Improved Bounds for Permutation Arrays Under Chebyshev Distance

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Abstract

Permutation arrays under the Chebyshev metric have been considered for error correction in noisy channels. Let $P(n, d)$ denote the maximum size of any array of permutations on n symbols with pairwise Chebyshev distance d. We give new techniques and improved upper and lower bounds on $P(n, d)$, including a precise formula for $P(n, 2)$.

1 Introduction

In [9] an interesting study of permutation arrays under the Chebyshev metric was presented. This complemented many studies of permutation arrays under other metrics, such as the Hamming metric [1] [2] [4], Kendall τ metric [7] [3], and several others [5]. The use of the Chebyshev metric was motivated by applications of error correcting codes and recharging in flash memories [7].

Let σ and π be two permutations (or strings) over an alphabet $\Sigma \subseteq [1...n] = \{1, 2, ..., n\}$. The Chebyshev distance between σ and π, denoted by $d(\sigma, \pi)$, is $\max\{\ |\sigma(i) - \pi(i)| \mid i \in \Sigma\ \}$. For an array (set) A of permutations (strings), the pairwise Chebyshev distance of A, denoted by $d(A)$, is $\min\{\ d(\sigma, \pi) \mid \sigma, \pi \in A\ \}$. An array A of permutations on $[1...n]$ with $d(A) = d$ will be called an (n, d) PA. Note that this includes the case when A is a set of integers, *i.e.* a set of strings of length one, where $d(A)$ corresponds to the minimum difference between integers in the set. Let $P(n, d)$ denote the maximum cardinality of any (n, d)-PA A. More generally, let $P_d(\Sigma)$ denote the maximum cardinality of any array of permutations over the alphabet $\Sigma \subseteq [1...n]$ with Chebyshev distance d. For example, $P_2(\{1, 3, 5, 7\}) = 4! = 24$, whereas $P(4, 2) = 6$.

We present several methods to improve on lower and upper bounds for $P(n, d)$. For comparison, we begin with the following theorem from [9].

Theorem 1. *([9]) If $n > d \geq 1$, then $P(n+1, d) \geq (\lfloor \frac{n}{d} \rfloor + 1)P(n, d)$.*

To generalize, let A be a subset of $[1...(n+1)]$ such that $d(A) \geq d$, then, for all $i \in A$, $P_d([1..(n+1)] - \{i\}) \geq P(n, d)$. Observe that the set $\{1, d+1, 2d+1, ..., \lfloor \frac{n}{d} \rfloor d + 1\}$ is a subset of $[1...(n+1)]$ with $\lfloor \frac{n}{d} \rfloor + 1$ elements with Chebyshev distance d and was used in [9] to prove Theorem 1.

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Theorem 2. *Let A be a subset of $[1...(n+1)]$ such that $d(A) \geq d$. If $n > d \geq 1$, then $P(n+1, d) \geq \sum_{i \in A} P_d([1..(n+1)] - \{i\})$.*

Theorem 2 is a generalization of Theorem 1 and often gives improved lower bounds. For example, using Theorem 1, one obtains $P(11, 3) \geq 36,132$, as $\lfloor \frac{10}{3} \rfloor + 1 = 4$ and the best lower bound currently known for $P(10, 3)$ is 9,033. Using Theorem 2 and choosing $A = \{3, 6, 9\}$, one obtains the lower bound 53,549, as $P_3([1..11] - \{3\}) = P_3([1..11] - \{9\} \geq 17,573$ and $P_3([1..11] - \{6\}) \geq 18,403$.

Another recursive technique in [9] gave the following result.

Theorem 3. *([9]) If $n > d$ and $r \geq 2$, then $P(rn, rd) \geq P(n, d)^r$.*

For example, we use Theorem 3 to get $P(18, 4) \geq P(9, 2)^2 = 2,520^2 = 514,382,400$. Theorem 3 is generalized by Theorem 4, which subsumes Theorem 3 and gives several new lower bounds. For example, we use Theorem 4, with a=3, to get $P(18, 5) \geq P(11, 3) * P(7, 2) \geq 53,549 * 630 = 33,735,870$.

Theorem 4. $P(n, d) \geq \max\{P(n_1, d_1) \cdot P(n_2, d_2) \mid d_1 + d_2 = d$ *and* $n_1 + n_2 = n$ *and, for some constant* a, $n_1 = ad_1 + r_1$ *and* $n_2 = ad_2 + r_2$, *with* $0 \leq\ r_1 \leq\ d_1$ *and with* $0 \leq\ r_2 \leq\ d_2\}$, *where the maximum is taken over all possible values of* n_1, n_2, d_1, d_2.

As another example, we use Theorem 4 to get the lower bound $P(16, 9) \geq P(9, 5) * P(7, 4) \geq 3,399$, where $a = 1$, $9 = 1 * 5 + 4$, $7 = 1 * 4 + 3$, and the best lower bounds known for $P(9, 5)$ and $P(7, 4)$ are 103 and 33, respectively.

For given n and d, Klove et al [9] defined $C = \{(\pi_1, ..., \pi_n) \in S_n \mid \pi_i = i \mod d$, for all $i \in [1..n]\}$ and gave the following theorems:

Theorem 5. *([9]) If $n = ad+b$, where $0 \leq b < d$, then C is an (n, d) PA and $|C| = ((a+1)!)^b (a!)^{d-b}$.*

Theorem 6. *([9]) If $n = ad + b$, where $0 \leq b < d$, then $P(n, d) \geq ((a + 1)!)^b (a!)^{d-b}$.*

Klove et al [9] gave, as an example, the lower bound $P(2a, 2) \geq (a!)^2$. They also gave the improvement, using Theorem 1 iteratively, $P(2a, 2) \geq \frac{97}{24}(a!)^2$. We give an exact equation for $P(n, 2)$. Specifically, we show $P(2a, 2) = \frac{(2a)!}{2^a}$.

Theorem 7. $P(n, 2) = \frac{n!}{2^{\lfloor n/2 \rfloor}}$.

The iterative use of Theorem 1 can be improved further by a generalization of Theorem 2 using strings of more than one symbol. Let A be a set of length m strings with no repeated symbols (permutations) over $[1..(n + m)]$ with $d(A) \geq d$. By an abuse of notation, for each $\sigma \in A$, let σ^C denote the complement in $[1..(n + m)]$ of the set of symbols used in σ. As in Theorem 2, we show that $P(n + m, d) \geq \sum_{\sigma \in A} P_d(\sigma^C)$. Let $Q((n + m), m, d)$ denote the collection of all sets A of permutations on a m symbol subset of $[1..(n + m)]$ with $d(A) \geq d$. Maximizing the sum over all such sets A yields the following.

