Computing Total Edge Irregularity Strength for Heptagonal Snake Graph and Related Graphs

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Computing Total Edge Irregularity Strength for Heptagonal Snake Graph and Related Graphs

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Abstract: A labeling of edges and vertices of a simple graph $G(V,E)$ by a mapping $\text{Ŧ}: V(G) \cup E(G) \rightarrow \{1,2,3,\dots,\text{Ћ}\}$ provided that any two pair of edges have distinct weights is called an edge irregular total Ћ-labeling. If Ћ is minimum and G admits an edge irregular total Ћ -labelling, then Ћ is called the total edge irregularity strength (TEIS) and denoted by $tes(G)$. In this paper, the definitions of the heptagonal snake graph $\mathrm{HPS_n}$,the double heptagonal snake graph $D(\mathrm{HPS_n})$ and an l −multiple heptagonal snake graph $L(\mathrm{HPS_n})$ have been introduced. The exact value of TEISs for the new family has also been investigated.

Keywords: Irregular labelling; Total edge irregularity strength; Edge irregular total labeling; Heptagonal snake graph.

2010 Mathematics subject classification: 05C78.

1. Introduction

In graph theory, graph labeling is an assignment of labels or weights to the vertices and/or edges of a graph. Graph labeling plays an important role in many fields such as computer science, coding theory, astronomy and physics. For a connected, simple and undirected graph $G(V,E)$, an edge I irregular I total Ћ-labeling $\text{Ŧ}: V(G) \cup E(G) \rightarrow \{1,2,3,\dots,\text{Ћ}\}$ has been defined in [1] as a labeling of its vertices and edges such that for any pair of edges pq and p^*q^* in a graph G their weights are distinct, i.e.$w_{\text{Ŧ}}(pq) \neq w_{\text{Ŧ}}(p^*q^*)$ where $w_{\text{Ŧ}}(pq) = \text{Ŧ}(pq) + \text{Ŧ}(p) + \text{Ŧ}(q)$. A minimum Ћ is a total edge irregularity strength of a graph G when G admits an edge irregular total Ћ -labelling. Also, the authors in [1] introduced an inequality that gives bounds of TEIS of a graph G

$$tes(G) \geq max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta G+1}{2}\right\rceil\right\} \qquad (1).$$

In [2], Ivanĉo and Jendroî has deduced TEIS for a tree. They also introduced the following conjecture

Conjecture 1. Let G be a graph different from K_5,then

$$tes(G) = max\left\{\left\lceil\frac{\Delta G+1}{2}\right\rceil, \left\lceil\frac{|E(G)|+2}{3}\right\rceil\right\}.$$

Since then, the conjecture has been verified for centralized uniform theta graphs in Putra et al. [3], for a polar grid graph in Salama [4], for hexagonal grid graphs in Al-Mushayt et al. [5], for Disjoint Union of Wheel Graphs in Jeyanthi [6], for strong product of two paths in Ahmad et al.[7], for subdivision of star S_n in Siddiqui [8], for generalized prism in Bača and Siddiqui [9], for helm and sun graphs in Ahmad et al.[10], for the corona product of paths with some graphs in Salman and Baskoro [11], for a wheel graph, a fan graph, a triangular Book graph and a friendship graph in Tilukay et al. [12], for a categorical product of two paths in Ahmad and Bača [13], for subdivision of star in Hinding et al. [14], for series parallel graphs in Rajasingh and Arockiamary [15], for categorical product of two cycles in Ahmad et al. [16].

In the current paper, the definitions of the heptagonal snake graph $\mathrm{HPS_n}$,the double heptagonal snake graph $D(\mathrm{HPS_n})$ and an l −multiple heptagonal snake graph $L(\mathrm{HPS_n})$ have been introduced. Moreover, the exact value of TEISs for a heptagonal snake graph, a double heptagonal snake graph and an l −multiple heptagonal snake graph has been investigated.

2. Main results:

In this section, we introduce the definition of the heptagonal snake graph $\mathrm{HPS_n}$ and also, we deduce the exact value of TEIS for it.

2.1 Computing TEIS of heptagonal snake graph HPS_n:

Definition1. A heptagonal snake graph HPS_n is a graph with $6n+1$ vertices and $7n$ edges obtained by interchanging every edge of a path $P_n$by a cycle graph C_7, as shown in Figure (1).

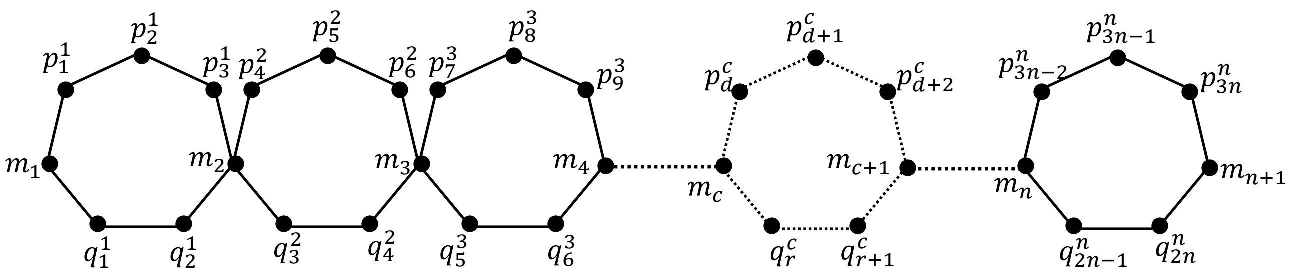

Figure (1) A heptagonal snake graph HPS_n

Therom1. If HPS_n is a heptagonal snake graph with $6n+1$ vertices, then

$$tes(HPS_n) = \left\lceil \frac{7n+2}{3} \right\rceil.$$

Proof: Since $HPS_n$is a heptagonal snake graph with maximum degree $\Delta(HPS_n) = 4$ and size $|E(HPS_n)| = 7n$, I thus (1) becomes

$$tes(HPS_n)) \geq \left\lceil \frac{7n+2}{3} \right\rceil.$$

To prove the equality by showing the existence of an edge irregular I total Ћ −labeling for HPS_n where Ћ $= \left\lceil \frac{7n+2}{3} \right\rceil$, we assume that a map Ŧ: $V(HPS_n) \cup E(HPS_n) \to \{1,2,\dots,$ Ћ $\}$ is a I total Ћ −labeling and Ћ $= \left\lceil \frac{7n+2}{3} \right\rceil$. The total Ћ −labeling Ŧ is defined for HPS_n for three cases as follows:

