distribution μ . Assume that μ is $\text{Aut}(\mathbb{L}_2)$ -ergodic. Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. infinite 1-clusters) in the percolation. Then*

$$\mu((s_0, s_1) = (1, 1)) = 0.$$

Proof. We may construct a $[3, 4, n, 4]$ lattice embedded into the hyperbolic plane \mathbb{H}^2 , such that the $[3, 4, n, 4]$ lattice, \mathbb{L}_1 and \mathbb{L}_2 satisfy the conditions as described in Section 2. Then each percolation configuration ω in $\{0, 1\}^{V(\mathbb{L}_2)}$ induces a constrained configuration $\tilde{\omega} \in \Omega$ by the condition that ω and $\tilde{\omega}$ has the same contour configuration; and that $\omega(v) = 1$ for $v \in V(\mathbb{L}_2)$ if and only if all the vertices of the $[3, 4, n, 4]$ lattice in the degree- n face of \mathbb{L}_1 containing v have state 1 in $\tilde{\omega}$. Then the lemma follows from Lemma 7.4. \square

Lemma 10.4. *Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider a site percolation ω on \mathbb{L}_2 with distribution μ . Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. infinite 1-clusters) in the percolation.*

Then it is not possible that

$$(s_0, s_1) = (0, k); \text{ for } 2 \leq k \leq \infty.$$

Proof. Assume that $s_0 = 0$, and that there exist at least two distinct infinite 1-clusters C_1 and C_2 . Let ℓ be a path consisting of edges of \mathbb{L}_2 joining a vertex $x \in C_1$ and a vertex $y \in C_2$. If ℓ does not cross infinite contours, then we can find another path ℓ' joining x and y such that ℓ' does not cross contours at all. Then $C_1 = C_2$. The contradiction implies that there exists at least one infinite contour in \mathbb{L}_1 . Since each infinite contour in \mathbb{L}_1 is a doubly infinite self-avoiding path, if there exists an infinite contour, then there exist at least one infinite 0-cluster and at least one infinite 1-cluster. But this is a contradiction to the fact that $s_0 = 0$. \square

Definition 10.5. Let $G = (V, E)$ be a graph. Given a set $A \in 2^V$, and a vertex $v \in V$, denote $\Pi_v A = A \cup \{v\}$. For $\mathcal{A} \subset 2^V$, we write $\Pi_v \mathcal{A} = \{\Pi_v A : A \in \mathcal{A}\}$. A site percolation process (\mathbf{P}, ω) on G is **insertion-tolerant** if $\mathbf{P}(\Pi_v \mathcal{A}) > 0$ for every $v \in V$ and every event $\mathcal{A} \subset 2^V$ satisfying $\mathbf{P}(\mathcal{A}) > 0$.

We can similarly define an insertion tolerant bond percolation by replacing a vertex with an edge in the above definition. A bond percolation is **deletion tolerant** if $\mathbf{P}[\Pi_{-e} \mathcal{A}] > 0$ whenever $e \in E(G)$ and $\mathbf{P}(\mathcal{A}) > 0$, where $\Pi_{-e} \omega = \omega \setminus \{e\}$.

Lemma 10.6. Let G be a graph with a transitive, unimodular, closed automorphism group $\Gamma \subset \text{Aut}(G)$. If (\mathbf{P}, ω) is a Γ -invariant, insertion-tolerant percolation process on G with infinitely many infinite clusters a.s., then a.s. every infinite cluster has infinitely many ends.

Proof. See Proposition 3.10 of [39]; see also [21] and [22]. \square

Lemma 10.7. Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider an $\text{Aut}(\mathbb{L}_2)$ -invariant, insertion-tolerant site percolation ω on \mathbb{L}_2 with distribution μ . Assume that μ is $\text{Aut}(\mathbb{L}_2)$ -ergodic. Let s_0 (resp. s_1) be the number of infinite 0-clusters (resp. infinite 1-clusters) in the percolation. Then

$$\mu((s_0, s_1) = (1, \infty)) = 0.$$

Proof. Assume that $\mu(s_0, s_1) = (1, \infty) = 1$; we shall obtain a contradiction. By Lemma 10.6, a.s. every infinite 1-cluster has infinitely many ends. Then the lemma follows from Lemma A.11. \square

Proposition 10.8. Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Consider an $\text{Aut}(\mathbb{L}_2)$ -invariant, $\text{Aut}(\mathbb{L}_2)$ -ergodic, insertion-tolerant site percolation ω on \mathbb{L}_2 with distribution μ . Let s_0, s_1, t be given as in Lemmas 10.1 and 10.2, then

$$(s_0, s_1, t) \in \{(0, 1, 0), (1, 0, 0), (\infty, \infty, \infty)\} \text{ a.s.}$$

Proof. By Lemma 6.1, we have μ -a.s. $s_0, s_1, t \in \{0, 1, \infty\}$. By Lemmas 10.1, 10.3, 10.4 and 10.7, we have μ -a.s. $(s_0, s_1) \in \{(0, 1), (1, 0), (\infty, \infty)\}$. By Lemma 10.2, μ -a.s. $t \in \{0, \infty\}$. Moreover, since each infinite contour must be a doubly infinite self-avoiding path,

if $t = \infty$, then $s_0 + s_1 = \infty$. This implies that if $s_0 + s_1 = 1$, then $t = 0$, a.s. Moreover, if $s_0 + s_1 \geq 2$, then $t \geq 1$. Then the proposition follows. \square

Example 10.9. Consider Example 2.3. We have

$$\frac{1}{n-1} \leq p_c < \frac{1}{2} < p_u = 1 - p_c \leq \frac{n}{n-1}$$

by Theorems 1.1, 1.2. and 1.3 of [6]. By Proposition 10.8, we have

- if $p \in [0, p_c]$, a.s. $(s_0, s_1, t) = (1, 0, 0)$;
- if $p \in (p_c, p_u)$, a.s. $(s_0, s_1, t) = (\infty, \infty, \infty)$;
- if $p \in [p_u, 1]$, a.s. $(s_0, s_1, t) = (0, 1, 0)$.

11. PROOF OF THEOREM 3.3 AND COROLLARY 3.4

In this section, we prove Theorem 3.3 and Corollary 3.4. The proof of Theorem 3.3 I. is based on a stochastic domination between the i.i.d. Bernoulli site percolation and the Ising model on the same graph, which correspond to random cluster models with $q = 1$ and $q = 2$, respectively. The proof of Theorem 3.3 II(a) is based on the ergodicity and symmetry of the measure μ^f , as well as Proposition 10.8, which lists all the possible numbers of infinite 0-clusters and infinite-1 clusters which have strictly positive probability to occur. Theorem 3.3 II (b)(c)(d) then follow from the fact that any one of the conditions (b)(c) and (d) implies (a). We then prove Corollary 3.4 by an inequality between $p_{c,2}^w$ and $p_{u,2}^f$, and Theorem 3.3 II(c). Theorem 3.3 III is proved by the well-known combinatorial correspondence between the Ising model and its random-cluster representation. Theorem 3.3 IV then follows from the decomposition of μ^f as a convex combination of μ^+ and μ^- and an inequality between $p_{u,2}^f$ and $p_{u,2}^w$.

We shall first review the stochastic domination we use to prove these results.

Definition 11.1. (*Stochastic Domination*) Let $G = (V, E)$ be a graph. Let $\Omega = \{0, 1\}^E$ (resp. $\Omega = \{0, 1\}^V$). Then the configuration space Ω is a partially ordered set with partial order given by $\omega_1 \leq \omega_2$ if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$ (resp. $\omega_1(v) \leq \omega_2(v)$ for all $v \in V$). A random variable $X : \Omega \rightarrow \mathbb{R}$ is called increasing if $X(\omega_1) \leq X(\omega_2)$ whenever $\omega_1 \leq \omega_2$. An event $A \subset \Omega$ is called increasing (respectively, decreasing) if its indicator function 1_A is increasing (respectively, decreasing). Given two probability measures μ_1, μ_2 on Ω , we write $\mu_1 \prec \mu_2$, and we say that μ_2 **stochastically dominates** μ_1 , if $\mu_1(A) \leq \mu_2(A)$ for all increasing events $A \subset \Omega$.

Lemma 11.2. (*Holley inequality*) Let $G = (V, E)$ be a finite graph. Let $\Omega = \{0, 1\}^E$ (resp. $\Omega = \{0, 1\}^V$). Let μ_1 and μ_2 be strictly positive probability measures on Ω such that

$$(32) \quad \mu_2(\max\{\omega_1, \omega_2\})\mu_1(\min(\omega_1, \omega_2)) \geq \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \Omega,$$

Then

$$\mu_1 \prec \mu_2.$$

Proof. See Theorem (2.1) of [17]; see also [26]. \square

Lemma 11.3. *Let $G = (V, E)$ be a finite graph. Let $\Omega = \{0, 1\}^E$. Let μ_1 and μ_2 be strictly positive probability measures on Ω such that*

$$(33) \quad \mu_2(\omega \cup \{e\})\mu_1(\omega \setminus \{e\}) \geq \mu_1(\omega \cup \{e\})\mu_2(\omega \setminus \{e\}), \quad \omega \in \Omega, e \in E.$$

Here ω is interpreted as the subset of E consisting of all the edges with state 1. If either μ_1 or μ_2 satisfies

$$(34) \quad \mu(\omega \cup \{e, f\})\mu(\omega \setminus \{e, f\}) \geq \mu(\omega \cup e \setminus \{f\})\mu(\omega \cup \{f\} \setminus \{e\}),$$

then (32) holds.

Proof. See Theorem 2.6 of [17]. □

For a planar graph G , let G_* be its planar dual graph. The following lemmas concerning planar duality, are proved in [6, 20].

Lemma 11.4. *Let G be a planar nonamenable quasi-transitive graph, and let $p, p_* \in (0, 1)$ satisfy*

$$(35) \quad p_* = \frac{(1-p)q}{p + (1-p)q}$$

In the natural coupling of $FRC_{p,q}^G$ and $WRC_{p_,q}^{G_*}$ as dual measures (i.e. a dual edge is present if and only if the corresponding primal edge is absent), the number of infinite clusters with respect to each is a.s. one of the following: $(0, 1)$, $(1, 0)$, (∞, ∞) .*

Proof. See Proposition 3.5 of [20], which is proved using the same technique as the proof of Theorem 3.7 of [6]. □

Lemma 11.5. *Let*

$$h(x) := \frac{x}{1-x}.$$

For any planar non-amenable quasi-transitive graph G ,

$$\begin{aligned} h(p_{c,q}^w(G))h(p_{u,q}^f(G_*)) &= h(p_{u,q}^w(G_*))h(p_{c,q}^f(G)) = 1, \\ 0 < p_{c,q}^w(G) \leq p_{c,q}^f(G) < 1, \text{ and } 0 < p_{u,q}^w(G) \leq p_{u,q}^f(G) < 1 \end{aligned}$$

Proof. See Corollary 3.6 of [20]. □

We start the proof of Theorem 3.3 with the following lemma.

