

A Dirac-type theorem for uniform hypergraphs *

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Abstract

Dirac (1952) proved that every connected graph of order $n > 2k + 1$ with minimum degree more than k contains a path of length at least $2k + 1$. Erdős and Gallai (1959) showed that every n -vertex graph G with average degree more than $k - 1$ contains a path of length k . The hypergraph extension of the Erdős-Gallai Theorem have been given by Györi, Katona, Lemons (2016) and Davoodi et al. (2018). Füredi, Kostochka, and Luo (2019) gave a connected version of the Erdős-Gallai Theorem for hypergraphs. In this paper, we give a hypergraph extension of the Dirac's Theorem: Given positive integers n, k and r , let H be a connected n -vertex r -graph with no Berge path of length $2k + 1$. We show that (1) If $k > r \geq 4$ and $n > 2k + 1$, then $\delta_1(H) \leq \binom{k}{r-1}$. Furthermore, the equality holds if and only if $S'_r(n, k) \subseteq H \subseteq S_r(n, k)$ or $H \cong S(sK_{k+1}^{(r)}, 1)$; (2) If $k \geq r \geq 2$ and $n > 2k(r - 1)$, then $\delta_1(H) \leq \binom{k}{r-1}$. The result is also a Dirac-type version of the result of Füredi, Kostochka, and Luo. As an application of (1), we give a better lower bound of the minimum degree than the ones in the Dirac-type results for Berge Hamiltonian cycle given by Bermond et al. (1976) and Clemens et al. (2016), respectively.

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1 Introduction

An r -uniform hypergraph, or r -graph, is a pair $H = (V, E)$, where V is a set of elements called vertices, and E is a collection of subsets of V with uniform size r called edges. In this article, all r -graphs H considered are simple, i.e. H contains no multiple edges. We call $|V|$ the order of H and $|E|$ the size of H , also denoted by $|H|$ or $e(H)$. We write graph for 2-graph for short. Given $S \subseteq V(H)$, the degree of S , denote by $d_H(S)$, is the number of edges of H containing S . The minimum s -degree $\delta_s(H)$ of H is the minimum of $d_H(S)$ over all $S \subseteq V(H)$ of size s . We call $\delta_1(H)$ the minimum degree of H , that is $\delta_1(H) = \min\{d_H(v) : v \in V(H)\}$. Let $N_H(S) = \{T : S \cup T \in E(H)\}$. Given two integers a, b with $a < b$, write $[a, b]$ for the set $\{a, a + 1, \dots, b\}$.

The following two theorems, due to Dirac [6] and Erdős and Gallai [7], are well-known in graph theory.

Theorem 1.1 (Dirac, 1952). *Let G be a connected graph on n vertices with minimum degree $\delta_1(G) > k$. If $n > 2k + 1$, then G contains a path of length at least $2k + 1$.*

Theorem 1.2 (Erdős-Gallai Theorem, 1959). *Let G be a graph on n vertices with $e(G) > \frac{(k-1)n}{2}$ (or $e(G) > \frac{(k-1)(n-1)}{2}$). Then G contains a path of length k (or a cycle of length at least k).*

The type of problems that relate the (d -)minimum degree (resp. the number of edges) in (hyper)graphs to the structure of the (hyper)graphs are often referred to as Dirac-type (resp. Turán-type) problems. These types of problems for hypergraphs have received much attention in recent years, see [15, 18, 19] for the surveys.

A Berge path P of length t in a hypergraph is a collection of $t + 1$ distinct vertices $\{v_0, v_1, \dots, v_t\}$ and t distinct edges $\{e_1, e_2, \dots, e_t\}$ such that $\{v_{i-1}, v_i\} \subseteq e_i$ for $1 \leq i \leq t$. A Berge cycle C of length t in a hypergraph is a collection of t distinct edges $\{e_1, e_2, \dots, e_t\}$ and t distinct vertices $\{v_1, \dots, v_t\}$ such that $\{v_{i-1}, v_i\} \subseteq e_i$ with indices taken modulo t . We call $\{v_0, \dots, v_t\}$ (resp. $\{v_1, \dots, v_t\}$) the key vertices of P (resp. C), denoted by $K(P)$ (resp. $K(C)$), and P is called a Berge path connecting v_0 and v_t . An r -graph H is called connected if for any two vertices $u, v \in V(H)$, there exists a Berge path P connecting u and v .

The hypergraph extension of the Erdős-Gallai Theorem (Theorem 1.2) have been solved completely in two recent papers by Győri, Katona, Lemons [13] and Davoodi et al. [5].

Theorem 1.3 (Theorem 1.3 in [13] and Theorem 3 in [5]). *Let H be an n -vertex r -graph with no Berge path of length k . If $r > k > 3$, then $e(H) \leq \frac{(k-1)n}{r+1}$. If $k \geq r+1 > 3$ then $e(H) \leq \frac{n}{k} \binom{k}{r}$. Furthermore, these bounds are sharp for each k and r for infinitely many n .*

We should mention that the extremal hypergraphs are disconnected in Theorem 1.3. The connected version of the Erdős-Gallai Theorem have also received much attention for graphs (see [1, 16, 17]) and hypergraphs (see [8, 9, 10, 11, 14]). Most of them focus on the size of a largest r -uniform connected n -vertex hypergraph with no Berge cycle of length at least k , among them, the best known result for the size of a largest connected n -vertex r -graph with no Berge path of length k was given by Füredi, Kostochka, and Luo [11]. We first define some constructions of extremal hypergraphs. Let $S_r(n, k)$ be the r -graph on vertex set $A \cup B$ with $|A| = k$ and $|B| = n - k$, and edge set

$$E = \{e : e \subset A \cup B \text{ with } |e| = r \text{ and } |e \cap B| \leq 1\};$$

let $S'_r(n, k)$ be the r -graph obtained from $S_r(n, k)$ by removing all the edges contained in A , i.e.

$$E(S'_r(n, k)) = \{e : e \subset A \cup B \text{ with } |e| = r, |e \cap B| = 1\}.$$

Let K_{k+1}^r be the complete r -graph on $k+1$ vertices and let $S(sK_{k+1}^r, 1)$ be the r -graph on $sk+1$ vertices consisting of s copies of K_{k+1}^r that intersect in exactly one common vertex called the center of $S(sK_{k+1}^r, 1)$. Let H_1 and H_2 be two hypergraphs. We write $H_1 \subseteq H_2$ for H_1 is a subgraph of H_2 and write $H_1 \cong H_2$ for H_1 is isomorphic to H_2 . From the definitions, one can directly check that the following proposition holds.

Proposition 1.4. *Suppose $k \geq r \geq 2$, $s \geq 2$, and $n \geq 2k + 1$. Then*

$$\delta_1(S'_r(n, k)) = \delta_1(S_r(n, k)) = \delta_1(S(sK_{k+1}^r, 1)) = \binom{k}{r-1},$$

$e(S_r(n, k)) = e(S'_r(n, k)) + \binom{k}{r} = (n-k) \binom{k}{r-1} + \binom{k}{r}$, $e(sK_{k+1}^r, 1) = s \binom{k+1}{r}$, and a longest Berge path in $S'_r(n, k)$, $S_r(n, k)$ and $S(sK_{k+1}^r, 1)$ has length $2k$.

Theorem 1.5 (Theorem 18 in [11]). *Let $k > 4r > 12$ and suppose n is larger than a proper function of k and r . If H is an n -vertex connected r -graph with no Berge path of length k , then*

$$e(H) \leq \left(n - \left\lceil \frac{k+1}{2} \right\rceil \right) \binom{\lfloor \frac{k-1}{2} \rfloor}{r-1} + \binom{\lceil \frac{k+1}{2} \rceil}{r}.$$

Furthermore, the $S_r(n, \lfloor \frac{k-1}{2} \rfloor)$ is an extremal hypergraph.

In this paper, we extend Theorem 1.1 to a hypergraph version for Berge paths, which is also a minimum degree version of Theorem 1.5.

Theorem 1.6. *Given positive integers n, k and r , let H be a connected n -vertex r -graph with no Berge path of length $2k + 1$.*

(1) *If $k > r \geq 4$ and $n > 2k + 1$, then $\delta_1(H) \leq \binom{k}{r-1}$. Furthermore, the equality holds if and only if $S'_r(n, k) \subseteq H \subseteq S_r(n, k)$ or $H \cong S(sK_{k+1}^{(r)}, 1)$.*

(2) *If $k \geq r \geq 2$ and $n > 2k(r - 1)$, then $\delta_1(H) \leq \binom{k}{r-1}$.*

Remarks: (1) Proposition 1.4 implies that the minimum degree threshold in Theorem 1.6 is optimal. For $n > 2k + 1 > k > r \geq 4$, we characterize all extremal graphs.

(2) From Proposition 1.4, we know also that the sizes of the extremal graphs in Theorem 1.6 are no more than the one in Theorem 1.5. It is surprise for us that $S(sK_{k+1}^r, 1)$ is also an extremal graph in Theorem 1.6, which has different type of structure from $S_r(n, k)$ and $S'_r(n, k)$.

A Berge cycle C in a hypergraph H is called a Berge Hamiltonian cycle if $K(C) = V(H)$. The Dirac-type results for Berge Hamiltonicity of hypergraphs have been studied in literatures.

Theorem 1.7. (a) (Bermond et al. [2]) *Let H be an r -uniform hypergraph on $n \geq r + 1$ vertices. If $\delta_1(H) \geq \binom{n-2}{r-1} + r - 1$, then H contains a hamiltonian Berge cycle.*

(b) (Clemens et al. [3]) *Let $r \geq 3$ and let H be an r -uniform hypergraph on $n > 2r - 2$ vertices. If $\delta_1(H) \geq \binom{\lceil \frac{n}{2} \rceil - 1}{r-1} + n - 1$, then H contains a hamiltonian Berge cycle.*

(c) (Coulson, Perarnau [4]) *Let H be an r -uniform hypergraph on n vertices, and suppose that $r = o(\sqrt{n})$. If $\delta_1(H) \geq \binom{\lceil \frac{n}{2} \rceil - 1}{r-1}$, then H contains a hamiltonian Berge cycle.*

Clearly, (b) and (c) are improvements of (a), as mentioned by the authors of [3], the term $n - 1$ in (b) can be reduced, (c) eliminated the term $n - 1$ from (b), but which requires that r is much smaller than n . As an application of Theorem 1.6, we reduce the term $n - 1$ to $\lceil \frac{n-1}{2} \rceil$ for $r \geq 4$ and $n \geq 2r + 4$.

Theorem 1.8. *For $r \geq 4$ and $n \geq 2r + 4$, let H be an r -graph with $\delta_1(H) > \binom{\lfloor \frac{n-1}{2} \rfloor}{r-1} + \lceil \frac{n-1}{2} \rceil$, then H contains a hamiltonian Berge cycle.*

The rest of this article is arranged as follows. In Section 2, we give some useful lemmas. We prove Theorems 1.6 and 1.8 in Sections 3 and 4, respectively. Some discussions and remarks will be given in the last section.

2 Preliminaries and lemmas

For an r -graph H , we use $\ell(H)$ (resp. $c(H)$) denote the length of a longest Berge path (resp. Berge cycle) contained in H .