Case 1: Ћ $\equiv 0 (mod\ 3)$,$1 \leq d \leq 3n$ and $1 \leq r \leq 2n$
Ŧ is defined as:

$$\text{Ŧ}(m_c) = \begin{cases} 1 & for\ c = 1 \\ 3c - 3 & for\ \ 2 \le c \le \frac{\text{Ћ}}{3} \\ \text{Ћ} & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n + 1 \end{cases},$$

$$\text{Ŧ}(p_d^c) = \begin{cases} d & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ \text{Ћ} & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(q_r^c) = \begin{cases} r + c - 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ \text{Ћ} & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(m_c p_d^c) = \begin{cases} 1 & for\ c = 1 \\ c + 1 & for\ \ 2 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} - 4 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(m_c q_r^c) = \begin{cases} 2 & for\ c = 1 \\ c + 2 & for\ \ 2 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} - 3 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(p_{d+2}^c m_{c+1}) = \begin{cases} c + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} + 2 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(q_{r+1}^c m_{c+1}) = \begin{cases} c + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} + 1 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(p_d^c p_{d+1}^c) = \begin{cases} c + 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} - 2 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(p_{d+1}^c p_{d+2}^c) = \begin{cases} c + 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$\text{Ŧ}(q_r^c q_{r+1}^c) = \begin{cases} c + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2\text{Ћ} - 1 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases}$$

Obviously, all edges and vertices labels are at most $\text{Ћ} = \left\lceil \frac{7n+2}{3} \right\rceil$. Now we introduce the edges' weights of HPS_n as follows:

$$w_{\text{Ŧ}}(m_c p_d^c) = \begin{cases} 3 & for\ c = 1 \\ 4c + d - 2 & for\ \ 2 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 4 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(m_c q_r^c) = \begin{cases} 4 & for\ c = 1 \\ 5c + r - 2 & for\ \ 2 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 3 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(p_{d+2}^c m_{c+1}) = \begin{cases} 4c + d + 4 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c + 2 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(q_{r+1}^c m_{c+1}) = \begin{cases} 5c + r + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c + 1 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(p_d^c p_{d+1}^c) = \begin{cases} 2d + c + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 2 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(p_{d+1}^c p_{d+2}^c) = \begin{cases} 2d + c + 4 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(q_r^c q_{r+1}^c) = \begin{cases} 3c + 2r + 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{3} \\ 7c - 1 & for\ \frac{\text{Ћ}}{3} + 1 \le c \le n \end{cases}$$

Case 2: $\text{Ћ} \equiv 1(mod\ 3)$,$1 \le d \le 3n$ and $1 \le r \le 2n$

Ƭ is defined as in case 1 but with some modifications in conditions. We put $\left\lceil\frac{\text{Ћ}}{3}\right\rceil$ instead of $\frac{\text{Ћ}}{3}$ in the definition of $m_c, m_c p_d^c$ and $m_c q_r^c$. In the others we put $1 \le c \le \left\lceil\frac{\text{Ћ}}{3}\right\rceil - 1$ instead of $1 \le c \le \frac{\text{Ћ}}{3}$ and $\left\lceil\frac{\text{Ћ}}{3}\right\rceil \le c \le n$ instead of $\frac{\text{Ћ}}{3} + 1 \le c \le n$. Obviously, all edges and vertices labels are at most $\text{Ћ} = \left\lceil\frac{7n+2}{3}\right\rceil$. The edges' weights for $m_c p_d^c$ and $m_c q_r^c$ are given by:

$$w_{\text{Ƭ}}(m_c p_d^c) = \begin{cases} 3 & for\ c = 1 \\ 4c + d - 2 & for\ \ 2 \le c \le \left\lceil\frac{\text{Ћ}}{3}\right\rceil - 1 \\ 4\left\lceil\frac{\text{Ћ}}{3}\right\rceil + \text{Ћ} - 2 & for\ \ c = \left\lceil\frac{\text{Ћ}}{3}\right\rceil \\ 7c - 4 & for\ \left\lceil\frac{\text{Ћ}}{3}\right\rceil + 1 \le c \le n \end{cases},$$

$$w_{\text{Ƭ}}(m_c q_r^c) = \begin{cases} 4 & for\ c = 1 \\ 5c + r - 2 & for\ \ 2 \le c \le \left\lceil\frac{\text{Ћ}}{3}\right\rceil - 1 \\ 4\left\lceil\frac{\text{Ћ}}{3}\right\rceil + \text{Ћ} - 1 & for\ \ c = \left\lceil\frac{\text{Ћ}}{3}\right\rceil \\ 7c - 3 & for\ \left\lceil\frac{\text{Ћ}}{3}\right\rceil + 1 \le c \le n \end{cases},$$

For other edges the weights are like in case 1, but we put $1 \le c \le \left\lceil\frac{\text{Ћ}}{3}\right\rceil - 1$ instead of $1 \le c \le \frac{\text{Ћ}}{3}$ and $\left\lceil\frac{\text{Ћ}}{3}\right\rceil \le c \le n$ instead of $\frac{\text{Ћ}}{3} + 1 \le c \le n$.

Case 3: $\text{Ћ} \equiv 2(mod\ 3)$,$1 \le d \le 3n$ and $1 \le r \le 2n$

Ƭ is defined as:

$$Ŧ(p_d^c)=\begin{cases} d & for\ 1\le c\le \left\lceil \frac{Ћ}{3}\right\rceil -1 \\ \left.\begin{matrix} d & for\ d,d+1 \\ Ћ & for\ \ d+2 \end{matrix}\right\} & c=\left\lceil \frac{Ћ}{3}\right\rceil \\ Ћ & for\ \left\lceil \frac{Ћ}{3}\right\rceil +1\le c\le n \end{cases}.$$