Lemma 11.6. *Let \mathbb{L}_2 be a regular tiling of the hyperbolic plane such that each vertex has degree n and each face has degree m . Let $q \geq 1$. Assume that $WRC_{p,q}^{\mathbb{L}_2}$ -a.s. there is a unique infinite open cluster for the random cluster model on \mathbb{L}_2 . Let τ be the unique infinite open cluster in the random cluster configuration ω . We define a site percolation configuration ξ on $V(\mathbb{L}_2)$, by letting all the vertices in τ have state 1, and all the other vertices have state 0. Then a.s. ξ has no infinite 0-cluster.*

Proof. Let \mathcal{A}_0 be the event that ξ has an infinite 0-cluster. By ergodicity of $WRC_{p,q}^{\mathbb{L}_2}$, either $WRC_{p,q}^{\mathbb{L}_2}(\mathcal{A}_0) = 0$ or $WRC_{p,q}^{\mathbb{L}_2}(\mathcal{A}_0) = 1$. Assume that $WRC_{p,q}^{\mathbb{L}_2}(\mathcal{A}_0) = 1$; we shall obtain a contradiction.

The dual configuration to the random cluster configuration on \mathbb{L}_2 is a bond configuration on \mathbb{L}_1 such that an edge in \mathbb{L}_1 is present if and only if its dual edge in \mathbb{L}_2 is absent in the random cluster configuration of \mathbb{L}_2 . Note also that the free boundary condition is dual to the wired boundary condition by the relation between dual configurations described above. Moreover, if the random cluster configuration on \mathbb{L}_2 has distribution $WRC_{p,q}^{\mathbb{L}_2}$, then its dual configuration on \mathbb{L}_1 has distribution $FRC_{p^*,q}^{\mathbb{L}_1}$; where p, p^* satisfy (35).

Let k be the number of infinite clusters in the bond configuration in \mathbb{L}_2 , and let k^* be the number of infinite clusters in the corresponding dual configuration in \mathbb{L}_1 . By Lemma 11.4, a.s. $(k, k^*) \in \{(0, 1), (1, 0), (\infty, \infty)\}$. (Indeed this is true for any $\text{Aut}(\mathbb{L}_2)$ -invariant, insertion tolerant and deletion tolerant bond configuration.) Hence if $WRC_{p,q}^{\mathbb{L}_2}$ -a.s. there is a unique infinite open cluster, then $FRC_{p^*,q}^{\mathbb{L}_1}$ -a.s. there is no infinite cluster in the corresponding dual configuration.

Since τ is an infinite cluster, there exists an infinite 1-cluster in ξ by construction. If there exists an infinite 0-cluster in ξ as well, by Lemma A.3, there exists an infinite contour C consisting of edges of \mathbb{L}_1 in which each edge has a dual edge joining a vertex of $V(\mathbb{L}_2)$ with state 1 in ξ and a vertex of $V(\mathbb{L}_2)$ with state 0 in ξ . Moreover, all the edges in C must be present in the dual configuration of ω , since every edge in C is dual to an edge of $V(\mathbb{L}_2)$ not open in ω . Then we have $k = 1$ and $k^* \geq 1$. But this is a contradiction to Lemma 11.4. Hence a.s. there are no infinite 0-clusters in ξ . \square

Lemma 11.7. *Let \mathbb{L}_2 be a regular tiling of the hyperbolic plane such that each vertex has degree n and each face has degree m . Then for the graph \mathbb{L}_2 ,*

$$p_{u,1} \leq p_{u,2}^w \leq p_{u,2}^f$$

Proof. The fact that $p_{u,2}^w \leq p_{u,2}^f$ follows from Lemma 11.5.

Now we prove that $p_{u,1} \leq p_{u,2}^w$. Note that the following stochastic monotonicity result holds:

$$(36) \quad WRC_{p,2} \prec WRC_{p,1} = FRC_{p,1},$$

by (4.1) of [45].

Let \mathcal{F}_1 be the event that there exists a unique infinite cluster in the random cluster configuration in \mathbb{L}_2 . By ergodicity of $WRC_{p,2}$ and $WRC_{p,1}$ and the monotonicity of $WRC_{p,2}(\mathcal{F}_1)$ and $WRC_{p,1}(\mathcal{F}_1)$ with respect to p , to show that $p_{u,1} \leq p_{u,2}^w$, it suffices to show that whenever $WRC_{p,2}(\mathcal{F}_1) = 1$, then $WRC_{p,1}(\mathcal{F}_1) = 1$.

Let $\mathcal{F}_{1,0} \subset \mathcal{F}_1$ be the event consisting of all the configurations in which both of the following two cases occur

- there exists a unique infinite cluster τ in the random cluster configuration on \mathbb{L}_2 ; and
- let $\xi \in \{0, 1\}^{V(\mathbb{L}_2)}$ be the site configuration obtained by assigning the state 1 to all the vertices in τ , and the state 0 to all the vertices not in τ ; then there exists no infinite 0-cluster in ξ .

By Lemma 11.6, if $WRC_{p,2}(\mathcal{F}_1) = 1$, then $WRC_{p,2}(\mathcal{F}_{1,0}) = 1$. Since $\mathcal{F}_{1,0}$ is an increasing event, by (36) we have $WRC_{p,2}(\mathcal{F}_{1,0}) = 1$, then $WRC_{p,1}(\mathcal{F}_{1,0}) = 1$. Since $\mathcal{F}_{1,0} \subset \mathcal{F}_1$, we have $WRC_{p,1}(\mathcal{F}_1) = 1$. This completes the proof. \square

Lemma 11.8. *Let \mathbb{L}_2 be a vertex-transitive, triangular tiling of the hyperbolic plane such that each vertex has degree $n \geq 7$. Consider the following Conditions (a),(b),(c) and (d) in Part II of Theorem 3.3. We have*

$$(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a);$$

where $A \Rightarrow B$ means that if A holds, then B holds.

Proof. The statement $(b) \Rightarrow (a)$ follows from Theorem 4.1 of [45].

The fact that $(c) \Rightarrow (a)$ follows from Theorem 3.2 (v) of [20]; while the fact that $(c) \Rightarrow (b)$ follows from Theorem 4.1 and Lemma 6.4 of [39].

The fact that $(d) \Rightarrow (c)$ follows from Lemma 11.7. \square

11.1. Proof of Part I of Theorem 3.3. First note that if (10) hold, then

$$(37) \quad \frac{e^h}{e^h + e^{-h}} = p_u.$$

by the fact that $p_c + p_u = 1$.

Let ν_1 (resp. ν_2) be the probability measure for the i.i.d. Bernoulli site percolation on \mathbb{L}_2 in which each vertex takes the value “+” with probability p_1 (resp. p_2) satisfying

$$\begin{aligned} \frac{e^{nJ}}{e^{nJ} + e^{-nJ}} &< p_1 < p_u \\ p_c < p_2 &< \frac{e^{-nJ}}{e^{nJ} + e^{-nJ}} \end{aligned}$$

and the value “-” with probability $1 - p_1$ (resp. $1 - p_2$). Such p_1 and p_2 exist by (11).

Fix a triangle face F_0 of \mathbb{L}_2 . Let $B_R = (V(B_R), E(B_R))$ be the finite subgraph of \mathbb{L}_2 consisting of all the faces of \mathbb{L}_2 whose graph distance to F_0 is at most R . Let $\nu_{1,R}$ (resp. $\nu_{2,R}$) be the restriction of ν_1 (resp. ν_2) on B_R . Let μ_R^+ (resp. μ_R^-) be the probability measure for the Ising model on B_R with respect to the coupling constant J and the “+” boundary condition (resp. the “-” boundary condition). Let ω_1, ω_2 be two configurations in $\{-1, 1\}^{V(B_R)}$. Then by Lemmas 11.2 and 11.3, we can check the F.K.G. lattice conditions below

$$\begin{aligned} \nu_{1,R}(\max\{\omega_1, \omega_2\})\mu_R^+(\min\{\omega_1, \omega_2\}) &\geq \nu_{1,R}(\omega_1)\mu_R^+(\omega_2) \\ \mu_R^-(\max\{\omega_1, \omega_2\})\nu_{2,R}(\min\{\omega_1, \omega_2\}) &\geq \mu_R^-(\omega_1)\nu_{2,R}(\omega_1). \end{aligned}$$

Then we obtain the following stochastic domination result:

$$\nu_{2,R} \prec \mu_R^- \prec \mu_R^+ \prec \nu_{1,R}.$$

Letting $R \rightarrow \infty$, for any $\text{Aut}(\mathbb{L}_2)$ -invariant Gibbs measure μ for the Ising model on \mathbb{L}_2 with coupling constant J , we have

$$\nu_2 \prec \mu^- \prec \mu \prec \mu^+ \prec \nu_1.$$

Since ν_2 -a.s. there are infinite “+”-clusters, μ -a.s. there are infinite “+”-clusters. Similarly, μ -a.s. there are infinite “-”-clusters, since ν_1 -a.s. there are infinite “-”-clusters. By Proposition 10.8, we conclude that when (11) hold, μ -a.s. there are infinitely many infinite

“+”-clusters, infinitely many infinite “-”-clusters and infinitely many infinite contours. This completes the proof of Part I.

11.2. Proof of Part II of Theorem 3.3. We first assume that μ^f is $\text{Aut}(\mathbb{L}_2)$ -ergodic. Since μ^f is also symmetric in switching “+” and “-” states, μ^f -a.s. the number of infinite “+”-clusters and the number of infinite “-”-clusters are equal. Then the conclusion follows from Proposition 10.8, which implies that if the number of infinite “+”-clusters and the number of infinite “-”-clusters are equal a.s., then both numbers are infinite.

By Lemma 11.8, if one of the conditions (a), (b), (c) and (d) in Lemma 11.8 holds, then μ^f almost surely there are infinitely many infinite “+”-clusters and infinitely many infinite “-”-clusters.

11.3. Proof of Corollary 3.4. By Proposition 3.2 (i) of [20], if (17) holds, then there is a unique infinite-volume Gibbs measure for the Ising model on \mathbb{L}_2 with coupling constant J . Since

$$p_{c,2}^w \leq p_{c,2}^f \leq p_{u,2}^f,$$

(17) implies Condition (c). Then Theorem 3.3 II(c) implies μ^f a.s. there are neither infinite “+”-clusters nor infinite “-”-cluster. The corollary now follows from the uniqueness of the infinite-volume Gibbs measure.

11.4. Proof of Part III of Theorem 3.3. Since $p = 1 - e^{-2J}$, when (12) holds, we have $p \geq p_{u,2}^w$. By Corollary 3.7 of [20] (see also (9)), there exists a unique infinite open cluster τ in the random cluster representation of the Ising model with wired boundary conditions $WRC_{p,2}$ -a.s.

By the correspondence of random-cluster configurations and Ising configurations, each infinite cluster in the random cluster representation must be a subset of an infinite cluster in the Ising model. Hence if $WRC_{p,2}$ -a.s. there is a unique infinite open cluster, μ^+ a.s. there exists an infinite “+”-cluster in the Ising model, and μ^- -a.s. there exists an infinite “-”-cluster in the Ising model.