Lemma 2.1. *Let H be a connected n -vertex r -graph. If $\ell(H) + 1 \leq c(H)$ then $n = c(H)$.*

Proof. Since every Berge cycle of length $t + 1$ has a Berge path of length t as its subgraph, $c(H) \leq \ell(H) + 1$, which implies that $\ell(H) + 1 = c(H)$. Choose a longest Berge cycle C in H with $K(C) = \{v_1, v_2, \dots, v_t\}$ and $E(C) = \{e_1, \dots, e_t\}$, i.e. $t = c(H)$. First, we claim that $V(C) = K(C)$. Otherwise, without loss of generality, suppose there is a vertex $v_0 \in e_t$ and $v_0 \notin K(C)$. Then we can find a Berge path P of length t with $K(P) = \{v_0, v_1, \dots, v_t\}$ and $E(P) = \{e_t, e_1, e_2, \dots, e_{t-1}\}$, which contradicts to $\ell(H) = t - 1$. Since $n = |V(H)| \geq |V(C)| = t$, we only need to prove that $n \leq t$. Suppose $n > t$. Since H is connected, there must be a vertex $u \in V(H) \setminus V(C)$ and an edge $e \in E(H) \setminus E(C)$ such that $u \in e$ and $e \cap V(C) \neq \emptyset$. Without loss of generality, assume $v_t \in e$. Then again we can find a Berge path P of length t with $K(P) = \{v_1, \dots, v_t, u\}$ and $E(P) = \{e_1, e_2, \dots, e_{t-1}, e\}$, which is a contradiction. \square

Lemma 2.2. *Let $k \geq r \geq 2$ and $n > 2k(r - 1)$ and let H be a connected n -vertex r -graph with $\delta_1(H) > \binom{k}{r-1}$. If $\ell(H) \leq c(H)$ then $c(H) \geq 2k + 1$.*

Proof. If $\ell(H) \leq c(H) - 1$, by Lemma 2.1, we have $c(H) = n \geq 2k(r - 1) + 1 \geq 2k + 1$. Now assume $\ell(H) = c(H) = t$. Suppose to the contrary that $t \leq 2k$. Pick a Berge cycle C of length t in H with $K(C) = \{v_1, v_2, \dots, v_t\}$ and $E(C) = \{e_1, \dots, e_t\}$.

Since

$$|V(C)| = \left| \bigcup_{i=1}^t e_i \right| \leq \sum_{i=1}^t |e_i \setminus \{v_i, v_{i+1}\}| + |K(C)| \leq 2k(r - 1),$$

where the indices take modulo t . Since H is connected and $n > 2k(r - 1)$, there is a vertex $u \in V(H) \setminus V(C)$ and an edge $e_0 \in E(H) \setminus E(C)$ such that $u \in e_0$ and $e_0 \cap V(C) \neq \emptyset$.

Claim 1. *For any edge e with $u \in e$, $e \setminus \{u\} \subseteq K(C)$.*

Suppose to the contrary that there is a vertex $w \in e \setminus \{u\}$ but $w \notin K(C)$. If $e \cap V(C) \neq \emptyset$, without loss of generality, assume $w \in e \cap e_t$. Then the path P with $K(P) = \{v_1, v_2, \dots, v_t, w, u\}$ and $E(P) = \{e_1, e_2, \dots, e_t, e\}$ is a Berge path of length $t + 1$, a contradiction. Now assume $e \cap V(C) = \emptyset$. Then $e \neq e_0$. Without loss of generality, suppose $v_t \in e_0$. Then the path P with $K(P) = \{v_1, \dots, v_t, u, w\}$ and $E(P) = \{e_1, \dots, e_{t-1}, e_0, e\}$ is a Berge path of length $t + 1$, a contradiction too.

Let $N = \cup_{u \in e} (e \setminus \{u\})$. By Claim 1, $N \subseteq K(C)$. We say $v_i, v_j \in N$ are equivalent if and only if for each $v_p \in \{v_i, v_{i+1}, \dots, v_j\}$ or $v_p \in \{v_j, v_{j+1}, \dots, v_i\}$, $v_p \in N$, where the indices take modulo t . Clearly, we can partition N into equivalent classes by the equivalent relation. Let \mathcal{C} be the set of equivalent classes of N . Let n_i and $n_{\geq i}$ be the numbers of the equivalent classes of size i and at least i , respectively. Since for each class $M = \{v_i, \dots, v_j\} \in \mathcal{C}$ we have $v_{j+1} \notin N$. So $|N| \leq t - |\mathcal{C}| = t - n_1 - n_{\geq 2}$. Also we have $|N| = \sum_{M \in \mathcal{C}} |M| \geq n_1 + 2n_{\geq 2}$. Therefore, $n_1 \leq \frac{t}{2} \leq k$ and $n_{\geq 2} \leq \frac{1}{3}(t - 2n_1) \leq \frac{2}{3}(k - n_1)$.

Now choose $M \in \mathcal{C}$ with $|M| \geq 2$. Assume $M = \{v_i, v_{i+1}, \dots, v_j\}$. Choose e such that $u, v_i \in e$. We claim that $M \subseteq e$ and for any edge $e' \neq e$ with $u \in e'$, $e' \cap M = \emptyset$. In fact, we show that for any $p \in \{i, i+1, \dots, j-1\}$ if $u, v_p \in e$ then for any edge $e' \neq e$ with $u \in e'$, $v_{p+1} \notin e'$. Otherwise, the cycle C with $K(C) = \{v_1, \dots, v_p, u, v_{p+1}, \dots, v_t\}$ and $E(C) = \{e_1, \dots, e_{p-1}, e, e', e_{p+1}, \dots, e_t\}$ is a Berge cycle of length $t + 1$, a contradiction. Therefore, $v_{p+1} \in e$ by the definition of N . This implies the claim. By the claim, an equivalent class M with $|M| \geq 2$ is contained in only one edge e with $u \in e$. So there are at most $n_{\geq 2}$ edges e such that $u \in e$ and e contains at least one equivalent class of size at least 2. Hence,

$$\binom{k}{r-1} < d_H(u) \leq n_{\geq 2} + \binom{n_1}{r-1} \leq \frac{2k}{3} + \binom{n_1}{r-1} - \frac{2n_1}{3}. \quad (1)$$

Let $f(x) = \binom{x}{r-1} - \frac{2x}{3}$. By inequality (1), we have $f(k) < f(n_1)$. But, by the convexity of $f(x)$ on $[0, k]$ and $n_1 \in [0, k]$, we have $f(n_1) \leq \max\{f(0), f(k)\} = f(k)$, a contradiction. \square

In the following, we give some properties of the longest Berge paths in a hypergraph. Let P be a Berge path with $K(P) = \{v_0, v_1, \dots, v_t\}$ and $E(P) = \{e_1, \dots, e_t\}$ in an r -graph H . For $a \in [0, t]$, we define $E_a^O(P) = \{e \in E(H) \setminus E(P) : v_a \in e\}$,

$$K_a(P) = \{v_i \in K(P) \setminus \{v_a\} : \text{there exists } e \in E_a^O(P) \text{ with } v_i \in e\},$$

$\kappa_a(P) = \{i : v_i \in K_a(P)\}$, $E_a^I(P) = \{e_i \in E(P) : v_a \in e_i\}$ and $\varepsilon_a^i(P) = \{i : e_i \in E_a^I(P)\}$. If the path P is clear from the context, write $K_a, \kappa_a, E_a^I, \varepsilon_a^i, E_a^O$ for $K_a(P), \kappa_a(P), E_a^I(P), \varepsilon_a^i(P), E_a^O(P)$ for short. Similarly, for $v \in V(H) \setminus K(P)$, define $E_v^O(P) = \{e \in E(H) \setminus E(P) : v \in e\}$,

$$K_v(P) = \{v_i : \text{there exists } e \in E_v^O(P) \text{ with } v_i \in e\},$$

$\kappa_v(P) = \{i : v_i \in K_v(P)\}$ and $E_v^I(P) = \{e_i \in E(P) : v \in e_i\}$, $\varepsilon_v^i(P) = \{i : e_i \in E_v^I(P)\}$. Also P will be omitted if the path P is clear from the context. A Berge path P is called *extendible* if we can get a longer Berge path from P by removing s edges from P and adding at least $s + 1$ new edges. Otherwise, we call P is *non-extendible*. Clearly, a longest Berge path in a hypergraph is non-extendible. For a family A of sets and a set B , let $A + B = \{a \cup B : a \in A\}$ and $A - B = \{a - B : a \in A\}$. For a set A of integers and an integer b , let $A + b = \{a + b : a \in A\}$ and $A - b = \{a - b : a \in A\}$.

The following is a simple observation.

Observation 2.3. *Let P be a Berge path in r -graph H with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. If $K_v(P) \cap \{v_0, v_t\} \neq \emptyset$ for some $v \notin K(P)$ then P is extendible.*

Corollary 2.4. *Let P be a longest Berge path in r -graph H with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. Then the following statements hold.*

- (a) *For each edge $e \in E_a^O(P)$, we have $e \setminus \{v_a\} \subseteq K_a(P)$ for $a \in \{0, t\}$.*
- (b) *$d_H(v_a) \leq \binom{|K_a|}{r-1} + |E_a^I|$ for $a \in \{0, t\}$. Moreover, the equality holds if and only if $N_H(v_a) = \binom{K_a}{r-1} \cup (E_a^I - \{v_a\})$ and $\binom{K_a}{r-1} \cap (E_a^I - \{v_a\}) = \emptyset$.*

Proof. (a) is clearly true. Otherwise, P will be an extendible Berge path by Observation 2.3.

(b) By (a), for any $e \in E_a^O(P)$, we have $e \setminus \{v_a\} \subseteq K_a(P)$, $a \in \{0, t\}$. So $N_H(v_a) \subseteq \binom{K_a}{r-1} \cup (E_a^I - \{v_a\})$. Thus $d_H(v_a) \leq \binom{|K_a|}{r-1} + |E_a^I|$. The equality holds if and only if $N_H(v_a)$ is the disjoint union of $\binom{K_a}{r-1}$ and $E_a^I - \{v_a\}$. \square

The following proposition plays an important role in the proof of the main theorem.

Proposition 2.5. *Suppose $k \geq r \geq 2$, $t \leq 2k$ and H is a connected n -vertex r -graph with $\ell(H) = t$. Let P be a longest Berge path with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. The following properties hold.*

- (1) *For $n > t + 1$ and $v \notin K(P)$, we have*

(1.1) $\kappa_0, \kappa_t, \kappa_v \subseteq [1, t-1]$, $1 \in \varepsilon_0^i$ and $t \in \varepsilon_t^i$;

(1.2) $(\kappa_0 - 1) \cap \kappa_t = \emptyset$, $(\kappa_0 - 1) \cap \kappa_v = \emptyset$, and $(\kappa_t + 1) \cap \kappa_v = \emptyset$;

(1.3) $(\varepsilon_0^i - 1) \cap \kappa_t = \varepsilon_t^i \cap \kappa_0 = \emptyset$;

(1.4) for $i \in \kappa_0$ and $j \in \kappa_t$, if $i \leq j$ then $E_{i-1}^O \cap E_{j+1}^O = \emptyset$, and if $i > j$ then $E_{i-1}^O \cap E_{j-1}^O = \emptyset$ and $E_{i+1}^O \cap E_{j+1}^O = \emptyset$;

(1.5) for $i \in \kappa_0$ and $j \in \kappa_t$, if $i \leq j$ then $v_{j+1} \notin e_i$ and $v_{i-1} \notin e_{j+1}$; if $i > j$ then $v_{j-1} \notin e_i$ and $v_{j+1} \notin e_{i+1}$.

