arXiv:1607.01591v1 [quant-ph] 6 Jul 2016

Ordering states of Tsallis relative α-entropies of coherence

Fu-Gang Zhang,[1,2] Lian-He Shao,[3] Yu Luo,[3] and Yongming Li[1,3,*]
[1]*School of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710119, China*
[2]*School of Mathematics and Statistics, HuangShan University, Huangshan, 245041, China*
[3]*College of Computer Science, Shaanxi Normal University, Xi'an, 710119, China*
(Dated: April 12, 2018)

In this paper, we study the ordering states with Tsallis relative α-entropies of coherence and l_1 norm of coherence for single-qubit states. We show that any Tsallis relative α-entropies of coherence and l_1 norm of coherence give the same ordering for single-qubit pure states. However, they don't generate the same ordering for some high dimensional pure states, even though these states are pure. We also consider three special Tsallis relative α-entropies of coherence, such as C_1 ,C_2 and $C_{\frac{1}{2}}$, and show any one of these three measures and C_{l_1} will not generate the same ordering for single-qubit mixed states. Furthermore, we find that any two of these three special measures generate different ordering for single-qubit mixed states.

PACS numbers: 03.65.Aa, 03.67.Mn

I. INTRODUCTION

Quantum coherence is one of the most important physical resources in quantum mechanics, which can be used in quantum optics [1], quantum information and quantum computation [2], thermodynamics [3, 4], and low temperature thermodynamics [5–7]. Many efforts have been made in quantifying the coherence of quantum states [8]. The authors of Ref. [9] proposed a rigorous framework to quantify coherence. The framework gave four conditions that any proper measure of the coherence must satisfy. Based on this framework, one can define suitable measures with respect to the prescribed orthonormal basis, including the relative entropy of coherence and the l_1 norm of coherence [9]. In addition, various other coherence measures were discussed [10–17]. Many further discussions about quantum coherence were aroused [18–29].

Up to now, many different coherence measures have been proposed based on different physical contexts. For the same state, different values of coherence will be obtained by different coherence measures. In this case, a very important question appears, that is, whether these measures generate the same ordering. We say that two coherence measures C_m and C_n generate the same ordering if they satisfy the condition

$$C_m(\rho) \leq C_m(\sigma) \Leftrightarrow C_n(\rho) \leq C_n(\sigma) \tag{1}$$

for any density operators ρ and σ. Liu et al. [18] showed that the relative entropy of coherence and the l_1 norm of coherence don't give the same ordering for some mixed states. The topics about ordering states were widely considered in entanglement measures [31–36] and quantum correlation measures [37–41]. Recently, the author of Ref. [17] proposed Tsallis relative α-entropies of coherence. The author proved Tsallis relative α-entropies of coherence satisfy the conditions of (C1),(C2a) and (C3). But the condition of (C2b), i.e. Monotonicity under incoherent selective measurements, seems to be more sophisticated. In fact, a counterexample showed that Tsallis relative α entropies of coherence may violate the condition (C2b) in some situations. Whereas, these coherence measures satisfy a generalized monotonicity for average coherence under subselection based on measurement [17].

In this paper, we study the ordering states with Tsallis relative α-entropies of coherence and l_1 norm of coherence for single-qubit states. First, we show that any Tsallis relative α-entropies of coherence and l_1 norm of coherence give the same ordering for single-qubit pure states. However, the condition (1) doesn't always satisfy for high dimensional pure states. Second, we consider three special Tsallis relative α-entropies of coherence, such as C_1 ,C_2 and $C_{\frac{1}{2}}$, and show any one of these three measures and C_{l_1} will not generate the same ordering for single-qubit mixed states. Furthermore, we find that any two of these three special measures generate different ordering for single-qubit mixed states.

This paper is organized as follows. In Sec. II, we briefly review some notions related to Tsallis relative α-entropies of coherence and l_1 norm of coherence. In Sec. III, we show that Tsallis relative α-entropies of coherence and l_1 norm of coherence generate the same ordering for single-qubit pure states. In Sec. IV, we show that they may not generate the same ordering for some single-qubit mixed states, and we give some examples to show our results. We summarize our results in Sec. V.