Let τ be the unique infinite open cluster in the random cluster configuration ω . We define a site percolation configuration ξ on $V(\mathbb{L}_2)$, by letting all the vertices in τ have state 1, and all the other vertices have state 0. By Lemma 11.6, a.s. ξ has no-infinite 0-clusters. Again by the correspondence of the random cluster configuration and the Ising configuration and proposition 10.8, we obtain $\mu^+(\mathcal{A}_+) = 1$, and $\mu^-(\mathcal{A}_-) = 1$.

11.5. Proof of Part IV of Theorem 3.3. The identities (13) and (14) follows Part I and the fact that

$$p_{u,2}^w \leq p_{u,2}^f;$$

and (16) follows from the fact that when (15) hold,

$$\mu^f = \frac{\mu^+ + \mu^-}{2};$$

The decomposition of μ^f as convex combination of the extremal measures μ^+ and μ^- it is a classical results in the case of regular lattices and for our lattices it was proven in Theorem 4.2 of [45] and expressions (17) (18) of [20].

12. PROOF OF THEOREMS 4.1 AND 4.2

In this section, we prove Theorems 4.1 and 4.2.

Proof of theorem 4.1. We first prove Part I of the theorem. Let s_+ (resp. s_-) be the number of infinite “+”-clusters (resp. infinite “-”-clusters). By ergodicity, $\text{Aut}(\mathbb{L}_2)$ -invariance and symmetry in “+” and “-” of $\mu_{1,f} \times \mu_{2,f}$, as well as Lemma 6.1, one of the following cases occurs:

- (i) $\mu_1^f \times \mu_2^f((s_+, s_-) = (0, 0)) = 1$; or
- (ii) $\mu_1^f \times \mu_2^f((s_+, s_-) = (1, 1)) = 1$; or
- (iii) $\mu_1^f \times \mu_2^f((s_+, s_-) = (\infty, \infty)) = 1$

Case (i) is impossible to occur by Lemma 10.1. Case (ii) is impossible to occur by Lemma 10.3. This completes the proof of Part I.

Now we show that Assumption II implies Assumption I. This follows from Theorem 4.1 of [44], and the fact that the XOR Ising measure is the product measure of two i.i.d Ising models.

The facts that Assumption IV implies Assumption III and Assumption III implies Assumption II follows from Lemma 11.8. This completes the proof of the theorem. \square

Before proving Theorem 4.2, we shall first introduce the following definition and proposition in [39] (see Theorem 3.3, Remark 3.4).

Definition 12.1. Let $G = (V, E)$ be a graph and Γ a transitive group acting on G . Suppose that X is either V , E or $V \cup E$. Let Q be a measurable space and $\Omega := 2^V \times Q^X$. A probability measure \mathbf{P} on Ω will be called a **site percolation with scenery** on G . The projection onto 2^V is the underlying percolation and the projection onto Q^X is the scenery. If $(\omega, q) \in \Omega$, we set $\Pi_v(\omega, q) = (\Pi_v\omega, q)$. We say the percolation with scenery \mathbf{P} is **insertion-tolerant** if $\mathbf{P}(\Pi_v\mathcal{B}) > 0$ for every measurable $\mathcal{B} \subset \Omega$ with positive measure. We say that \mathbf{P} has **indistinguishable infinite clusters** if for every $\mathcal{A} \subset 2^V \times 2^V \times Q^X$ that is invariant under diagonal actions of Γ , for \mathbf{P} -a.e. (ω, q) , either all infinite clusters C of ω satisfy $(C, \omega, q) \in \mathcal{A}$, or they all satisfy $(C, \omega, q) \notin \mathcal{A}$.

Proposition 12.2. Let \mathbf{P} be a site percolation with scenery on a graph $G = (V, E)$ with state space $\Omega := 2^V \times Q^X$, where Q is a measurable space and X is either V , E or $V \cup E$. If \mathbf{P} is Γ -invariant and insertion tolerant, then \mathbf{P} has indistinguishable infinite clusters.

Proof of Theorem 4.2 Let $\Lambda = (V_\Lambda, E_\Lambda)$ be a subgraph of \mathbb{L}_2 consisting of faces of \mathbb{L}_2 . Let $\Lambda_* = (V_{\Lambda_*}, E_{\Lambda_*})$ be the dual graph of Λ , such that there is a vertex in V_Λ corresponding to each triangle face in Λ , as well as the unbounded face; the edges in E_Λ and E_{Λ_*} are in 1-1 correspondence by duality.

Consider an XOR Ising model on Λ with respect to two i.i.d. Ising models σ_3, σ_4 with free boundary conditions and coupling constants $J \geq 0$ satisfying the assumption of Theorem 4.1. The partition function of the XOR Ising model can be computed by

$$Z_{\Lambda, f} = \sum_{\sigma_3, \sigma_4 \in \{\pm 1\}^{V_\Lambda}} \prod_{(u, v) \in E_\Lambda} e^{J(\sigma_{3,u}\sigma_{3,v} + \sigma_{4,u}\sigma_{4,v})}.$$

Following the same computations as in [9], we obtain

$$(38) \quad Z_{\Lambda, f} = C_1 \sum_{P_* \in \mathcal{P}_*, P \in \mathcal{P}, P \cap P_* = \emptyset} \left(\frac{2e^{-2J}}{1 + e^{-4J}} \right)^{|P_*|} \left(\frac{1 - e^{-4J}}{1 + e^{-4J}} \right)^{|P|}.$$

where \mathcal{P}_* (resp. \mathcal{P}) consists of all the contour configurations on E_{Λ_*} (resp. E_Λ) such that each vertex of V_{Λ_*} (resp. V_Λ) has an even number of incident present edges, and $C_1 = 2^{|V_\Lambda| - |E_\Lambda| + 2} (e^{2J} - e^{-2J})^{|E_\Lambda|}$ is a constant.

When J, K satisfies (18), we have

$$\begin{aligned} \frac{2e^{-2J}}{1 + e^{-4J}} &= \frac{1 - e^{-4K}}{1 + e^{-4K}}; \\ \frac{2e^{-2K}}{1 + e^{-4K}} &= \frac{1 - e^{-4J}}{1 + e^{-4J}}. \end{aligned}$$

Thus the partition function $Z_{\Lambda, f}$, up to a multiplicative constant, is the same as the partition function of the XOR Ising model on Λ_* with coupling constant K .

Recall that there is exactly one vertex $v_\infty \in V_{\Lambda_*}$ corresponding to the unbounded face in Λ . The XOR Ising model $\sigma_{XOR} = \sigma_1 \sigma_2$ on Λ_* , corresponds to an XOR Ising model on $\Lambda_* \setminus \{v_\infty\}$ (which is a subgraph of \mathbb{L}_1) with the boundary condition that all the boundary vertices have the same state in σ_1 and all the boundary vertices have the same state in σ_2 . Hence the boundary condition must be a mixture of $++$, $+-$, $-+$ and $--$. However, each one of the 4 possible boundary conditions gives the same distribution of contours in the XOR Ising model. From the expression (38), we can see that there is a natural probability measure on the set of contours $\Phi = \{(P, P_*) : P \in \mathcal{P}, P_* \in \mathcal{P}_*, P \cap P_* = \emptyset\}$, such that the probability of each pair of contours $(P, P_*) \in \Phi$ is proportional to $\left(\frac{2e^{-2J}}{1 + e^{-4J}} \right)^{|P_*|} \left(\frac{1 - e^{-4J}}{1 + e^{-4J}} \right)^{|P|}$, and the marginal distribution on \mathcal{P} is the distribution of contours for the XOR Ising model on \mathbb{L}_2 with coupling constant J and free boundary conditions, while the marginal distribution on \mathcal{P}_* is the distribution of contours for the XOR Ising model on \mathbb{L}_1 with coupling constant K and $++$ boundary conditions.

We let Λ and $\Lambda_* \setminus \{v_\infty\}$ increase and approximate the graph \mathbb{L}_2 and \mathbb{L}_1 , respectively. If with a positive μ_{++} probability, there exists exactly one infinite contour C in \mathbb{L}_2 , then μ_f -a.s. there exists an infinite cluster in \mathbb{L}_2 containing C , since contours in \mathbb{L}_1 and \mathbb{L}_2 are disjoint. Consider the XOR Ising spin configuration as a site percolation on \mathbb{L}_2 , with scenery given by contour configurations of \mathbb{L}_2 within the “+” clusters of \mathbb{L}_2 . In the notation of Definition 12.1, $Q = \{0, 1\}$, and $X = E(\mathbb{L}_2)$. An edge in $E(\mathbb{L}_2)$ is present (has state “1”) if and only if it is in a “+”-cluster of the XOR Ising configuration on \mathbb{L}_2 and present in the contour configuration of \mathbb{L}_2 . This way we obtain an automorphism-invariant and

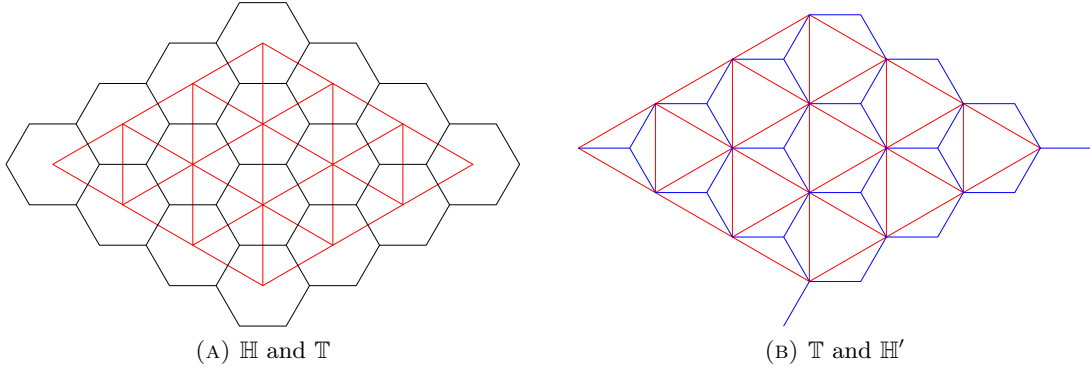


FIGURE 11. Hexagonal lattice \mathbb{H} (represented by black lines), dual triangular lattice \mathbb{T} (represented by red lines) and hexagonal lattice \mathbb{H}' obtained from the star-triangle transformation (represented by blue lines).

insertion-tolerant percolation with scenery. Let $\mathcal{A} \subset 2^V(\mathbb{L}_2) \times 2^V(\mathbb{L}_2) \times 2^E(\mathbb{L}_2)$ be the triple (C, ω, q) such that

- ω is an XOR Ising spin configuration on \mathbb{L}_2 ; and
- C is an infinite “+”-cluster; and
- q is the \mathbb{L}_2 -contour configuration within “+”-clusters of ω ; and
- C contains an infinite contour in q .

We can see that \mathcal{A} is invariant under diagonal actions of automorphisms. By Theorem 4.1, μ_f -a.s. there exists infinitely many infinite “+”-clusters in \mathbb{L}_2 . By Proposition 12.2, either all the infinite clusters are in \mathcal{A} , or no infinite clusters are in \mathcal{A} . Similar arguments applies for “-”-clusters of \mathbb{L}_2 . Hence almost surely the number of infinite contours in \mathbb{L}_2 is 0 or ∞ . Since the distribution of infinite contours in \mathbb{L}_2 is exactly that of contours for the XOR Ising model on \mathbb{L}_1 with coupling constant K and ++ (or +-, -+, --) boundary condition, the theorem follows. \square

13. PROOF OF THEOREMS 5.1 AND 5.2

In this section, we prove Theorems 5.1 and 5.2.

