Minimal Logarithmic Signatures for one type of Classical Groups

MLSs for one type of Classical Groups

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Abstract As a special type of factorization of finite groups, logarithmic signature (LS) is used as the main component of cryptographic keys for secret key cryptosystems such as PGM and public key cryptosystems like MST_1, MST_2 and MST_3. An LS with the shortest length, called a minimal logarithmic signature (MLS), is even desirable for cryptographic applications. The MLS conjecture states that every finite simple group has an MLS. Recently, the conjecture has been shown to be true for general linear groups $GL_n(q)$, special linear groups $SL_n(q)$, and symplectic groups $Sp_n(q)$ with q a power of primes and for orthogonal groups $O_n(q)$ with q as a power of 2. In this paper, we present new constructions of minimal logarithmic signatures for the orthogonal group $O_n(q)$ and $SO_n(q)$ with q as a power of odd primes. Furthermore, we give constructions of MLSs for a type of classical groups — projective commutator subgroup $P\Omega_n(q)$.

Keywords (Minimal) logarithmic signature · Orthogonal group · Projective commutator subgroup · Stabilizer · Spreads

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1 Introduction

The security of many public key cryptosystems is based on the hardness assumptions of certain problems over large finite abelian algebraic structures such as cyclic groups, rings and finite fields. Two well-known hard problems are integers factorization problem (IFP) and discrete logarithm problem (DLP).

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However, these hardness assumptions would be broken if quantum computers become practical. For instance, Shor's quantum algorithms [17] solve IFP and DLP very efficiently. The security status of currently used cryptosystems, mainly based on IFP and DLP or their variants, becomes even worse due to the known great progress on finding possible solutions for building quantum computers on practical scales.

Therefore, it is imminent to design effective and practical cryptographic schemes that have the potential for resisting quantum algorithm attacks. Actually, several attempts using non-abelian algebraic structures were made and some available cryptographic schemes such as PGM, MST_1, MST_2 and MST_3 [3,16,14,11,9,24] were developed during the past decades. In particular, as a natural analogy of the hardness assumption of IFP, the group factorization problem (GFP)[10,15] and its hardness assumption over certain factorization basis, referred as logarithmic signature, play a core role in the security arguments for the family of MST cryptosystems.

Security is not the unique goal of designing a cryptosytem. Instead, efficiency is also a major issue. With the purpose for minimizing the parameter sizes, a natural question comes to mind: How to make the factorization basis known as logarithmic signature (LS), as short as possible in the family of MST cryptosystems? A minimal logarithmic signature (MLS) is an LS with the shortest length. In other word, further shorten an MLS would make it no longer an LS. New question arises: Does any finite (non-abelian) group has MLSs?

In fact, some encouraging work has been done in searching the MLSs for finite groups. According to the pioneering work due to Vasco et al. [18], Holmes [20] and Lempken et al. [8], we know that, with few exceptions, MLSs exist for all groups of order $\leq 10^{10}$. Most recently, Nikhil Singhi, Nidhi Singhi, and Magliveras [21,23] made another breakthrough: MLSs exist for the groups $GL_n(q)$, $SL_n(q)$, $Sp_n(q)$ with q as a power of a prime and $O_n(q)$ with q as a power of 2. As far as we know, this is the first result not constrained by a specified boundary on group orders. Besides, Nikhil Singhi and Nidhi Singhi [23] also pointed out, without any proof, that the MLSs *should also exist* for $O_n(q)$ with q as a power of odd primes.

Therefore, in this paper, our main motivation is to present new constructions of minimal logarithmic signatures for the orthogonal group $O_n(q)$, the special orthogonal group $SO_n(q)$, the projective special orthogonal group $PSO_n(q)$, the commutator subgroup $\Omega_n(q)$ and one type of classical groups [5] — projective commutator subgroup $P\Omega_n(q)$ with q as a power of odd primes. For $O_n(q)$ and $SO_n(q)$, the proposed MLSs have the similar structure $[A, B', G_w]$, where $A = \langle a \rangle$, and $B' = \{hC \mid h \in B\}$, $C \leq B$, $B = \langle b \rangle$, while $G_w = P : Q$ is a semi-direct product of a p-group P and a direct product $Q = GL_1(q) \times Y$ (see Table 1). We employ two canonical homomorphisms $\eta : SO_n(q) \to PSO_n(q)$ and $\theta : \Omega_n(q) \to P\Omega_n(q)$ for proving the existence of MLSs for $PSO_n(q)$ and $P\Omega_n(q)$, respectively.

The rest of contents are organized as follows: Necessary preliminaries are presented in Section 2; In Section 3, we utilize the Levi decomposition of

Table 1 MLSs for $O_n(q)$ and $SO_n(q)$

	$A = \langle a \rangle$	$B' = \{hC \mid h \in B\}$ with $C \le B = \langle b \rangle, |C| = q-1$	$G_w = P : Q$ with $Q = GL_1(q) \times Y$	
	a	b	$|P|$	Y
$O_n(q)$	$x_1^{q^m-1}$ for $x_1 \in GL_{2m}(q)$	$\begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (D_1^t)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for $D_1 \in GL_{2m-2}(q)$	q^{2m-2}	$O^-_{2m-2}(q)$
	$x_2^{q^{m-1}-1}$ for $x_2 \in GL_{2m}(q)$	$\begin{pmatrix} D_2 & 0 \\ 0 & (D_2^t)^{-1} \end{pmatrix}$ for $D_2 \in GL_{2m}(q)$	q^{2m-2}	$O^+_{2m-2}(q)$
	$x_3^{q^m-1}$ for $x_3 \in GL_{2m+1}(q)$	$\begin{pmatrix} D_3 & 0 & 0 \\ 0 & (D_3^t)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $D_3 \in GL_{2m}(q)$	q^{2m-1}	$O_{2m-1}(q)$
$SO_n(q)$	$x_1^{*q^m-1}$ for $x_1^* \in O^-_{2m}(q)$	$\begin{pmatrix} D_1^* & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D_1^{*t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for $D_1^* \in O_{m-1}(q)$	q^{2m-2}	$SO^-_{2m-2}(q)$
	$x_2^{*q^{m-1}-1}$ for $x_2^* \in O^+_{2m}(q)$	$\begin{pmatrix} D_2^* & 0 \\ 0 & D_2^{*t} \end{pmatrix}$ for $D_2^* \in O_m(q)$	q^{2m-2}	$SO^+_{2m-2}(q)$
	$x_3^{*q^m-1}$ for $x_3^* \in O_{2m+1}(q)$	$\begin{pmatrix} D_3^* & 0 & 0 \\ 0 & D_3^{*t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $D_3^* \in O_m(q)$	q^{2m-1}	$SO_{2m-1}(q)$

parabolic subgroups to construct LSs for the parabolic subgroups of $O_n(q)$ and $SO_n(q)$; In Section 4, analogous to methods in [23], we use the totally isotropic subspaces to prove the existence of MLSs for $O^-_{2m}(q)$ and $SO^-_{2m}(q)$; In Section 5, we utilize suitable spread in the $P(V)$ [2,7,12] to accomplish the proof for $O^+_{2m}(q)$ and $SO^+_{2m}(q)$; In Section 6, we make further efforts to present the constructions of MLSs for $PSO^\pm_{2m}(q)$ and $P\Omega^\pm_{2m}(q)$; In Section 7, we take account of MLSs for $O_{2m+1}(q)$, $SO_{2m+1}(q)$, $PSO_{2m+1}(q)$, $\Omega_{2m+1}(q)$ and $P\Omega_{2m+1}(q)$.