Theorem 8. *For any $n \geq d \geq 1, m \geq 1$, $P(n + m, d) \geq \max_{A \in Q((n+m),m,d)} \ \sum_{\sigma \in A} P_d(\sigma^C)$.*

In [9] a 3-fold iterative use of Theorem 1, for $d = 3$ and $n = 5$ gives a set $S \in Q(8, 3, 3)$ with $|S| = 18$. That is, $(\lfloor \frac{5}{3} \rfloor + 1)(\lfloor \frac{6}{3} \rfloor + 1)(\lfloor \frac{7}{3} \rfloor + 1) = 18$. However, by computation one can obtain a set $T \in Q(8, 3, 3)$ with $|T| = 24$. Thus, not only can one obtain a larger subset of $[1..(n + m)]$ than the iterative use of Theorem 1, but also larger sets than $P(n, d)$ by the use of complement alphabets. For $m < n$, let $P(n, m, d)$ denote the maximum cardinality of any set A in $Q(n, m, d)$. We have computed several lower bounds for $P(n, m, d)$. See, for example, Tables 4 and 5 in Section 4.

Corollary 9. *For any* $n \geq d \geq 1, m \geq 1$, $P(n+m,d) \geq P(n+m,m,d) * P(n,d)$.

Proof. That is, for any set $A \in Q((n+m),m,d)$, and any $\sigma \in A$, $P_d(\sigma^C) \geq P(n,d)$. □

We have shown in previous examples that Corollary 9 gives improved lower bounds, by computation, over an iterative use of Theorem 1. The next theorem show that such improvements exist even for arbitrarily large n. For example, if $d=5$ and $k=2$, an iterative use of Theorem 1 gives $P(dk+d-1,d) = P(14,5) \geq (\lfloor \frac{13}{5} \rfloor + 1)(\lfloor \frac{12}{5} \rfloor + 1)(\lfloor \frac{11}{5} \rfloor + 1)(\lfloor \frac{10}{5} \rfloor + 1)P(10,5) = 3^4 P(10,5) = 81P(10,5)$. By Theorem 10, $P(dk+d-1) = P(14,5) \geq (3^5 - \binom{6}{4})P(10,5) = 228P(10,5)$.

Theorem 10. *For any* $d \geq 3$ *and* $k \geq 1$, $P(dk+d-1,d) \geq \left((k+1)^d - \binom{k+d-1}{d-1}\right) P(dk-1,d)$.

As another example of the improvement shown by Theorem 10 consider the case when $k=3$ and $d=3$. The theorem states that $P(11,3) \geq 54 \cdot P(8,3)$, whereas the three fold iterative use of Theorem 1 gives $P(11,3) \geq (\lfloor \frac{10}{3} \rfloor + 1) \cdot (\lfloor \frac{9}{3} \rfloor + 1) \cdot (\lfloor \frac{8}{3} \rfloor + 1) \cdot P(8,3) = 48 \cdot P(8,3)$. By computational methods, we show that $P(11,3,3) \geq 59$ and hence, by Theorem 8, we have $P(11,3) \geq 59 \cdot P(8,3)$. In fact, as shown in Table 1, $P(11,3) \geq 53,549$.

Let $V(n,d)$ be the number of permutations on $\{1,2,\ldots,n\}$ within distance d of the identity permutation.

Kløve *et al.* [9] also gave general lower and upper bounds.

Theorem 11. *[9] For* $n > d \geq 2$, $P(n,d) \geq \frac{n!}{V(n,d-1)}$

Theorem 12. *[9] For even* d *and* $2d \geq n \geq d \geq 2$, $P(n,d) \leq \frac{(n+1)!}{V(n+1,d/2)}$,

In Theorem 13 we give a better upper bound. Using Theorem 13 we show, for example, $P(11,6) \leq 462$. Kløve [8] also proved lower bounds on the size of spheres of permutations under the Chebyshev distance.

Theorem 13. *For* $1 \leq k \leq d < n$,

$$P(n,d) \leq P(n-k,d) \cdot \binom{n}{k}.$$

In [9] there is also the following interesting theorem.

Theorem 14. *[9] For fixed r, there exist constants* c_r *and* d_r *such that* $P(d+r,d) = c_r$, *for* $d \geq d_r$.

Moreover, an upper bound on the constants c_r and d_r is given in [9]. The proof uses the concept of *potent* symbols. Basically, an integer is potent for Chebyshev distance d if there is another integer, say j, in the given alphabet, such that $|j-i| \geq d$. That is, the symbol can be used in permutations to achieve distance d.

Definition 15. *If A is a PA on* $d+r$ *symbols with Chebyshev distance d, then the integers 1,2, … ,r and d+1,d+2, … ,d+r are potent.*

The following theorem provides improved upper bounds for the constants c_r and d_r of Theorem 14.

Theorem 16. *Suppose that* $P(n_0, n_0 - k) \le m$ *such that*

$$2k(m+1) < (n_0+1)(1 + \lfloor n_0/(2k-1) \rfloor). \tag{1}$$

Then $P(n, n-k) \le m$*, for all* $n \ge n_0$.

As an example, Theorem 16 can be used to show that the constants c_2, d_2 in Theorem 14 are $d_2 = 3$ and $c_2 = 10$.

Corollary 17. $P(n, n-2) = 10$*, for all* $n \ge 5$.

As part of the proof of Corollary 17, we have computed a PA A on [1..5] with $d(A) = 3$, so $P(5,3) \ge 10$. In [9], $P(5,3) \le 9$ was claimed, but was apparently due to a computational error.

Theorem 16 can also be used to show improved bounds for c_r and d_r, for $r \ge 3$. For example, by Theorem 13, we have $P(n, n-3) \le P(n-1, n-3) \cdot \binom{n}{1} = 10 \cdot n$, for all $n \ge 6$. Observe that, for $k = 3$, $n_0 = 295$, and $m = 2950$, the inequality of Equation (1) is true. So, $P(n, n-3) \le 2{,}950$, for all $n \ge 295$. Thus, $c_3 \le 2{,}950$ and $d_3 \le 295$, which improves the bounds $c_3 \le 46{,}080$ and $d_3 \le 230{,}401$ given in [9].

In [9] a few additional recursive constructions were described to obtain lower bounds for $P(n,d)$. For example, for any permutation $\sigma \in S_n$ and any m $(1 \le m \le n)$, define $\phi_m(\sigma) = (m, \pi_1, \pi_2, ..., \pi_n)$, where:
$\pi_i = \sigma_i$, if $i < m$, and
$\pi_i = \sigma_i + 1$, if $i \ge m$.