(1.6) for $a \in \{0, t\}$, $\kappa_a \cap (\kappa_a + 1) \cap \varepsilon_0^i \cap \varepsilon_v^i = \emptyset$.

(2) If $n > 2k(r-1)$ and $\delta_1(H) > \binom{k}{r-1}$, then

(2.1) $(\varepsilon_0^i - 1) \cap \varepsilon_t^i = \emptyset$;

(2.2) $(\varepsilon_0^i - 2) \cap \kappa_t = (\varepsilon_t^i + 1) \cap \kappa_0 = \emptyset$.

Proof. (1.1) If $t \in \kappa_0$, then there is an edge $e \in E(H) \setminus E(P)$ with $v_0, v_t \in e$. Thus $C = \{e_1, e_2, \dots, e_t, e\}$ is a Berge cycle of length $t+1$, which implies that $\ell(H) + 1 \leq c(H)$. By Lemma 2.1, $n = t+1$, a contradiction to $n > t+1$. So $t \notin \kappa_0$. Similarly, $0 \notin \kappa_t$. Therefore, $\kappa_0, \kappa_t \subseteq [1, t-1]$. $\kappa_v \subseteq [1, t-1]$ follows directly from Observation 2.3.

By the definitions of ε_0^i and ε_t^i , we have $1 \in \varepsilon_0^i$ and $t \in \varepsilon_t^i$.

(1.2) Suppose to the contrary that there exists $i \in (\kappa_0 - 1) \cap \kappa_t \subseteq [1, t-2]$. Then there are edges $e \in E_0^O(P)$ with $v_{i+1} \in e$ and $f \in E_t^O(P)$ with $v_i \in f$. We claim that $e \neq f$. Otherwise, we have $t \in \kappa_0$, a contradiction to (1.1). So the cycle C with $E(C) = \{e_1, \dots, e_i, f, e_t, \dots, e_{i+2}, e\}$ and $K(C) = \{v_0, \dots, v_i, v_t, \dots, v_{i+1}\}$ is a Berge cycle of length $t+1$, which again implies that $\ell(H) + 1 \leq c(H)$. By Lemma 2.1, $n = t+1$, a contradiction too.

Now suppose to the contrary that there is an $i \in (\kappa_0 - 1) \cap \kappa_v$. Then there are edges $e \in E_0^O(P)$ with $v_{i+1} \in e$ and $f \in E_v^O(P)$ with $v_i \in f$. Note that $e \neq f$ by Observation 2.3. So P' with $K(P') = \{v, v_i, v_{i-1}, \dots, v_0, v_{i+1}, \dots, v_{2k}\}$ and $E(P) = \{f, e_i, \dots, e_1, e, e_{i+2}, \dots, e_{2k}\}$ is a Berge path of length $2k+1$, a contradiction. Similarly, we can show $\kappa_v \cap (\kappa_t + 1) = \emptyset$.

(1.3) Suppose to the contrary that there is an $i \in (\varepsilon_0^i - 1) \cap \kappa_t \subseteq [1, t-1]$. Then $v_0 \in e_{i+1}$ and there exists an edge $e \in E_t^O(P)$ with $v_t, v_i \in e$. So the cycle C with $E(C) = \{e_1, \dots, e_i, e, e_t, \dots, e_{i+2}, e_{i+1}\}$ and $K(C) = \{v_0, \dots, v_i, v_t, \dots, v_{i+1}\}$ is a Berge cycle of length $t+1$. We get a contradiction with a same reason as in Case (1.1). With similar arguments, we have $\varepsilon_t^i \cap \kappa_0 = \emptyset$.

(1.4) Let $e \in E_0^O(P)$ with $v_i \in e$ and $f \in E_t^O(P)$ with $v_j \in f$. If $i \leq j$, suppose that there exists $g \in E_{i-1}^O \cap E_{j+1}^O$, i.e. $g \notin E(P)$ and $v_{i-1}, v_{j+1} \in g$. We claim that e, f, g are pairwise different. In fact, if $e = f$ then $t \in \kappa_0$, a contradiction to (1.1); if $e = g$ then $j + 1 \in \kappa_0$, a contradiction to $(\kappa_0 - 1) \cap \kappa_t = \emptyset$; if $f = g$ then $i - 1 \in \kappa_t$, again a contradiction to $(\kappa_0 - 1) \cap \kappa_t = \emptyset$. So the cycle C with $E(C) = \{e_1, \dots, e_{i-1}, g, e_{j+2}, \dots, e_t, f, e_j, \dots, e_{i+1}, e\}$ and $K(C) = \{v_0, \dots, v_{i-1}, v_{j+1}, \dots, v_t, v_j, \dots, v_i\}$ is a Berge cycle of length $t + 1$, which again implies that $\ell(H) + 1 \leq c(H)$. By Lemma 2.1, $n = t + 1$, a contradiction too. With similar discussion, we have $E_{i-1}^O \cap E_{j-1}^O = \emptyset$ for $i > j$.

(1.5) It can be proved similarly as (1.4), just replace g by e_i .

(1.6) If not, suppose $j \in \kappa_a \cap (\kappa_a + 1) \cap \varepsilon_0^i \cap \varepsilon_v^i$, then there is edges $e \notin E(P)$ with $v_{j-1}, v_j \in e$ and $v_a, v \in e_j$. Without loss of generality, assume $a = 0$. So P' with $E(P') = \{e_j, e_1, \dots, e_{j-1}, e, e_{j+1}, \dots, e_t\}$ and $K(P') = \{v, v_0, v_1, \dots, v_t\}$ is a Berge path of length $t + 1$, a contradiction.

Now suppose $n > 2k(r - 1)$ and $\delta_1(H) > \binom{k}{r-1}$.

(2.1) Suppose to the contrary that there is an $i \in (\varepsilon_0^i - 1) \cap \varepsilon_t^i \subseteq [1, t - 1]$. Then $v_0 \in e_{i+1}$ and $v_t \in e_i$. So the cycle C with $E(C) = \{e_1, \dots, e_i, e_t, e_{t-1}, \dots, e_{i+1}\}$ and $K(C) = \{v_0, \dots, v_{i-1}, v_t, v_{t-1}, \dots, v_{i+1}\}$ is a Berge cycle of length t , which implies that $\ell(H) \leq c(H) = t$. By Lemma 2.2, $t \geq 2k + 1$, a contradiction.

(2.2) Suppose to the contrary that there is an $i \in (\varepsilon_0^i - 2) \cap \kappa_t \subseteq [1, t - 2]$. Then $v_0 \in e_{i+2}$ and there exists an edge $e \in E_t^O(P)$ with $v_t, v_i \in e$. So the cycle C with $E(C) = \{e_1, \dots, e_i, e, e_t, \dots, e_{i+3}, e_{i+2}\}$ and $K(C) = \{v_0, \dots, v_i, v_t, \dots, v_{i+2}\}$ is a Berge cycle of length t , a contradiction again. Similarly, we have $(\varepsilon_t^i + 1) \cap \kappa_0 = \emptyset$. \square

Remark of Proposition 2.5: By the proof of (1), we directly have (1.1), (1.2) and (1.3) still hold for $n = t + 1$ provided that H does not contain a Berge Hamiltonian cycle.

3 The Proof of Theorem 1.6

3.1 The Proof of Theorems 1.6 (1)

We first prove a weak version.

Theorem 3.1. *Let $k > r \geq 4$ and $n > 2k + 1$. If H is a connected n -vertex r -graph with $\delta_1(H) \geq \binom{k}{r-1}$ then $\ell(H) \geq 2k$. Moreover, if $\ell(H) = 2k$ then for any longest*

Berge path P with $E(P) = \{e_1, e_2, \dots, e_{2k}\}$ and $K(P) = \{v_0, v_1, \dots, v_{2k}\}$, we have $|K_0(P)| = |K_{2k}(P)| = k$.

Proof. Let P be a longest Berge path in H with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. By (b) of Corollary 2.4, $d_H(v_a) \leq \binom{|K_a|}{r-1} + |E_a^I|$ for $a \in \{0, t\}$. We prove by contradiction. Suppose to the contrary that $t = \ell(H) \leq 2k$. By (1.1) and (1.2) of Proposition 2.5,

$$|K_0| + |K_t| = |\kappa_0 - 1| + |\kappa_t| = |(\kappa_0 - 1) \cup \kappa_t| \leq |[0, t - 1]| = t \leq 2k.$$

Similarly, by (1.3) of Proposition 2.5, we get

$$|E_0^I| = |\varepsilon_0^i| = |\varepsilon_0^i - 1| \leq t - |\kappa_t| \leq 2k - |\kappa_t| \quad (2)$$

and $|E_t^I| = |\varepsilon_t^i| \leq t - |\kappa_0| \leq 2k - |\kappa_0|$. Without loss of generality, assume $|K_0| \leq |K_t|$. Then $|K_0| \leq k$ and

$$\begin{aligned} \binom{k}{r-1} &\leq d_H(v_0) \leq \binom{|K_0|}{r-1} + |E_0^I| \\ &\leq \binom{|K_0|}{r-1} + 2k - |K_t| \\ &\leq \binom{|K_0|}{r-1} + 2k - |K_0|. \end{aligned} \quad (3)$$

If $|K_0| = k$ then $|K_t| = k$ and $t = 2k$, we are done. So now we assume $|K_0| < k$.

Case 1. $(k, r) \neq (5, 4)$.

By the convexity of the function $f(x) = \binom{x}{r-1} + 2k - x$ for $x \in [0, k-1]$, we have

$$\begin{aligned} \binom{k}{r-1} &\leq \binom{|K_0|}{r-1} + 2k - |K_0| \\ &\leq \max\{f(0), f(k-1)\} = \binom{k-1}{r-1} + k + 1 \\ &= \binom{k}{r-1} + \left(k + 1 - \binom{k-1}{r-2}\right). \end{aligned} \quad (4)$$

Thus $k + 1 - \binom{k-1}{r-2} \geq 0$. But $k + 1 - \binom{k-1}{r-2} \leq k + 1 - \binom{k-1}{2} = k(5 - k)/2 < 0$ when $(k, r) \neq (5, 4)$, a contradiction.

Case 2. $(k, r) = (5, 4)$.