For $m_c, q_r^c, m_c p_d^c\ \ m_c q_r^c, p_d^c p_{d+1}^c$ and $q_r^c q_{r+1}^c$ we find that $Ŧ$ is defined as in case 1 but we put $\left\lceil \frac{Ћ}{3}\right\rceil$ instead of $\frac{Ћ}{3}$,for the others we put $1\le c\le \left\lceil \frac{Ћ}{3}\right\rceil -1$ instead of $1\le c\le \frac{Ћ}{3}$ and $\left\lceil \frac{Ћ}{3}\right\rceil \le c\le n$ instead of $\frac{Ћ}{3}+1\le c\le n$. Clearly, all edges and vertices labels are at most $Ћ=\left\lceil \frac{7n+2}{3}\right\rceil$. Also, the weights for all edges are given as in case 1 such that for $m_c p_d^c$,$m_c q_r^c\ p_d^c p_{d+1}^c$ and $q_r^c q_{r+1}^c$ but with $\left\lceil \frac{Ћ}{3}\right\rceil$ instead of $\frac{Ћ}{3}$ and $\left\lceil \frac{Ћ}{3}\right\rceil$. For others we put $1\le c\le \left\lceil \frac{Ћ}{3}\right\rceil -1$ instead of $1\le c\le \frac{Ћ}{3}$ and $\left\lceil \frac{Ћ}{3}\right\rceil \le c\le n$ instead of $\frac{Ћ}{3}+1\le c\le n$.
Obviously, for all cases the edge's weights are different. Hence, $Ŧ$ is I an edge I irregular itotal $Ћ$ −labeling. Thus

$$tes(HPS_n)=\left\lceil \frac{7n+2}{3}\right\rceil.$$

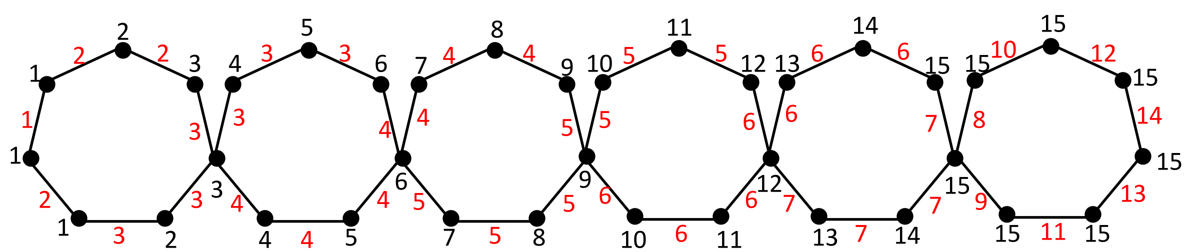

Figure (2) $tes(HPS_6)=15$ for $n=6$

Figure (2) is an illustration of Theorem 1, where all edge vertex and vertex labels are at most $Ћ=15$ and any two different edges have distinct weights.

2.2 Computing TEIS of heptagonal snake graph $D(HPS_n)$:

Definition2. A double heptagonal snake graph $D(HPS_n)$ consists of two heptagonal snake graphs that have a common path P_n, as shown in Figure (3).

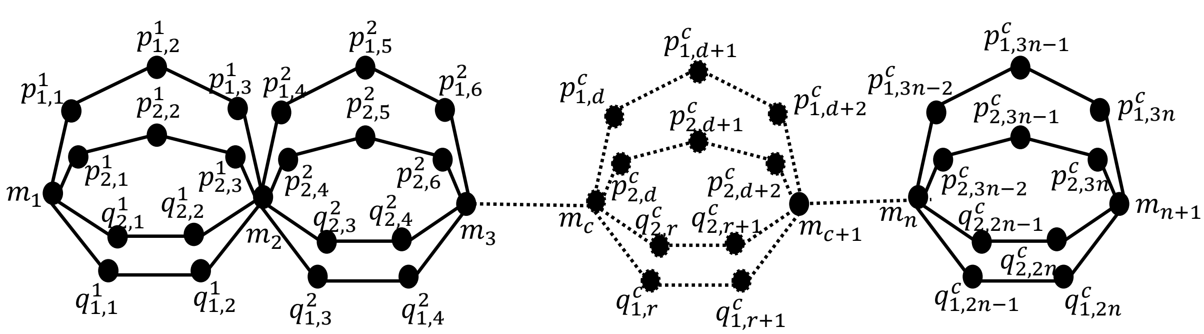

Figure (3) A double heptagonal snake graph $D(HPS_n)$

Therom2. Let $D(HPS_n)$ be a double heptagonal snake graph on n+1 vertices, $n \geq 2$. Then

$$tes(D(HPS_n)) = \left\lceil \frac{14n+2}{3} \right\rceil.$$

Proof: Since $|E(D(HPS_n)| = 14n$ and the maximum degree of a double heptagonal snake graph is given by $\Delta(D(HPS_n)) = 8$, I then from (1) we have

$$tes(D(HPS_n)) \geq \left\lceil \frac{14n+2}{3} \right\rceil.$$

To verify equality by showing the existence of an edge irregular I total Ћ −labeling for $D(HPS_n)$ with $Ћ = \left\lceil \frac{14n+2}{3} \right\rceil$, we assume that a map $Ŧ: V(D(HPS_n)) \cup E(D(HPS_n)) \rightarrow \{1,2,\dots,Ћ\}$ is a I total Ћ −labeling and define the total Ћ −labeling Ŧ for $D(HPS_n)$ in the following cases as:

Case 1: $Ћ \equiv 0 (mod\ 6)$,$1 \leq d \leq 3n$ and $1 \leq r \leq 2n$.

Ŧ is defined as:

$$Ŧ(m_c) = \begin{cases} 1 & for\ c = 1 \\ 6c-6 & for\ 2 \leq c \leq \frac{Ћ}{6} \\ Ћ & for\ \frac{Ћ}{6}+1 \leq c \leq n+1 \end{cases},$$

$$Ŧ(p_{1,d}^c) = Ŧ(p_{2,d}^c) = \begin{cases} 2d-1 & for\ 1 \leq c \leq \frac{Ћ}{6} \\ Ћ & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(q_{1,r}^c) = Ŧ(q_{2,r}^c) = \begin{cases} 2r+2c-2 & for\ 1 \leq c \leq \frac{Ћ}{6} \\ Ћ & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(m_c p_{1,d}^c) = \begin{cases} 1 & for\ c = 1 \\ 2c & for\ 2 \leq c \leq \frac{Ћ}{6} \\ 14c-2Ћ-11 & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(m_c p_{2,d}^c) = \begin{cases} 2 & for\ c = 1 \\ 2c+1 & for\ 2 \leq c \leq \frac{Ћ}{6} \\ 14c-2Ћ-10 & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(m_c q_{1,r}^c) = \begin{cases} 2 & for\ c = 1 \\ 2c+1 & for\ 2 \leq c \leq \frac{Ћ}{6} \\ 14c-2Ћ-9 & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(m_c q_{2,r}^c) = \begin{cases} 3 & for\ c = 1 \\ 2c+2 & for\ 2 \leq c \leq \frac{Ћ}{6} \\ 14c-2Ћ-8 & for\ \frac{Ћ}{6}+1 \leq c \leq n \end{cases},$$