II. PRELIMINARIES

In this section, we review some notions related to quantifying quantum coherence. Considering a finite-

*Electronic address: liyongm@snnu.edu.cn

dimensional Hilbert space H with $d = dim(H)$. Fix a basis $\{|i\rangle\}$, let I be a set of incoherence states, which is of the form

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|,$$

where $\delta_i \in [0,1]$, and $\sum_{i=1}^{d} \delta_i = 1$. Baumgratz et al. [9] proposed that any proper measure of the coherence C must satisfy the following four conditions:

$(C1) : C(\rho) \geq 0$ and $C(\rho) = 0$ if and only if $\rho \in I$;

$(C2a) : C(\rho) \geq (\Phi(\rho))$, where Φ is any incoherent completely positive and trace preserving maps;

$(C2b) : C(\rho) \geq \sum_i p_i C(\rho_i)$, where $p_i = Tr(K_i \rho K_i^\dagger)$, $\rho_i = \frac{K_i \rho K_i^\dagger)}{Tr(K_i \rho K_i^\dagger)}$, for all K_i with $\sum_i K_i K_i^\dagger = I$ and $K_i I K_i^\dagger \subseteq I$.

$(C3) : \sum_i p_i C(\rho_i) \geq C(p_i \rho_i)$ for any ensemble $\{p_i, \rho_i\}$.

It has been shown that l_1 norm of coherence and relative entropy of coherence satisfy these four conditions [9]. l_1 norm of coherence [9] is defined as

$$C_{l_1}(\rho) = \sum_{i \neq j} \mid \rho_{ij} \mid, \tag{2}$$

here ρ_{ij} are entries of ρ. The coherence measure defined by the l_1 norm is based on the minimal distance of ρ to the set of incoherent states I, $C_D(\rho) = min_{\delta \in I} D(\rho, \delta)$ with D being the l_1 norm, and there is $0 \leq C_{l_1}(\rho) \leq d-1$. The upper bound is attained for the maximally coherent state $\mid \varphi_{max}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \mid i\rangle$.

Tsallis relative α-entropies [42, 43] for the density matrices ρ and δ, denoted by $D_\alpha(\rho\|\delta)$, is defined as

$$D_\alpha(\rho\|\delta) = \frac{Tr(\rho^\alpha \delta^{1-\alpha}) - 1}{\alpha - 1}$$

for $\alpha \in (0,1) \sqcup (1,\infty)$. $D_\alpha(\rho\|\delta)$ reduces to the von Neumann relative entropy when $\alpha \to 1$ [42], i.e.,

$$\lim_{\alpha \to 1} D_\alpha(\rho\|\delta) = S(\rho\|\delta) = Tr[\rho(\ln\rho - \ln\delta)].$$

Tsallis relative α-entropies of coherence [17], denoted by $C_\alpha(\rho)$, is defined as

$$C_\alpha(\rho) = \min_{\delta \in I} D_\alpha(\rho\|\delta).$$

$C_\alpha(\rho)$ reduces to relative entropy of coherence $C_r(\rho)$ when $\alpha \to 1$ [9], i.e., $C_1(\rho) = C_r(\rho) = S(\rho_{diag}) - S(\rho)$. The author of Ref. [17] proved that Tsallis relative α-entropies of coherence satisfy the conditions of (C1), (C2a) and (C3) for all $\alpha \in (0,2]$, but it may violate (C2b) in some situations. However, these measures satisfy a generalized monotonicity for average coherence under subselection based on measurement as the following form [17].

For all $\alpha \in (0,2]$, Tsallis relative α-entropies of coherence $C_\alpha(\rho)$ satisfy

$$\sum_i p_i^\alpha q_i^{1-\alpha} C_\alpha(\rho_i) \leq C_\alpha(\rho) \tag{3}$$

where $p_i = Tr(K_i \rho K_i^\dagger)$, $q_i = Tr(K_i \delta_\rho K_i^\dagger)$, and $\rho_i = \frac{K_i \rho K_i^\dagger}{p_i}$.