2 Preliminaries

2.1 Classical Spreads and Quadratic Spaces in Finite Fields

Let K be a finite field and V a n-dimensional vector space over K. For $v_1 \in V$, $\langle v_1 \rangle$ denotes the one-dimensional subspace generated by v_1. $P(V)$ denotes the projective space on V, which is the set of all one-dimensional subspaces of V [13].

Now, we describe the classical spread [12,7,4,2,23]. An r-partial spread in V is a set $S = \{W_i \mid 1 \le i \le t\}$ of r-dimensional subspaces W_i such that $W_i \cap W_j = \langle 0 \rangle$ for $i \ne j$. If $\cup_{i=1}^t W_i = V$, then S is an r-spread in V. Besides, when S is an r-(partial) spread in $P(V)$, it partitions $P(V)$ into $(r-1)$-dimensional subspaces of $P(V)$.

Suppose that $V = F_{q^{2m}}$ is a finite field, α is a primitive element of field $F_{q^{2m}}$ and $W = F_{q^m}$ is an m-subspace of V. For every $x \in V$, $W_x = \{wx \mid w \in W\}$,

the set $S = \{W_x \mid x \in V\}$ forms a m-spread in V [23]. Meanwhile, we have the following remark.

Remark 1 [21,23] Let $W_i = W\alpha^{(q^m-1)i} = \{w\alpha^{(q^m-1)i} | w \in W\}$, $0 \leq i \leq q^m$. Then, the spread S can also be described as $S = \{W_i | 0 \leq i \leq q^m\}$.

Let V be an n-dimensional vector space over the finite field $K = F_q$ with $q = p^e$ for some prime p and a positive integer e. Let $\mathcal{B} = \{e_1, \cdots, e_n\}$ be an ordered basis for V. Then a *bilinear form* over a vector space V is a map $f : V \times V \to K$ satisfying:

$$f(\lambda u + v, w) = \lambda f(u, w) + f(v, w),$$
$$f(u, \lambda v + w) = \lambda f(u, v) + f(u, w).$$

The radical of f, denoted by $rad(f)$, is $V^\perp = \{u \in V \mid f(u, v) = 0, \forall v \in V\}$. f is called *non-singular* if $rad(f) = \langle 0 \rangle$.

A map $Q : V \to K$ is called a *quadratic form* if it satisfies:

$$Q(\lambda u + v) = \lambda^2 Q(u) + \lambda f(u, v) + Q(v),$$

where f is a symmetric bilinear form. The radical of Q is $rad(Q) = \{v \in rad(f) \mid Q(v) = 0\}$. Q is called *non-singular* if $rad(Q) = \langle 0 \rangle$.

An *isometry* on a quadratic space (V, Q) is a non-singular linear map $g : V \to V$ such that $Q(g(v)) = Q(v)$ for all $v \in V$. Two quadratic spaces (V, Q) and (V, Q') are said to be *equivalent*, if there is an isometry $g : V \to V$ [13,23]. Besides, the group of all isometries of an inner-product space (V, f) is denoted by *Isom(V, f)* and that of all isometries of a quadratic space (V, Q) is denoted by *Isom(V, Q)*. When q is a power of odd primes then we have, *Isom(V, f)= Isom(V, Q)* [13,23].

A vector $v \in V$ is said to be *isotropic* if $f(v, v) = 0$, *singular* if $Q(v) = 0$ and *non-singular* if $Q(v) \neq 0$. A subspace W of V is called *totally isotropic* if $f(u, v) = 0$ for all $u, v \in W$ and *totally singular* if $Q(v) = 0$ for all $v \in W$. Besides, a point $\langle v \rangle \in P(V)$ is called a *singular point* if v is a singular vector, and $\langle v \rangle$ is called an *isotropic point* in $P(V)$ if v is an isotropic vector.

2.2 Logarithmic Signatures and Minimal Logarithmic Signatures for Finite Groups

Definition 1 (Logarithmic Signature) [8] Let G be a finite group, $A \subseteq G$. Let $\alpha = [A_1, \cdots, A_s]$ be a sequence of ordered subsets A_i of G such that $A_i = [a_{i1}, \cdots, a_{ir_i}]$ with $a_{ij} \in G$ $(1 \leq j \leq r_i)$. Then α is called a logarithmic signature for G (or A) if each $g \in G$ (or A) is uniquely represented as a product

$$g = a_{1j_1} \cdots a_{sj_s}$$

with $a_{ij_i} \in A_i$ $(1 \leq i \leq s)$.

The sequences A_i are called the blocks of α, the length of α is defined to be $l(\alpha) = \sum_{i=1}^{s} r_i$. Let $|G| = \prod_{j=1}^{k} p_j^{a_j}$ (or $|A| = \prod_{j=1}^{k} p_j^{a_j}$) be the prime power decomposition of $|G|$ (or $|A|$) and $\alpha = [A_1, A_2, \ldots, A_s]$ be an LS for G (or A). From [6], we have $l(\alpha) \geq \sum_{j=1}^{k} a_j p_j$.

Definition 2 (Minimal Logarithmic Signature) [8] A logarithmic signature α for a finite group G (or A) with $l(\alpha) = \sum_{j=1}^{k} a_j p_j$ is called a minimal logarithmic signature (MLS) for G (or A).

Lemma 1 *[21] Let $A, B \leq G$, if A and B satisfy any one of the following two conditions:*

(i) $G = A \times B$ is a direct product of A and B, $A \cap B = \{1\}$
(ii) $G = A : B$ is a semi-direct product of A and B, $A \cap B = \{1\}$.

Then, [A, B] is an LS for G.