$$Ŧ(p_{1,d+2}^{c}m_{c+1}) = \begin{cases} 2c+2 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ+1 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(p_{2,d+2}^{c}m_{c+1}) = \begin{cases} 2c+3 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ+2 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(q_{1,r+1}^{c}m_{c+1}) = \begin{cases} 2c+1 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-1 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(q_{2,r+1}^{c}m_{c+1}) = \begin{cases} 2c+2 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(p_{1,d}^{c}p_{1,d+1}^{c}) = \begin{cases} 2c+1 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-7 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(p_{1,d+1}^{c}p_{1,d+2}^{c}) = \begin{cases} 2c+1 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-3 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(p_{2,d}^{c}p_{2,d+1}^{c}) = \begin{cases} 2c+2 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-6 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(p_{2,d+1}^{c}p_{2,d+2}^{c}) = \begin{cases} 2c+2 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-2 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(q_{1,r}^{c}q_{1,r+1}^{c}) = \begin{cases} 2c+1 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-5 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$Ŧ(q_{2,r}^{c}q_{2,r+1}^{c}) = \begin{cases} 2c+2 & for\ 1 \le c \le \frac{Ћ}{6} \\ 14c-2Ћ-4 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

Obviously, all edges and vertices labels are at most $Ћ = \left\lceil \frac{14n+2}{3} \right\rceil$. Now we introduce the edges' weights of $D(HPS_n)$ as follows:

$$w_{Ŧ}(m_c p_{1,d}^{c}) = \begin{cases} 3 & for\ c = 1 \\ 8c+2d-7 & for\ 2 \le c \le \frac{Ћ}{6} \\ 14c-11 & for\ \frac{Ћ}{6}+1 \le c \le n \end{cases},$$

$$w_{\text{Ŧ}}(m_c p_{2,d}^c) = \begin{cases} 4 & for\ c = 1 \\ 8c + 2d - 6 & for\ \ 2 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 10 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(m_c q_{1,r}^c) = \begin{cases} 5 & for\ c = 1 \\ 10c + 2r - 7 & for\ \ 2 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 9 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(m_c q_{2,r}^c) = \begin{cases} 6 & for\ c = 1 \\ 10c + 2r - 6 & for\ \ 2 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 8 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{1,d+2}^c m_{c+1}) = \begin{cases} 8c + 2d + 5 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c + 1 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{2,d+2}^c m_{c+1}) = \begin{cases} 8c + 2d + 6 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c + 2 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(q_{1,r+1}^c m_{c+1}) = \begin{cases} 10c + 2r + 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 1 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(q_{2,r+1}^c m_{c+1}) = \begin{cases} 10c + 2r + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{1,d}^c p_{1,d+1}^c) = \begin{cases} 4d + 2c + 1 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 7 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{1,d+1}^c p_{1,d+2}^c) = \begin{cases} 4d + 2c + 5 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 3 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{2,d}^c p_{2,d+1}^c) = \begin{cases} 4d + 2c + 2 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 6 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(p_{2,d+1}^c p_{2,d+2}^c) = \begin{cases} 4d + 2c + 6 & for\ \ 1 \le c \le \frac{\text{Ћ}}{6} \\ 14c - 2 & for\ \frac{\text{Ћ}}{6} + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}\left(q_{1,r}^{c}q_{1,r+1}^{c}\right)=\begin{cases}6c+4r-1 & for\ \ 1\le c\le \frac{\text{Ћ}}{6}\\ 14c-5 & for\ \frac{\text{Ћ}}{6}+1\le c\le n\end{cases},$$

$$w_{\text{Ŧ}}\left(q_{2,r}^{c}q_{2,r+1}^{c}\right)=\begin{cases}6c+4r & for\ \ 1\le c\le \frac{\text{Ћ}}{6}\\ 14c-4 & for\ \frac{\text{Ћ}}{6}+1\le c\le n\end{cases}.$$

Case 2: $\text{Ћ}\equiv 1(mod\ 6)$,$1\le d\le 3n$ and $1\le r\le 2n$.

Ŧ is defined as:

$$\text{Ŧ}\left(m_c q_{1,r}^{c}\right)=\begin{cases}2 & for\ c=1\\ 2c+1 & for\ \ 2\le c\le \left\lceil\frac{\text{Ћ}}{6}\right\rceil-1\\ 2\left\lceil\frac{\text{Ћ}}{6}\right\rceil+2 & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c-2\text{Ћ}-9 & for\ \left\lceil\frac{\text{Ћ}}{6}\right\rceil+1\le c\le n\end{cases},$$

$$\text{Ŧ}\left(m_c q_{1,r}^{c}\right)=\begin{cases}3 & for\ c=1\\ 2c+2 & for\ \ 2\le c\le \left\lceil\frac{\text{Ћ}}{6}\right\rceil-1\\ 2\left\lceil\frac{\text{Ћ}}{6}\right\rceil+3 & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c-2\text{Ћ}-8 & for\ \left\lceil\frac{\text{Ћ}}{6}\right\rceil+1\le c\le n\end{cases}.$$

For $m_c, m_c p_{1,d}^{c}$ and $m_c p_{2,d}^{c}$ we find that Ŧ is defined as in case 1 but with $\left\lceil\frac{\text{Ћ}}{6}\right\rceil$ instead of $\frac{\text{Ћ}}{6}$. For others we put $1\le c\le \left\lceil\frac{\text{Ћ}}{6}\right\rceil-1$ instead of $1\le c\le \frac{\text{Ћ}}{6}$ and $\left\lceil\frac{\text{Ћ}}{6}\right\rceil\le c\le n$ instead of $\frac{\text{Ћ}}{6}+1\le c\le n$. Clearly, all edges and vertices labels are at most $\text{Ћ}=\left\lceil\frac{14n+2}{3}\right\rceil$.The edges weights of $D(HPS_n)$ are given as:

$$w_{\text{Ŧ}}\left(m_c p_{1,d}^{c}\right)=\begin{cases}3 & for\ c=1\\ 8c+2d-7 & for\ \ 2\le c\le \left\lceil\frac{\text{Ћ}}{6}\right\rceil-1\\ 8\left\lceil\frac{\text{Ћ}}{6}\right\rceil+\text{Ћ}-6 & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c-11 & for\ \left\lceil\frac{\text{Ћ}}{6}\right\rceil+1\le c\le n\end{cases},$$