A .E. Rastegin [17] gave an elegant mathematical analytical expression of Tsallis relative α-entropies of coherence. For all $\alpha \geq 0$ and $\alpha \neq 1$, the Tsallis relative α-entropies of coherence $C_\alpha(\rho)$, for a state ρ, can be expressed as

$$C_\alpha(\rho) = \frac{1}{\alpha - 1}\{r^\alpha - 1\} \tag{4}$$

where $r = \sum_i \langle i|\rho^\alpha|i\rangle^{\frac{1}{\alpha}}$. For the given ρ and α, based on this coherence measure, the nearest incoherence state from ρ is the state

$$\delta_\rho = \frac{1}{r} \sum_i \langle i|\rho^\alpha|i\rangle\}^{\frac{1}{\alpha}} |i\rangle\langle i|.$$

Considering an interesting case $\alpha = 2$, we get

$$C_2(\rho) = (\sum_j \sqrt{\sum_i |\rho_{i,j}|^2})^2 - 1 \tag{5}$$

where $\rho_{i,j} = \langle i|\rho|j\rangle$. C_2 is a function of squared module $|\rho_{i,j}|^2$, we should distinguish it from l_2 norm of coherence C_{l_2}. C_{l_2} is defined as

$$C_{l_2}(\rho) = \sum_{i \neq j} \mid \rho_{ij} \mid^2 .$$

It has been shown that C_{l_2} doesn't satisfy the condition (C2b) [9]. Although C_2 also violates the condition (C2b), but it obeys a generalized monotonicity property Eq. (3) [17].

III. ORDERING STATES WITH C_α AND C_{l_1} FOR SINGLE-QUBIT PURE STATES

In this section, we show that Tsallis relative α-entropies of coherence and l_1 norm of coherence generate the same ordering for single-qubit pure states.

Let $\mid \psi\rangle = \sqrt{p} \mid 0\rangle + e^{i\varphi}\sqrt{1-p} \mid 1\rangle$ be a single-qubit pure state, where $p \in [0,1]$. It is easy to calculate that l_1 norm of coherence of $\mid \psi\rangle$ is equal to $C_{l_1} = 2\sqrt{p(1-p)}$. Tsallis relative α-entropies of coherence is equal to $C_\alpha = \frac{1}{\alpha-1}\{r^\alpha - 1\}$, where $r = p^{\frac{1}{\alpha}} + (1-p)^{\frac{1}{\alpha}}$. So we have the following proposition.

Proposition 1: (1) C_{l_1} is an increasing function for $p \leq \frac{1}{2}$, and it is a decreasing function for $p \geq \frac{1}{2}$.

(2) C_α is an increasing function for $p \leq \frac{1}{2}$, and it is a decreasing function for $p \geq \frac{1}{2}$.

Proof: (1) It is clear that $C_{l_1} = 2\sqrt{p(1-p)}$ is an increasing function for $p \leq \frac{1}{2}$, and is a decreasing function for $p \geq \frac{1}{2}$.

(2) We first consider the derivation of r with respect to p. It is obvious that

$$\frac{\partial r}{\partial p} = \frac{1}{\alpha}[p^{\frac{1-\alpha}{\alpha}} + (1-p)^{\frac{1-\alpha}{\alpha}}] \begin{cases} > 0, & \alpha < 1, p > \frac{1}{2}, \\ < 0, & \alpha < 1, p < \frac{1}{2} \\ < 0, & \alpha > 1, p > \frac{1}{2}, \\ > 0, & \alpha > 1, p < \frac{1}{2}. \end{cases}$$

So we can know

$$\frac{\partial C_\alpha}{\partial p} = \frac{\alpha}{\alpha - 1} r^{\alpha-1} \frac{\partial r}{\partial p} \begin{cases} < 0, & \alpha < 1, p > \frac{1}{2}, \\ > 0, & \alpha < 1, p < \frac{1}{2} \\ < 0, & \alpha > 1, p > \frac{1}{2}, \\ > 0, & \alpha > 1, p < \frac{1}{2}. \end{cases}$$

Therefore, C_α is an increasing function for $p \leq \frac{1}{2}$, and is a decreasing function for $p \geq \frac{1}{2}$.