Then all the equalities hold in the inequalities (2), (3) and (4), that is $d_H(v_a) = \binom{5}{3} = 10$, $t = 2k = 10$, $|K_t| = |K_0| = k - 1 = 4$, and by (b) of Corollary 2.4, $N_H(v_a) =$

$\binom{K_a}{3} \cup (E_a^I - \{v_a\})$ for $a \in \{0, 10\}$. So $|E_a^I| = |E_a^I - \{v_a\}| = d_H(v_a) - |\binom{K_a}{3}| = 6$. By (1.1), (1.2) and (1.3) of Proposition 2.5, $\kappa_0, \kappa_{10} \subseteq [1, 9]$, $(\kappa_0 - 1) \cap \kappa_{10} = \emptyset$, and $(\varepsilon_0^i - 1) \cap \kappa_{10} = \varepsilon_{10}^i \cap \kappa_0 = \emptyset$. Therefore, $(\varepsilon_0^i - 1) \cup \kappa_{10} = (\varepsilon_{10}^i - 1) \cup (\kappa_0 - 1) = [0, 9]$. So $\kappa_0 - 1 \subseteq [0, 9] \setminus \kappa_{10} = \varepsilon_0^i - 1$.

Claim 2. $\kappa_0 \cap (\kappa_0 - 1) = \emptyset$ and $\kappa_0 \subseteq [2, 9]$. Similarly, $\kappa_{10} \cap (\kappa_{10} - 1) = \emptyset$ and $\kappa_{10} \in [1, 8]$.

If $\kappa_0 \cap (\kappa_0 - 1) \neq \emptyset$, say $i \in \kappa_0 \cap (\kappa_0 - 1)$, then $i - 1, i \in \kappa_0 - 1 \subseteq \varepsilon_0^i - 1$. So $i, i + 1 \in \kappa_0 \subseteq \varepsilon_0^i$ and $i + 1 \notin \varepsilon_{10}^i$. Let $\kappa_0 = \{i, i + 1, j_1, j_2\}$. Then $e' = \{v_0, v_i, v_{i+1}, v_{j_1}\} \in E_0^O(P)$. Since $\binom{K_0}{3} \cap (E_0^I - \{v_0\}) = \emptyset$, $e_{i+1} \setminus \{v_0\} \notin \binom{K_0}{3}$. Assume $e_{i+1} = \{v_0, v_i, v_{i+1}, u\}$. Then $u \neq v_{j_1}, v_{j_2}, v_{10}$. If $u \notin K(P)$ then the path P' with $K(P') = \{u, v_0, \dots, v_{10}\}$ and $E(P') = \{e_{i+1}, e_1, \dots, e_i, e', e_{i+2}, \dots, e_{10}\}$ is a Berge path of length 11, a contradiction. So we assume $u \in K(P)$. Then the path P'' with $K(P'') = K(P)$ and $E(P'') = (E(P) \setminus \{e_{i+1}\}) \cup \{e'\}$ is also a Berge path of length 10. But $K_0(P'') = \{v_i, v_{i+1}, v_{j_1}, v_{j_2}, u\}$ and $E_{10}^I(P'') = E_{10}^I(P)$. By (1.3) of Proposition 2.5, $|K_0(P'') \cup E_{10}^I(P'')| = |K_0(P'')| + |E_{10}^I(P'')| = 11$, but $|K_0(P'') \cup E_{10}^I(P'')| \leq |[1, 10]| = 10$, this is a contradiction.

Now suppose $1 \in \kappa_0$. Denote $\kappa_0 = \{1, j_1, j_2, j_3\}$. Then $e' = \{v_0, v_1, v_{j_1}, v_{j_2}\} \in E_0^O(P)$. Assume $e_1 = \{v_0, v_1, u, v\}$. With a similar discussion with the above case, if $e_1 \not\subseteq K(P)$ then we have a Berge path of length 11 by adding e' to P ; if $e_1 \subseteq K(P)$ then we obtain a Berge path Q of length 10 with $|K_0(Q)| > |K_0(P)|$ and $E_{10}^I(Q) = E_{10}^I(P)$ by replacing e_1 by e' . In each case, we get a contradiction. Therefore, $\kappa_0 \cap (\kappa_0 - 1) = \emptyset$ and $\kappa_0 \subseteq [2, 9]$. Similarly, we have $\kappa_{10} \cap (\kappa_{10} - 1) = \emptyset$ and $\kappa_{10} \subseteq [1, 8]$.

By Claim 2, $\kappa_0 - 1, \kappa_{10} \subseteq [1, 8]$ and $\kappa_a - 1$ contains no consecutive integers for $a \in \{0, 1\}$. Since $(\kappa_0 - 1) \cap \kappa_{10} = \emptyset$ and $|\kappa_0| = |\kappa_{10}| = 4$, we have $(\kappa_0 - 1) \cup \kappa_{10} = [1, 8]$. This forces that $\kappa_0 = \kappa_{10} = \{2, 4, 6, 8\}$. So $\varepsilon_0^i = [0, 9] \setminus \kappa_{10} + 1 = \{1, 2, 4, 6, 8, 10\}$ and $\varepsilon_{10}^i = [0, 9] \setminus (\kappa_0 - 1) + 1 = \{1, 3, 5, 7, 9, 10\}$. This implies that $\{v_0, v_1, v_2\} \subseteq e_2$ and $\{v_1, v_{10}\} \subseteq e_1$. Set $e' = \{v_0, v_2, v_4, v_6\}$. Then $e' \in E_0^O(P)$ because of $\binom{K_0}{3} \subseteq N_H(v_0)$. Therefore, the cycle C with $E(C) = \{e', e_3, \dots, e_{10}, e_1, e_2\}$ and $K(C) = \{v_0, v_2, v_3, \dots, v_{10}, v_1\}$ is a Berge cycle of length 11. Since $c(H) \leq \ell(H) + 1 = 11$, we have $c(H) = 11$. By Lemma 2.1, $n = c(H) = 11$, a contradiction to $n > 2k + 1 = 11$. \square

The following theorem characterize the extremal graphs in Theorem 1.6 (1).

Theorem 3.2. For $k \geq r \geq 2$, $n > 2k + 1$ and $d \in \{\binom{k}{r-1}, \binom{k}{r-1} + 1\}$, let H be a connected n -vertex r -graph with $\delta_1(H) \geq d$. If $\ell(H) = 2k$ and for any longest Berge path P in H , we have $|\kappa_0(P)| = |\kappa_{2k}(P)| = k$, then $d = \binom{k}{r-1}$ and either $S'_r(n, k) \subseteq H \subseteq S_r(n, k)$ or $H \cong S(sK_{k+1}^{(r)}, 1)$ with $n = sk + 1$.

Proof. Let P be a longest Berge path in H with $E(P) = \{e_1, \dots, e_{2k}\}$ and $K(P) = \{v_0, v_1, \dots, v_{2k}\}$. By (1.1) and (1.2) of Proposition 2.5, $\kappa_0 - 1 \subseteq [0, 2k - 2]$, $\kappa_{2k} \subseteq [1, 2k - 1]$ and $(\kappa_0 - 1) \cap \kappa_{2k} = \emptyset$. By Theorem 3.1, $|\kappa_0 - 1| = |\kappa_{2k}| = k$. Thus $(\kappa_0 - 1) \cup \kappa_{2k} = [0, 2k - 1]$ and $0 \in \kappa_0 - 1$, $2k - 1 \in \kappa_{2k}$.

Claim 3. $\kappa_v \cap (\kappa_v - 1) = \emptyset$ for any $v \notin K(P)$. Furthermore, $K(P) = V(P)$.

Suppose to the contrary that there is an $i \in \kappa_v \cap (\kappa_v - 1)$ for some $v \notin K(P)$. Then $i, i + 1 \in \kappa_v$. Since $i \in [0, 2k - 1] = (\kappa_0 - 1) \cup \kappa_{2k}$, we have either $i \in \kappa_0 - 1$ or $i \in \kappa_{2k}$. But this is impossible since $\kappa_v \cap (\kappa_0 - 1) = \emptyset$ or $\kappa_v \cap (\kappa_{2k} + 1) = \emptyset$ from (1.2) of Proposition 2.5. Now suppose there is a vertex $v \in V(P) \setminus K(P)$. Suppose $v \in e_{j+1}$. Then $v_j, v_{j+1} \in K_v(P)$, i.e. $j, j + 1 \in \kappa_v(P)$. This is impossible since $\kappa_v \cap (\kappa_v - 1) = \emptyset$.

Case 1. $\max(\kappa_0 - 1) > \min \kappa_{2k}$.

Since $(\kappa_0 - 1) \cup \kappa_{2k} = [0, 2k - 1]$, we can pick $1 \leq i_0 \leq 2k - 3$ such that $i_0 \in \kappa_{2k}$ and $i_0 + 1 \in \kappa_0 - 1$. Thus $i_0 \in \kappa_{2k}$ and $i_0 + 2 \in \kappa_0$. Let $e_0 \in E_0^O(P)$ and $f_0 \in E_{2k}^O(P)$ be the edges such that $v_0, v_{i_0+2} \in e_0$ and $v_{2k}, v_{i_0} \in f_0$.

Since $V(P) = K(P)$, $n > 2k + 1 = |V(P)|$ and H is connected, we have $V(H) \setminus V(P) \neq \emptyset$ and $E(H) \setminus E(P) \neq \emptyset$.

Claim 4. For any $v \in V(H) \setminus V(P)$ and any $e \in E(H)$ with $v \in e$, $e \setminus \{v\} \subseteq V(P)$.

Let $v \in V(H) \setminus V(P)$. We first claim that: (*) there is no Berge path Q of length at least two connecting v and some vertex of P such that $E(Q) \cap E(P) = \emptyset$ and $|K(Q) \cap V(P)| = 1$. Suppose to the contrary that there is such a Berge path Q connecting v and some vertex $v_j \in V(P)$. Then $|E(Q)| \geq 2$. Set $K(Q) = \{x, \dots, v_j\}$ such that $K(Q) \cap V(P) = \{v_j\}$. If $j \leq i_0$, then P' with $K(P') = K(Q) \cup \{v_{j+1}, \dots, v_{i_0}, v_{2k}, \dots, v_{i_0+2}, v_0, \dots, v_{j-1}\}$ and $E(P') = E(Q) \cup \{e_{j+1}, \dots, e_{i_0}, f_0, e_{2k}, \dots, e_{i_0+3}, e_0, e_1, \dots, e_{j-1}\}$ is a Berge path of at least $2k + 1$, a contradiction. If $j = i_0 + 1$, then P' with $K(P') = K(Q) \cup \{v_{i_0+2}, v_0, \dots, v_{i_0}, v_{2k}, \dots, v_{i_0+3}\}$ and $E(P') = E(Q) \cup \{e_{i_0+2}, e_0, e_1, \dots, e_{i_0}, f_0, e_{2k}, \dots, e_{i_0+4}\}$ is a Berge path of length at least $2k + 2$, a contradiction. So assume $j \geq i_0 + 2$. Then P' with $K(P') = K(Q) \cup \{v_{j-1}, \dots, v_{i_0+2}, v_0, \dots, v_{i_0}, v_{2k}, \dots, v_{j+1}\}$ and $E(P') = E(Q) \cup \{e_{j-1}, \dots, e_{i_0+3}, e_0, e_1, \dots, e_{i_0}, f_0, e_{2k}, \dots, e_{j+2}\}$

is a Berge path of length at least $2k + 1$, a contradiction too. $(*)$ holds. The claim $(*)$ also implies that for any edge $e \in E(H)$ with $v \in e$, $e \cap V(P) \neq \emptyset$. Now suppose that there is a vertex $w \in e \setminus \{v\}$ with $w \notin V(P)$. Choose an edge f with $w \in f$ (f exists since $d_H(w) \geq d \geq 2$). So $f \cap V(P) \neq \emptyset$. Assume $v_j \in f \cap V(P)$. Then Q with $K(Q) = \{v, w, v_j\}$ and $E(Q) = \{e, f\}$ is a Berge path of length two connecting v and v_j . Clearly, $E(Q) \cap E(P) = \emptyset$ and $|K(Q) \cap V(P)| = 1$. By $(*)$, this is impossible.