For any PA A and symbols $1 \le s_1 < s_2 < ... < s_t \le n+1$, define $A[s_1, s_2, ..., s_t]$ to be $\{\phi_m(\sigma) \mid \sigma \in A,\ m \in \{s_1, s_2, ..., s_t\}\}$

Theorem 18. *([9]) If A is an* (n,d) *PA of size M and* $s_j + d \le s_{j+1}$*, for* $1 \le j \le$ *t-1, then* $A[s_1, s_2, ..., s_t]$ *is an (n + 1,d) PA of size* tM*.*

Theorem 19. *([9]) If A is an* (n,d) *PA of size M and* $n \le 2d$*, then A[d] is an (n+1,d+1) PA of size M.*

Theorem 18 implies the following:

Theorem 20. *([9]) If* $d < n \le 2d$*, then* $P(n+1, d+1) \ge P(n,d)$*.*

In Table 1 we give several lower bounds for $P(n,d)$ and in Table 3 we give several upper bounds for $P(n,d)$.

2 Lower Bounds

In [9] a greedy algorithm was used to find a PA C on $[1..n]$ with $d(C) \ge d$:

> Let the identity permutation in S_n be the first permutation in C. For any set of permutations chosen, choose as the next permutation in C the lexicographically next permutation in S_n with distance at least d to the chosen permutations in C if such a permutation exists.

We modified this greedy algorithm by choosing an initial set C of pairwise distance d permutations randomly. Because of the randomness, we also allowed the algorithm to automatically start again and repeat the process while recording the best result. We call this the *Random/Greedy* strategy.

Many of the lower bounds in Table 1, for small values of n, were obtained by this modified greedy algorithm. A few were found by computing a largest clique in a graph, whose nodes are all permutations, and edges are between nodes at Chebyshev distance $\geq d$, called the *Clique* approach. Others were found using Theorems 2, 4, 7, or 8. Computations using the ideas of Theorem 8 were often done with a Max Weighted Clique solver tool [6] [10]. That is, to compute a lower bound for $P(n+m, d)$, a graph G was created with nodes labeled by permutations on m symbols of [1..(n+m)], and whose edges connect two nodes with labels L_1 and L_2, where $d(L_1, L_2) \geq d$. A node with label L is given a weight of $P_d(L^C)$, where the complement is taken with respect to the set [1..(n+m)]. Values for $P_d(L^C)$ were pre-computed, using a modification of the Random/Greedy algorithm. A maximum weighted clique of G corresponds to the lower bound given in Theorem 8. As the set of all permutations on a m symbol subset of [1..(n+m)] gets very large as m and n get large, heuristics were sometimes used to decide which permutations to use as labels in the graph G.

Table 1: Lower Bounds for $P(n, d)$.

n/d	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	**3**	1	1	1	1	1	1	1	1
4	**6**	**3**	1	1	1	1	1	1	1
5	**30**	**10**	**3**	1	1	1	1	1	1
6	**90**	**20**	**10**	**3**	1	1	1	1	1
7	**630**	100	33	**10**	**3**	1	1	1	1
8	**2,520**	430	**70**	33	**10**	**3**	1	1	1
9	**22,680**	1,654	295	103	33	**10**	**3**	1	1
10	**113,400**	9,033	1,336	247	103	33	**10**	**3**	1
11	see Thm 7	53,549	6,397	998	326	103	33	**10**	**3**
12	see Thm 7	317,728	26,678	4,355	842	330	103	33	**10**
13	see Thm 7	1,642,473	114,720	17,049	3,294	978	330	103	33
14	see Thm 7	11,081,916	647,420	81,888	10,709	2,805	1,089	330	103
15	see Thm 7	55,409,580	3,887,796	392,033	50,283	8,604	3,144	1,089	330
16	see Thm 7	332,457,480	15,551,184	1,898,103	250,867	37,017	9,379	3,399	1,089
17	see Thm 7	1,994,744,880	77,755,920	7,592,412	1,261,267	174,655	30,106	10,374	3,399
18	see Thm 7	11,968,469,280	514,382,400	33,735,870	3,783,801	862,566	129,756	31,779	10,758

We have, $P(n, d) = 1$, for all $d \geq n$, as a single permutation is a (n, d)-PA. That $P(n, n-1) = 3$, for all $n \geq 3$ was shown in [9]. We show $P(n, n-2) = 10$, for all $n \geq 5$ by Corollary 17 and the *Clique* approach ([9] incorrectly gave $P(n, n-2) \leq 9$). The bound $P(4, 2) = 6$ was cited in [9].

We show in Theorem 7 that $P(n, 2) = \frac{n!}{2^{\lceil n/2 \rceil}}$. $P(6, 3) \geq 20$ was cited in [9]. We computed $P(7, 4) \geq 33$ by the *Random/Greedy* strategy, which improved on the previous lower bound of 28 [9]. It follows from Theorem 19 that $P(n, n-3) \geq 33$, for all $n \geq 7$.

The bounds $P(7, 3) \geq 100$, $P(8, 4) \geq 70$, and $P(9, 5) \geq 103$ were found by the *Random/Greedy* strategy, whereas [9] gave lower bounds of 84, 70 and 95, respectively. That $P(n, n-4) \geq 103$, for all $n \geq 9$ follows from Theorem 19. The bounds $P(8, 3) \geq 430, P(9, 4) \geq 295, P(10, 5) \geq$

$247, P(11,6) \geq 326$ and $P(12,7) \geq 330$ were all found by the *Random/Greedy* strategy, whereas [9] gave lower bounds of $401, 283, 236, 236$ and 236, respectively. That $P(n, n-5) \geq 330$, for all $n \geq 12$, follows from Theorem 19. The bounds $P(9,3) \geq 1,654, P(10,4) \geq 1,336, P(11,5) \geq 998, P(12,6) \geq 842$ and $P(13,7) \geq 978$ were all found by the *Random/Greedy* strategy and $P(14,8) \geq 1,089$ was obtained by Theorem 3. That $P(n, n-6) \geq 1,089$, for all $n \geq 14$ follows from Theorem 19. The bounds $P(10,3) \geq 9,033, P(11,4) \geq 6,397, P(12,5) \geq 4,355, P(13,6) \geq 3,294, P(14,7) \geq 2,805, P(15,8) \geq 3,144$ were all found by Theorem 8. $P(16,9) \geq 3,399$ was found by Theorem 4, using $P(9,5)$ and $P(7,4)$.

Theorem 2 was used to obtain the current lower bound $P(11,3) \geq 53,549$. That is, by computation we found $P_3([1..11] - \{3\}) = P_3([1..11] - \{9\} \geq 17,573$ and $P_3([1..11] - \{6\}) \geq 18,403$. So, $P(11,3) \geq 2 * 17,573 + 18,403 = 53,549$. Here is a proof of Theorem 2.

Theorem 2. Let A be a subset of $[1...(n+1)]$ such that $d(A) \geq d$. If $n > d \geq 1$, then $P(n+1, d) \geq \sum_{i \in A} P_d([1..(n+1)] - \{i\})$.