Lemma 2 *[21, 23] Let H be a normal subgroup of G, $A \subseteq G$ and η the canonical homomorphism $\eta : G \to G/H$ such that $a, b \in A$, $a \neq b$ imply that $aH \neq bH$. Let $A' = \eta(A)$, and suppose hat $[A_1, A_2, \cdots, A_k]$ is an LS for A. Let $B_i = \eta(A_i) \subseteq G/H$ for $1 \leq i \leq k$. Then, $[B_1, B_2, \cdots, B_k]$ is an LS for A'.*

Now, let V be a finite dimensional vector space over F_q, f be a bilinear form and Q be a quadratic form. We call $L \subseteq P(V)$ a *Singhi subset* [23] if L is one of the following sets [21, 23]:

(i) the set of all isotropic points of $P(V)$ with respect to the bilinear form f,
(ii) the set of all non-isotropic points of $P(V)$ with respect to the bilinear form f,
(iii) the set of all singular points of $P(V)$ with respect to the quadratic form Q,
(iv) the set of all non-singular points of $P(V)$ with respect to the quadratic form Q.

(Note that a *Singhi subset* L that meets condition (i) will be used in our later construction.)

Lemma 3 *[21, 23] Suppose that $G|L$ is a transitive permutation group action such that G is a subgroup of $GL(V)$ and $L \subseteq P(V)$ is a Singhi subset. Let S be an r-partial spread in V, which partitions L. Let $W \in S$, $w \in P(W)$ and G_w be the stabilizer of w in G. Suppose there are sets $A, B \subseteq G$ such that*

(i) A acts sharply transitive on S with respect to W under the action of G on the set of all r-dimensional subspaces of V.
(ii) B acts sharply transitive on $L \cap P(W)$ with respect to w under the action of G on $P(W)$.

Then, $[A, B, G_w]$ is an LS for G.

Lemma 4 *[6, 21, 23] If G is a solvable group, then G has an MLS.*

Lemma 5 *[23] Let G be a finite group and $x \in G$ be an element of order t. For $s \in N, s \leq t$, let $S = \{x^i | 0 \leq i < s\} = \{1, x^1, x^2, \cdots, x^{s-1}\}$ be a cyclic set generated by x. Then S has an MLS $\beta = [A_1, A_2, \cdots, A_k]$ satisfying the following condition:*

For any list $[j_i, j_2, \cdots, j_k]$, such that $x^{j_i} \in A_i, 1 \leq i \leq k, \sum_{i=1}^{k} j_i < s$.

Lemma 6 *[23] Let G be a finite group and $[A_1, \cdots, A_r]$ be an LS for G such that for each subset A_j, $1 \leq j \leq r$, an MLS exists. Then G has an MLS.*

3 Construction I: LSs for Parabolic Subgroups of $O_n(q)$ and $SO_n(q)$

Throughout this paper, we assume that q is a power of odd primes. First, we construct the LSs for parabolic subgroups in $O_{2m+1}(q)$ and $O_{2m}^{\pm}(q)$.

Let W be an isotropic k-space of $V = F_{q^{2m}}$, then the stabilizer of W is the maximal parabolic subgroup P_{max} in $O_{2m}^{\pm}(q)$ of shape $q^{k(k-1)/2+k(2m-2k)}$: $(GL_k(q) \times O_{2m-2k}^{\pm}(q))$. Specifically, P_{max} has the shape

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ * & \cdots & 1 & \cdots & 0 \\ 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & C & D \end{pmatrix}$$

the normal p-subgroup R is a group of shape

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ * & \cdots & 1 & \cdots & & 0 \\ 0 & 0 & 0 & I_k & & 0 \\ 0 & 0 & 0 & C' & I_{2m-2k} \end{pmatrix}$$

with center of order $q^{k(k-1)/2}$ and the subset Q of matrices of the shape

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & D \end{pmatrix}$$

is a subgroup isomorphic to $GL_k(q) \times O_{2m-2k}^{\pm}(q)$. Moreover, $R \cap Q = \{I_n\}$, $P_{max} = R : Q$, therefore, P_{max} has an LS $[R, Q]$ (see Lemma 1). Also, P_{max} is of shape $q^{k(k-1)/2+k(2m+1-2k)}$: $(GL_k(q) \times O_{2m+1-2k}(q))$ in $O_{2m+1}(q)$.

Similarly, for $SO_{2m}^{\pm}(q)$ and $SO_{2m+1}(q)$, P'_{max} is of shape $q^{k(k-1)/2+k(2m-2k)}$: $(GL_k(q) \times SO_{2m-2k}^{\pm}(q))$ in $SO_{2m}^{\pm}(q)$ and $q^{k(k-1)/2+k(2m+1-2k)}$: $(GL_k(q) \times SO_{2m+1-2k}(q))$ in $SO_{2m+1}(q)$.

4 Construction II: MLSs for $O_{2m}^-(q)$ and $SO_{2m}^-(q)$

Now, we construct MLSs for $O_{2m}^-(q)$ and $SO_{2m}^-(q)$. Here, our fundamental tools are Lemma 3 and Lemma 6. Choosing suitable quadratic form Q of minus type, we take advantage of all isotropic points of $P(V)$ for constructing the corresponding MLSs.

First, we observe $O_{2m}^-(q)$. Suppose q is a power of odd primes, $V = F_{q^{2m}}$ is a $2m$-dimensional vector space over F_q, and L is the set of all isotropic points of $P(V)$. For $y \in V$, $\overline{y}$ denotes y^{q^m}. $T_s : V \to V$ is a linear transformation defined by $T_s(v) = sv$ for a given $s \in V$ and all $v \in V$. Let α be a primitive element of the field $F_{q^{2m}}$ and $x \in GL_{2m}(q)$ be the matrix corresponding to the linear transformation T_α [23]. We define a bilinear map $f : V \times V \to F_q$ by $f(x,y) = tr_{F_{q^m}/F_q}(x\overline{y} + \overline{x}y) = \sum_{i=0}^{m-1}(x\overline{y} + \overline{x}y)^{q^i}$ and a quadratic form $Q : V \to F_q$ by $Q(x) = tr_{F_{q^m}/F_q}(x\overline{x}) = \sum_{i=0}^{m-1}(x\overline{x})^{q^i}$ [21,23]. Then, we observe that the number of non-zero isotropic points in $P(V)$ with respect to the quadratic space (V, Q) are $(q^m + 1)(q^{m-1} - 1)/(q-1)$ and the quadratic form Q is of minus type [13,23]. Let G be the group of all isometries of (V, Q), then $G \cong O_{2m}^-(q)$ and G is a permutation group acting transitively on isotropic points [13,23].