Now choose $v \notin K(P)$. By Claim 4, we have $e \setminus \{v\} \subseteq K_v(P)$ for every e with $v \in e$. Therefore,

$$d_H(v) \leq \left| \binom{K_v(P)}{r-1} \right| = \binom{|\kappa_v(P)|}{r-1}.$$

Since $\kappa_v(P) \cap (\kappa_v(P) - 1) = \emptyset$ and $\kappa_v(P) \subseteq [1, 2k - 1]$, we get $|\kappa_v(P)| \leq k$ and the equality holds if and only if $\kappa_v(P) = \{1, 3, 5, \dots, 2k - 1\}$, which also implies that $\kappa_0 = \kappa_{2k} = \{1, 3, 5, \dots, 2k - 1\}$ by (1.2) of Proposition 2.5. Since $d_H(v) \geq \delta_1(H) \geq \binom{k}{r-1}$, we have $|\kappa_v(P)| = k$. Therefore, $N_H(v) = \binom{K_v(P)}{r-1} = \binom{\{v_1, v_3, \dots, v_{2k-1}\}}{r-1}$ for all $v \in V(H) \setminus V(P)$. For $v_{2i} \in \{v_0, v_2, \dots, v_{2k}\}$, choose $v \in V(H) \setminus V(P)$. Since $N_H(v) = \binom{\{v_1, v_3, \dots, v_{2k-1}\}}{r-1}$, there are two distinct edges $e', e'' \in E(H)$ such that $v_{2i-1}, v \in e'$ and $v, v_{2i+1} \in e''$. So, by replacing v_{2i} with v and e_{2i}, e_{2i+1} with e', e'' in P , we get a new Berge path P' of length $2k$. By the symmetry of v_{2i} and v , we have $N_H(v_{2i}) = \binom{\{v_1, v_3, \dots, v_{2k-1}\}}{r-1}$. Therefore for all $v \in V(H) \setminus \{v_1, v_3, \dots, v_{2k-1}\}$, $N_H(v) = \binom{\{v_1, v_3, \dots, v_{2k-1}\}}{r-1}$. This implies that $S'_r(n, k) \subseteq H \subseteq S_r(n, k)$.

Case 2. For any longest Berge path P in H with $E(P) = \{e_1, \dots, e_{2k}\}$ and $K(P) = \{v_0, v_1, \dots, v_{2k}\}$, $\max(\kappa_0(P) - 1) < \min \kappa_{2k}(P)$.

Fix a longest Berge path P with $E(P) = \{e_1, \dots, e_{2k}\}$ and $K(P) = \{v_0, \dots, v_{2k}\}$ in H . Since $(\kappa_0 - 1) \cap \kappa_{2k} = \emptyset$, $(\kappa_0 - 1) \cup \kappa_{2k} = [0, 2k - 1]$ and $|\kappa_0| = |\kappa_{2k}| = k$, we have $\kappa_0 - 1 = [0, k - 1]$ and $\kappa_{2k} = [k, 2k - 1]$. By Claim 3, we have $\kappa_v \cap (\kappa_v - 1) = \emptyset$ for any $v \notin K(P)$ and $V(P) = K(P)$.

Since H is connected and $n > 2k + 1$, we have $V(H) \setminus V(P) \neq \emptyset$ and $E(H) \setminus E(P) \neq \emptyset$. We claim that for every edge $e \in E_v^O(P)$ with $v \notin V(P)$, if $e \cap V(P) \neq \emptyset$ then $e \cap V(P) = \{v_k\}$. Suppose there is $v_i \in e \cap V(P)$ for some $i \neq k$. Then $i \in \kappa_v(P)$. If $i < k$ then $i \in [0, k - 1] = \kappa_0 - 1$, a contradiction to $\kappa_v \cap (\kappa_0 - 1) = \emptyset$ ((1.2) of Proposition 2.5). Now assume $i > k$. Then $i - 1 \in [k, 2k - 1] = \kappa_{2k}$, a contradiction to $\kappa_v \cap (\kappa_{2k} + 1) = \emptyset$ ((1.2) of Proposition 2.5). The claim follows.

Since $n > 2k + 1$, there exists integer $s \geq 3$ such that $(s - 1)k + 2 \leq n \leq sk + 1$. In the following we will show $H \cong S(sK_{k+1}^{(r)}, 1)$ with v_k as the center vertex by induction on s . For the base case $s = 3$, denote $V(H) = V(P) \cup R = \{v_0, v_1, \dots, v_{2k}\} \cup$

R . So $|R| = n - 2k - 1 \leq k$. And for any $v \in R$, by the above claim, we have $N_H(v) \subseteq \binom{(R \setminus \{v\}) \cup \{v_k\}}{r-1}$. Since $d_H(v) \geq \binom{k}{r-1}$ and $|(R \setminus \{v\}) \cup \{v_k\}| \leq k$, we have $|R| = k$ and $N_H(v) = \binom{(R \setminus \{v\}) \cup \{v_k\}}{r-1}$. To show $H \cong S(3K_{k+1}^r, 1)$, it is sufficient to show that $H[V(P)] \cong S(2K_{k+1}^r, 1)$. For $i \in [0, k-1] = \kappa_0 - 1$, let $e \in E(H)$ with $v_i \in e$, we claim that $e \subseteq \{v_0, \dots, v_k\}$. If not, suppose there is $v_j \in e$ with $j \in [k+1, 2k] = \kappa_{2k} + 1$. If $e \notin E(P)$ then $e \in E_i^O(P) \cap E_j^O(P)$. This is impossible since by (1.4) of Proposition 2.5, for $i+1 \in \kappa_0$, $j-1 \in \kappa_{2k}$ and $i+1 \leq j-1$, we have $E_i^O(P) \cap E_j^O(P) = \emptyset$. Now suppose $e \in E(P)$. Let $e = e_h$ for some $h \in [1, 2k]$. If $h \in [1, k] = \kappa_0$, by (1.5) of Proposition 2.5, $v_{s+1} \notin e_h$ for any $s \in \kappa_{2k}$, a contradiction to $v_j \in e$ since $j-1 \in \kappa_{2k}$. If $h \in [k+1, 2k] = \kappa_{2k} + 1$, then $h-1 \in \kappa_{2k}$. By (1.5) of Proposition 2.5, $v_{s-1} \notin e_h$ for any $s \in \kappa_0$, a contradiction to $v_i \in e$ since $i+1 \in \kappa_0$. With similar discussion, we have for $i \in [k+1, 2k] = \kappa_{2k} + 1$ and all edges $e \in E(H)$ with $v_i \in e$, $e \subseteq \{v_k, \dots, v_{2k}\}$. Therefore, $N_H(v_i) \subseteq \binom{\{v_0, \dots, v_k\} \setminus \{v_i\}}{r-1}$ for $i \in [0, k-1]$, and $N_H(v_i) \subseteq \binom{\{v_k, \dots, v_{2k}\} \setminus \{v_i\}}{r-1}$ for $i \in [k+1, 2k]$. Since $\delta_1(H) \geq \binom{k}{r-1}$, we have all ' \subseteq 's are '='s. So $H[V(P)] \cong S(2K_{k+1}^r, 1)$ with v_k as the center vertex.

Now assume the statement holds for $s \geq 3$. For $s+1$, let $H' = H - \{v_0, \dots, v_{k-1}\}$. Then $|V(H')| \in [(s-1)k+2, sk+1]$ and $\delta_1(H') \geq d$ since $d_{H'}(v_k) > \binom{|\{v_{k+1}, \dots, v_{2k}\}|}{r-1} = \binom{k}{r-1}$ and the degrees of the rest vertices remain unchanged in H' . Since H' is the subgraph of H and $\ell(H) = 2k$, we have $\ell(H') = 2k$ and for any longest Berge path Q in H' with $K(Q) = \{u_0, u_1, \dots, u_{2k}\}$, $\max(\kappa_0(Q) - 1) < \min \kappa_{2k}(Q)$ (in fact, it is easy to show that $u_k = v_k \in K(Q)$ by the connectivity of H'). By induction hypothesis, we have $|V(H')| = sk+1$ and $H' \cong S(sk_{k+1}^{(r)}, 1)$ with v_k as the center, which implies $H \cong S((s+1)K_{k+1}^{(r)}, 1)$ with the center v_k . We are done. \square

Theorem 1.6 (1) follows from Theorems 3.1 and 3.2 immediately. The following corollary of Theorem 3.2 will be used in the proof of Theorem 1.6 (2).

Corollary 3.3. *For $k \geq r \geq 2$ and $n > 2k+1$, let H be a connected n -vertex r -graph with $\delta_1(H) > \binom{k}{r-1}$. If $\ell(H) = 2k$ then for any longest Berge path P in H , $|\kappa_0(P)| = |\kappa_{2k}(P)| = k$ does not hold.*

Now it is ready to prove Theorem 1.6 (2). We only need to cope with the cases $k > r = 3$ and $k = r \geq 3$. The following is a simple observation.