Proof. Let $A = \{a_1, a_2, ..., a_k\}$ be a subset of $[1...(n+1)]$ such that $d(A) \geq d$. For $a_i \neq a_j$, and permutations σ and τ in $[1..(n+1)] - \{a_i\}$ and $[1..(n+1)] - \{a_j\}$, respectively, $a_i\sigma$ and $a_j\tau$ are permutations on [1..(n+1)] such that $d(a_i\sigma, a_j\tau) \geq d$. It follows that $\bigcup_{a_i \in A} a_i B$, with B a set of permutations over $[1..(n+1)] - \{a_i\}$ with Chebyshev distance $\geq d$, is a set of permutations on $[1..(n+1)]$ with Chebyshev distance $\geq d$. □

Here is a proof for Theorem 4.

Theorem 4. $P(n,d) \geq \max\{P(n_1, d_1) \cdot P(n_2, d_2) \mid d_1 + d_2 = d$ and $n_1 + n_2 = n$ and, for some constant a, $n_1 = ad_1 + r_1$ and $n_2 = ad_2 + r_2$, with $0 \leq r_1 \leq d_1$ and with $0 \leq r_2 \leq d_2\}$, where the maximum is taken over all possible values of n_1, n_2, d_1, d_2.

Proof. Let $n = n_1 + n_2$ and $d = d_1 + d_2$. Let A be a PA on the n_1 symbols in $\Sigma_1 = [1...n_1]$ with Hamming distance d_1 and let B be a PA on the n_2 symbols in $\Sigma_2 = [1..., n_2]$ with Hamming distance d_2. Let $\Sigma = [1...n = n_1 + n_2]$. Define the function F_1 mapping Σ_1 into Σ by:

$$F_1(x) = \begin{cases} x & \text{if } 1 \leq x \leq r_1, \\ x + sd_2 & \text{if } (s-1)d_1 + r_1 + 1 \leq x \leq sd_1 + r_1, \text{ for some } 1 \leq s \leq a. \end{cases}$$

and define the function F_2 mapping Σ_2 into Σ by:

$$F_2(x) = \begin{cases} x + (t-1)d_1 + r_1 & \text{if } (t-1)d_2 < x \leq td_2, \text{ for some } 1 \leq t \leq a, \\ x + n_1, & \text{if } ad_2 < x \leq ad_2 + r_2. \end{cases}$$

Construct the PA C = $\{ F_1(\sigma)F_2(\tau) \mid \sigma \in A$ and $\tau \in B \}$.

C is a set of $|A| \cdot |B|$ permutations on the alphabet Σ of n symbols. We show that the Chebyshev distance between permutations in C is at least $d = d_1 + d_2$. Consider two different permutations $\pi_1 = F_1(\sigma_1)F_2(\tau_1)$ and $\pi_2 = F_1(\sigma_2)F_2(\tau_2)$ in C, where $\sigma_1, \sigma_2 \in A$ and $\tau_1, \tau_2 \in B$. Since $\pi_1 \neq \pi_2$, either $\sigma_1 \neq \sigma_2$ or $\tau_1 \neq \tau_2$. Due to the similarity of the argument we only explicitly examine the case when $\sigma_1 \neq \sigma_2$. So, the Chebyshev distance between σ_1 and σ_2 is at least d_1. That is, there is a position i ($1 \leq i \leq n_1$) such that $|\sigma_1(i) - \sigma_2(i)| \geq d_1$. Assume, without loss of generality, that $\sigma_1(i) > \sigma_2(i)$. In other words, $\sigma_1(i)$ and $\sigma_2(i)$ are in different intervals of d_1 symbols in Σ_1, i.e. $\sigma_2(i)$ is in the interval $[(s-1)d_1 + r_1, sd_1 + r_1]$, for some s, and $\sigma_1(i)$ is in the interval $[(s'-1)d_1 + r_1, s'd_1 + r_1]$, for some $s' > s$. Hence, F_1 maps $\sigma_1(i)$ to $\sigma_1(i) + s'd_2$ and maps $\sigma_2(i)$ to $\sigma_2(i) + sd_2$. So, $|(\sigma_1(i) + s'd_2) - (\sigma_2(i) + sd_2)| = |\sigma_1(i) - \sigma_2(i) + s'd_2 - sd_2| = |\sigma_1(i) - \sigma_2(i)| + |s'd_2 - sd_2| \geq d_1 + d_2$. □

Example 1. For the example $P(16,9) \geq P(9,5) * P(7,4) \geq 3,399$, we see that

$$F_1(x) = \begin{cases} x & \text{if } 1 \leq x \leq 4, \\ x+4 & \text{if } 5 \ \leq x \leq 9 \end{cases}$$

and

$$F_2(x) = \begin{cases} x+4 & \text{if } 1 \leq x \leq 4, \\ x+9 & \text{if } 5 \ \leq x \leq 7 \end{cases}.$$

Consider two permutations, say ρ = 1,2,3,4,5,6,7,8,9 and σ = 6,1,4,3,2,5,8,9,7, which are at Chebyshev distance 5, and a permutation, say τ = 1,2,3,4,5,6,7. Then, $F_1(\rho)$= 1,2,3,4,9,10,11,12,13 and $F_1(\sigma)$= 10,1,4,3,2,9,12,13,11. So,

$F_1(\rho)F_2(\tau)$ = 1,2,3,4,9,10,11,12,13,5,6,7,8,14,15,16, and
$F_1(\sigma)F_2(\tau)$= 10,1,4,3,2,9,12,13,11,5,6,7,8,14,15,16
are permutations on [1..16] and at Chebyshev distance 9.

Using the construction given in Theorem 4, we can obtain a PA for $P(3n,3)$ from PAs for $P(2n,2)$ and $P(n,1)$, respectively, which is of size $P(2n,2)*P(n,1)$. As we show in Corollary 22 that $P(2n,2) \geq \frac{(2n)!}{2^n}$ and, clearly, $P(n,1) = n!$, we have, for example, the lower bound $P(3n,3) \geq \frac{(2n)!n!}{2^n}$.

Turning now to the specific case of d=2. We first prove a recursive lower bound for P(n,2).

Theorem 21. *For all $n \geq 4$, $P(n,2) \geq P(n-2,2)\binom{n}{2}$.*

Proof. Let A be a PA on the $n-2$ symbols $\{1,...,n-2\}$ with Chebyshev distance 2. Take new symbols $a = n-1$, $b = n$, and insert them into each permutation of A in each of the possible $\binom{n}{2}$ positions such that a precedes b. If in the resulting permutation, the symbols appear in the order $a, n-2, b$, possibly separated by other symbols, then swap the positions of a and b. Let the resulting PA be B. Clearly, B has $\binom{n}{2}$ times as many permutations as A. We show that B has Chebyshev distance 2.