Now, we roughly explain the main idea for constructing the MLSs. As described above, since the number of non-zero isotropic points in $P(V)$ with respect to the quadratic space (V, Q) are $(q^m+1)(q^{m-1}-1)/(q-1)$, therefore, we need to construct a cyclic group A of order $q^m + 1$, which must be sharply transitive on a partial spread S and a cyclic set B of cardinality $(q^{m-1} - 1)/(q - 1)$, which must be sharply transitive on the projective subspace of $P(V)$. Then, we take advantage of the Lemma 3, Lemma 5 and Lemma 6 for proving the existence of MLSs for $O_{2m}^-(q)$.

First, we define two special cyclic subgroups of $O_{2m}^-(q)$. Let $a_1 = x_1^{q^{m-1}} \in GL_{2m}(q)$ be the matrix corresponding to the linear transformation $T_{\alpha^{q^m-1}}$. Let $W_1' = \{e_1, e_2, \cdots, e_{m-1}\}$ is an $(m-1)$-dimensional totally isotropic subspace of V, $D_1 \in GL(W_1')$ be a generator of the *Singer cyclic subgroup* of $GL(W_1')$ [19] . Then $b_1 \in GL_{2m}(q)$ can be well defined as follows:

$$b_1 = \begin{pmatrix} D_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (D_1^t)^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Meanwhile, we get that $a_1, b_1 \in G$ [1,23]. Let $A_1 = \langle a_1 \rangle$, $B_1 = \langle b_1 \rangle$ be the cyclic subgroups of G generated by a_1 and b_1, respectively. Then, A_1 is of order $q^m + 1$ and B_1 is of order $q^{m-1} - 1$.

Let $C_1 = \langle b_1^{\frac{q^{m-1}-1}{q-1}} \rangle$ be the subgroup of order $q-1$ of B_1 and $B_1' = \{gC_1 | g \in B_1\}$ be the left coset of C_1 in B_1. Thus, $|B_1'| = \frac{q^{m-1}}{q-1}$. Furthermore, A_1 and B_1' are chosen so that $A_1 \cap B_1' = \{1\}$ and both are cyclic sets. Then, from Lemma 5, it follows that A_1 and B_1' have MLSs.

Now, let $S_1' = \{W_i' \mid 0 \le i \le q^m\}$ be the classical spread as described in Remark 1. W_i' are $(m-1)$-dimensional totally isotropic subspaces of V for $0 \le i \le q^m$. Clearly, the partial spread S_1' partitions the set of all isotropic points of $P(V)$.

Also, we observe that the group A_1 is sharply transitive on S_1' with respect to W_1'. Also, it is clear that B_1 is isomorphic to the Singer cyclic subgroup of $GL_{m-1}(q)$ and B_1' is sharply transitive on $P(W_1')$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_1'$.

Now, we consider $G^* = SO_{2m}^-(q)$. Being similar to the case of $O_{2m}^-(q)$, let $a_1^* = x_1^{*q^m-1} \in SO_{2m}^-(q)$ be the matrix corresponding to the linear transformation $T_{\alpha^{q^m-1}}$. Let $W_1' = \{e_1, e_2, \cdots, e_{m-1}\}$ is an $(m-1)$-dimensional totally isotropic subspace of V. Let $D_1^* \in O_{m-1}(q) \le GL(W_1')$, $b_1^* \in SO_{2m}^-(q)$ is presented well defined as follows:

$$b_1^* = \begin{pmatrix} D_1^* & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D_1^{*t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

thus, $A_1^* = \langle a_1^* \rangle$, and $B_1^* = \langle b_1^* \rangle$ are the cyclic subgroups of $SO_{2m}^-(q)$ generated by a_1^* and b_1^* with order $q^m + 1$ and $q^{m-1} - 1$, respectively.

Let $C_1^* = \langle b_1^{* \frac{q^{m-1}-1}{q-1}} \rangle$ be the subgroup of order $q-1$ of B_1^* and $B_1'^* = \{gC_1^* \mid g \in B_1^*\}$ be the left coset of C_1^* in B_1^*. Thus, $|B_1'^*| = \frac{q^{m-1}}{q-1}$. Furthermore, the cyclic sets A_1^* and $B_1'^*$ are chosen so that $A_1^* \cap B_1'^* = \{1\}$. Consequently, from Lemma 5, it follows that A_1^* and $B_1'^*$ have MLSs.

Also, let $S_1' = \{W_i' \mid 0 \le i \le q^m\}$ be the classical spread as described in Remark 1. W_i' are $(m-1)$-dimensional totally isotropic subspaces of V for $0 \le i \le q^m$. It's clear that the partial spread S_1' partitions the set of all isotropic points of $P(V)$.

We observe that the group A_1^* is sharply transitive on S_1' with respect to W_1'. Also, B_1^* is isomorphic to the *Singer cyclic subgroup* of $GL_{m-1}(q)$ and $B_1'^*$ is sharply transitive on $P(W_1')$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_1'$. Hence, we have the following lemma from the fact above.

Lemma 7 *Let $A_1, B_1' \subseteq O_{2m}^-(q)$ (resp. $A_1^*, B_1'^* \subseteq SO_{2m}^-(q)$), S_1' be the partial spread, W_1' be the subspace of V and $w_1 = \langle e_1 \rangle$. Then,*

(i) A_1 (resp. A_1^*) is a sharply transitive set on S_1' with respect to W_1'.
(ii) B_1' (resp. $B_1'^*$) is a sharply transitive set on $P(W_1')$ with respect to w_1.

Theorem 1 *Let q be a power of odd primes. Then, the orthogonal group $O_{2m}^-(q)$ has an MLS.*

Proof In case, when $m = 1$, $G = O_2^-(q) \cong D_{q+1}$ which is a dihedral group of order $2(q+1)$. From Lemma 4, $O_2^-(q)$ has an MLS. When $m > 1$, let $A = A_1$, $B = B_1'$, $w_1 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$. Then, from Lemma 3 and Lemma 7, $[A_1, B_1', G_w]$ is an LS for G. The stabilizer G_w is

a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times O^-_{2m-2}(q)$. Now from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $O^-_{2m-2}(q)$ has an MLS. Thus, G_w has an MLS. Also, from Lemma 5, A_1 and B'_1 have MLSs. Hence, using Lemma 6, G has an MLS.