Consider the XOR Ising model with spins located on vertices of the hexagonal lattice \mathbb{H} with coupling constants J_a, J_b, J_c on horizontal, NW/SE, NE/SW edges, see Figure 11 for an embedding of \mathbb{H} into the plane such that all the edges are either horizontal, NW/SE or NE/SW.

A [4,6,12] lattice is a graph that can be embedded into the Euclidean plane \mathbb{R}^2 such that each vertex is incident to 3 faces with degrees 4, 6, and 12, respectively. See Figure 12. We shall explain the relation between perfect matchings on the [4,6,12] lattice and constrained percolation configurations in the [3,4,6,4] lattice as discussed in Section 2.

A **perfect matching**, or a **dimer configuration** on a [4,6,12] lattice is a subset of edges such that each vertex of the [4,6,12] lattice is incident to exactly one edge in the subset.

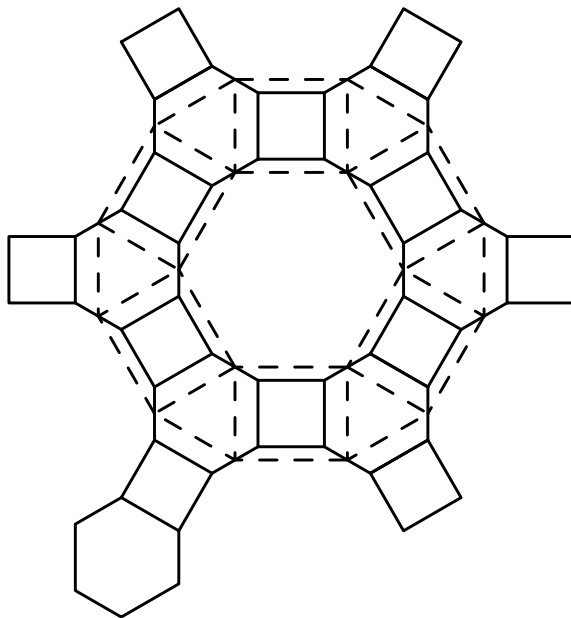


FIGURE 12. $[3,4,6,4]$ lattice (represented by dashed lines) and the $[4,6,12]$ lattice \mathbb{A} (represented by black lines).

A **Type-I** edge of a $[4,6,12]$ lattice is an edge of a square face. Any other edge of the $[4,6,12]$ lattice is a **Type-II** edge. We say two Type-II edges e_1, e_2 are **adjacent** if there exists a Type-I edge e_3 , such that both e_1 and e_2 share a vertex with e_3 in the $[4,6,12]$ lattice. A subset of Type-II edges is **connected** if for any two edges e and f in the subset, there exist a sequence of Type-II edges $e_0(= e), e_1, \dots, e_n(= f)$ in the subset, such that e_i and e_{i-1} are adjacent, for $1 \leq i \leq n$. A **Type-II cluster** is a maximal connected set of present Type-II edges in a perfect matching.

The $[3,4,6,4]$ lattice is constructed as follows. A vertex of the $[3,4,6,4]$ lattice is placed at the midpoint of each Type-II edge of the $[4,6,12]$ lattice. Two vertices of the $[3,4,6,4]$ lattices are joined by an edge if and only if they are midpoints of two adjacent Type-II edges. See Figure 12. The restriction of any dimer configuration on the $[4,6,12]$ lattice to Type-II edges naturally correspond to a constrained percolation configuration on the $[3,4,6,4]$ lattice in Ω . A Type-II edge is present in a dimer configuration if and only if its midpoint has state “1” in the corresponding constrained percolation configuration. It is straightforward to check that this way we obtain a bijection between restrictions to Type-II edges of dimer configurations on the $[4,6,12]$ lattice and constrained percolation configurations on the $[3,4,6,4]$ lattice in Ω . Recall that constrained percolation configurations on the $[3,4,6,4]$ lattice induces contour configurations on the hexagonal lattice \mathbb{H} and the triangular lattice \mathbb{T} by a 2-to-1 mapping $\phi : \Omega \rightarrow \Phi$. See Figure 6.

From the connection of the $[4,6,12]$ lattice and the $[3,4,6,4]$ lattice, as well as the connection of the $[3,4,6,4]$ lattice with the hexagonal lattice \mathbb{H} and the dual triangular lattice \mathbb{T} as described in Section 2, we can see that each square face of the $[4,6,12]$ lattice is crossed by a unique edge of \mathbb{H} and a unique edge of \mathbb{T} . Each vertex of \mathbb{H} is located at the center

of a hexagon face of the $[4,6,12]$ lattice, and each vertex of \mathbb{T} is located in the center of a degree-12 face of the $[4,6,12]$ lattice.

Note that the \mathbb{H} is a bipartite graph; i.e. all the vertices can be colored black and white such that the vertices of the same color are not adjacent. Let Γ be the translation group of the hexagonal lattice generated by translations along two different directions, such that the set of black vertices and the set of white vertices form two distinct orbits under the action of Γ . Note that Γ acts transitively on the dual triangular lattice \mathbb{T} .

In order to define a probability measure for perfect matchings on the $[4,6,12]$ lattice, we introduce edge weights. We assign weight 1 to each Type-II edge, and weight w_e to the Type-I edge e . Assume that the edge weights of the $[4,6,12]$ lattice satisfy the following conditions.

- (B1) The edge weights are Γ -invariant.
- (B2) If e_1, e_2 are two parallel Type-I edges around the same square face, then $w_{e_1} = w_{e_2}$.
- (B3) If e_1, e_2 are two perpendicular Type-I edges around the same square face, then $w_{e_1}^2 + w_{e_2}^2 = 1$.

The reason we assume (B1) is to define a Γ -translation-invariant probability measure. The reason we assume (B2) and (B3) is to define a probability measure for dimer configurations of the $[4,6,12]$ lattice, which, under the connection described above to constrained percolation configurations in Ω , will induce a probability measure on Ω satisfying the symmetry assumption (A3).

Under (B1)–(B3), the edge weights are described by three independent parameters. We may sometimes assume that the parameters satisfy the identity below, which reduces the number of independent parameters to two.

- (B4) Let

$$(39) \quad h(x, y, z) = x + y + z + xy + xz + yz - xyz - 1.$$

For each edge e of the hexagonal lattice \mathbb{H} , let e_1 (resp. e_2) be a Type-I edge of the $[4,6,12]$ lattice parallel (resp. perpendicular) to e . Let

$$t_e = \frac{1 - w_{e_1}}{w_{e_2}},$$

where w_{e_1} (resp. w_{e_2}) is the edge weight of e_1 (resp. e_2) for dimer configurations on the $[4,6,12]$ lattice. Under the assumption (B1), t_e is uniquely defined independent of the e_1, e_2 chosen - as long as e_1 is parallel to e and e_2 is perpendicular to e . Let e_a, e_b, e_c be three edges of \mathbb{H} with distinct orientations in the embedding of \mathbb{H} into \mathbb{R}^2 . Then $h(t_{e_a}, t_{e_b}, t_{e_c}) = 0$.

In [30], the authors define a probability measure for any bi-periodic, bipartite, 2-dimensional lattice. Specializing to our case, let $\mu_{n,D}$ be the probability measure of dimer configurations on a toroidal $n \times n$ $[4,6,12]$ lattice \mathbb{A}_n (see [30] for details). Let \mathcal{M}_n be the set of all perfect matchings on \mathbb{A}_n , and let $M \in \mathcal{M}_n$ be dimer configuration, then

$$(40) \quad \mu_{n,D}(M) = \frac{\prod_{e \in M} w_e}{\sum_{M \in \mathcal{M}_n} \prod_{e \in M} w_e},$$

where w_e is the weight of the edge e . As $n \rightarrow \infty$, $\mu_{n,D}$ converges weakly to a translation-invariant measure μ_D (see [30]).

Theorem 13.1. *For the dimer model on the $[4,6,12]$ lattice.*

- I If the edge weights satisfy (B1)-(B4), μ_D almost surely there are neither infinite Type-II clusters nor infinite contours.*
- II If the edge weights satisfy (B1)-(B3), μ_D almost surely there exists at most one infinite contour.*

Proof. Let $\tilde{\mu}_D$ be the marginal distribution of μ_D restricted on Type-II edges. Recall that the restriction of dimer configurations to Type-II edges on the $[4,6,12]$ lattice, are in 1-1 correspondence with constrained percolation configurations on the $[3,4,6,4]$ lattice in Ω , as described before. See also Figure 12.

Also recall that $\tilde{\mu}_D$ is the weak limit of measures on larger and larger tori; since the edge weights are translation-invariant, the measures on tori are translation-invariant. Hence $\tilde{\mu}_D$ satisfies (A1) if edge weights satisfies (B1).

The measure $\tilde{\mu}_D$ is both translation-invariant and mixing (see [30]), hence $\tilde{\mu}_D$ is totally ergodic and satisfies (A2).

If the edge weights satisfy (B2) and (B3), the measures on tori are symmetric under θ . Hence $\tilde{\mu}_D$ satisfies (A3).

By the results in [9], the marginal distribution of contours in \mathbb{H} (resp. \mathbb{T}) under $\tilde{\mu}_D$ is the same as the distribution of contours of an XOR Ising model $\sigma_{XOR,\mathbb{T}}$ (resp. $\sigma_{XOR,\mathbb{H}}$) with spins located on vertices of \mathbb{T} (resp. \mathbb{H}), if the dimer edge weights and Ising coupling constants satisfy the following conditions:

- each Type-II edge has weight 1;
- each Type-I edge parallel to an edge of e with coupling constants J_e has weight w_e such that $w_e = \frac{1-e^{-4J_e}}{1+e^{-4J_e}}$;
- each Type-I edge perpendicular to an edge e with coupling constants J_e has weight w_e such that $w_e = \frac{2e^{-2J_e}}{1+e^{-4J_e}}$.

Moreover, when the edge weights satisfy (B1)-(B3), $\sigma_{XOR,\mathbb{T}}$ and $\sigma_{XOR,\mathbb{H}}$ are dual to each other, i.e. the coupling constants J_τ and K_τ on a pair of dual edges $e \in \mathbb{H}, e^* \in \mathbb{T}$ satisfy (21). The finite energy assumptions (A4) and (A5) follows from the finite energy of $\sigma_{XOR,\mathbb{H}}$ and $\sigma_{XOR,\mathbb{T}}$.

Since $\sigma_{XOR,\mathbb{H}}$ and $\sigma_{XOR,\mathbb{T}}$ are dual to each other, one of the following cases might occur

- I $\sigma_{XOR,\mathbb{H}}$ is in the low-temperature state, and $\sigma_{XOR,\mathbb{T}}$ is in the high-temperature state;
- II $\sigma_{XOR,\mathbb{H}}$ is in the high-temperature state, and $\sigma_{XOR,\mathbb{T}}$ is in the low-temperature state;
- III both $\sigma_{XOR,\mathbb{H}}$ and $\sigma_{XOR,\mathbb{T}}$ are in the critical state.