By the above proposition, we can show that C_α and C_{l_1} give the same ordering for single-qubit pure states. Let $|\psi\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$ and $|\varphi\rangle = \sqrt{q}\,|0\rangle + \sqrt{1-q}\,|1\rangle$ be two single-qubit pure states. The following result can be obtained.

Result 1: $C_\alpha(|\psi\rangle) \leq C_\alpha(|\varphi\rangle)$ if and only if $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\varphi\rangle)$.

Proof: It is easy to know $C_\alpha(p) = C_\alpha(1-p)$, $C_{l_1}(p) = C_{l_1}(1-p)$. Without loss of generality, we can set $p, q \leq \frac{1}{2}$. In line with proposition 1, we have $C_\alpha(|\psi\rangle) \leq C_\alpha(|\varphi\rangle)$ if and only if $p \leq q$, and $p \leq q$ if and only if $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\varphi\rangle)$. Therefore, $C_\alpha(|\psi\rangle) \leq C_\alpha(|\varphi\rangle)$ if and only if $C_{l_1}(|\psi\rangle) \leq C_{l_1}(|\varphi\rangle)$.

Result 1 shows, for any $\alpha \in (0, 2]$, C_α and C_{l_1} generate the same ordering for single-qubit pure states. Moreover, for any two $\alpha_1, \alpha_2 \in (0, 2]$, C_{α_1} and C_{α_2} also generate the same ordering for single-qubit pure states. Some explicit examples as Fig. 1 can intuitively show our conclusion.

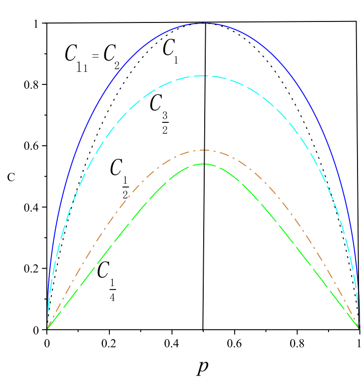

Fig. 1. Tsallis relative α-entropies of coherence versus l_1 norm of coherence for single-qubit pure states.

It is worth noting that result 1 is only effective for single-qubit pure states. We find this result may be invalid for states in high dimensional systems, even though these states are pure. We give a counterexample. Two qutrit pure states [18] are given as follow,

$$|\psi_1\rangle = \sqrt{\tfrac{12}{25}}\,|0\rangle + \sqrt{\tfrac{12}{25}}\,|1\rangle + \sqrt{\tfrac{12}{25}}\,|2\rangle,$$
$$|\psi_2\rangle = \sqrt{\tfrac{7}{10}}\,|0\rangle + \sqrt{\tfrac{2}{10}}\,|1\rangle + \sqrt{\tfrac{1}{10}}\,|2\rangle.$$

It is easy to calculate that $C_{l_1}(|\psi_1\rangle) = 1.5143$, $C_{\frac{1}{2}}(|\psi_1\rangle) = 0.6400$, $C_{l_1}(|\psi_2\rangle) = 1.5603$ $C_{\frac{1}{2}}(|\psi_2\rangle) = 0.5303$. It is clear that $C_{l_1}(|\psi_1\rangle) < C_{l_1}(|\psi_2\rangle)$, and $C_{\frac{1}{2}}(|\psi_1\rangle) > C_{\frac{1}{2}}(|\psi_2\rangle)$. So we know that C_{l_1} and $C_{\frac{1}{2}}$ generate different ordering for single-qutrit pure states $|\psi_1\rangle$ and $|\psi_2\rangle$.