$$w_{\text{Ŧ}}\left(m_c p_{2,d}^{c}\right)=\begin{cases}4 & for\ c=1\\ 8c+2d-6 & for\ \ 2\le c\le \frac{\text{Ћ}}{6}\\ 8\left\lceil\frac{\text{Ћ}}{6}\right\rceil+\text{Ћ}-5 & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c-10 & for\ \frac{\text{Ћ}}{6}+1\le c\le n\end{cases},$$

$$w_{\text{Ŧ}}(m_c q_{1,r}^c) = \begin{cases} 5 & for\ c = 1 \\ 10C + 2r - 7 & for\ \ 2 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1 \\ 8\left\lceil \frac{\text{Ћ}}{6} \right\rceil + \text{Ћ} - 4 & for\ \ \ c = \left\lceil \frac{\text{Ћ}}{6} \right\rceil \\ 14c - 9 & for\ \ \left\lceil \frac{\text{Ћ}}{6} \right\rceil + 1 \le c \le n \end{cases} ,$$

$$w_{\text{Ŧ}}(m_c q_{2,r}^c) = \begin{cases} 6 & for\ c = 1 \\ 10C + 2r - 6 & for\ \ 2 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1 \\ 8\left\lceil \frac{\text{Ћ}}{6} \right\rceil + \text{Ћ} - 3 & for\ \ \ c = \left\lceil \frac{\text{Ћ}}{6} \right\rceil \\ 14c - 8 & for\ \ \left\lceil \frac{\text{Ћ}}{6} \right\rceil + 1 \le c \le n \end{cases} .$$

For others the edges' weights are given as in case 1 but with $1 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1$ instead of $1 \le c \le \frac{\text{Ћ}}{6}$ and $\left\lceil \frac{\text{Ћ}}{6} \right\rceil \le c \le n$ instead of $\frac{\text{Ћ}}{6} + 1 \le c \le n$.

Case 3: $\text{Ћ} \equiv 2(mod\ 6)$,$1 \le d \le 3n$ and $1 \le r \le 2n$.

Ŧ is defined as:

$$\text{Ŧ}(p_{1,d}^c) = \text{Ŧ}(p_{2,d}^c) = \begin{cases} 2d - 1 & for\ \ 1 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1 \\ \left.\begin{array}{ll} 2d - 1 & for\ \ d \\ \text{Ћ} & for\ \ d + 1, d + 2 \end{array}\right\} & c = \left\lceil \frac{\text{Ћ}}{6} \right\rceil \\ \text{Ћ} & for\ \left\lceil \frac{\text{Ћ}}{6} \right\rceil + 1 \le c \le n \end{cases} ,$$

$$\text{Ŧ}(p_{1,d}^c p_{1,d+1}^c) = \begin{cases} 2c + 1 & for\ \ 1 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1 \\ 2\left\lceil \frac{\text{Ћ}}{6} \right\rceil + 2 & for\ \ c = \left\lceil \frac{\text{Ћ}}{6} \right\rceil \\ 14c - 2\text{Ћ} - 7 & for\ \ \left\lceil \frac{\text{Ћ}}{6} \right\rceil + 1 \le c \le n \end{cases} ,$$

$$\text{Ŧ}(p_{2,d}^c p_{2,d+1}^c) = \begin{cases} 2c + 2 & for\ \ 1 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1 \\ 2\left\lceil \frac{\text{Ћ}}{6} \right\rceil + 3 & for\ \ c = \left\lceil \frac{\text{Ћ}}{6} \right\rceil \\ 14c - 2\text{Ћ} - 6 & for\ \ \left\lceil \frac{\text{Ћ}}{6} \right\rceil + 1 \le c \le n \end{cases} .$$

For $m_c, m_c p_{1,d}^c$, $m_c p_{2,d}^c$, $m_c q_{1,r}^c$ and $m_c q_{2,r}^c$ we find that Ŧ is defined as in case 1 but with $\left\lceil \frac{\text{Ћ}}{6} \right\rceil$ instead of $\frac{\text{Ћ}}{6}$. For others we put $1 \le c \le \left\lceil \frac{\text{Ћ}}{6} \right\rceil - 1$ instead of $1 \le c \le \frac{\text{Ћ}}{6}$ and $\left\lceil \frac{\text{Ћ}}{6} \right\rceil \le c \le n$ instead of $\frac{\text{Ћ}}{6} + 1 \le c \le n$. Clearly, all edges and vertices labels are at most $\text{Ћ} = \left\lceil \frac{14n+2}{3} \right\rceil$.The edges' weights of $D(HPS_n)$ are given as:

$$w_{Ŧ}(m_c p_{1,d}^c) = \begin{cases} 3 & for\ c = 1 \\ 8c + 2d - 7 & for\ \ 2 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ \left.\begin{matrix} 8\left\lceil \frac{Ћ}{6} \right\rceil + 2d - 7 & for\ \ d \\ 8\left\lceil \frac{Ћ}{6} \right\rceil + Ћ - 6 & for\ \ d+1, d+2 \end{matrix}\right\} & c = \left\lceil \frac{Ћ}{6} \right\rceil \\ 14c - 11 & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases},$$

$$w_{Ŧ}(m_c p_{2,d}^c) = \begin{cases} 4 & for\ c = 1 \\ 8c + 2d - 6 & for\ \ 2 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ \left.\begin{matrix} 8\left\lceil \frac{Ћ}{6} \right\rceil + 2d - 6 & for\ \ d \\ 8\left\lceil \frac{Ћ}{6} \right\rceil + Ћ - 5 & for\ \ d+1, d+2 \end{matrix}\right\} & c = \left\lceil \frac{Ћ}{6} \right\rceil \\ 14c - 10 & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases},$$

$$w_{Ŧ}(p_{1,d}^c p_{1,d+1}^c) = \begin{cases} 4d + 2c + 1 & for\ \ 1 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ 2\left\lceil \frac{Ћ}{6} \right\rceil + Ћ + 2\mathrm{d} + 1 & for\ \ c = \left\lceil \frac{Ћ}{6} \right\rceil \\ 14c - 7 & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases},$$

$$w_{Ŧ}(p_{2,d}^c p_{2,d+1}^c) = \begin{cases} 4d + 2c + 2 & for\ \ 1 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ 2\left\lceil \frac{Ћ}{6} \right\rceil + Ћ + 2\mathrm{d} + 2 & for\ \ c = \left\lceil \frac{Ћ}{6} \right\rceil \\ 14c - 6 & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases}.$$

For $m_c q_{1,r}^c$ and $m_c q_{2,r}^c$ the edges' weights are given as in case 1 but with $\left\lceil \frac{Ћ}{6} \right\rceil$ instead of $\frac{Ћ}{6}$. For others we put $1 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1$ instead of $1 \le c \le \frac{Ћ}{6}$ and $\left\lceil \frac{Ћ}{6} \right\rceil \le c \le n$ instead of $\frac{Ћ}{6} + 1 \le c \le n$.