Theorem 2 *Let q be a power of odd primes. Then, the special orthogonal group $SO^-_{2m}(q)$ has an MLS.*

Proof In case, when $m = 1$, $G^* = SO^-_2(q)$ is a solvable group of order $q + 1$. Lemma 4 implies that $SO^-_2(q)$ has an MLS. When $m > 1$, let $A = A^*_1$, $B = B*'_1$, $w_1 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$. Thus, from Lemma 3 and Lemma 7, $[A^*_1, B'^*_1, G^*_w]$ is an LS for $SO^-_{2m}(q)$. The stabilizer G^*_w is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times SO^-_{2m-2}(q)$. Now, from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $SO^-_{2m-2}(q)$ has an MLS. Thus, G^*_w has an MLS. Also, from Lemma 5, A^*_1 and B'^*_1 have MLSs. Hence, using Lemma 6, $SO^-_{2m}(q)$ has an MLS.

5 Construction III: MLSs for $O^+_{2m}(q)$ and $SO^+_{2m}(q)$

Similarly, we construct MLSs for $O^+_{2m}(q)$ and $SO^+_{2m}(q)$. Also, L is the set of all isotropic points of $P(V)$.

We at first consider $O^+_{2m}(q)$. Let $V = F_{q^{2m}}$, α a primitive element of field $F_{q^{2m}}$ and q be a power of odd primes. The corresponding bilinear map and quadratic form are described as $f(x, y) = f(x_1+x_2\beta, y_1+y_2\beta) = tr_{F_{q^m}/F_q}(x_1y_2 + x_2y_1) = \sum_{i=0}^{m-1}(x_1y_2 + x_2y_1)^{q^i}$ and $Q(x) = Q(x_1 + x_2\beta) = tr_{F_{q^m}/F_q}(x_1x_2) = \sum_{i=0}^{m-1}(x_1x_2)^{q^i}$, respectively, where $\beta = \alpha^{q^m-1}$ [23]. We observe that the number of non-zero isotropic points in $P(V)$ with respect to the quadratic space (V, Q) are $(q^m - 1)(q^{m-1} + 1)/(q - 1)$ and the quadratic form Q is of plus type [13,23]. Let G be the group of all isometries of (V, Q), then, $G \cong O^+_{2m}(q)$ and G is a permutation group acting transitively on isotropic points [13,23].

Similarly, we must have to construct a cyclic group A of order $q^{m-1} + 1$ which is sharply transitive on a partial spread S and a cyclic set B of cardinality $(q^m - 1)/(q - 1)$ which is sharply transitive on $P(W)$, the projective subspace of $P(V)$. Furthermore, we also need to use the Lemma 3, Lemma 5 and Lemma 6 to prove the existence of MLSs.

First, we define two special cyclic subgroups of $O^+_{2m}(q)$. Let $a_2 = x_2^{q^{m-1}-1} \in GL_{2m}(q)$ be the matrix corresponding to the linear transformation $T_{\alpha^{q^{m-1}-1}}$, where $x_2 = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x \in GL_{2m-2}(q)$. Let $W_2 = \{e_1, e_2, \cdots, e_m\}$ be an m-dimensional totally isotropic subspace of V, $D_2 \in GL(W_2)$ be a generator of the *Singer cyclic subgroup* of $GL(W_2)$. Then, $b_2 \in GL_{2m}(q)$ is defined as follows: [1,23]

$$b_2 = \begin{pmatrix} D_2 & 0 \\ 0 & (D_2^t)^{-1} \end{pmatrix}$$

Meanwhile, we have that $a_2, b_2 \in G$ [1,23]. Let $A_2 = \langle a_2 \rangle$ and $B_2 = \langle b_2 \rangle$ be the cyclic subgroups of G generated by a_2 and b_2, respectively. Then, A_2 is of order $q^{m-1} + 1$ and B_2 is of order $q^m - 1$.

Let $C_2 = \langle b_2^{\frac{q^m-1}{q-1}} \rangle$ be the subgroup of order $q-1$ of B_2 and $B_2' = \{gC_2 | g \in B_2\}$ be the left coset of C_2 in B_2. Thus, $|B_2'| = \frac{q^m-1}{q-1}$. Furthermore, the cyclic sets A_2 and B_2' are chosen so that $A_2 \cap B_2' = \{1\}$. Then from Lemma 5, it follows that A_2 and B_2' have MLSs.

Now, let $S_2 = \{W_i \mid 0 \leq i \leq q^m\}$ be the classical spread as described in Remark 1. W_i are m-dimensional totally isotropic subspaces of V for $0 \leq i \leq q^m$. Then, the partial spread S_2 clearly partitions the set of all isotropic points of $P(V)$.

Also, we observe that the group A_2 is sharply transitive on S_2 with respect to W_2. Also, it is clear that B_2' is sharply transitive on $P(W_2)$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_2$.

Now, we take account for $G^* = SO_{2m}^+(q)$. Being similar to $O_{2m}^+(q)$, $a_2^* = x_2^{*q^{m-1}-1} \in SO_{2m}^+(q)$ is also the matrix corresponding to the linear transformation $T_{\alpha^{q^{m-1}-1}}$, where $x_2^* = \begin{pmatrix} x^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x^* \in SO_{2m-2}^+(q)$. Let $W_2 = \{e_1, e_2, \cdots, e_m\}$ be an m-dimensional totally isotropic subspace of V and $D_2^* \in GL(W_2)$ be a generator of the *Singer cyclic subgroup* of $GL(W_2)$. Then $b_2^* \in SO_{2m}^+(q)$ is defined as follows: [23]

$$b_2^* = \begin{pmatrix} D_2^* & 0 \\ 0 & D_2^{*t} \end{pmatrix}$$

Hence, $A_2 = \langle a_2 \rangle$ and $B_2 = \langle b_2 \rangle$ are the cyclic subgroups of G with order $q^{m-1} + 1$ and $q^m - 1$, respectively.

Let $C_2^* = \langle b_2^{*\frac{q^m-1}{q-1}} \rangle$ be the subgroup of order $q-1$ of B_2^* and $B_2'^* = \{gC_2^* | g \in B_2^*\}$ be the left coset of C_2^* in B_2^*. Thus, $|B_2'^*| = \frac{q^m-1}{q-1}$. Furthermore, the cyclic sets A_2^* and $B_2'^*$ are chosen so that $A_2^* \cap B_2'^* = \{1\}$. Then from Lemma 5, it follows that A_2^* and $B_2'^*$ have MLSs.

Now, let $S_2 = \{W_i \mid 0 \leq i \leq q^m\}$ the classical spread as described in Remark 1. W_i are m-dimensional totally isotropic subspaces of V for $0 \leq i \leq q^m$. Thus,, the partial spread S_2 clearly partitions the set of all isotropic points of $P(V)$.

Consequently, we observe that the group A_2^* is sharply transitive on S_2 with respect to W_2. Also, it is clear that $B_2'^*$ is sharply transitive on $P(W_2)$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_2$. Hence, we have the following lemma.