Observation 3.4. *Let A be a set of integers. If $(A-1) \cup \{a_1, \dots, a_s\} = A \cup \{b_1, \dots, b_s\}$ (resp. $(A+1) \cup \{a_1, \dots, a_s\} = A \cup \{b_1, \dots, b_s\}$) with $a_1 < \dots < a_s$ and $b_1 < \dots < b_s$, then $A = \cup_{i=1}^s [b_i + 1, a_i]$ (resp. $A = \cup_{i=1}^s [a_i, b_i - 1]$).*

3.2 Proof of $k > r = 3$.

Theorem 3.5. *Let $k > 3$ and $n > 4k$. If H is a connected n -vertex 3-graph with $\delta_1(H) > \binom{k}{2}$, then $\ell(H) \geq 2k + 1$.*

Proof. Suppose to the contrary that $\ell(H) \leq 2k$. Let P be a longest Berge path in H with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. Then $t \leq 2k$. Moreover, if $t = 2k$ then $|\kappa_0| \neq |\kappa_{2k}|$ by Corollary 3.3. By (2.1) of Proposition 2.5, $|\varepsilon_0^i| + |\varepsilon_t^i| = |(\varepsilon_0^i - 1) \cup \varepsilon_t^i| \leq |[0, t]| \leq 2k + 1$. Without loss of generality, assume $|\varepsilon_0^i| \geq |\varepsilon_t^i|$. So $|\varepsilon_t^i| \leq k$. By Corollary 2.4, we have

$$\binom{k}{2} < d_H(v_a) \leq \binom{|\kappa_a|}{2} + |\varepsilon_a^i| \quad (5)$$

for $a \in \{0, t\}$. This implies that $|\kappa_t| \geq k - 1$. By (1.3) of Proposition 2.5,

$$(\varepsilon_0^i - 1) \cap \kappa_t = \varepsilon_t^i \cap \kappa_0 = \emptyset \text{ and so } |\kappa_a| \leq t - |\varepsilon_{t-a}^i| \text{ for } a \in \{0, t\}. \quad (6)$$

Therefore, we have $|\varepsilon_t^i| \leq |\varepsilon_0^i| \leq t - k + 1 \leq k + 1$ and the equality holds if and only if $t = 2k$, $|\kappa_t| = k - 1$ and $|\varepsilon_t^i| = k$. Again by (5) and $|\varepsilon_0^i| \leq k + 1$, we have $|\kappa_0| \geq k - 1$. By (1.2) of Proposition 2.5,

$$(\kappa_0 - 1) \cap \kappa_t = \emptyset \text{ and } (\kappa_0 - 1) \cup \kappa_t \subseteq [0, t - 1]. \quad (7)$$

If $|\varepsilon_0^i| = k + 1$, then we have $t = 2k$, $|\kappa_t| = k - 1$ and $|\varepsilon_t^i| = k$. By (6), we have $\varepsilon_0^i - 1 = [0, 2k - 1] \setminus \kappa_{2k}$. By (2.2) of Proposition 2.5, $(\varepsilon_0^i - 2) \cap \kappa_{2k} = \emptyset$. So $\varepsilon_0^i - 2 \subseteq [-1, 2k - 1] \setminus \kappa_{2k} = (\varepsilon_0^i - 1) \cup \{-1\}$. By Observation 3.4, $\varepsilon_0^i - 1 = [0, \max(\varepsilon_0^i - 1)]$. Since $|\varepsilon_0^i - 1| = |\varepsilon_0^i| = k + 1$, we have $\varepsilon_0^i = [1, k + 1]$. Combining with $(\varepsilon_0^i - 1) \cup \varepsilon_t^i = [0, t]$ and $(\varepsilon_0^i - 1) \cap \varepsilon_t^i = \emptyset$, we have $\varepsilon_{2k}^i = [k + 1, 2k]$. So $v_0, v_{2k} \in e_{k+1}$. But H is a 3-graph means $e_{k+1} = \{v_0, v_k, v_{k+1}\} = \{v_{2k}, v_k, v_{k+1}\}$, this is impossible.

If $|\varepsilon_0^i| = k$, we claim that $|\varepsilon_t^i| = k$. If not, by (5) again, we have $|\kappa_t| \geq k$. By (6), we get $|\kappa_t| = k$ and $t = 2k$. Again by (6) and (2.2) of Proposition 2.5 and Observation 3.4, we have $\varepsilon_0^i = [1, k]$ and $\kappa_{2k} = [k, 2k - 1]$. By (7), we have $\kappa_0 - 1 \subseteq [0, 2k - 1] \setminus \kappa_{2k} = [0, k - 1]$, i.e. $\kappa_0 \subseteq [1, k] = \varepsilon_0^i$. Since $t = 2k$, $|\kappa_0| \neq |\kappa_t| = k$. Hence $|\kappa_0| = k - 1$. By (5), we have $N_H(v_0) = \binom{K_0}{2} \cup (E_0^I - \{v_0\})$ and $\binom{K_0}{2} \cap (E_0^I - \{v_0\}) = \emptyset$, which implies that $\{v_0, v_s, v_{s+1}\} \in E_0^O(P)$ for some $s, s + 1 \in \kappa_0 \subseteq \varepsilon_0^i$, a contradiction to $\{v_0, v_s, v_{s+1}\} = e_{s+1} \in E(P)$. Therefore, if $|\varepsilon_0^i| = k$ then $|\varepsilon_t^i| = k$. Moreover, the above discussion also implies that the case $|\varepsilon_0^i| = k$, $|\kappa_t| = k$ and $|\kappa_0| = k - 1$ does not happen. Since $|\varepsilon_t^i| = k$, (6) implies $|\kappa_0| \leq k$. If $|\kappa_t| = k$, by (6), we have $t = 2k$.

So $|\kappa_0| \neq |\kappa_{2k}| = k$. This forces $|\kappa_0| = k - 1$. But this case does not happen by the above statement. So we assume $|\kappa_t| = k - 1$. Since $|\varepsilon_0^i| = |\varepsilon_t^i| = k$, by the symmetry of 0 and t , the case $|\varepsilon_t^i| = k$, $|\kappa_0| = k$ and $|\kappa_t| = k - 1$ does not hold too. Thus, we only need to consider the case $|\kappa_0| = |\kappa_{2k}| = k - 1$. By (5), we have

$$N_H(v_a) = \binom{K_a}{2} \cup (E_a^I - \{v_a\}) \text{ and } \binom{K_a}{2} \cap (E_a^I - \{v_a\}) = \emptyset \text{ for } a \in \{0, t\}.$$

By (1.3),(2.2) and (2.1) of Proposition 2.5, we have $\kappa_0, \kappa_0 - 1, \varepsilon_0^i - 1 \subseteq [0, t - 1] \setminus \varepsilon_t^i$. So $|(\kappa_0 - 1) \cap \kappa_0 \cap (\varepsilon_0^i - 1)| \geq |\kappa_0 - 1| + |\kappa_0| + |\varepsilon_0^i - 1| - 2|[0, t - 1] \setminus \varepsilon_t^i| = k - 2 \geq 2$. Since $\kappa_0 \subseteq [1, t - 1]$, $0, t - 1 \notin (\kappa_0 - 1) \cap \kappa_0 \cap (\varepsilon_0^i - 1)$. So we can pick $s \in (\kappa_0 - 1) \cap \kappa_0 \cap (\varepsilon_0^i - 1)$ with $s \in [1, t - 2]$, i.e. $s, s + 1 \in \kappa_0$ and $s + 1 \in \varepsilon_0^i$. Hence $\{v_0, v_s, v_{s+1}\} \in E_0^O(P)$, but $e_{s+1} = \{v_0, v_s, v_{s+1}\} \in E_0^I(P)$, a contradiction.

Now we assume $|\varepsilon_0^i| \leq k - 1$, i.e. $|\varepsilon_t^i| \leq |\varepsilon_0^i| \leq k - 1$. By (5), we have $|\kappa_0|, |\kappa_t| \geq k$. On the other hand, by (7), $2k \leq |\kappa_0| + |\kappa_t| \leq |(\kappa_0 - 1) \cup \kappa_t| \leq |[0, t - 1]| = t \leq 2k$. So we have $t = 2k$ and $|\kappa_0| = |\kappa_t| = k$, this is impossible. \square

3.3 Proof of $k = r \geq 3$.

Theorem 3.6. *For $r \geq 3$ and $n > 2r(r - 1)$, let H be a connected r -graph on n vertices. If $\delta_1(H) > r$ then $\ell(H) \geq 2r + 1$.*

Proof. Suppose to the contrary that $\ell(H) \leq 2r$. Let P be a longest Berge path in H with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. Then $t \leq 2r$, and if $t = 2r$ then $|\kappa_0| \neq |\kappa_{2r}|$ by Corollary 3.3. Let $a_s(P) = d_H(v_s) - |E_s^I(P)|$ for $s \in \{0, t\}$. Then we have the following observations: If $a_s(P) = 0$ then $|\kappa_s| = 0$ and $|E_s^I| \geq r + 1$; if $a_s(P) = 1$ then $|\kappa_s| = r - 1$ and $|E_s^I| \geq r$; and if $a_s(P) \geq 2$ then $|\kappa_s| \geq r$. Without loss of generality, assume $a_0(P) \geq a_t(P)$. Note that the corresponding version of (5)

$$r < d_H(v_a) \leq \binom{|\kappa_a|}{r - 1} + |\varepsilon_a^i|, \quad (8)$$

for $a \in \{0, t\}$, (6) and (7) still hold.

Case 0. $a_0(P) = a_t(P) = 0$.

Then $|E_0^I| + |E_t^I| \geq 2r + 2$. But, by (2.1) of Proposition 2.5, $|E_0^I| + |E_t^I| = |(\varepsilon_0^i - 1) \cup \varepsilon_t^i| \leq t + 1 \leq 2r + 1$, a contradiction.

Claim 5. *If $|\kappa_s| = r - 1$ then $|\varepsilon_{t-s}^i| \neq t - r + 1$ for $s \in \{0, t\}$.*

We give the proof for $s = 0$, the case $s = t$ can be proved similarly. Suppose to the contrary that $|\varepsilon_t^i| = t - r + 1$. Then $|\kappa_0| + |\varepsilon_t^i| = t$. By (6) and (2.2) of Proposition 2.5, $\varepsilon_t^i = [1, t] \setminus \kappa_0$ and $\varepsilon_t^i + 1 \subseteq [1, t + 1] \setminus \kappa_0 = \varepsilon_t^i \cup \{t + 1\}$ i.e. $(\varepsilon_t^i + 1) \cup \{\min \varepsilon_t^i\} = \varepsilon_t^i \cup \{t + 1\}$. By Observation 3.4, $\varepsilon_t^i = [\min \varepsilon_t^i, t]$. Since $|\varepsilon_t^i| = t - r + 1$, we have $\varepsilon_t^i = [r, t]$ and so $\kappa_0 = [1, r - 1]$. Thus $\varepsilon_0^i = [0, t] \setminus \varepsilon_t^i + 1 = [1, r]$, $E_0^O(P) = \{e_0\}$, where $e_0 = \{v_0\} \cup K_0$. We claim that for any e_i with $i \in [1, r - 1]$, $e_i \subseteq K(P)$. Otherwise, suppose there is $v \notin K(P)$ and $i \in [1, r - 1]$ such that $v \in e_i$. Let P_i be obtained from P by replacing e_i by e_0 . Note that $v_{i-1}, v_i \in e_0$. So P_i is a longest Berge path in H with $K(P_i) = K(P)$. Since $i \in \varepsilon_0^i(P)$, $v_0 \in e_i$ but $e_i \notin E(P_i)$. Thus $v_0 \in K_v(P_i)$. So P_i is extendible by Observation 2.3, which is a contradiction. The claim also implies that for each $i \in [1, r - 1]$, there exists a $j_i \in [r, t]$ such that $v_{j_i} \in e_i$. But this is impossible. Otherwise, $v_0 \in e_i$ since $i \in \varepsilon_0^i$. Since $j_i \in \varepsilon_t^i$, we have C with $E(C) = \{e_i, e_{j_i+1}, \dots, e_t, e_{j_i}, \dots, e_{i+1}, e_0, e_{i-1}, \dots, e_1\}$ and $K(C) = \{v_0, v_{j_i}, \dots, v_t, v_{j_i-1}, \dots, v_1\}$ is a Berge cycle of length $t + 1$, which is a contradiction to $n > 2r(r - 1)$ (by Lemma 2.1, we have $n = c(H) = t + 1 \leq 2r + 1$).

Case 1. $a_0(P) = 1$ and $a_t(P) = 0$.