See Section 5 for definitions of the low-temperature state, high-temperature state and critical state for XOR Ising models.

In Case III, both (A6) and (A7) are satisfied because of the ergodicity of measures for the critical XOR Ising model on \mathbb{H} and \mathbb{T} ; see Lemma 13.2. Moreover, Case III occurs if

and only if the edge weights satisfy (B4). Hence when the edge weights satisfy (B1)-(B4), $\tilde{\mu}_D$ satisfy (A1)-(A7). Theorem 13.1 I follows from Theorem 2.4II(c).

Note that the measures for the high-temperature XOR Ising models on \mathbb{H} and \mathbb{T} are also ergodic; see Lemma 13.3. Therefore, in each case of I, II and III, at least one of (A6) and (A7) is satisfied. Then Theorem 13.1 II follows from Theorem 2.4 II(a)(b) and Lemmas 8.1 and 8.2. □

In order to prove the Theorems 5.1 and 5.2, we prove the following lemmas.

Lemma 13.2. *The measure for the critical XOR Ising model on \mathbb{H} (resp. \mathbb{T}), obtained as the weak limit of measures on tori, is ergodic.*

Proof. Let $\rho = \frac{\sigma+1}{2}$, where $\sigma : V_{\mathbb{H}} \rightarrow \{\pm 1\}$ is the spin configuration for an Ising model on \mathbb{H} . Following the same arguments as in the proof of Lemma 10.2 in [27], it suffices to show that for the critical Ising model on \mathbb{H} (resp. \mathbb{T}), we have

$$\lim_{|u-v| \rightarrow \infty} |\langle \rho(u)\rho(v) \rangle - \langle \rho(u) \rangle \langle \rho(v) \rangle| = 0,$$

which is equivalent to show that

$$(41) \quad \lim_{|u-v| \rightarrow \infty} \langle \sigma(u)\sigma(v) \rangle = 0.$$

Note that $\langle \sigma(u)\sigma(v) \rangle$ is an even spin correlation function (i.e. the expectation of the product of spins on an even number of vertices), and hence for all the infinite-volume, translation-invariant Gibbs measures of the Ising model on \mathbb{H} (resp. \mathbb{T}) corresponding to the given coupling constant, $\langle \sigma_u \sigma_v \rangle$ has a unique value; see [33].

Consider the FK random cluster representation of the Ising model with $q = 2$, the two-point spin correlation $\langle \sigma(u)\sigma(v) \rangle$ is exactly the connectivity probability of u and v in the random cluster model, up to a multiplicative constant; see Chapter 1.4 of [17]. Therefore in order to show (41), it suffices to show that the connectivity probabilities of two vertices in the corresponding random cluster model, as the distances of the two vertices go to infinity, converge to zero.

By Theorem 4 of [7], we infer that the connectivity probabilities of two vertices in the $q = 2$ random cluster model corresponding to the critical Ising model on the triangular lattices with coupling constants K_a, K_b, K_c satisfying $g(K_a, K_b, K_c) = 0$ converge to zero as the distances of the two vertices go to infinity.

Note that the hexagonal lattice is a bipartite graph, i.e., all the vertices can be colored black and white such that vertices of the same color can never be adjacent. Recall that the star-triangle transformation is a replacement of each black vertex of \mathbb{H} , as well as its incident edges, into a triangle. The resulting graph is a triangular lattice \mathbb{T}' ; see the right graph of Figure 11. The parameters of the random cluster model on \mathbb{H} and the random cluster model \mathbb{T}' satisfy certain identities, such that the probabilities of connections of any two adjacent vertices in \mathbb{T}' (which are also vertices in \mathbb{H}) internal to each triangle face in \mathbb{T}' which has a black vertex of \mathbb{H} in the center are the same for the random cluster model on \mathbb{H} and the random cluster model on \mathbb{T}' ; see Page 160-161 of [17].

Using a star-triangle transformation (see the right graph of Figure 11), and (6.69) of [17], we deduce that the connectivity probabilities of two vertices in the $q = 2$ random cluster model on the hexagonal lattice corresponding to the critical Ising model on \mathbb{H} with coupling constants J_a, J_b, J_c satisfying $f(J_a, J_b, J_c) = 0$ converge to zero as the distances of the two vertices go to infinity. Note that the weak limit of measures with free boundary conditions is known to exist and translation-invariant, see Theorem (4.19) of [17]. By the uniqueness of $\langle \sigma(u)\sigma(v) \rangle$ under all the translation-invariant measures, we obtain that (41) holds under the measure obtained as the weak limit of measures with periodic boundary conditions. \square

Lemma 13.3. *The measure for the high-temperature XOR Ising model on \mathbb{H} (resp. \mathbb{T}), obtained as the weak limit of measures on tori, is ergodic.*

Proof. The identity (41) holds under the measure for the high-temperature Ising model on \mathbb{H} (resp. \mathbb{T}); see [14, 35]. \square

Proof of Theorem 5.1. We first show that in the critical XOR Ising model on \mathbb{H} or \mathbb{T} , almost surely there are no infinite contours. It is proved in [9] that the contours of XOR Ising model with spins located on \mathbb{H} (resp. \mathbb{T}) have the same distribution as contours in \mathbb{T} (resp. \mathbb{H}) for the Type-II clusters of dimer configurations on the $[4, 6, 12]$ lattice, if the coupling constants of the XOR Ising model and the edge weights of the $[4, 6, 12]$ lattice satisfy certain conditions. It is not hard to check that when the coupling constants of the XOR Ising model on \mathbb{T} (resp. \mathbb{H}) are critical, then the edge weights of the corresponding dimer model on the $[4,6,12]$ lattice satisfy (B1)-(B4). By Theorem 13.1 I, almost surely there are no infinite contours in the critical XOR Ising model on \mathbb{T} or \mathbb{H} .

Now we prove that almost surely there are no infinite clusters for the critical XOR Ising model on \mathbb{H} or \mathbb{T} . We write down the proof for the critical XOR Ising model on \mathbb{H} here, the case for the XOR Ising model on \mathbb{T} can be proved in a similar way.

Let \mathcal{A} be the event that there exists an infinite cluster for the XOR Ising model on \mathbb{H} . Assume that $\mu(\mathcal{A}) > 0$; we will obtain a contradiction. By translation-invariance of \mathcal{A} and Lemma 13.2, if $\mu(\mathcal{A}) > 0$ then $\mu(\mathcal{A}) = 1$. Let \mathcal{A}_1 (resp. \mathcal{A}_2) be the event that there exists an infinite “+”-cluster (resp. “-”-cluster) for the XOR Ising model on \mathbb{H} , then

$$(42) \quad \mu(\mathcal{A}_1 \cup \mathcal{A}_2) = 1.$$

By symmetry $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2)$. By translation-invariance of $\mathcal{A}_1, \mathcal{A}_2$ and Lemma 13.2, either $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 1$, or $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 0$. By (42), we have $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) = 1$, hence $\mu(\mathcal{A}_1 \cap \mathcal{A}_2) = 1$, i.e. μ -a.s. there exist both an infinite “+”-cluster and an infinite “-”-cluster in the critical XOR Ising configuration on \mathbb{H} .

Let $\phi_{\mathbb{T}}$ be the contour configuration associated to the critical XOR Ising configuration on \mathbb{H} . Let $\omega \in \phi^{-1}(\phi_{\mathbb{T}})$ be a constrained percolation configuration on the $[3, 4, 6, 4]$ lattice whose contour configuration is $\phi_{\mathbb{T}}$. It is not hard to check that in ω there exist both an infinite 1-cluster and an infinite 0-cluster if in the original XOR Ising model on \mathbb{H} , there exist both an infinite “+”-cluster and an infinite “-”-cluster. By Lemma A.3, μ -a.s. there exists an infinite contour in $\phi_{\mathbb{T}}$. The contradiction implies that μ -a.s. there are no infinite clusters in the critical XOR Ising model on \mathbb{H} . \square

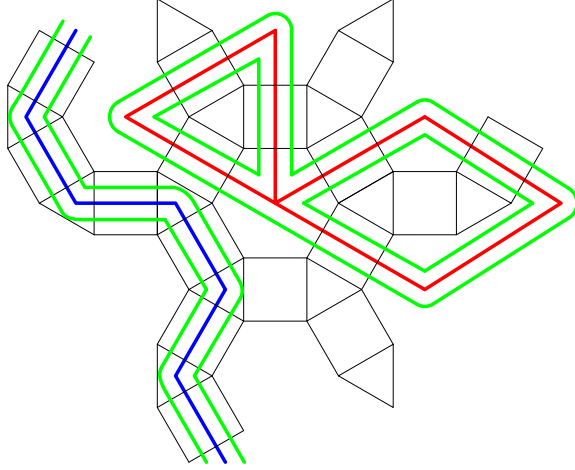


FIGURE 13. Contour configuration and interfaces. Blue lines represent contours of \mathbb{H} . Red lines represent contours of \mathbb{T} . Green lines represent interfaces.

Proof of Theorem 5.2. By the correspondence between contours in an XOR Ising model with spins located on vertices of \mathbb{H} (resp. \mathbb{T}) and contours on \mathbb{T} (resp. \mathbb{H}) for the Type-II clusters of dimer configurations on the $[4, 6, 12]$ lattice, as proved in [9], as well as correspondence between Type-II clusters of dimer configurations on the $[4, 6, 12]$ lattice and clusters of constrained configurations on the $[3, 4, 6, 4]$ lattice and Theorem 2.4, it suffices to show that the probability measure for the low-temperature XOR Ising model on \mathbb{H} (resp. \mathbb{T}) satisfies (A1)-(A5) and (A7) (resp. (A1)-(A6)), for $\mathbb{L}_1 = \mathbb{H}$.

It is straightforward to verify (A1)-(A5). The assumption (A6) (resp. (A7)) follows from Lemma 13.3. \square

APPENDIX A. DETERMINISTIC RESULTS ABOUT CONTOURS AND CLUSTERS

In this section, we prove deterministic results concerning contours and clusters for the constrained percolation model on the $[m, 4, n, 4]$ lattice in preparation to prove the main theorems.

Let P be the underlying plane into which the $[m, 4, n, 4]$ lattice is embedded. Recall that when $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, P is the Euclidean plane \mathbb{R}^2 and the graph G is amenable. When $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$, P is the hyperbolic plane \mathbb{H}^2 and the graph G is non-amenable.

We consider an embedding of the $[m, 4, n, 4]$ lattice into P in such a way that each edge has length 1. Let $\phi \in \Phi$ be a contour configuration, and let C be a contour in ϕ . To each component of $P \setminus \phi$, we associate an **interface**, which is a closed set consisting of all the points in the component whose distance to C is $\frac{1}{4}$. Here by distance, we mean either Euclidean distance or hyperbolic distance depending on whether P is \mathbb{R}^2 or \mathbb{H}^2 . Obviously each interface is either a self-avoiding cycle or a doubly infinite self-avoiding walk. See Figure 13 for an example of interfaces on the $[3, 4, 6, 4]$ lattice.