Lemma 8 *Let A_2, $B_2' \subseteq O_{2m}^+(q)$ (resp. A_2^*, $B_2'^* \subseteq SO_{2m}^+(q)$), S_2 be the partial spread, W_2 be the subspace of V, and $w_2 = \langle e_1 \rangle$. Then,*

(i) A_2 (resp. A_2^*) is a sharply transitive set on S_2 with respect to W_2.

(ii) B_2' (resp. $B_2'^*$) is a sharply transitive set on $P(W_2)$ with respect to w_2.

Theorem 3 *Let q be a power of odd primes. Then, the orthogonal group $O_{2m}^+(q)$ has an MLS.*

Proof Let $G = O_{2m}^+(q)$. In case,when $m = 1$, $O_2^+(q)$ is of order $2(q-1)$. Then, by using Lemma 4, $O_2^+(q)$ has an MLS. When $m > 1$, let $A = A_2$, $B = B_2'$, $w_2 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$. Hence, from Lemma 5 and Lemma 8, $[A_2, B_2', G_{w_2}]$ is an LS for G. Besides, the stabilizer G_{w_2} is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times O_{2m-2}^+(q)$. Now from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $O_{2m-2}^+(q)$ has an MLS, therefore, $O_{2m}^+(q)$ also has an MLS. Thus, G_{w_2} has an MLS. Also, from Lemma 5, the cyclic sets A_2 and B_2' have MLSs. Therefore, using Lemma 6, G has an MLS.

Theorem 4 *Let q be a power of odd primes. Then, special orthogonal group $SO_{2m}^+(q)$ has an MLS.*

Proof In case, when $n = 1$, $G^* = SO_2^+(q)$ is a solvable group of order $q - 1$. Then by using Lemma 4, $SO_2^+(q)$ has an MLS. When $m > 1$, let $A = A_2^*$, $B = B_2'^*$, $w_2 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$. Hence, from Lemma 5 and Lemma 8, $[A_2^*, B_2'^*, G_{w_2}^*]$ is an LS for G. Besides, the stabilizer $G_{w_2}^*$ is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times SO_{2m-2}^+(q)$. Now from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $SO_{2m-2}^+(q)$ has an MLS, therefore, $G_{w_2}^*$ has an MLS. Also, from Lemma 5, the cyclic sets A_2^* and $B_2'^*$ have MLSs. Finally,, using Lemma 6, $SO_{2m}^+(q)$ has an MLS.

6 Construction IV: MLSs for $PSO_{2m}^{\pm}(q)$ and $P\Omega_{2m}^{\pm}(q)$

In this section, we consider the MLSs for $PSO_{2m}^{\pm}(q)$ and $P\Omega_{2m}^{\pm}(q)$. Being different from $O_{2m}^{\pm}(q)$ and $SO_{2m}^{\pm}(q)$, our technique is based on some canonical homomorphisms.

Theorem 5 *Let q be a power of odd primes. Then, $PSO_{2m}^{\pm}(q)$ has an MLS.*

Proof (1)In case, when $G^* = SO_{2m}^-(q)$ and $G' = PSO_{2m}^-(q)$, let $A = A_1^*$, $B = B_1'^*$, $w_1 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$ as described as Section 3. Suppose $\eta_1 : SO_{2m}^-(q) \to PSO_{2m}^-(q) \cong SO_{2m}^-(q)/Z(SO_{2m}^-(q))$ is the canonical homomorphism onto $PSO_{2m}^-(q)$, and let $\overline{A_1^*} = \eta(A_1^*)$, $\overline{B_1'^*} = \eta(B_1'^*)$ and $\overline{G_{w_1}^*} = \eta(G_{w_1}^*)$, then $[\overline{A_1^*}, \overline{B_1'^*}, \overline{G_{w_1}^*}]$ is the corresponding LS for $PSO_{2m}^-(q)$ from Lemma 2. Also from Section 3, the stabilizer $\overline{G_{w_1}^*}$ is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times PSO_{2m-2}^-(q)$. Thus, using the same induction as used in Theorem 2, we get that $PSO_{2m}^-(q)$ has an MLS.

(2)In case, when $G^* = SO_{2m}^+(q)$ and $G' = PSO_{2m}^+(q)$, let $A = A_2^*$, $B = B_2'^*$, $w_2 = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$ as described as

Section 4. Suppose $\eta_2 : SO_{2m}^+(q) \to PSO_{2m}^+(q) \cong SO_{2m}^+(q)/Z(SO_{2m}^+(q))$ is the canonical homomorphism onto $PSO_{2m}^+(q)$, and let $\overline{A_2^*} = \eta(A_2^*)$, $\overline{B_2^{\prime*}} = \eta(B_2^{\prime*})$ and $\overline{G_{w_2}^*} = \eta(G_{w_2}^*)$, then $[\overline{A_2^*}, \overline{B_2^{\prime*}}, \overline{G_{w_2}^*}]$ is the corresponding LS for $PSO_{2m}^+(q)$ from Lemma 2. Hence, the stabilizer $\overline{G_{w_2}^*}$ is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times PSO_{2m-2}^+(q)$. Thus, using the same induction as used in Theorem 2, we get that $PSO_{2m}^+(q)$ has an MLS.

Theorem 6 *Let q be a power of odd primes. Then, $P\Omega_{2m}^\pm(q)$ has an MLS.*

Proof (1)In case, when $q^m \equiv -1 \bmod 4$, from [13],we have that $PSO_{2m}^\pm(q) = P\Omega_{2m}^\pm(q)$. Therefore, being similar to the case in $PSO_{2m}^\pm(q)$, $[\overline{A_1^*}, \overline{B_1^{\prime*}}, \overline{G_{w_1}^*}]$ is the corresponding LS for $P\Omega_{2m}^-(q)$ and $[\overline{A_2^*}, \overline{B_2^{\prime*}}, \overline{G_{w_2}^*}]$ is the corresponding LS for $P\Omega_{2m}^+(q)$. Meanwhile, as described in Theorem 5, $P\Omega_{2m}^-(q)$ and $P\Omega_{2m}^+(q)$ both have MLSs.