Then $|\kappa_0| = r - 1$, $|E_0^I| \geq r$ and $|\kappa_t| = 0$, $|E_t^I| \geq r + 1$. By $2r + 1 \leq |\varepsilon_0^i| + |\varepsilon_t^i| \leq t + 1 \leq 2r + 1$, we have $|\varepsilon_0^i| = r$, $|\varepsilon_t^i| = r + 1$, and $t = 2r$. By Claim 5, this is impossible.

Case 2. $a_0(P) = 1$ and $a_t(P) = 1$.

Then $|\varepsilon_s^i| \geq r$, $|\kappa_s| = r - 1$ for $s \in \{0, t\}$, and $t \geq 2r - 1$. If $t = 2r - 1$ then $|\varepsilon_t^i| \geq r = t - r + 1$. By Claim 5, we get a contradiction.

Now suppose $t = 2r$. If $|\varepsilon_s^i| \geq r + 1 (= 2r - r + 1)$ for some $s \in \{0, t\}$, again by Claim 5, we have a contradiction. Thus $|\varepsilon_0^i| = |\varepsilon_{2r}^i| = r$. By (2.1) of Proposition 2.5, there exists an integer $z \in [1, 2r - 1]$ such that $[0, 2r] = (\varepsilon_0^i - 1) \cup \varepsilon_{2r}^i \cup \{z\}$. Without loss of generality, we may assume $\max(\varepsilon_0^i - 1) < z$ or $\min \varepsilon_{2r}^i < z$. Otherwise, we have $\max(\varepsilon_0^i - 1) > z$ and $\min \varepsilon_{2r}^i > z$. Then we reverse the order of P , i.e. relabel the vertices v_i by v_{2r-i} for $0 \leq i \leq 2r$ and edges e_j by e_{2r+1-j} for $1 \leq j \leq 2r$, and denote the reversed path by P' . Hence $\varepsilon_0^i(P') = 2r + 1 - \varepsilon_{2r}^i(P)$, $\varepsilon_{2r}^i(P') = 2r + 1 - \varepsilon_0^i(P)$, and $[0, 2r] = (\varepsilon_0^i(P') - 1) \cup \varepsilon_{2r}^i(P') \cup \{z'\}$, where $z' = 2r - z$. So,

$$\max(\varepsilon_0^i(P') - 1) = \max(2r - \varepsilon_{2r}^i(P)) = 2r - \min \varepsilon_{2r}^i(P) < 2r - z = z'$$

and

$$\min \varepsilon_{2r}^i(P') = \min(2r + 1 - \varepsilon_0^i(P)) = 2r - \max(\varepsilon_0^i(P)) < 2r - z = z'$$

Therefore, we can reverse the order of P instead if any.

Claim 6. $\min \varepsilon_{2r}^i < \max(\varepsilon_0^i - 1)$.

If not, then $\varepsilon_{2r}^i = [r + 1, 2r]$ ($z \leq r$) or $[r, 2r] \setminus \{z\}$ (if $z > r$).

If $\varepsilon_{2r}^i = [r + 1, 2r]$, then $\kappa_0 = [1, r] \setminus \{x\}$ for some $1 \leq x \leq r$. Let $e_0 = \{v_0\} \cup K_0$. Then $e_0 \notin E(P)$. Since $\min \varepsilon_{2r}^i = r + 1 > z$, we have $\max(\varepsilon_0^i - 1) < z$, which implies $z = r$, $\varepsilon_0^i - 1 = [0, r - 1]$ (or equivalently, $\varepsilon_0^i = [1, r]$). We claim that for any e_i with $i \in [1, r - 1]$ or $[1, r] \setminus \{x + 1\}$ (if $x < r$), $e_i \subseteq K(P)$. Otherwise, suppose there is $v \notin K(P)$ and $i \in [1, r - 1]$ or $[1, r] \setminus \{x + 1\}$ (if $x < r$) such that $v \in e_i$. Let P_i be obtained from P by replacing e_i by e_0 if $i \neq x$ and let P_i be obtained from P by replacing e_{i+1} by e_0 and reversing the order of e_1, \dots, e_i and their corresponding key vertices if $i = x$. Note that $v_{i-1}, v_i \in e_0$ if $i \neq x$ and $v_0, v_{i+1} \in e_0$ if $i = x$ (since $x < r$). So P_i is a longest Berge path in H with $K(P_i) = K(P)$ (remark: the first element of $K(P_i)$ is v_i for $i = x$). Since $i \in \varepsilon_0^i(P)$, $v_0 \in e_i$ but $e_i \notin E(P_i)$. Thus $v_0 \in K_v(P_i)$ if $i \neq x$ or $v_i \in K_v(P_i)$ if $i = x$. So P_i is extendible by Observation 2.3, which is a contradiction. Next we claim that for each $i \in [1, r - 1]$ or $[1, r] \setminus \{x + 1\}$, $e_i \setminus \{v_0\} \subseteq \{v_1, \dots, v_r\}$. If not, suppose there exists such an i and a $j_i \in [r + 1, 2r]$ such that $v_{j_i} \in e_i$. Since $i \in \varepsilon_0^i$, we have $v_0 \in e_i$. Since $j_i \in \varepsilon_{2r}^i$, we have C with $E(C) = \{e_i, e_{j_i+1}, \dots, e_{2r}, e_{j_i}, \dots, e_{i+1}, e_0, e_{i-1}, \dots, e_1\}$ and $K(C) = \{v_0, v_{j_i}, \dots, v_{2r}, v_{j_i-1}, \dots, v_1\}$ is a Berge cycle of length $2r + 1$, which is a contradiction to $n > 2r(r - 1)$ (by Lemma 2.1, we have $n = c(H) \leq 2r + 1$). Let $A = [1, r - 1]$ or $[1, r] \setminus \{x + 1\}$. Then $|A \cup \{0\}| = r$ and for each $i \in A \cup \{0\}$, we have $e_i \setminus \{v_0\} \subseteq \{v_1, \dots, v_r\}$. Since $\{v_1, \dots, v_r\}$ has exactly r subsets of size $r - 1$, we have $\{e_i : i \in A \cup \{0\}\} = \binom{\{v_1, \dots, v_r\}}{r-1}$. So replacing $\{e_1, \dots, e_r\}$ by $\{e_i : i \in A \cup \{0\}\}$ with suitable order in P , we get another longest Berge path Q with $K(Q) = K(P)$ in H . Clearly, $[r + 1, 2r] \subseteq \varepsilon_{2r}^i(Q)$. So, $\kappa_0(Q) \subseteq [1, r]$. Denote $\{y\} = [1, r] \setminus A$. Then $e_y \notin E(Q)$ and $v_0 \in e_y$. This forces that $e_y = K_0(Q) \cup \{v_0\} \subseteq \{v_0, v_1, \dots, v_r\}$. But this is a contradiction to $\{e_i : i \in A \cup \{0\}\} = \binom{\{v_1, \dots, v_r\}}{r-1}$.

So $\varepsilon_{2r}^i = [r, 2r] \setminus \{z\}$. Thus $\varepsilon_0^i - 1 = [0, r - 1]$. This forces that $\kappa_{2r} \subseteq [r, 2r - 1]$, a contradiction to $\max(\varepsilon_0^i - 1) > \min \kappa_{2r}$. The claim follows.

So ε_{2r}^i (resp. $\varepsilon_0^i - 1$) consists of at least two consecutive intervals. Hence $|\varepsilon_{2r}^i \cup (\varepsilon_{2r}^i + 1)| \geq r + 2$ (resp. $|(\varepsilon_0^i - 1) \cup (\varepsilon_0^i - 2)| \geq r + 2$). By (6) and (2.2) of Proposition 2.5, $|\varepsilon_{2r}^i \cup (\varepsilon_{2r}^i + 1)| = r + 2$ and $\kappa_0 \cup \varepsilon_{2r}^i \cup (\varepsilon_{2r}^i + 1) = [1, 2r + 1]$. Denote $\min \varepsilon_{2r}^i = a$. By Observation 3.4, ε_{2r}^i consists of exactly two consecutive intervals. So we may assume $\varepsilon_{2r}^i = [a, a + p] \cup [r + p + 2, 2r]$ for some integer $p \geq 0$. Thus $\kappa_0 =$

$[1, a - 1] \cup [a + p + 2, r + p + 1]$. Let $e' = K_0 \cup \{v_0\}$. Then $e' \notin E(P)$. Since $|e'| = |\kappa_0 \cup \{0\}| = |[0, a - 1] \cup [a + p + 2, r + p + 1]| = r \geq 3$, there exists $i \in \kappa_0$ such that $v_{i-1}, v_i \in e'$. Now let P' be obtained from P by replacing e_i with e' . Then P' is another longest Berge path with $K(P') = K(P)$ and the same ε_{2r}^i with P . So $\kappa_0(P') = \kappa_0(P)$ too. This implies that $e_i = e'$, a contradiction.

Case 3. $a_0 \geq 2$.

Then $|\kappa_0| \geq r$.

If $a_t = 0$ then $|\varepsilon_t^i| \geq r + 1$. This is impossible, since, by (6), $|\kappa_0| + |\varepsilon_t^i| = |\kappa_0 \cup \varepsilon_t^i| \leq t \leq 2r$.

If $a_t = 1$ then $|\kappa_t| = r - 1$ and $|\varepsilon_t^i| \geq r$. By (6), we have $\varepsilon_t^i \cap \kappa_0 = \emptyset$. So $|\kappa_0| = |\varepsilon_t^i| = r$, $\varepsilon_t^i \cup \{2r + 1\} = [1, 2r + 1] \setminus \kappa_0$ and $t = 2r$. By (2.2) of Proposition 2.5, $(\varepsilon_t^i + 1) \cup \{\min \varepsilon_t^i\} = [1, 2r + 1] \setminus \kappa_0$. By Observation 3.4, $\varepsilon_t^i = [r + 1, 2r]$ and so $\kappa_0 = [1, r]$. By (7), $\kappa_t = [r, 2r - 1] \setminus \{i_0\}$ for some $i_0 \in [r, 2r - 1]$. So $E_t^O = \{e_0\}$, where $e_0 = \{v_t\} \cup K_t(P) = \{v_r, \dots, v_{2r}\} \setminus \{v_{i_0}\}$.