For a proof by contradiction, assume $\sigma, \tau \in B$ have $d(\sigma,\tau) \leq 1$. If σ, τ are such that, $\sigma(i), \tau(i) \in \{a,b\}$ and $\sigma(j), \tau(j) \in \{a,b\}$, for some i,j, then, $d(\sigma,\tau) \geq 2$, because removing symbols a,b gives a permutation in A and all permutations in A have distance at least 2. It follows that two permutations σ,τ have at most one position, say i, such that $\sigma(i), \tau(i) \in \{a,b\}$. If there is no position i such that $\sigma(i), \tau(i) \in \{a,b\}$, then $d(\sigma,\tau) \geq 2$, as the symbol b is at distance at least 2 with all symbols except a and itself. Similarly, it follows that there cannot be a position i such that $\sigma(i) = \tau(i) = a$ or $\sigma(i) = a$ and $\tau(i) = b$, as this means $\sigma(j) = b$, for some j, and $\tau(j) \notin \{a,b\}$, *i.e.* $|\sigma(j) - \tau(j)| \geq 2$.

There is one remaining case, namely, $\sigma(i) = \tau(i) = b$, for some i, then, for some $j \neq k$, $\sigma(j) = a$ and $\tau(k) = a$. As we are assuming $d(\sigma,\tau) \leq 1$, we must have $\tau(j) = n-2$ and $\sigma(k) = n-2$. Now consider the order of the positions i,j, and k. If both j and k are less than i, say in the order $j < k < i$. Then, the permutation σ has symbols in the order $a, n-2, b$, which contradicts the requirement that the symbols a and b are swapped. If both j and k are greater than i, say in the order $i < j < k$, then the permutation σ has the symbols in the order b, a, $n-2$, which contradicts the requirement that the symbols a and b not be swapped. Lastly, if we have the order, say $j < i < k$, then the permutation σ has the symbols in the order $n-2$, b, a, which contradicts the requirement that the symbols a and b not be swapped. ∎

The following gives a lower bound for $P(n,2)$ which is larger than the bound $P(2a,2) \geq \frac{97}{24}(a!)^2$ in [9] by an exponential factor. It is proven by induction using Theorem 21.

Corollary 22. $P(n, 2) \geq \frac{n!}{2^{\lceil n/2 \rceil}}$.

Proof. This is shown by induction on n. First observe that $P(3, 2) = 3$ and $P(2, 2) = 1$. For the inductive step, assume $P(n, 2) \geq \frac{n!}{2^{\lceil n/2 \rceil}}$. By Theorem 21, $P(n + 2, 2) \geq P(n, 2) * \binom{n+2}{2}$. By the inductive hypothesis, we obtain $P(n + 2, 2) \geq \frac{n!}{2^{\lceil n/2 \rceil}} \frac{(n+2)(n+1)}{2} = \frac{(n+2)!}{2^{\lceil (n+2)/2 \rceil}}$ □

Here is a proof for Theorem 8.

Theorem 8 For any $n \geq d \geq 1$, $P(n + m, d) \geq \max_{A \in Q((n+m),m,d)} \sum_{\sigma \in A} P_d(\sigma^C)$.

Proof. Let σ_1 and σ_2 be permutations of length m over the alphabet $[1...n]$ with Chebyshev distance at least d. We call these *prefixes*. Let τ_1 and τ_2 be permutations over $\Sigma_n^{-\sigma_1}$ with Chebyshev distance at least d. We call these *suffixes*. The Chebyshev distance between $\sigma_1\tau_1$ and $\sigma_1\tau_2$ is at least d and the Chebyshev distance between $\sigma_1\tau$ and $\sigma_2\tau$ is at least d, for any τ. So, for any set $U \in Q(n, m, d)$, the set $\{ \sigma\tau \mid \sigma \in U$ and $\tau \in V$, where $V \in Q_d(\Sigma_n^{-\sigma})\}$, is a PA on n symbols with pairwise Chebyshev distance at least d and has $\sum_{\sigma \in U} P_d(\Sigma_n^{-\sigma})$ permutations. □

As an example, we show that $P(12, 4) \geq 26,678$. Create a graph, say G, whose nodes are all prefixes of length three and whose edges connect such nodes with Chebyshev distance at least four. Furthermore, a node σ, a prefix of length three, is given the weight $P_4(\Sigma_{14}^{-\sigma})$. That is, the weight of a node is the maximum number of suffixes for the given prefix. By Theorem 5, the size of a maximum weighted clique of G is a lower bound for $P(12, 4)$. Using a MaxClique solver [10] [6] we obtaind the lower bound 26,678.

We now give a proof for Theorem 10.

Theorem 10. For any $d \geq 3$ and $k \geq 1$,

$$P(dk + d - 1, d) \geq \left((k + 1)^d - \binom{k + d - 1}{d - 1} \right) P(dk - 1, d).$$

Proof. Let $\Phi(a_1, a_2, \ldots, a_s)$ denote the alphabet $[1..(dk+d-1)] - \{a_1, a_2, \ldots, a_s\}$, for $a_1, a_2, \ldots, a_s \in [1..(dk + d - 1)]$. By Theorem 2, $P(dk + d - 1, d) \geq \sum_{a_1 \in A_1} P_d(\Phi(a_1))$, where $A_1 = \{d - 1, 2d - 1, \ldots, kd + d - 1\}$. Note that $|\Phi(a_1)| = dk + d - 2$. Similarly, for each $\Phi(a_1)$, by Theorem 2, $P_d(\Phi(a_1)) \geq \sum_{a_2 \in A_2} P_d(\Phi(a_1, a_2))$, where $A_2 = \{d - 2, 2d - 2, \ldots, kd + d - 2\}$. Note that $|\Phi(a_1, a_2)| = dk + d - 3$. By applying Theorem 2 $d - 1$ times, $P_d(\Phi(a_1, a_2, \ldots, a_{d-2})) \geq \sum_{a_{d-1} \in A_{d-1}} P_d(\Phi(a_1, a_2, \ldots, a_{d-1}))$, where $A_{d-1} = \{1, d + 1, \ldots, kd + 1\}$. Note that $|\Phi(a_1, a_2, \ldots, a_{d-1})| = dk$.

$$P(dk + d - 1, d) \geq \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \cdots \sum_{a_{d-1} \in A_{d-1}} P_d(\Phi(a_1, a_2, \ldots, a_{d-1})). \tag{2}$$

Note that there are $k + 1$ choices for each of the symbols a_i, $1 \leq i \leq d - 1$, with the property that any two choices are at distance at least d. Consider a sequence $\alpha = (a_1, a_2, \ldots, a_{d-1})$ with $a_i \in A_i, 1 \leq i \leq d - 1$. We call such a sequence $a_1, a_2, \ldots, a_{d-1}$ *monotone* if $a_1 > a_2 > \cdots > a_{d-1}$; otherwise, the sequence is *mixed*.