   (2)In case, when $q^m \equiv 1 \bmod 4$, we must consider the reflection in $G^* = SO_n(q)$. For the isotropic 1-space $w = \langle e_1 \rangle$, the reflection $r_w : V \to V$ is defined by $r_w(v) = v - 2\frac{f(v,w)}{f(w,w)}w$ for each $v \in V$. Also, the linear transformation r_w is an element of G_w^*. For$G' = \Omega_{2m}^-(q)$ (resp. $\Omega_{2m}^+(q)$), each element of G_w' is a product of an even number of reflections and G_w' is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times \Omega_{2m-2}^-(q)$ (resp. $GL_1(q) \times \Omega_{2m-2}^+(q)$). Besides, an element x in $SO_{2m}^\pm(q)$ is in $\Omega_{2m}^\pm(q)$ if and only if the rank of $I_{2m} + x$ is even. From the construction of A_1^* (resp. A_2^*) and $B_1^{\prime*}$ (resp. $B_2^{\prime*}$) in Section 3 and Section 4, we get that the ranks of $I_{2m} + x_1^*$ (resp. $I_{2m} + x_2^*$) and $I_{2m} + b_1^*$ (resp. $I_{2m} + b_2^*$) are both even, Thus, A_1^* (resp. A_2^*) $\le G'$, $B_1^{\prime*}$ (resp. $B_2^{\prime*}$) $\subseteq G'$. As described in Theorem 3 and Theorem 4, we have $\Omega_{2m}^-(q)$ (resp. $\Omega_{2m}^+(q)$) has an MLS.

   Furthermore,, let $\theta_1 : \Omega_{2m}^-(q) \to P\Omega_{2m}^-(q)$ and $\theta_2 : \Omega_{2m}^+(q) \to P\Omega_{2m}^+(q)$ be the canonical homomorphisms onto $P\Omega_{2m}^-(q)$ and $P\Omega_{2m}^+(q)$, respectively. Therefore, $[\theta_1(A_1^*), \theta_1(B_1^{\prime*}), \theta_1(G_{w_1}^*)]$ is the corresponding LS for $P\Omega_{2m}^-(q)$ and $[\theta_2(A_2^*), \theta_2(B_2^{\prime*}), \theta_2(G_{w_2}^*)]$ is the corresponding LS for $P\Omega_{2m}^+(q)$. Also, for $P\Omega_{2m}^-(q)$, the stabilizer $\theta_1(G_{w_1}^*)$ is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times P\Omega_{2m-2}^-(q)$; for $P\Omega_{2m}^+(q)$. the stabilizer $\theta_2(G_{w_2}^*)$ is a semi-direct product of a p-group of order q^{2m-2} and $GL_1(q) \times P\Omega_{2m-2}^+(q)$. Thus, using the same induction as used in Theorem 2 and Theorem 4, we get that $P\Omega_{2m}^-(q)$ and $P\Omega_{2m}^+(q)$ both have MLSs.

7 Construction V: MLSs for $O_{2m+1}(q)$, $SO_{2m+1}(q)$, $PSO_{2m+1}(q)$ and $P\Omega_{2m+1}(q)$

In this section, we first construct MLSs for $O_{2m+1}(q)$ and $SO_{2m+1}(q)$. Then we consider the MLSs for $PSO_{2m+1}(q)$ and $P\Omega_{2m+1}(q)$. Here, L is also the set of all isotropic points of $P(V)$.

   Let $V = F_{q^{2m+1}}$ and q be a power of odd primes. The corresponding non-singular alternating bilinear map is $f(x,y) = tr_{F_{q^{2m+1}}/F_q}(ax\overline{y}) = \sum_{i=1}^{2m+1}(ax\overline{y})^{q^i}$,

where $a \in F^*_{q^{2m+1}}$ and $a + \overline{a} = 0$ [23]. Also, $G = O_{2m+1}(q)$ is the isometry group of the inner product space (V, f) and G is a permutation group acting transitively on isotropic points [13,23]. Then we observe that the number of non-zero isotropic points in $P(V)$ with respect to the inner product space (V, f) are $(q^m - 1)(q^m + 1)/(q - 1)$ [13,23].

Similarly, we construct a cyclic group A of order $q^m + 1$, which is sharply transitive on a partial spread S and a cyclic set B of cardinality $q^m - 1/(q-1)$, which is sharply transitive on $P(W)$, the projective subspace of $P(V)$. Then we use the Lemma 3, Lemma 5 and Lemma 6 to prove the existence of MLSs.

Let $a_3 = x_3^{q^m-1} \in GL_{2m+1}(q)$ be the matrix corresponding to the linear transformation $T_{\alpha^{q^m-1}}$, where $x_3 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, $x \in GL_{2m}(q)$. Let $W_3 = \{e_1, e_2, \cdots, e_m\}$ be an m-dimensional totally isotropic subspace of V, $D_3 \in GL(W_3)$ be a generator of the *Singer cyclic subgroup* of $GL(W_3)$. Then $b_3 \in GL_{2m+1}(q)$ defined as follows: [23]

$$b_3 = \begin{pmatrix} D_3 & 0 & 0 \\ 0 & (D_3^t)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From [1,23], we have that $a_3, b_3 \in G$. Let $A_3 = \langle a_3 \rangle$, $B_3 = \langle b_3 \rangle$ be cyclic subgroups of G. Thus, $|A_3| = q^m + 1$ and $|B_3| = q^m - 1$.

Let $C_3 = \langle b_3^{\frac{q^m-1}{q-1}} \rangle$ be the subgroup of order $q - 1$ of B_3, $B_3' = \{gC_3 | g \in B_3\}$ the left coset of C_3 in B_3. Therefore, $|B_3'| = \frac{q^m-1}{q-1}$. Furthermore, the cyclic sets A_3 and B_3' are chosen so that $A_3 \cap B_3' = \{1\}$. Hence, from Lemma 5, it follows that A_3 and B_3' have MLSs.

Now, let $S_3 = \{W_i \mid 0 \leq i \leq q^m\}$ be the classical spread as described in Remark 1, W_i be m-dimensional totally isotropic subspaces of V for $0 \leq i \leq q^m$. Therefore, the partial spread S_3 partitions the set of all isotropic points of $P(V)$.

We observe that the group A_3 is sharply transitive on S_3 with respect to W_3. Also, B_3' is sharply transitive on $P(W_3)$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_3$.

In case, when $G^* = SO_{2m+1}(q)$, let $a_3^* = x_3^{*q^m-1} \in SO_{2m+1}(q)$ be the matrix corresponding to the linear transformation $T_{\alpha^{q^m-1}}$, where $x_3^* = \begin{pmatrix} x^* & 0 \\ 0 & 1 \end{pmatrix}$, $x^* \in SO_{2m}^{\pm}(q)$. Let $W_3 = \{e_1, e_2, \cdots, e_m\}$ be an m-dimensional totally isotropic subspace of V and $D_3^* \in O_m(q) \leq GL(W_3)$. Then, $b_3^* \in SO_{2m+1}(q)$ is defined as follows: [23]

$$b_3 = \begin{pmatrix} D_3^* & 0 & 0 \\ 0 & D_3^{*t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From [1,23], we have that $a_3^*, b_3^* \in G$. Let $A_3^* = \langle a_3^* \rangle$, $B_3^* = \langle b_3^* \rangle$ be cyclic subgroups of G. Then, $|A_3^*| = q^m + 1$ and $|B_3^*| = q^m - 1$.