Case 4: $Ћ \equiv 3(mod\ 6)$,$1 \le d \le 3n$ and $1 \le r \le 2n$.

Ŧ is defined as:

$$Ŧ(p_{1,d}^c) = Ŧ(p_{2,d}^c) = \begin{cases} 2d - 1 & for\ \ 1 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ \left.\begin{matrix} 2d - 1 & for\ \ d \\ Ћ & for\ \ d+1, d+2 \end{matrix}\right\} & c = \left\lceil \frac{Ћ}{6} \right\rceil \\ Ћ & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases},$$

$$Ŧ(q_{1,r}^c) = Ŧ(q_{2,r}^c) = \begin{cases} 2r + 2d - 2 & for\ \ 1 \le c \le \left\lceil \frac{Ћ}{6} \right\rceil - 1 \\ \left.\begin{matrix} 2r + 2d - 2 & for\ \ r \\ Ћ & for\ \ r+1 \end{matrix}\right\} & c = \left\lceil \frac{Ћ}{6} \right\rceil \\ Ћ & for\ \left\lceil \frac{Ћ}{6} \right\rceil + 1 \le c \le n \end{cases},$$

$$Ŧ\left(q_{1,r}^{c}q_{1,r+1}^{c}\right)=\begin{cases}2c+1 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+2 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ-5 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

$$Ŧ\left(q_{2,r}^{c}q_{2,r+1}^{c}\right)=\begin{cases}2c+2 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+3 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ-4 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases}.$$

For $m_c, m_c p_{1,d}^{c},\ m_c p_{2,d}^{c},\ m_c q_{1,r}^{c}, m_c q_{2,r}^{c}, p_{1,d}^{c}p_{1,d+1}^{c}$ and $p_{2,d}^{c}p_{2,d+1}^{c}$ we find that Ŧ is defined as in case 1 but we put $\left\lceil\frac{Ћ}{6}\right\rceil$ instead of $\frac{Ћ}{6}$. For the others we put $1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1$ instead of $1\le c\le\frac{Ћ}{6}$ and $\left\lceil\frac{Ћ}{6}\right\rceil\le c\le n$ instead of $\frac{Ћ}{6}+1\le c\le n$. Clearly, all edges and vertices labels are at most $Ћ=\left\lceil\frac{14n+2}{3}\right\rceil$.The edges' weights of $D(HPS_n)$ are given as:

$$w_{Ŧ}\left(q_{1,r}^{c}q_{1,r+1}^{c}\right)=\begin{cases}6c+4r-1 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 4\left\lceil\frac{Ћ}{6}\right\rceil+Ћ+2r & for\ \ c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-5 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

$$w_{Ŧ}\left(q_{2,r}^{c}q_{2,r+1}^{c}\right)=\begin{cases}6c+4r & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 4\left\lceil\frac{Ћ}{6}\right\rceil+Ћ+2r+1 & for\ \ c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-4 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases}.$$

For $m_c p_{1,d}^{c},\ m_c p_{2,d}^{c},\ m_c q_{1,r}^{c}, m_c q_{2,r}^{c}, p_{1,d}^{c}p_{1,d+1}^{c}$ and $p_{2,d}^{c}p_{2,d+1}^{c}$ the edges' weights are given as in case 1 but with $\left\lceil\frac{Ћ}{6}\right\rceil$ instead of $\frac{Ћ}{6}$. For others we put $1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1$ instead of $1\le c\le\frac{Ћ}{6}$ and $\left\lceil\frac{Ћ}{6}\right\rceil\le c\le n$ instead of $\frac{Ћ}{6}+1\le c\le n$.

Case 5: $Ћ\equiv 4(mod\ 6)$,$1\le d\le 3n$ and $1\le r\le 2n$.

Ŧ is defined as:

$$Ŧ\left(p_{1,d}^{c}\right)=Ŧ\left(p_{2,d}^{c}\right)=\begin{cases}2d-1 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ \left.\begin{array}{ll}2d-1 & for\ \ d\\ Ћ & for\ \ d+1,d+2\end{array}\right\} & c=\left\lceil\frac{Ћ}{6}\right\rceil\\ Ћ & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

$$Ŧ\left(p_{1,d+1}^{c}p_{1,d+2}^{c}\right)=\begin{cases}2c+1 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+2 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ-3 & for\ \left\lceil\frac{Ћ}{6}\right\rceil\le c\le n\end{cases},$$

$$Ŧ\left(p_{2,d+1}^{c}p_{2,d+2}^{c}\right)=\begin{cases}2c+2 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+3 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ-2 & for\ \left\lceil\frac{Ћ}{6}\right\rceil\le c\le n\end{cases}.$$

For $m_c, q_{1,r}^c, q_{2,r}^c,\ m_c p_{1,d}^c,\ m_c p_{2,d}^c,\ m_c q_{1,r}^c,\ m_c q_{2,r}^c, p_{1,d}^c p_{1,d+1}^c, p_{2,d}^c p_{2,d+1}^c,\ q_{1,r}^c q_{1,r+1}^c$ and $q_{2,r}^c q_{2,r+1}^c$we find that Ŧ is defined as in case 1 but we put $\left\lceil\frac{Ћ}{6}\right\rceil$ instead of $\frac{Ћ}{6}$. For the others we put $1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1$ instead of $1\le c\le\frac{Ћ}{6}$ and $\left\lceil\frac{Ћ}{6}\right\rceil\le c\le n$ instead of $\frac{Ћ}{6}+1\le c\le n$. Clearly, all edges and vertices labels are at most $Ћ=\left\lceil\frac{14n+2}{3}\right\rceil$.The edges weights of $D(HPS_n)$ are given as:

$$w_{Ŧ}\left(p_{1,d+1}^{c}p_{1,d+2}^{c}\right)=\begin{cases}4d+2c+5 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+Ћ+2\mathrm{d}+3 & for\ \ c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-3 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

$$w_{Ŧ}\left(p_{2,d+1}^{c}p_{2,d+2}^{c}\right)=\begin{cases}4d+2c+6 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+Ћ+2\mathrm{d}+4 & for\ \ c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases}.$$