So far, we have sequences, such as α, of length $d-1$. We now consider sequences of length d obtained by adding an extra symbol to α (at the end). Since $|\Phi(a_1, a_2, \ldots, a_{d-1})| = dk$, by Theorem 1

$$P_d(\Phi(a_1, a_2, \ldots, a_{d-1}), d) \geq kP(dk-1, d).$$

That is, the proof of Theorem 1 shows there are always k symbols one can add to the end of such sequences α and preserve distance d. We show that $P_d(\Phi(a_1, a_2, \ldots, a_{d-1})) \geq (k+1)P(dk-1, d)$, if the sequence $a_1, a_2, \ldots, a_{d-1}$ is mixed. That is, there are always $k+1$ symbols at pairwise distance d to add to the end of α, if α is mixed. Note that, for symbols x and y, such that $d(x, y) \geq d$, $d(\alpha x, \alpha y|) \geq d$.

Assume $a_1, a_2, \ldots, a_{d-1}$ is mixed. We construct a sequence $S = s_1, s_2, \ldots s_{k+1}$ of elements in $P_d(\Phi(a_1, a_2, \ldots, a_{d-1}))$ with $d(s_i, s_{i+1}) \geq d$, for all i. Using S we get $k+1$ sequences, say $\tau_1, \tau_2, \ldots, \tau_{k+1}$, where τ_i consists of $a_1, a_2, \ldots, a_{d-1}$ followed by s_i. It follows that $P_d(\Phi(\tau_i)) \geq (k+1)P(dk-1, d)$.

Consider a table T with $d-1$ columns and $k+1$ rows, where row i of T contains the i^{th} element of A_j and column j of T, $1 \leq j \leq d-1$ contains the elements of A_{d-j} in sorted order. In particular, row i and column j of T contains the element $(i-1)d + j$. See Table 2 for an example when $d = 6$ and $k = 5$.

The desired sequence $S = s_1, s_2, \ldots, s_{k+1}$ is obtained from Table 2 by choosing one element from each row with the property that the element chosen from row $i+1$ must come from a column whose index is at least as large as the index of the column chosen for row i. (This is to ensure distance at least d.) Also, an element must be chosen from each row in order to get a sequence of length $k+1$. In addition, one cannot choose any of the elements in the sequence $a_1, a_2, \ldots, a_{d-1}$, which are already in α, and so are numbers deleted from the alphabet, There is one and only one such symbol in each column. For example, consider the mixed sequence 17, 22, 15, 8, 1 shown (in bold) in Table 2 (represented in the table in right-to-left order). In this example a desired sequence S can be chosen to be 4, 10, 16, 23, 29, 35. In the mixed sequence 17, 22, 15, 8, 1 we have $a_1 = 17 < a_2 = 22$.

In every mixed sequence $a_1, a_2, \ldots, a_{d-1}$ there must be a j such that $a_j \leq a_{j+1}$. The desired sequence S can be chosen by taking elements in order in column $d-j-1$ until (but not including) a_{j+1}, say in row i), followed by elements in column $d-j$ starting in row i and continuing through all remaining rows. This always works as (1) each column has one and only one deleted element and (2) the condition $a_j \leq a_{j+1}$ ensures that the deleted element in column $d-j$ occurs in a row with index smaller than i.

Observe that, if $a_1, a_2, \ldots, a_{d-1}$ is monotone, there is no j such that $a_j < a_{j+1}$. Consequently, there is no way to construct the desired sequence S by moving to a higher index column when a deleted symbol is encountered. That is, the higher index column always has a different deleted symbol in the given row or a latter row.

Let M be the set of all sequences $m_j = a_1, a_2, \ldots, a_{d-1}$ with $a_i \in \{d-i, 2d-i, \ldots, kd+d-i\}$, for all $i, 1 \leq i \leq d-1$, with the property that, for $j \neq k$, $d(m_j, m_k) \geq d$. Map each sequence $m_i = a_1, a_2, \ldots,\ a_{d-1}$ to $x = (x_1, x_2, \ldots, x_{d-1}) \in [0..k]^{d-1}$ using

$$x = (\lfloor a_1/d \rfloor, \lfloor a_2/d \rfloor, \lfloor a_3/d \rfloor, \ldots, \lfloor a_{d-1}/d \rfloor).$$

A sequence $a_1, a_2, \ldots, a_{d-1}$ is monotone if and only if $x_1 \geq x_2 \geq \cdots \geq x_{d-1}$. The number of such vectors x is $\binom{k+d-1}{d-1}$. (This is the number of ways of choosing a set of $d-1$ elements from $k+1$ sets of $d-1$ indistinguishable items.) So, the number of monotone sequences $a_1, a_2, \ldots, a_{d-1}$ is

1	2	3	4	5
7	**8**	9	10	11
13	14	**15**	16	**17**
19	20	21	**22**	23
25	26	27	28	29
31	32	33	34	35

Table 2: An example of a mixed sequence (in bold), for $d = 6$ and $k = 5$. The sequence 17,22,15,8,1 is shown right-to-left.

$n_{mon} = \binom{k+d-1}{d-1}$. The number of mixed sequences $a_1, a_2, \ldots, a_{d-1}$ is $n_{mix} = (k+1)^{d-1} - \binom{k+d-1}{d-1}$. That is, the number of choices for $a_1 \in A_1, a_2 \in A_2, ..., a_{d-1} \in A_{d-1}$ is $(k+1)^{d-1}$, and

$$P(dk+d-1,d) \geq \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \cdots \sum_{a_{d-1} \in A_{d-1}} P_d(\Phi(a_1, a_2, \ldots, a_{d-1})) \quad (3)$$

$$\geq (kn_{mon} + (k+1)n_{mix})P(dk-1,d) \quad (4)$$

$$\geq \left((k+1)^d - \binom{k+d-1}{d-1}\right) P(dk-1,d). \quad (5)$$

The theorem follows. □

Lower bounds for $P(n,d)$ are given in Table 1. The values in bold are exact. Precise lower bounds for $P(n,2)$ are given in Theorem 14. Other lower bounds are from Theorems 2, 3, 4 and 7, and from the Random/Greedy algorithm. We offer some side-by-side comparisons with results from Table II in [9] shown below in parentheses.

$P(5,2) \geq 30$ *(29)*
$P(n,n-2) = 10$, for all $n \geq 5$ *(9)*
$P(8,3) \geq 430$ *(401)*
$P(8,4) \geq 70$ *(68)*
$P(n,n-4) \geq 103$, for all $n \geq 9$ *(95)*
$P(11,6) \geq 326$ *(236)*

$P(7,2) \geq 630$ *(582)*
$P(7,3) \geq 100$ *(84)*
$P(n,n-3) \geq 33$, for all $n \geq 7$ *(28)*
$P(9,4) \geq 295$ *(283)*
$P(10,5) \geq 247$ *(236)*
$P(n,n-5) \geq 330$, for all $n \geq 12$ *(236)*

3 Upper Bounds

We begin with a proof of Theorem 13, which is an improvement on Theorem 12.