Let $C_3^* = \langle b_3^{*\frac{q^m-1}{q-1}} \rangle$ be the subgroup of order $q-1$ of B_3^* and $B_3^{'*} = \{gC_3^* | g \in B_3^*\}$ be the left coset of C_3^* in B_3^*. Thus, $|B_3^{'*}| = \frac{q^m-1}{q-1}$. Furthermore, the cyclic sets A_3^* and $B_3^{'*}$ are chosen so that $A_3^* \cap B_3^{'*} = \{1\}$. Then from Lemma 5, it follows that A_3^* and $B_3^{'*}$ have MLSs.

Now, let $S_3 = \{W_i \mid 0 \le i \le q^m\}$ be the classical spread as described in Remark 1, W_i be m-dimensional totally isotropic subspaces of V for $0 \le i \le q^m$. Then, the partial spread S_3 partitions the set of all isotropic points of $P(V)$.

Moreover, we observe that the group A_3^* is sharply transitive on S_3 with respect to W_3. Also, $B_3^{'*}$ is sharply transitive on $P(W_3)$ with respect to $\langle e_1 \rangle$, where $e_1 \in W_3$. Hence, we have the following lemma.

Lemma 9 *Let A_3, $B_3' \subseteq O_{2m+1}(q)$ (resp. A_3^*, $B_3^{'*} \subseteq SO_{2m+1}(q)$), S_3 be the partial spread, W_3 be the subspace of V, and $w = \langle e_1 \rangle$. Then,*

(i) A_3 (resp. A_3^*) is a sharply transitive set on S_3 with respect to W_3.
(ii) B_3' (resp. $B_3^{'*}$) is a sharply transitive set on $P(W_3)$ with respect to w_3.

Theorem 7 *Let q be a power of odd primes. Then, the orthogonal group $O_{2m+1}(q)$ has an MLS.*

Proof Let $G = O_{2m+1}(q)$, $A = A_3$, $B = B_3'$, $w = \langle e_1 \rangle$. When $m = 0$, $O_1(q) \cong C_2$. Lemma 4 implies that $O_1(q)$ has an MLS. When $m \ge 1$, from Lemma 5 and Lemma 9, $[A_3, B_3', G_w]$ is an LS for G. Hence, the stabilizer G_w is a semi-direct product of a p-group of order q^{2m-1} and $GL_1(q) \times O_{2m-1}(q)$. Now from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $O_{2m-1}(q)$ has an MLS, therefore, $O_{2m+1}(q)$ also has an MLS. Thus, G_w has an MLS. Also, from Lemma 5, the cyclic sets A_3 and B_3' have MLSs. Therefore, using Lemma 6, G has an MLS.

Theorem 8 *Let q be a power of odd primes. Then, $SO_{2m+1}(q)$ has an MLS.*

Proof Let $G^* = SO_{2m+1}(q)$, $A = A_3^*$, $B = B_3^{'*}$, $w = \langle e_1 \rangle$. In case, when $m = 0$, $SO_1(q) \cong C_2$. Lemma 4 implies that $SO_1(q)$ has an MLS. In case, when $m \ge 1$, from Lemma 5 and Lemma 9, $[A_3^*, B_3^{'*}, G_w]$ is an LS for G. Then the stabilizer G_w^* is a semi-direct product of a p-group of order q^{2m-1} and $GL_1(q) \times SO_{2m-1}(q)$. Now from Lemma 4, p-groups and $GL_1(q)$ have MLSs. Furthermore, by the induction hypothesis, we assume that $SO_{2m-1}(q)$ has an MLS, therefore, G_w^* has an MLS. Also, from Lemma 5, the cyclic sets A_3^* and $B_3^{'*}$ have MLSs. Therefore, using Lemma 6, $SO_{2m+1}(q)$ has an MLS.

Theorem 9 *Let q be a power of odd primes. Then, $PSO_{2m+1}(q)$ has an MLS.*

Proof In case, when $G^* = SO_{2m+1}(q)$ and $G'^* = PSO_{2m+1}(q)$, let $A = A_3^*$, $B = B_3^{'*}$, $w = \langle e_1 \rangle$, L be the set of all isotropic points of $P(V)$ as described above. Suppose $\eta_3 : SO_{2m+1}(q) \to PSO_{2m+1}(q) \cong SO_{2m+1}(q)/Z(SO_{2m+1}(q))$ is the canonical homomorphism onto $PSO_{2m+1}(q)$, and let $\overline{A_3^*} = \eta(A_3^*)$, $\overline{B_3^*} = \eta(B_3^*)$ and $\overline{G_w} = \eta(G_w)$, then $[\overline{A_3^*}, \overline{B_3^*}, \overline{G_w}]$ is the corresponding LS

for $PSO_{2m+1}(q)$ from Lemma 2. Also from Section 3, the stabilizer $\overline{G_w}$ is a semi-direct product of a p-group of order q^{2m-1} and $GL_1(q) \times PSO_{2m-1}(q)$. Thus, using the same induction as used in Theorem 2, we get that $PSO_{2m+1}(q)$ has an MLS.

Theorem 10 *Let q be a power of odd primes. Then, $P\Omega_{2m+1}(q)$ has an MLS.*

Proof In case, when $n = 2m + 1$, from [13], we observe that $\Omega_{2m-1}(q) \cong Sp_{2m-2}(q)$. Also from [23], we have that $Sp_{2m-2}(q)$ has an MLS. Using the induction, we get that $\Omega_{2m+1}(q)$ has an MLS. Then, we utilize the canonical homomorphism $\theta_3 : \Omega_{2m+1}(q) \to P\Omega_{2m+1}(q)$ for proving that $P\Omega_{2m+1}(q)$ has LS . Consequently, using the same induction as used in Section 5, we get that $P\Omega_{2m+1}(q)$ has an MLS. Here, we omit the corresponding proof.

Conclusion

We utilize partial spreads of totally isotropic subspaces, stabilizers of isotopic 1-subspaces and linear transformations in corresponding vector spaces to construct MLSs for $O_n(q)$, $SO_n(q)$, $PO_n(q)$, $PSO_n(q)$, $\Omega_n(q)$ and $P\Omega_n(q)$ with q as a power of odd primes. Meanwhile, our methods can be used to construct MLSs for other finite simple groups.

Acknowledgements

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