Now we claim that for any $i \in [r + 1, 2r]$, $e_i \subset \{v_r, \dots, v_{2r}\}$. Else, suppose $j \in [r + 1, 2r]$ is a counterexample, i.e., there exists $v \notin \{v_r, \dots, v_{2r}\}$ with $v \in e_j$. If $j \neq i_0 + 1$ and $v \notin K(P)$, then we get a longer Berge path P' of length $2r + 1$ with $E(P') = \{e_1, \dots, e_{j-1}, e_0, e_{2r}, \dots, e_{j+1}, e_j\}$ and $K(P') = \{v_0, \dots, v_{j-1}, v_{2r}, \dots, v_j, v\}$, a contradiction. If $j \neq i_0 + 1$ but $v \in K(P)$, assume $v = v_s$ with $s \in [0, r - 1]$. Since $s + 1 \in \kappa_0 = [1, r]$, there is an edge $e \in E(H) \setminus E(P)$ containing v_0 and v_{s+1} . Hence we get a Berge cycle C of length $2r + 1$ with $K(C) = \{v_0, v_{s+1}, v_{s+2}, \dots, v_{j-1}, v_{2r}, \dots, v_j, v_s, \dots, v_1\}$ and $E(C) = \{e, e_{s+2}, \dots, e_{j-1}, e_0, e_{2r}, \dots, e_j, e_s, \dots, e_1\}$, which is a contradiction by Lemma 2.1. Let

$$A = \left\{ e \in E(H) : e = \{v_{2r}\} \cup X, \text{ where } X \in \binom{\{v_r, \dots, v_{2r-1}\}}{r-1} \right\}.$$

Note that $\varepsilon_{2r}^i = [r + 1, 2r]$. So we have, for all $i \in \{0\} \cup [r + 1, 2r] \setminus \{i_0 + 1\}$, $e_i \setminus \{v_{2r}\} \in \binom{\{v_r, \dots, v_{2r-1}\}}{r-1}$. Since $|\{0\} \cup [r + 1, 2r] \setminus \{i_0 + 1\}| = r = |\binom{\{v_r, \dots, v_{2r-1}\}}{r-1}|$, we get $\{e_0\} \cup E_{2r}^I \setminus \{e_{i_0+1}\} = A$. Note that A induces an almost complete r -graph on $\{v_r, \dots, v_{2r}\}$, i.e. $H[A] \cong K_{r+1}^{(r)} - e$, where $e = \{v_r, \dots, v_{2r-1}\}$. By the symmetry of v_i 's for $i \in [r, 2r - 1]$, it is easy to check that for any order of the vertices in $\{v_r, \dots, v_{2r}\}$, there exists a Berge path of length r corresponding to it. Now suppose $j = i_0 + 1 \in [r + 1, 2r]$. If $v \notin K(P)$, we have a Berge path P' of length $2r + 1$ with $K(P') = \{v_0, \dots, v_{j-1}, v_{2r}, \dots, v_j, v\}$, a contradiction. Similarly, if $v = v_s \in K(P)$ for some $s \in [0, r - 1]$, we get a Berge cycle C of length $2r + 1$ with $K(C) =$

$\{v_0, v_{s+1}, \dots, v_{j-1}, v_{2r}, \dots, v_j, v_s, \dots, v_1\}$, also a contradiction by Lemma 2.1. Therefore, we have the claim. But the claim implies $\{e_0\} \cup E_{2r}^I \subseteq A$, this is impossible since $|\{e_0\} \cup E_{2r}^I| = r + 1 > r = |A|$, a contradiction.

Now assume $a_t \geq 2$. Then $|\kappa_t| \geq r$. By (7), $|\kappa_t| + |\kappa_0| \leq t \leq 2r$. This forces that $t = 2r$ and $|\kappa_t| = |\kappa_0| = r$. But this is impossible by Corollary 3.3.

The proof is completed. \square

4 An application

To prove Theorem 1.8, we first give a lemma.

Lemma 4.1. *Suppose $r \geq 2$ and H is a connected n -vertex r -graph with $\ell(H) = t$. If $\delta_1(H) > \binom{\lfloor \frac{t}{2} \rfloor}{r-1} + \lceil \frac{t}{2} \rceil$, then $n = t + 1$ and H contains a Berge Hamiltonian cycle.*

Proof. Suppose to the contrary that $n > t + 1$ or $n = t + 1$ but H does not contain a Berge Hamiltonian cycle. Let P be a longest Berge path with $E(P) = \{e_1, \dots, e_t\}$ and $K(P) = \{v_0, v_1, \dots, v_t\}$. By the Remark of Proposition 2.5, (1.1), (1.2) and (1.3) hold. Now by (1.1) and (1.2) of Proposition 2.5, we have $(\kappa_0 - 1) \cap \kappa_t = \emptyset$ and $(\kappa_0 - 1) \cup \kappa_t \subseteq [0, t - 1]$, which implies that $|K_0| + |K_t| \leq t$. Similarly by (1.3) of Proposition 2.5, we have $|E_0^I| \leq t - |K_t|$. Without loss of generality, suppose $|K_0| \leq |K_t|$. Then $|K_0| \leq \lfloor \frac{t}{2} \rfloor$. By (b) of Corollary 2.4,

$$d_H(v_0) \leq \binom{|K_0|}{r-1} + |E_0^I| \leq \binom{|K_0|}{r-1} + t - |K_t| \leq \binom{|K_0|}{r-1} + t - |K_0|.$$

By the convexity of the function $f(x) = \binom{x}{r-1} + t - x$ for $x \in [0, \lfloor \frac{t}{2} \rfloor]$, we have

$$d_H(v_0) \leq \max \left\{ f(0), f \left(\left\lfloor \frac{t}{2} \right\rfloor \right) \right\} = \binom{\lfloor \frac{t}{2} \rfloor}{r-1} + \left\lceil \frac{t}{2} \right\rceil < \delta_1(H),$$

a contradiction. \square

Now, it is ready to give the proof of Theorem 1.8.

Proof of Theorem 1.8. First, we claim that H is connected. If not, then there are two distinct vertices $u_1, u_2 \in V(H)$ such that there is no Berge path connecting them. For any $v \in V(H)$, let

$$A_v := \{u \in V(H) \setminus \{v\} : d_2(\{u, v\}) \geq 1\}.$$

Then $d_H(v) \leq \binom{|A_v|}{r-1}$ for any $v \in V(H)$. Clearly, $u_1 \notin A_{u_2}$ and $u_2 \notin A_{u_1}$ and $A_{u_1} \cap A_{u_2} = \emptyset$. Therefore, $|A_{u_1}| + |A_{u_2}| \leq |V(H) \setminus \{u_1, u_2\}| = n - 2$. Without loss of generality, assume $|A_{u_1}| \leq |A_{u_2}|$. Then $|A_{u_1}| \leq \lfloor \frac{n-2}{2} \rfloor$. So $d_H(u_1) \leq \binom{\lfloor \frac{n-2}{2} \rfloor}{r-1} < \delta_1(H)$, a contradiction.

If $n \geq 2r+4$ is even, then there exists an integer $k > r$ so that $n = 2k+2 > 2k+1$. Since $\delta_1(H) > \binom{k}{r-1}$, by Theorem 1.6, we get $\ell(H) \geq 2k+1$. Since a Berge path P of length $2k+1$ contains $2k+2 = n$ key vertices, P is a longest Berge path, i.e. $\ell(H) = 2k+1$. Since $\delta_1(H) > \binom{k}{r-1} + k+1$, by Lemma 4.1, H contains a Berge Hamiltonian cycle.

If $n \geq 2r+5$ is odd, then there exists an integer $k > r$ such that $n = 2k+3 > 2k+1$. Similarly, since $\delta_1(H) > \binom{k+1}{r-1} + k+1 > \binom{k}{r-1} + k+1$ and by Theorem 1.6, we have $\ell(H) \geq 2k+1$. If $\ell(H) = 2k+1$, by Lemma 4.1, we have $n = 2k+1+1 = n-1$, a contradiction. Thus $\ell(H) = 2k+2 = n-1$. By Lemma 4.1, H contains a Berge Hamiltonian cycle. \square

5 Remarks

In this paper, we give the minimum degree threshold for r -uniform hypergraphs on n vertices containing no Berge path of length $2k+1$. Furthermore, for $k > r \geq 4$ and $n > 2k+1$, we characterize the extremal graphs. However, we know nothing about $k < r$, we leave this as a problem.

Recently, Füredi, Kostochka, and Luo [12] gave minimum degree thresholds for non-uniform hypergraphs containing no long Berge paths and cycles. But as we have known there is no Dirac-type minimum degree condition for uniform hypergraphs containing no long cycles so far. It is also an interesting problem to determine the minimum degree condition for uniform hypergraphs containing no long cycles.

References

- [1] P. N. Balister, E. Gyóri, J. Lehel, R. H. Schelp, Connected graphs without long paths, *Discrete Math.*, 308(19) (2008) 4487–4494.
- [2] J.-C. Bermond, A. Germa, M.-C. Heydemann, D. Sotteau, Hypergraphes Hamiltoniens, in *Problèmes combinatoires et théorie des graphes (Colloq. Internat.*

- CNRS, Univ. Orsay, Orsay, 1976). Colloq. Internat. CNRS, vol. 260 (CNRS, Paris, 1978), pp. 39–43
- [3] D. Clemens, J. Ehrenmüller, and Y. Person, A Dirac-type theorem for Hamilton Berge cycles in random hypergraphs, *Electron. Notes Discrete Math.*, 54 (2016), pp. 181–186.
- [4] M. Coulson, G. Perarnau: A Rainbow Dirac’s Theorem, *SIAM J. Discrete Math.* Vol. 34 (2020), No. 3, pp. 1670–1692.
- [5] A. Davoodi, E. Győri, A. Methuku, C. Tompkins, An Erdős-Gallai type theorem for uniform hypergraphs, *European J. Combin.* 69 (2018) 159–162.
- [6] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (3) 2 (1952) 69–81.
- [7] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* 10 (1959) 337–356.
- [8] B. Ergemlidze, E. Gyori, A. Methuku, N. Salia, C. Tompkins and O. Zamora. Avoiding long Berge cycles, the missing cases $k = r + 1$ and $k = r + 2$, (2018) arXiv:1808.07687
- [9] Z. Füredi, A. Kostochka, R. Luo: Avoiding long Berge cycles. *J. Comb. Theory, Ser. B* 137 (2019): 55–64
- [10] Z. Füredi, A. Kostochka, and R. Luo, Avoiding long Berge cycles II, exact bounds for all n , (2018), arXiv:1807.06119.
- [11] Z. Füredi, A. Kostochka, R. Luo: On 2-connected hypergraphs with no long cycles. *Electr. J. Comb.* 26(4) (2019): P4.31.
- [12] Z. Füredi, A. Kostochka, R. Luo: Berge cycles in non-uniform hypergraphs. *Electr. J. Comb.* 27(3) (2020), P3.9.
- [13] E. Győri, G. Y. Katona, N. Lemons, Hypergraph extensions of the Erdős-Gallai theorem, *European J. Combin.* 58 (2016) 238–246.
- [14] E. Győri, A. Methuku, N. Salia, C. Tompkins, M. Vizer: On the maximum size of connected hypergraphs without a path of given length. *Discrete Math.* 341(9) (2018): 2602–2605.

- [15] P. Keevash, Hypergraph Turán problems, in: Surveys in Combinatorics 2011, in: London Math. Soc. Lecture Note Ser., vol.392, Cambridge Univ. Press, Cambridge, 2011, pp.83–139.
- [16] G. N. Kopylov, Maximal paths and cycles in a graph, Dokl. Akad. Nauk SSSR, 234(1) (1977) 19–21.
- [17] R. Luo, The maximum number of cliques in graphs without long cycles, J. Combin. Theory Ser.B, 128(2018) 219–226.
- [18] V. Rödl, A. Ruciński, Dirac-type questions for hypergraphs a survey (or more problems for endre to solve). An Irregular Mind. Bolyai Soc. Math. Stud. 21, 561–590 (2010).
- [19] Y. Zhao, Recent advances on Dirac-type problems for hypergraphs, A. Beveridge et al. (eds.), Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications 159, 145–165 (2016).