IV. ORDERING STATES WITH C_α AND C_{l_1} FOR SINGLE-QUBIT MIXED STATES

We consider ordering states with C_α and C_{l_1} for single-qubit mixed states. Any single-qubit state ρ can be written as [2]

$$\rho(x, y, z) = \begin{bmatrix} \frac{1+z}{2} & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} \end{bmatrix}$$

with $x^2 + y^2 + z^2 \leq 1$. By a diagonal and unitary matrix [18], $\rho(x, y, z)$ can be transformed into a state with the form

$$\rho(t, z) = \begin{bmatrix} \frac{1+z}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{1-z}{2} \end{bmatrix} \tag{6}$$

with $t^2 + z^2 \leq 1$. It has been shown that $\rho(x, y, z)$ and $\rho(t, z)$ can be transformed into each other by an incoherent operation. Therefore, we only need to consider states with the form $\rho(t, z)$. Next, we show C_α and C_{l_1} generate the different ordering for some single-qubit mixed states with the form Eq. (6). Based on the Eq. (2), we get the l_1 norm of coherence of $\rho(t, z)$, $C_{l_1}(\rho(t, z)) = t$. Because the expression of C_α is sophisticated for any $\alpha \in (0, 2]$, so we only consider three special Tsallis relative α-entropies of coherence, C_2, C_1 and $C_{\frac{1}{2}}$.

By substituting Eq. (6) into Eq. (5), we obtain the expression of C_2.

$$C_2(\rho) = r_2^2 - 1, \tag{7}$$

where

$$r_2 = \frac{1}{2}\sqrt{(1+z)^2 + t^2} + \frac{1}{2}\sqrt{(1-z)^2 + t^2}. \tag{8}$$

The authors of Ref. [18] considered the ordering states with the relative entropy of coherence C_r and the l_1 norm of coherence C_{l_1}, and obtained many useful results. In this section, we see C_r as a special C_α of $\alpha = 1$. We again discuss this question from our perspective. For $\alpha \to 1$, Tsallis α-relative entropies reduce to the von Neumann relative entropy [42]

$$\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = S(\rho\|\delta) = Tr[\rho(\ln\rho - \ln\delta)].$$

Thus $C_\alpha(\rho)$ reduce to $C_r(\rho)$. We can denote

$$\begin{aligned} C_1(\rho) &= C_r(\rho) = S(\rho_{diag}) - S(\rho) \\ &= \frac{1+\sqrt{t^2+z^2}}{2}\ln\frac{1+\sqrt{t^2+z^2}}{2} \\ &+ \frac{1-\sqrt{t^2+z^2}}{2}\ln\frac{1-\sqrt{t^2+z^2}}{2} \\ &- \frac{1+z}{2}\ln\frac{1+z}{2} - \frac{1-z}{2}\ln\frac{1-z}{2}. \end{aligned} \tag{9}$$

For $\alpha = \frac{1}{2}$, in order to calculate $C_{\frac{1}{2}}$ of a mixed state $\rho(t,z)$ with the form Eq. (6), we need eigenvalues and eigenvectors of this state. With an easy calculation, we can obtain the eigenvalues of $\rho(t,z)$,

$$\lambda_1 = \tfrac{1+\sqrt{t^2+z^2}}{2},\ \lambda_2 = \tfrac{1-\sqrt{t^2+z^2}}{2}.$$

Their norm eigenvectors are

$$|\lambda_1\rangle = \Big[\tfrac{t}{(\sqrt{t^2+z^2}-z)^{\frac{1}{2}}*2\sqrt{t^2+z^2})^{\frac{1}{2}}}, \tfrac{(\sqrt{t^2+z^2}-z)^{\frac{1}{2}}}{(2\sqrt{t^2+z^2})^{\frac{1}{2}}}\Big]^T,$$

$$|\lambda_2\rangle = \Big[\tfrac{-t}{(\sqrt{t^2+z^2}+z)^{\frac{1}{2}}*2\sqrt{t^2+z^2})^{\frac{1}{2}}}, \tfrac{(\sqrt{t^2+z^2}+z)^{\frac{1}{2}}}{(2\sqrt{t^2+z^2})^{\frac{1}{2}}}\Big]^T.$$

Substituting its eigenvalues and eigenvectors into Eq. (4), the expression of $C_{\frac{1}{2}}(\rho)$ can be given as:

$$C_{\frac{1}{2}}(\rho) = -2\big(r_{\frac{1}{2}}^{\frac{1}{2}} - 1\big), \tag{10}$$