For $m_c p_{1,d}^c,\ m_c p_{2,d}^c,\ m_c q_{1,r}^c,\ m_c q_{2,r}^c, p_{1,d}^c p_{1,d+1}^c,\ p_{2,d}^c p_{2,d+1}^c\ q_{1,r}^c q_{1,r+1}^c$ and $q_{2,r}^c q_{2,r+1}^c$the edges' weights are given as in case 1 but we put $\left\lceil\frac{Ћ}{6}\right\rceil$ instead of $\frac{Ћ}{6}$. For the others we put $1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1$ instead of $1\le c\le\frac{Ћ}{6}$ and $\left\lceil\frac{Ћ}{6}\right\rceil\le c\le n$ instead of $\frac{Ћ}{6}+1\le c\le n$.

Case 6: $Ћ\equiv 5(mod\ 6)$,$1\le d\le 3n$ and $1\le r\le 2n$.

Ŧ is defined as:

$$Ŧ\left(q_{1,r+1}^{c}m_{c+1}\right)=\begin{cases}2c+1 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+2 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ-1 & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

$$Ŧ\left(q_{2,r+1}^{c}m_{c+1}\right)=\begin{cases}2c+2 & for\ \ 1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1\\ 2\left\lceil\frac{Ћ}{6}\right\rceil+3 & for\quad c=\left\lceil\frac{Ћ}{6}\right\rceil\\ 14c-2Ћ & for\ \left\lceil\frac{Ћ}{6}\right\rceil+1\le c\le n\end{cases},$$

For $p_{1,d+2}^c m_{c+1}$ and $p_{2,d+2}^c m_{c+1}$we find that Ŧ is defined as in case 1 but we put $1\le c\le\left\lceil\frac{Ћ}{6}\right\rceil-1$ instead of $1\le c\le\frac{Ћ}{6}$ and $\left\lceil\frac{Ћ}{6}\right\rceil\le c\le n$ instead of $\frac{Ћ}{6}+1\le c\le n$.. For the others we put $\left\lceil\frac{Ћ}{6}\right\rceil$ instead of $\frac{Ћ}{6}$. Clearly, all edges and vertices labels are at most $Ћ=\left\lceil\frac{14n+2}{3}\right\rceil$.The edges' weights of $D(HPS_n)$ are given as:

$$w_{\text{Ŧ}}\left(q_{1,r+1}^{c}m_{c+1}\right)=\begin{cases}10c+2r+1 & for\ \ 1\le c\le\left\lceil\frac{\text{Ћ}}{6}\right\rceil-1\\ 4\left\lceil\frac{\text{Ћ}}{6}\right\rceil+\text{Ћ}+2\text{r} & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c-1 & for\ \left\lceil\frac{\text{Ћ}}{6}\right\rceil+1\le c\le n\end{cases},$$

$$w_{\text{Ŧ}}\left(q_{2,r+1}^{c}m_{c+1}\right)=\begin{cases}10c+2r+2 & for\ \ 1\le c\le\left\lceil\frac{\text{Ћ}}{6}\right\rceil-1\\ 4\left\lceil\frac{\text{Ћ}}{6}\right\rceil+\text{Ћ}+2\text{r}+1 & for\ \ c=\left\lceil\frac{\text{Ћ}}{6}\right\rceil\\ 14c & for\ \left\lceil\frac{\text{Ћ}}{6}\right\rceil+1\le c\le n\end{cases},$$

For $p_{1,d+2}^{c}m_{c+1}$ and $p_{2,d+2}^{c}m_{c+1}$ the edges' weights are given as in case 1 but we put $1\le c\le\left\lceil\frac{\text{Ћ}}{6}\right\rceil-1$ instead of $1\le c\le\frac{\text{Ћ}}{6}$ and $\left\lceil\frac{\text{Ћ}}{6}\right\rceil\le c\le n$ instead of $\frac{\text{Ћ}}{6}+1\le c\le n$.. For the others we put $\left\lceil\frac{\text{Ћ}}{6}\right\rceil$ instead of $\frac{\text{Ћ}}{6}$. clearly, for all cases any pairs of edges the weights are different. Hence, Ŧ is I an edge I irregular itotal Ћ −labeling. Thus

$$tes(D(HPS_n))=\left\lceil\frac{14n+2}{3}\right\rceil.$$

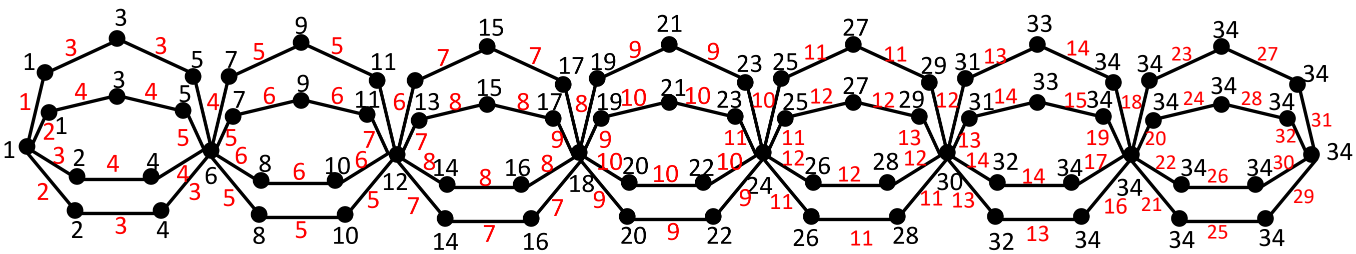

Figure (4) $tes(D(HPS_n))=34$,for $n=7$

Figure (4) is an illustration of Theorem 2, where all edge vertex and vertex labels are at most $\text{Ћ}=34$ and any two different edges have distinct weights.

2.3 Computing TEIS of heptagonal snake graph $L(HPS_n)$:

Definition3. An $\boldsymbol{l}$-multiple heptagonal snake graph $L(\mathrm{HPS_n})$ is a graph consists of l heptagonal snake graphs that have a common path P_n.