Theorem 13. For $1 \leq k \leq d < n$,

$$P(n,d) \leq P(n-k,d) \cdot \binom{n}{k}.$$

Proof. Consider any PA on n symbols with distance d. Partition the PA into subsets determined by the positions of the highest k symbols, $\{n-k+1, n-k+2, \ldots, n\}$. Two permutations are in the same subset if their highest k symbols occur in the same subset of k positions, though not necessarily with the same symbol in the same position. For example if $n = 5, d = 2$, and $k = 2$,

Table 3: Upper Bounds for $P(n,d)$.

n/d	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	3	1	1	1	1	1	1	1	1
4	6	3	1	1	1	1	1	1	1
5	30	10	3	1	1	1	1	1	1
6	90	20	10	3	1	1	1	1	1
7	630	105	35	10	3	1	1	1	1
8	2,520	560	70	56	10	3	1	1	1
9	22,680	1,680	378	126	84	10	3	1	1
10	113,400	12,600	2,100	256	210	100	10	3	1
11	see Thm 7	92,400	11,550	1,386	462	330	110	10	3
12	see Thm 7	369,600	34,650	7,920	924	792	495	120	10
13	see Thm 7	3,603,600	270,270	72,072	5,148	1,716	1,287	715	130
14	see Thm 7	33,633,600	2,102,100	252,252	30,030	3,432	3,003	2,002	910
15	see Thm 7	168,168,000	15,765,750	768,768	420,420	19,305	6,435	5,005	3,003

then the permutations 54321 and 45132 would be in the same subset since the symbols 4 and 5 both occur in positions 1 and 2. Observe that there can be at most $\binom{n}{k}$ subsets since that is the number of ways to choose k positions.

Since any two permutations must have distance at least d, and there is no way for any pair of the highest $k \leq d$ symbols to satisfy this distance, within a single subset the Chebyshev distance must be satisfied by the remaining $n-k$ symbols, $\{1, 2, \ldots, n-k\}$. Assume each of the $\binom{n}{k}$ subsets contains $P(n-k,d)$ permutations. If we add one additional permutation to the PA, it will belong to exactly one of these subsets. If we take that subset and delete the highest k symbols from each permutation, we are left with a contracted PA on $n-k$ symbols and distance d, however it now contains more than $P(n-k,d)$ permutations, giving us a contradiction. Therefore we can have no more than $P(n-k,d) \cdot \binom{n}{k}$ permutations in the original PA. □

Note that the best results from Theorem 13 typically come from choosing $k = d$.

Example 2. By Theorem 13, $P(11,6) \leq P(5,6)\binom{11}{6}$. Since $P(5,6) = 1$, this means $P(11,6) \leq \binom{11}{6} = 462$. In [9], Example 3, they gave $P(11,6) \leq 850$.

Again, we turn to d=2.

Corollary 23. $P(n,2) \leq \frac{n!}{2^{\lfloor n/2 \rfloor}}$.

Proof. This is shown by induction on n. First observe that $P(3,2) = 3$ and $P(2,2) = 1$. For the inductive step, assume $P(n,2) \leq \frac{n!}{2^{\lfloor n/2 \rfloor}}$. By Theorem 13, $P(n+2,2) \leq P(n,2) * \binom{n+2}{2}$. By the inductive hypothesis, we obtain $P(n+2,2) \leq \frac{n!}{2^{\lfloor n/2 \rfloor}} \frac{(n+2)(n+1)}{2} = \frac{(n+2)!}{2^{\lfloor (n+2)/2 \rfloor}}$ □

Theorem 7. $P(n,2) = \frac{n!}{2^{\lfloor n/2 \rfloor}}$.

Theorem 7) follows directly from Corollaries 22 and 23.

Upper bounds, for small values of n and d, shown in Table 3 were computed by determining the largest clique in a "distance" graph, *i.e.* a graph with a node for each permutation and an edge between pairs of nodes at distance at least d. Others are computed by Theorem 13. We offer some side-by-side comparisons with results from Table II in [9] shown in parentheses below.

$$P(4,2) \leq 6 \quad \textit{(24)} \qquad P(5,2) \leq 30 \quad \textit{(120)}$$
$$P(6,2) \leq 90 \quad \textit{(720)} \quad P(7,2) \leq 630 \quad \textit{(5040)}$$
$$P(5,3) \leq 10$$

We give next a proof for Theorem 16. The basic idea is that if $P(n_0, n_0 - k) \leq m$, and m is small enough compared to n_0, then one can prove that the diagonal in the lower bound table, such as Table 1, *i.e.* $P(n, n-k)$, for all $n \geq n_0$, is also m. The argument is a counting argument based on the number of potent symbols and the length of the permutation.

Theorem 16. Suppose that $P(n_0, n_0 - k) \leq m$ such that

$$2k(m+1) < (n_0 + 1)(1 + \lfloor n_0/(2k-1) \rfloor). \tag{6}$$

Then $P(n, n-k) \leq m$, for all $n \geq n_0 \geq 2k$.

Proof. Suppose to the contrary that $P(n, n-k) \geq m+1$, for some $n > n_0$. Let n be the smallest such number. Let $A = \{\pi_1, \pi_2, \ldots, \pi_{m+1}\}$ be a PA on n symbols with distance $n-k$. Let k_i denote the number of potent symbols in position i, taken over all permutations in A. Let $z = 1 + \lfloor n_0/(2k-1) \rfloor$, so $n_0 \geq (z-1)(2k-1)$. We show that $k_i \geq z$, for all i. Suppose, by symmetry of argument, that $k_1 \leq z-1$ and (by rearranging permutation order) only π_i, $1 \leq i \leq k_1$, have potent symbols in the first position. Observe that each permutation has $2k$ potent symbols, *i.e.* the symbols in $[1..k] \cup [n-k+1..n]$, and that, by our assumption, all of the first k_1 permutations, and only the first k_1 permutations, have a potent symbol in position 1. So, if there are $z-1$ permutations, each adding $2k-1$ potent symbols to some position $j > 1$, the total number of potent symbols (other than the one in position 1) is $(2k-1)(z-1)$. Since the number of positions, namely, $n > n_0$, is greater than $(2k-1)(z-1)$, by the pigeonhole principle, there is a position $j > 1$ where all π_i, $1 \leq i \leq k_1$, do not have potent symbols. Merge columns 1 and j and decrease n. That is, do the following:

- for each permutation π_i, $1 \leq i \leq k_1$, exchange the potent symbol in position 1 with the symbol in position j.
- delete the symbol in position 1 in all permutations (they are no longer potent) and appropriately modify the symbols in each permutation so that they are consecutive integers (deletions may have created gaps).