$r_{\frac{1}{2}}$ is expressed as:

$$\begin{aligned} r_{\frac{1}{2}} &= \Big[\big(\frac{1+\sqrt{t^2+z^2}}{2}\big)^{\frac{1}{2}}\frac{\sqrt{t^2+z^2}+z}{2\sqrt{t^2+z^2}} \\ &+ \big(\frac{1-\sqrt{t^2+z^2}}{2}\big)^{\frac{1}{2}}\frac{\sqrt{t^2+z^2}-z}{2\sqrt{t^2+z^2}}\Big]^2 \\ &+ \Big[\big(\frac{1+\sqrt{t^2+z^2}}{2}\big)^{\frac{1}{2}}\frac{\sqrt{t^2+z^2}-z}{2\sqrt{t^2+z^2}} \\ &+ \big(\frac{1-\sqrt{t^2+z^2}}{2}\big)^{\frac{1}{2}}\frac{\sqrt{t^2+z^2}+z}{2\sqrt{t^2+z^2}}\Big]^2. \end{aligned}$$

In the following, let us consider the monotonicity of expressions of these three coherence measures with respect to variable z.

Proposition 2: For a fixed value t, $C_\alpha(\rho(t,z))$ is an increasing function with respect to z for $0 \le z \le \sqrt{1-t^2}$, and it is a decreasing function with respect to z for $-\sqrt{1-t^2} \le z \le 0$. i.e. $\frac{\partial C_\alpha(\rho(t,z))}{\partial z} \ge 0$, for $0 \le z \le \sqrt{1-t^2}$, and $\frac{\partial C_\alpha(\rho(t,z))}{\partial z} \le 0$, for $-\sqrt{1-t^2} \le z \le 0$, where $\alpha = 2, 1, \frac{1}{2}$ and $\rho(t,z)$ is single-qubit mixed state with the form Eq. (6).

Proof: Through analyzing the expression of $C_\alpha(\rho)$, we find $C_\alpha(\rho(t,z)) = C_\alpha(\rho(t,-z))$, where $\alpha = 2, 1, \frac{1}{2}$, $-\sqrt{1-t^2} \le z \le \sqrt{1-t^2}$. Thus, we only need to show that $C_\alpha(\rho)$ is an increasing function for $0 \le z \le \sqrt{1-t^2}$.

(1) We consider the derivation of $C_2(\rho)$ related to z.

$$\frac{\partial C_2(\rho)}{\partial z} = r.\Big[\frac{1+z}{\sqrt{t^2+(1+z)^2}} - \frac{1-z}{\sqrt{t^2+(1-z)^2}}\Big].$$

It is obvious that $\frac{\partial C_2(\rho)}{\partial z} \ge 0$ for $0 \le z \le \sqrt{1-t^2}$.

(2) For $0 \le z \le \sqrt{1-t^2}$, we consider the derivation of $C_{\frac{1}{2}}(\rho)$ related to z,

$$\frac{\partial C_1(\rho)}{\partial z} = \frac{1}{2}\ln\frac{1-z}{1+z} + \frac{z}{2\sqrt{t^2+z^2}}\ln\frac{1+\sqrt{z^2+t^2}}{1-\sqrt{z^2+t^2}} \ge 0.$$

If $f(x) = \frac{1}{x}\ln\frac{1+x}{1-x}$ is an increasing function for $x \ge 0$, then it is obvious that $\frac{\partial C_1(\rho)}{\partial z} \ge 0$. Next, let us prove this fact.

$$\begin{aligned} f'(x) &= -\frac{1}{x^2}\ln\frac{1+x}{1-x} + \frac{2}{x(1-x^2)} \\ &= \frac{1}{x^2}\Big(\frac{2x}{1-x^2} - \ln\frac{1+x}{1-x}\Big). \end{aligned}$$

let $g(x) = \frac{2x}{1-x^2} - \ln\frac{1+x}{1-x}$. If $x = 0$, then $g(x) = 0$, and $\forall x \ge 0$, we have

$$g'(x) = \frac{2+2x^2}{(1-x^2)^2} - \frac{2}{1-x^2} = \frac{4x^2}{(1-x^2)^2} \ge 0.$$

So, $g(x) \ge 0$ for all $x \ge 0$. Therefore, $f'(x) \ge 0$. According to the above fact, we can easily know $\frac{\partial C_1(\rho)}{\partial z} \ge 0$.