Therom3. If $L(\mathrm{HPS_n})$ is an l −multiple heptagonal snake graph. Then

$$tes(L(\mathrm{HPS_n}))=\left\lceil\frac{7ln+2}{3}\right\rceil.$$

Conclusion

In this paper, the definitions of the heptagonal snake graph $\mathrm{HPS_n}$,the double heptagonal snake graph $D(\mathrm{HPS_n})$ and an l −multiple heptagonal snake graph $L(\mathrm{HPS_n})$ have been introduced. The exact values of TEISs for a heptagonal snake graph, a double heptagonal snake graph and an l −multiple heptagonal snake graph have also been investigated and given in the forms

$$tes(\mathrm{HPS_n})=\left\lceil\frac{7n+2}{3}\right\rceil,$$

$$tes(\mathrm{D}(\mathrm{HPS_n})) = \left\lceil \frac{14n+2}{3} \right\rceil,$$
$$tes(L(\mathrm{HPS_n})) = \left\lceil \frac{7ln+2}{3} \right\rceil.$$

Ethics approval and consent to participate: This article does not contain any studies with human participants or animals performed by any of the authors.

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Figures

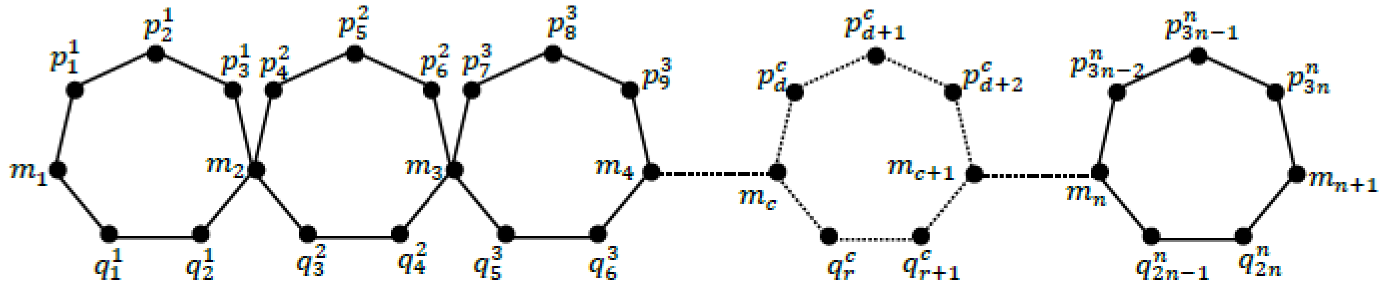

Figure 1

A heptagonal snake graph

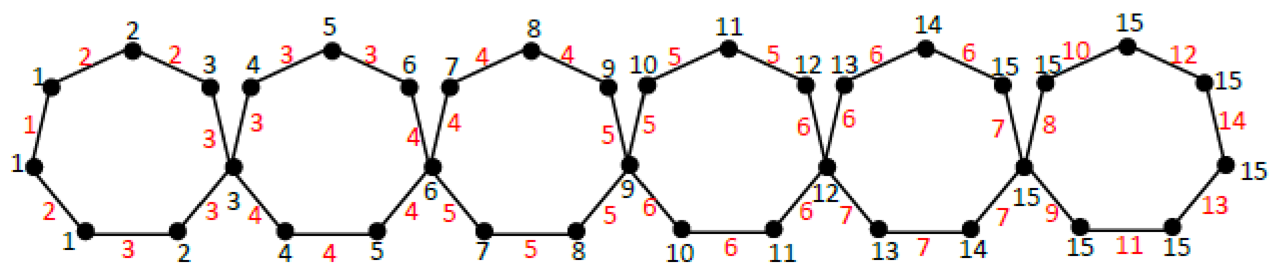

Figure 2

An illustration of Theorem 1, where all edge vertex and vertex labels are at most Ћ=15 and any two different edges have distinct weights.

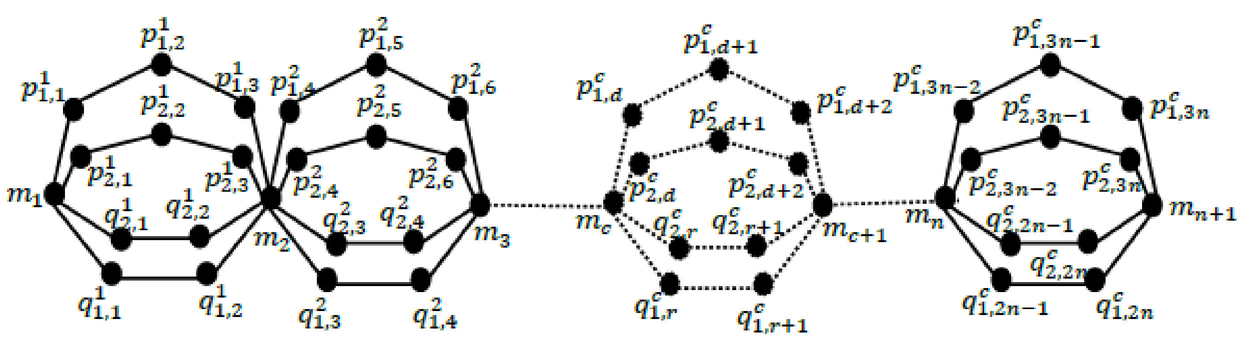

Figure 3

A double heptagonal snake graph

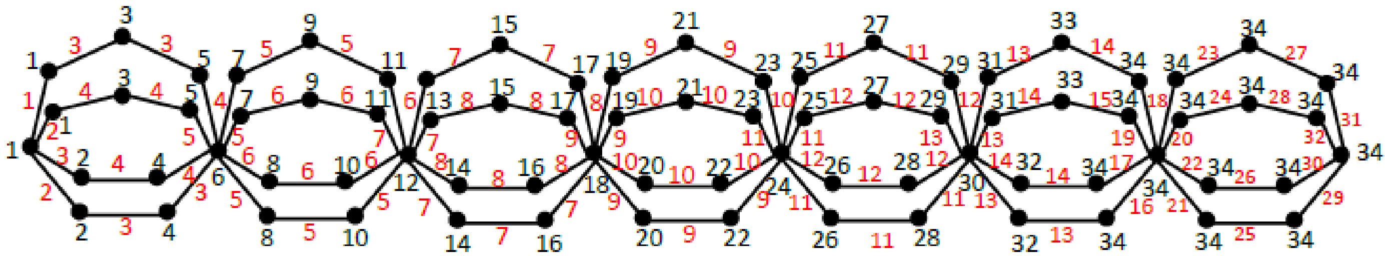

Figure 4

□□□ (□(□□□□))=34 for □=7