Note that when $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$ and $\min\{m, n\} \geq 3$, both \mathbb{L}_1 and \mathbb{L}_2 are vertex-transitive, non-amenable, planar graphs with one end.

Lemma A.1. *Whenever we have an interface I , let F_I be the set consisting of all the vertices of G whose (Euclidean or hyperbolic) distance to the interface is $\frac{1}{4}$. Then all the vertices of F_I are in the same cluster. If I is a doubly-infinite self-avoiding path, then F_I is part of an infinite cluster.*

Proof. Recall that the interface I is either a self-avoiding cycle or a doubly-infinite self-avoiding walk. Give I a fixed direction. Moving along I following the fixed direction, let $\{S_j\}_{j \in J}$ be the set of faces crossed by I in order, where $J \subseteq \mathbb{Z}$ is a set of integers, such that for any $j_1 < j_2$, $j_1, j_2 \in J$, I crosses S_{j_1} first, and then crosses S_{j_2} . Note that it is possible to have $S_{j_1} = S_{j_2}$ for $j_1 \neq j_2$.

For any two vertices $u, v \in F_I$, we can find a sequence of indices $j_1 < j_2 < \dots < j_k \in J$, and a sequence of vertices of G ,

$$(43) \quad u = v_{j_1,1}, v_{j_1,2}(= v_{j_2,1}), v_{j_2,2}(= v_{j_3,1}) = \dots = v_{j_k,2} = v,$$

such that for any $1 \leq i \leq k$, $v_{j_i,1}$ and $v_{j_i,2}$ are two vertices (which may not be distinct) in $F_I \cap \partial S_{j_i}$, and there exists a path $\ell_{v_{j_i,1}v_{j_i,2}}$ connecting $v_{j_i,1}$ and $v_{j_i,2} \subset \partial S_{j_i}$, and $\ell_{v_{j_i,1}v_{j_i,2}} \cap I = \emptyset$. Note that any two consecutive vertices in (43) are in the same cluster, therefore u and v are in the same cluster.

If I is a doubly-infinite self-avoiding path, then I crosses infinitely many faces. Each face crossed by I has at least one boundary vertex in F_I . Each vertex of F_I is a boundary vertex of at most 4 faces. Therefore $|F_I| = \infty$. Since all the vertices in F_I are in the same cluster, F_I is part of an infinite cluster. \square

In the following lemma, contours may be primal or dual as usual.

Lemma A.2. *If there exist at least two infinite contours, then there exists an infinite 0-cluster or an infinite 1-cluster. Moreover, if C_1 and C_2 are two infinite contours, then there exists an infinite cluster incident to C_1 .*

Proof. If there exist at least two infinite contours, then we can find two distinct infinite contours C_1 and C_2 , two points $x \in C_1$ and $y \in C_2$ and a self-avoiding path p_{xy} , consisting of edges of G and two half-edges, one starting at x and the other ending at y , and connecting x and y , such that p_{xy} does not intersect any infinite contours except at x and at y . Indeed, we may take any path intersecting two distinct contours, and then take a minimal subpath with this property.

Let $v \in V$ be the first vertex of G along p_{xy} starting from x . Let u be the point along the line segment $[v, x]$ lying on an interface of C_1 . Let ℓ_u be the interface of C_1 containing u . Then ℓ_u is either a doubly-infinite self-avoiding path or a self-avoiding cycle.

We consider these two cases separately. Firstly, if ℓ_u is a doubly-infinite self-avoiding path, then we claim that v is in an infinite (0 or 1-)cluster of the constrained site configuration on G . Indeed, this follows from Lemma A.1.

Secondly, if ℓ_u is a self-avoiding cycle, then $P \setminus \ell_u$ has two components, Q_v and Q'_v , where Q_v is the component including v . Since ℓ_u is a cycle, exactly one of Q_v and Q'_v is bounded, the other is unbounded. Since $C_1 \subseteq Q'_v$, and C_1 is an infinite contour, we deduce that Q'_v is unbounded, and Q_v is bounded. Since $y \notin \ell_u$, either $y \in Q_v$, or $y \in Q'_v$. If $y \in Q'_v$, then any path, consisting of edges of G and one half-edge incident to y , connecting

v and y must cross C_1 . In particular, p_{xy} crosses C_1 not only at x , but also at some point other than x . This contradicts the definition of p_{xy} . Hence $y \in Q_v$. Since $C_1 \cap C_2 = \emptyset$, this implies $C_2 \subseteq Q_v$; because if $C_2 \cap Q'_v \neq \emptyset$, then $C_2 \cap C_1 \neq \emptyset$. But $C_2 \subseteq Q_v$ is impossible since C_2 is infinite and Q_v is bounded. Hence this second case is impossible.

Therefore we conclude that if there exist at least two infinite contours, then there exists an infinite (0 or 1)-cluster. \square

Lemma A.3. *Let $x \in V$ be in the infinite 0-cluster, let $y \in V$ be in the infinite 1-cluster, and let ℓ_{xy} be a path, consisting of edges of G and connecting x and y . Then ℓ_{xy} has an odd number of crossings with infinite contours in total.*

In particular, if there exist both an infinite 0-cluster and an infinite 1-cluster in a constrained percolation configuration $\omega \in \Omega$, then there exists an infinite contour in $\phi(\omega) \in \Phi$.

Proof. Same as Lemma 2.8 of [27]. \square

Lemma A.4. *Let C_∞ be an infinite contour. Then each infinite component of $G \setminus C_\infty$ contains an infinite cluster that is incident to C_∞ .*

Proof. The lemma can be proved using similar technique as in Lemma 2.7 of [27]. \square

Lemma A.5. *Let $\omega \in \Omega$. Assume that there is exactly one infinite 0-cluster and exactly one infinite 1-cluster in ω . Assume that there exist a vertex x in the infinite 0-cluster, a vertex y in the infinite 1-cluster, and a path ℓ_{xy} , consisting of edges of G and joining x and y , such that ℓ_{xy} crosses exactly one infinite contour, C_∞ . Then C_∞ is incident to both the infinite 0-cluster and the infinite 1-cluster.*

Proof. By Lemma A.4, there is an infinite cluster in each infinite component of $G \setminus C_\infty$. Since there are exactly two infinite clusters, $G \setminus C_\infty$ has at most 2 infinite components. Since each infinite cluster lies in some infinite component of $G \setminus C_\infty$, the number of infinite components of $G \setminus C_\infty$ is at least one.

If $G \setminus C_\infty$ has exactly two infinite components, then we can construct two infinite connected set of vertices in the two infinite components of $G \setminus C_\infty$, as a consequence of Lemma A.4, denoted by W_1 and W_2 , such that C_∞ is incident to both W_1 and W_2 . Moreover, W_1 and W_2 are exactly part of the infinite 0-cluster and part of the infinite 1-cluster. Therefore C_∞ is incident to both the infinite 0-cluster and the infinite 1-cluster.

If $G \setminus C_\infty$ has exactly one infinite component, denoted by R , then both the infinite 0-cluster and the infinite 1-cluster lie in R , and in particular $x, y \in R$. We can find a path ℓ'_{xy} , connecting x and y , using edges of G , such that ℓ'_{xy} does not cross C_∞ at all. We can change path from ℓ'_{xy} to ℓ_{xy} by choosing finitely many faces S_1, S_2, \dots, S_k of G ; along the boundary of each face, make every present edge in the path absent and every absent edge in the path present; and we perform this procedure for S_1, \dots, S_k one by one. Such a the path modification procedure does not change the parity of the number of crossings of the path with C_∞ . Hence we infer that ℓ_{xy} intersects C_∞ an even number of times. But this is a contradiction to Lemma A.3 which says ℓ_{xy} crosses C_∞ an odd number of times, since we assume that ℓ_{xy} crosses exactly one infinite contour C_∞ . \square

Lemma A.6. *Assume that ξ is an infinite cluster, and C is an infinite contour. Assume that x is a vertex of G in ξ , and let $y \in C$ be the midpoint of an edge of G . Assume that there exists a path p_{xy} connecting x and y , consisting of edges of G and a half-edge incident to y , such that p_{xy} crosses no infinite contours except at y . Let z be the first vertex of G along p_{xy} starting from y . Then $z \in \xi$.*

Proof. The proof is an adaptation of the proof of Lemma 2.9 in [27]. Lemma 2.9 in [27] applies when the graph G is a square grid embedded in the Euclidean plane \mathbb{R}^2 . This lemma applies when the graph G is a general $[m, 4, n, 4]$ lattice embedded in either the Euclidean plane or the hyperbolic plane.

Since p_{xy} crosses no infinite contours except at y , let C_1, \dots, C_m be all the finite contours crossing p_{xy} . We claim that $P \setminus \cup_{i=1}^m C_i$ has a unique unbounded component, which contains both x and y . Indeed, since $x \in \xi$ and $y \in C$; neither the infinite cluster ξ nor the infinite contour C can lie in a bounded component of $P \setminus \cup_{i=1}^m C_i$.

Let I be the intersection of the union of the interfaces of C_1, \dots, C_m with the unique unbounded component of $P \setminus \cup_{i=1}^m C_i$. Since each C_i , $1 \leq i \leq m$, is a finite contour, each interface of C_i is finite. In particular, I consists of finitely many disjoint self-avoiding cycles, denoted by D_1, \dots, D_t . For $1 \leq i \leq t$, $P \setminus D_i$ has exactly one unbounded component, and one bounded component. Moreover, for $i \neq j$, D_i and D_j come from interfaces of distinct contours.

Let B_i be the bounded component of $P \setminus D_i$. We claim that each B_i is simply-connected, and $B_i \cap B_j = \emptyset$, for $i \neq j$. Indeed, B_i is simply connected, since the boundary of B_i , D_i is a self-avoiding cycle, whose embedding on the plane is a simple closed curve, for $1 \leq i \leq t$. Let $1 \leq i < j \leq t$. Since D_i and D_j are disjoint, either $B_i \cap B_j = \emptyset$, or one of B_i and B_j is a proper subset of the other. Without loss of generality, assume B_i is a proper subset of B_j . Then D_i is a proper subset of B_j . Hence D_i is in a bounded component of $P \setminus \cup_{k=1}^m C_k$, which contradicts the definition of D_i .

Let R_i be the set of faces F of G , for which $B_i \cap F \neq \emptyset$. Let $\tilde{B}_i = \cup_{F \in R_i} F$. Note that for $1 \leq i \leq t$, each \tilde{B}_i is a simply-connected, closed set. Let B'_i be the interior of \tilde{B}_i . Then each B'_i is a simply-connected, open set; moreover, $B'_i \cap B'_j = \emptyset$, if $i \neq j$. This follows from the fact that for $i \neq j$, D_i and D_j come from interfaces of distinct contours, and the fact that $B_i \cap B_j = \emptyset$, for $i \neq j$.

Let $B' = \cup_{i=1}^t B'_i$. Then B' is open, and $x, y, z \in P \setminus B'$, although x and z may be on the boundary of B' .