The result is a PA of $m+1$ permutations on $n-1$ symbols with Chebyshev distance $n-k$. This contradicts our choice of n being smallest.

Note that the total number of potent symbols in the PA A is $2k(m+1)$. Since $k_i \geq z$, for all $1 \leq i \leq n$, $2k(m+1) \geq nz \geq (n_0+1)(1 + \lfloor n_0/(2k-1) \rfloor)$ which contradicts Inequality 6. □

Corollary 17. $P(n, n-2) = 10$, for all $n \geq 5$.

Table 4: Lower bounds for $P(n, m, 2)$ (left) and $P(n, m, 3)$ (right). The tight bounds are in bold.

n/m	2	3	4	5
4	**4**	**6**	**1**	**1**
5	6	15	23	30
6	**9**	24	53	78
7	12	42	104	234
8	**16**	59	187	479
9	20	88	306	979
10	**25**	115	478	1,732
11	30	158	709	3,002
12	**36**	202	1,028	4,805
13	42	261	1,430	7,490
14	**49**	322	1,953	11,165
15	56	400	2,600	16,291

n/m	2	3	4	5
4	**2**	**3**	**1**	**1**
5	**4**	6	6	**10**
6	**4**	**8**	14	19
7	6	15	30	49
8	**9**	24	49	107
9	**9**	**27**	78	181
10	12	40	118	313
11	**16**	59	177	530
12	**16**	**64**	245	817
13	20	85	333	1,232
14	**25**	116	466	1,838
15	**25**	**125**	601	2,620

Proof. $P(n, n-2) = 10$, for all $5 \leq n \leq 11$, by the clique approach. In Theorem 16, set $n_0 = 11, k = 2$, and $m = 10$. Then $z = 1 + \lfloor n_0/(2k-1) \rfloor = 4$ and $2k(m+1) = 44 < 48 = (n_0+1)z$. So, $P(n, n-2) \leq 10$, for all $n \geq 11$, follows by Theorem 16. By Theorem 20 , $P(n, n-2) \geq 10$, for all $n \geq 5$. Therefore $P(n, n-2) = 10$, for all $n \geq 5$. ☐

Theorem 14 states that $P(n, d)$ values along the diagonal $n = d + r$ in Table 1 are all equal to c_r, if $n \geq d_r$, for some constants c_r and d_r. Corollary 17 shows that these constants for $r = 2$ are $c_2 = 10$ and $d_2 = 3$.

4 Prefixes

Computed values for $P(n, m, d)$, for $2 \leq d \leq 5$, $4 \leq n \leq 15$, and $2 \leq m \leq 5$ are given in Tables 4, 5. For example, $P(9, 3, 4) \geq 15$, as shown in Table 5, means there is a set of 15 prefix strings of three symbols over the alphabet [1..9] with pairwise Chebyshev distance 4. For example, $\{795, 451, 125, 129, 165, 169, 291, 512, 516, 569, 691, 851, 912, 916, 956\}$ is such a set. Our computations use a modification of the Random/Greedy algorithm to compute $Q(n, m, d)$. These sets are useful in applications of Theorem 8 toward obtaining improved lower bounds. Our computed sets are available on our web site.

Theorem 24. *If $d \mid n$ and $d \geq m \geq 2$, then $P(n, m, d) = (n/d)^m$.*

Proof. Let $k = n/d$. First, we show that $P(n, m, d) \leq k^m$. Let A be an array of size $P(n, m, d)$ in $Q(n, m, d)$. Map each permutation π in A to $[0..k^m - 1]$ using $f(\pi) = \sigma$ where $\sigma(i) = j$ if $\pi(i) \in [jd + 1 \ldots jd + d - 1]$. Since $d(A) = d$, map f is injective. Therefore $P(n, m, d) \leq k^m$.

To show the lower bound $P(n, m, d) \geq k^m$, consider set $A \in Q(n, m, d)$ of permutations π such that $\pi(i) \in \{i, i + d, \ldots, i + (k-1)d\}$ for all $i \in [1..m]$. Then $|A| = k^m$ and $d(A) = d$. The theorem follows. ☐

Theorem 25. *If $d \mid n$ and $d \geq m \geq 2$, then $P(n - i, m, d) = (n/d)^m$ for any $i \in [0..d - m]$.*

Proof. Let $k = n/d$. By Theorem 24, the theorem follows for $i = 0$. Then $P(n-i, m, d) \leq k^m$ for $i \geq 1$.

To show lower bound $P(n-i, m, d) \geq k^m$, consider set $A \in Q(n, m, d)$ of all permutations π such that $\pi(j) \in \{j, j+d, \ldots, j+(k-1)d\}$, for all $j \in [1..m]$. All numbers in π are $\leq m+(k-1)d = kd + m - d = n + m - d \leq n - i$. The theorem follows. □

Table 5: Lower bounds for $P(n, m, 4)$ (left) and $P(n, m, 5)$ (right). The tight bounds are in bold.

n/m	2	3	4	5
4	**1**	**1**	**1**	**1**
5	**2**	**3**	**3**	**3**
6	**4**	6	6	9
7	**4**	**8**	14	18
8	**4**	**8**	**16**	30
9	6	15	28	55
10	**9**	24	50	97
11	**9**	**27**	76	174
12	**9**	**27**	**81**	234
13	12	41	116	334
14	**16**	58	176	512
15	**16**	**64**	243	803

n/m	2	3	4	5
4	**1**	**1**	**1**	**1**
5	**1**	**1**	**1**	**1**
6	**2**	**3**	**3**	**3**
7	**4**	6	6	9
8	**4**	**8**	14	18
9	**4**	**8**	**16**	30
10	**4**	**8**	**16**	**32**
11	6	15	28	55
12	**9**	24	49	95
13	**9**	**27**	77	173
14	**9**	**27**	**81**	236
15	**9**	**27**	**81**	**243**

5 Conclusion and Open Problems

We have given several new lower and upper bounds (See Tables 1 and 3) for $P(n, d)$ as well as several new techniques for their computation. We conjecture that the bounds for c_r and d_r in Theorem 14 can be improved. For example, from Table 1 it appears that $c_3 \geq 33$ and $c_4 \geq 103$. Is it true that $c_3 = 33, d_3 = 4$, and $c_4 = 103, d_4 = 5$?

We computed lower bounds for $P(n, m, d)$ for $n \leq 15$ and $m \leq 5$ (see Tables 4 and 5). The computation of bounds for $P(n, m, d)$ is significantly faster than the computation of bounds for $P(n, d)$ if m is small. Is there a polynomial time algorithm for computing $P(n, m, d)$, for $m = O(1)$?

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