There is a path $p'_{xy} \subseteq [p_{xy} \cap (P \setminus B')] \cup \partial B'$, connecting x and y , where $\partial B'$ is the boundary of B' . More precisely, p_{xy} is divided by $\partial B'$ into segments; on each segment of p_{xy} in $P \setminus B'$, p'_{xy} follows the path of p_{xy} ; for each segment of p_{xy} in B' , p'_{xy} follows the boundary of B' to connect the two endpoints of the segment. This is possible since B' consists of bounded, disjoint, simply-connected, open sets B'_i , for $1 \leq i \leq t$, and both x and v are in the complement of B' in P .

All the vertices along p'_{xy} are in the same cluster. In particular, this implies that x and z are in the same infinite cluster ξ . \square

Lemma A.7. *If there exist exactly two infinite contours, then there exists an infinite cluster incident to both infinite contours.*

Proof. Let C_1, C_2 be the two infinite contours. Since there exist only two infinite contours, we can find two points $x \in C_1, y \in C_2$, and a self-avoiding path p_{xy} , consisting of edges of G and two half-edges, one starting at x and the other ending at y , and connecting x and y , such that p_{xy} does not intersect infinite contours except at x and at y .

Let $v \in V$ be the first vertex of G along p_{xy} starting from x . By the proof of Lemma A.2, v is in an infinite cluster ξ incident to C_1 . Let z be the first vertex of G along p_{xy} . By Lemma A.6, $z \in \xi$, and therefore ξ is an infinite cluster incident to both C_1 and C_2 . \square

Lemma A.8. *Let $\omega \in \Omega$. If there is exactly one infinite 0-cluster and exactly one infinite 1-cluster in ω , then there exists an infinite contour that is incident to both the infinite 0-cluster and the infinite 1-cluster in ω .*

Proof. Let x be a vertex in the infinite 0-cluster, and let y be a vertex in the infinite 1-cluster. Let ℓ_{xy} be a path joining x and y and consisting of edges of G .

By Lemma A.3, ℓ_{xy} must cross infinite contours an odd number of times. By Lemma A.5, if ℓ_{xy} crosses exactly one infinite contour, C_∞ , then C_∞ is incident to both the infinite 0-cluster and the infinite 1-cluster, and so the lemma is proved in this case.

Suppose that there exist more than one infinite contour crossing ℓ_{xy} . Let C_1 and C_2 be two distinct infinite contours crossing ℓ_{xy} .

Let $u \in C_1 \cap \ell_{xy}$ and $v \in C_2 \cap \ell_{xy}$ (Here we interpret the contours and the paths as their embeddings to P , so that u, v are points in P), such that the portion of ℓ_{xy} between u and v , p_{uv} , does not cross any infinite contours except at u and at v . As in the proof of Lemma A.2, let u_1 be the first vertex of G along p_{uv} , starting from u ; and let v_1 be the first vertex of G along p_{uv} starting from v . Let u_2 (resp. v_2) be the point along the line segment $[u, u_1]$ (resp. $[v, v_1]$) lying on an interface. Following the procedure in the proof of Lemma A.2, we can find an infinite cluster ξ_1 , such that $u_1 \in \xi_1$. The following cases might happen:

- I $x \notin \xi_1$ and $y \notin \xi_1$;
- II $x \notin \xi_1$ and $y \in \xi_1$;
- III $x \in \xi_1$ and $y \notin \xi_1$;
- IV $x \in \xi_1$ and $y \in \xi_1$.

First of all, Case IV is impossible because we assume x and y are in two distinct infinite clusters. Secondly, if Case I is true, then there exist at least three infinite clusters, which is a contradiction to our assumption.

Case II and Case III can be handled using similar arguments, and we write down the proof of Case II here.

If Case II is true, first note that $y \in \xi_1$ implies that C_1 is incident to the infinite 1-cluster. Let z be the first point in $C_1 \cap \ell_{xy}$ (again interpret edges as line segments), when traveling along ℓ_{xy} starting from x . Let p_{xz} be the portion of ℓ_{xy} between x and z .

Next, we will prove the following claim by induction on the number of complete edges of G along p_{xz} (in contrast to the half edge along p_{xz} with an endpoint z).

Claim A.9. *Under Case II, there is an infinite contour incident to both the infinite 0-cluster and the infinite 1-cluster.*

Assume that the number of complete edges of G along p_{xz} is n , where $n = 0, 1, 2, \dots$

First of all, consider the case when $n = 0$. This implies that C_1 is incident to the infinite 0-cluster at x . Recall that C_1 is also incident to the infinite 1-cluster at y , and so Claim A.9 is proved.

We make the following induction hypothesis:

- Claim A.9 holds for $n \leq k$, where $k \geq 0$.

Now we consider the case when $n = k + 1$. The **interior points** of p_{xz} are all points along p_{xz} except x and z . We consider two cases:

- (a) at interior points, p_{xz} crosses only finite contours but not infinite contours;
- (b) at interior points, p_{xz} crosses infinite contours.

We claim that if Case (a) occurs, then C_1 is incident to both the infinite 0-cluster and the infinite 1-cluster. It suffices to show that C_1 is incident to the infinite 0-cluster.

Let z_1 be the first vertex in V along p_{xz} starting from z . According to Lemma A.6, both x and z_1 are in the infinite 0-cluster. We infer that C_1 is incident to the infinite 0-cluster at x , if p_{xz} intersects only finite contours at interior points.

Now we consider Case (b). Let C_3 be an infinite contour crossing p_{xz} at interior points. Obviously, C_3 and C_1 are distinct, because C_1 crosses p_{xz} only at z . Let w be the last point in $C_3 \cap p_{xz}$, when traveling along p_{xz} , starting from x , and let p_{wz} be the portion of p_{xz} between w and z . Assume p_{wz} does not cross infinite contours at interior points.

Let w_1 be the first vertex of G along p_{wz} , starting from w , and let w_2 be the midpoint of w and w_1 . According to the proof of Lemma A.2, we can find an infinite cluster ξ_3 including w_1 . The following cases might happen:

- i $x \notin \xi_3$, and $y \notin \xi_3$;
- ii $x \in \xi_3$, and $y \notin \xi_3$;
- iii $x \notin \xi_3$, and $y \in \xi_3$;
- iv $x \in \xi_3$, and $y \in \xi_3$.

First of all, Case iv is impossible because we assume x and y are in two distinct infinite clusters. Secondly, if Case i is true, then there exist at least three infinite clusters, which is a contradiction to the assumption that there exists exactly one infinite 0-cluster and one infinite 1-cluster.

If Case ii is true, then C_3 is incident to the infinite 0-cluster including x . Since $w_1 \in \xi_3$, and p_{wz} does not cross infinite contours except at w and z , by Lemma A.6, we infer that $z \in \xi_3$, and ξ_3 is exactly the infinite 0-cluster including x . We conclude that C_1 is incident to the infinite 0-cluster including x as well, and Claim A.9 is proved.

If Case iii is true, then C_3 is incident to the infinite 1-cluster including y . Let t be the first vertex in $p_{xz} \cap C_3$, when traveling from p_{xz} , starting at x , and let p_{xt} be the portion of p_{xz} between x and t . We explore the path p_{xt} as we have done for p_{xz} . Since the length of p_{xz} is finite, and the number of full edges of G along p_{xt} is less than that of p_{xz} by at least 1, we apply the induction hypothesis with C_1 replaced by C_3 , C_2 replaced by C_1 , ξ_1 replaced by ξ_3 , p_{xz} replaced by p_{xt} , and we conclude that there exists an infinite contour adjacent to both the infinite 0-cluster and infinite 1-cluster. \square

Lemma A.10. *Let C_1 and C_2 be two infinite contours, and let ξ_0 and ξ_1 be two infinite clusters. The following two cases cannot occur simultaneously.*

- ξ_0 is incident to both C_1 and C_2 ;
- ξ_1 is incident to both C_1 and C_2 .

Proof. Same arguments as Lemma 6.3 of [27]. □

Lemma A.11. *Let \mathbb{L}_2 be the regular tiling of the hyperbolic plane with triangles, such that each vertex has degree $n \geq 7$. Let $\omega \in \{0, 1\}^{V(\mathbb{L}_2)}$ be a site percolation configuration on \mathbb{L}_2 . Assume that there exists an infinite 1-cluster $\eta \subseteq \omega$ with infinitely many ends. Then there exist at least two infinite 0-clusters in ω .*

Proof. Since $\eta \subseteq \omega$ has infinitely many ends, there exists a finite box B of \mathbb{L}_1 , such that $\eta \setminus B$ has at least two distinct infinite components. Let X, Y be two distinct infinite components of $\eta \setminus B$. Define the boundary of X (resp. Y), ∂X (resp. ∂Y) to be the set of all edges in \mathbb{L}_1 , such that each dual edge has exactly one endpoint in X (resp. Y), and one endpoint in $\mathbb{L}_2 \setminus B$ and not in X (resp. not in Y). Then ∂X and ∂Y are part of contours - each vertex of \mathbb{L}_1 in ∂X and ∂Y has degree 1 or 2, whose degree-1 vertices are along ∂B (here ∂B consists of all the edges of \mathbb{L}_1 on the boundary of the finite box B).

Each component of ∂X or ∂Y (here we assume that points on ∂B are not included in ∂X or ∂Y) must be one of the following three cases:

- I a finite component; or
- II a doubly infinite self-avoiding path which does not intersect ∂B ;
- III a singly-infinite self-avoiding path starting from a vertex along ∂B .

Let $B_1 \supset B$ be a box of \mathbb{L}_2 containing B . Then the embedding of ∂B_1 into \mathbb{H}^2 is a simple closed curve consisting of edges in \mathbb{L}_2 . Since X and Y are two infinite components of $\eta \setminus B$, we deduce that $\partial B_1 \cap X \neq \emptyset$ and $\partial B_1 \cap Y \neq \emptyset$. Since ∂B_1 is a closed curve, there exist $x_1, x_2 \in \partial B_1 \cap X$ and $y_1, y_2 \in \partial B_1 \cap Y$, such that there are segments $p_{x_1 y_1} \subset \partial B_1$ joining x_1 and y_1 , and $p_{x_2 y_2} \subset \partial B_1$ joining x_2 and y_2 such that $p_{x_1 y_1}$ and $p_{x_2 y_2}$ does not intersect each other except possibly at x_1, x_2, y_1, y_2 ; $p_{x_1 y_1}$ does not intersect $X \cup Y$ except at x_1, y_1 ; and that $p_{x_2 y_2}$ does not intersect $X \cup Y$ except at x_2, y_2 .

Then we claim that both $p_{x_1 y_1}$ and $p_{x_2 y_2}$ cross a component of ∂X of Type III, and a component of ∂Y of Type III. To see why that is true, assume that $p_{x_1 y_1}$ crosses only components of ∂X of Type I or II, then we can find a path $q_{x_1 y_1}$ consisting of edges in $\mathbb{L}_2 \setminus B$ and joining x_1 and y_1 such that $q_{x_1 y_1}$ does not cross ∂X at all. Then Y and X are the same component of $\eta \setminus B$. The contradiction implies that $p_{x_1 y_1}$ must cross a component of ∂X of Type III. Similarly, $p_{x_1 y_1}$ must cross a component of ∂Y of Type III; $p_{x_2 y_2}$ must cross a component of ∂X of Type III, and a component of ∂Y of Type III.

Let ℓ_1 (resp. ℓ_2) be a component of ∂X of Type III crossed by $p_{x_1 y_1}$ (resp. $p_{x_2 y_2}$). Let V_1 (resp. V_2) consist of all the vertices of \mathbb{L}_2 on a triangle face crossed by ℓ_1 (resp. ℓ_2) but not in X . Then V_1 is part of an infinite 0-cluster ξ , and V_2 is part of an infinite 0-cluster ζ ; and moreover, ξ and ζ are distinct since both $p_{x_1 y_1}$ and $p_{x_2 y_2}$ also cross components of ∂Y of Type III. This completes the proof. □

Lemma A.12. *Let $G = (V, E)$ be a square tiling of the hyperbolic plane satisfying I and II. Let $\omega \in \Omega$ be a constrained percolation configuration on G . If there exists a contour in the corresponding contour configuration of ω , then there exist at least one infinite 1-cluster and at least one infinite 0-cluster in ω .*

Proof. Let C be a contour in the corresponding contour configurations. By Lemma 6.5, C is an infinite tree in which each vertex is incident to 2 or 4 edges. Since C has no cycles, the complement $\mathbb{H}^2 \setminus C$ of C in the hyperbolic plane \mathbb{H}^2 has no bounded components.

We claim that each unbounded component of $\mathbb{H}^2 \setminus C$ contains at least one infinite cluster. Let Λ be an unbounded component of $\mathbb{H}^2 \setminus C$. Let $V_{\Lambda, C} \subset V$ consist of all the vertices in Λ that are also in a face of G intersecting C . Then all the vertices in $V_{\Lambda, C}$ are in the same cluster of ω and $|V_{\Lambda, C}| = \infty$. Hence there exists an infinite cluster in ω containing every vertex in $V_{\Lambda, C}$.

Let $e \in C$ be an edge crossing a pair of opposite edges of a black face b of G . Let v_1, v_2, v_3, v_4 be the 4 vertices of b . Assume that v_1 and v_2 are on one side of e while v_3 and v_4 are on the other side of e . By the arguments above, v_1 and v_2 are in an infinite cluster ξ_1 of ω ; similarly, v_3 and v_4 are in an infinite cluster ξ_2 of ω . Moreover, $e \in C$ implies that v_1 and v_3 have different state; and therefore exactly one of ξ_1 and ξ_2 is an infinite 0-cluster, and the other is an infinite 1-cluster. \square

Lemma A.13. *Let $G = (V, E)$ be a square tiling of the hyperbolic plane satisfying I and II. Let $\omega \in \Omega$ be a constrained percolation configuration on G , and let ϕ be the corresponding contour configuration of ω . Then each component of $\mathbb{H}^2 \setminus \phi$ contains an infinite cluster in ω .*

Proof. By Lemma 6.5, ϕ is the disjoint union of infinite trees, in which each vertex has degree 2 or 4. Since ϕ contains no cycles, each component of $\mathbb{H}^2 \setminus \phi$ is unbounded.

Let Λ be an unbounded component of $\mathbb{H}^2 \setminus \phi$. Let $V_{\Lambda, \phi} \subset V$ consist of all the vertices in Λ that are also in a face of G intersecting ϕ . Then all the vertices in $V_{\Lambda, \phi}$ are in the same cluster of ω and $|V_{\Lambda, \phi}| = \infty$. Hence there exists an infinite cluster in ω containing every vertex in $V_{\Lambda, \phi}$. Since every vertex in $V_{\Lambda, \phi}$ is in Λ , and any cluster intersecting Λ is completely in Λ , we conclude that Λ contains an infinite cluster of ω . \square

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REFERENCES

- [1] M. Aizenman, *Translation invariance and instability of phase coexistence in the two-dimensional Ising system*, Comm. Math. Phys. **73** (1980), 83–94.
- [2] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, 2008.
- [3] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Critical percolation on any non-amenable group has no infinite clusters*, Ann. Probab. **27** (1999), 1347–1356.

- [4] ———, *Group-invariant percolation on graphs*, *Geom. funct. anal.* **9** (1999), 29–66.
- [5] I. Benjamini and O. Schramm, *Percolation beyond \mathbb{Z}^d many questions and a few answers*, *Electron. Commun. Probab.* **1** (1996), 71–82.
- [6] ———, *Percolation in the hyperbolic plane*, *Journal of the American Mathematical Society* **14** (2000), 487–507.
- [7] B. Bollobás and O. Riordan, *Percolation on dual lattices with k -fold symmetry*, *Random Structures and Algorithms* **32** (2008), 463–472.
- [8] A. Borodin, I. Corwin, and V. Gorin, *Stochastic six-vertex model*, *Duke Math. J.* **165** (2016), 563–624.
- [9] C. Boutillier and B. de Tilière, *Height representation of XOR-Ising loops via bipartite dimers*, *Electron. J. Probab.* **19** (2014), 1–33.
- [10] R.M. Burton and M. Keane, *Density and uniqueness in percolation*, *Commun. Math. Phys.* **121** (1989), 501–505.
- [11] J.W. Cannon, W. J. Floyd, R. Kenyon, and Parry W.R., *Hyperbolic geometry*, *Flavors of Geometry (Math. Sci. Res. Inst. Publ., 31)*, Cambridge University Press, Cambridge, 1997, pp. 59–115.
- [12] A. Coniglio, CR. Nappi, F. Peruggi, and R. Russo, *Percolation and phase transitions in the Ising model*, *Commun. Math. Phys.* **51** (1976), 315–323.
- [13] J. Dubédat, *Exact bosonization of the Ising model*, arXiv:1112.4399.
- [14] H. Duminil-Copin and D. Cimasoni, *The critical temperature for the Ising model on planar doubly periodic graphs*, *Electron. J. Probab.* **44** (2013), 1–18.
- [15] R. Fitzner and R. van der Hofstad, *Mean-field behavior for nearest-neighbor percolation in $d > 10$* , arXiv:1506.07977v2.
- [16] G. Grimmett, *Percolation*, Springer, 1999.
- [17] ———, *The random cluster model*, Springer-Verlag, 2006.
- [18] G. Grimmett and Z. Li, *The 1-2 model*, *Contemporary Mathematics* **696** (2017), 139–152.
- [19] ———, *Critical surface of the 1-2 model*, *International Mathematics Research Notices* **21** (2018), 6617–6672.
- [20] O. Häggstöm, J. Jonasson, and R. Lyons, *Explicit isoperimetric constants and phase-transitions in the random cluster model*, *Ann. Probab.* **30** (2002), 443–473.
- [21] O. Häggstöm and Y. Peres, *Monotonicity of uniqueness for percolation on cayley graphs: All infinite clusters are born simultaneously*, *Probab. Theory and Related Fields* **113** (1999), 273–285.
- [22] O. Häggstöm, Y. Peres, and R.H. Schonmann, *Percolation on transitive graphs as a coalescent process: relentless merging followed by simultaneous uniqueness*, *Perplexing Probability Problems :Papers in Honor of H. Kesten (M. Bramson and R. Durrett, eds.)*, Birkhäuser, Boston, 1999, pp. 69–90.
- [23] T. Hara and G. Slade, *Mean-field behavior and the lace expansion*, *Probability and Phase Transition (G. R. Grimmett, ed.)*, Kluwer, Dordrecht, 1994, pp. 87–122.
- [24] T. E. Harris, *A lower bound for the critical probability in a certain percolation process*, *Proc. Cambridge Philos. Soc.* **56** (1960), 13–20.

- [25] Y. Higuchi, *A sharp transition for the two-dimensional Ising percolation*, Probab. Theory Relat. Fields **97** (1993), 489–514.
- [26] R. Holley, *Remarks on the FKG inequalities*, Commun. Math. Phys. **36** (1974), 227–231.
- [27] A. Holroyd and Z. Li, *Constrained percolation in two dimensions*, ANNALES DE L'INSTITUT HENRI POINCARÉ D (2020), arXiv:1510.03943v2.
- [28] R. Kenyon, *Lectures on dimers*, Statistical Mechanics IAS/Park City Math.Ser.,**16** (2009), 191–230.
- [29] R. Kenyon, J. Miller, S. Sheffield, and D.B. Wilson, *The six-vertex model and Schramm-Loewner evolution*, arXiv:1605.06471v1.
- [30] R. Kenyon, A. Okounkov, and S. Sheffield, *Dimers and amoeba*, Ann. Math. **163** (2006), 1019–1056.
- [31] H. Kesten, *The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$* , Comm. Math. Phys. **74** (1980), 41–59.
- [32] J. Lebowitz, *On the uniqueness of the equilibrium state for Ising spin systems*, Commun. Math. Phys. **25** (1972), 276–282.
- [33] ———, *Coexistence of phases in Ising ferromagnets*, J. Statist. Phys. **16** (1977), 453–461.
- [34] Z. Li, *Local statistics of realizable vertex models*, Commun. Math. Phys. **304** (2011), 723–763.
- [35] ———, *Critical temperature of periodic Ising models*, Comm. Math. Phys. **315** (2012), 337–381.
- [36] ———, *1-2 model, dimers and clusters*, Electr. J. Probab. **19** (2014), 1–28.
- [37] ———, *Uniqueness of the infinite homogeneous cluster in the 1-2 model*, Electron. Commun. Probab. **19** (2014), 1–8.
- [38] R. Lyons and Y. Peres, *Probability on trees and networks*, Cambridge University Press, 2016, Available at <http://pages.iu.edu/~rdlyons/>.
- [39] R. Lyons and O. Schramm, *Indistinguishability of percolation clusters*, Ann. Probab. **27** (1999), 1809–1836.
- [40] A. Nachmias and Y. Peres, *Non-amenable graphs of high girth have $p_c < p_u$ and mean field exponents*, Electron. Commun. Probab. **17** (2012), 1–8.
- [41] C.M. Newman and L.S. Schulman, *Infinite clusters in percolation models*, J. Statist. Phys. **26** (1981), 613–628.
- [42] I. Pak and T. Smirnova Nagnibeda, *On non-uniqueness of percolation in non-amenable cayley graphs*, C.R.Acad.Sci.Paris **330** (2000), 495–550.
- [43] D. Renault, *The vertex-transitive TLF-planar graphs*, Discrete Mathematics **303** (2009), 2815–2833.
- [44] R.H. Schonmann, *Stability of infinite clusters in supercritical percolation*, Probab. Theory and Related Fields **113** (1999), 287–300.
- [45] ———, *Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs*, Commun. Math. Phys. **219** (2001), 271–322.
- [46] M. Schwartz and J. Bruck, *Constrained codes as network of relations*, Information theory, IEEE transactions (2008).

- [47] A. Thom, *A remark about the spectral radius*, Int. Math. Res. Notices **10** (2015), 2856–2864.
- [48] L. G. Valiant, *Holographic algorithms (extended abstract)*, Proc. 45th IEEE Symposium on Foundations of Computer Science (2004), 306–315.
- [49] D. B. Wilson, *XOR-Ising loops and the Gaussian free field*, arXiv:1102.3782.
- [50] C. Wu, *Ising model on hyperbolic graphs II*, J Stats. Phys. **100** (2000), 893–904.

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