(3) We consider the derivation of $C_{\frac{1}{2}}(\rho)$ with z. $\frac{\partial C_{\frac{1}{2}}(\rho)}{\partial z} = -r_{\frac{1}{2}}^{-\frac{1}{2}}\frac{\partial r}{\partial z} \ge 0$, the proof of $\frac{\partial r_{\frac{1}{2}}}{\partial z} \le 0$ will be provided in appendix. Since $\frac{\partial r_{\frac{1}{2}}}{\partial z} \le 0$, then we have $\frac{\partial C_{\frac{1}{2}}(\rho)}{\partial z} \ge 0$.

In accordance with the above discussion, for a fixed t, $C_\alpha(\rho(t,z))$ have maximum when $z = \sqrt{1-t^2}$, and have minimum when $z = 0$, where $\alpha = 2, 1, \frac{1}{2}$. Therefore, we consider two special states.

$$\rho_{max}(t) = \rho(t, \sqrt{1-t^2}) = \begin{bmatrix} \frac{1+\sqrt{1-t^2}}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{1-\sqrt{1-t^2}}{2} \end{bmatrix} \tag{11}$$

$$\rho_{min}(t) = \rho(t, 0) = \begin{bmatrix} \frac{1}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{1}{2} \end{bmatrix}. \tag{12}$$

For any single-qubit mixed state $\rho(t,z)$ with the form Eq. (6), $C_\alpha(\rho_{min}(t))$ is the lower bound of $C_\alpha(\rho(t,z))$, $C_\alpha(\rho_{max}(t))$ is the upper bound of $C_2(\rho(t,z))$, where $\alpha = 2, 1, \frac{1}{2}$. Before analyzing the ordering states with C_α and C_{l_1}, we calculate these three Tsallis relative α-entropies of coherence of these two special states, where $\alpha = 2, 1, \frac{1}{2}$. By substituting Eq. (11), Eq. (12) into Eq. (7), Eq. (9), Eq. (10), We get:

$$
\begin{aligned}
C_{2,max}(t) =& C_2(\rho_{max}(t)) = t,\\
C_{2,min}(t) =& C_2(\rho_{min}(t)) = t^2.\\
C_{1,max}(t) =& C_r(\rho_{max})\\
=& -\frac{1+\sqrt{1-t^2}}{2}\ln\frac{1+\sqrt{1-t^2}}{2}\\
& -\frac{1-\sqrt{1-t^2}}{2}\ln\frac{1-\sqrt{1-t^2}}{2},\\
C_{1,min}(t) =& C_r(\rho_{min})\\
=& \frac{1+t}{2}\ln\frac{1+t}{2} + \frac{1-t}{2}\ln\frac{1-t}{2} + \ln 2.\\
C_{\frac{1}{2},max}(t) =& -2\big[(\frac{2-t^2}{2})^{\frac{1}{2}} - 1\big],\\
C_{\frac{1}{2},min}(t) =& -2\big[(\frac{1+\sqrt{1-t^2}}{2})^{\frac{1}{2}} - 1\big].
\end{aligned}
$$

For any $t \in [0,1]$, and $\alpha = 2, 1, \frac{1}{2}$, $C_\alpha(\rho_{max})$, $C_\alpha(\rho_{min})$ are two functions related to variable t, and l_1 norm of coherence of state $\rho(t,z)$ is equal to t. These two functions will form a closed region. For any state $\rho(t,z)$ with the form Eq. (6), $(t, C_\alpha(\rho(t,z)))$ will correspond to a point in closed region. Our main result will be obtained as the following.

Result 2: C_α and C_{l_1} don't generate the same ordering for some single-qubit states with form Eq. (6), where $\alpha = 2, 1, \frac{1}{2}$.

We will only analyze the ordering states with C_2 and C_{l_1} as presented in Fig. 2. C_1, $C_{\frac{1}{2}}$ and C_{l_1} are similar as presented in Fig. 3, Fig. 4. Let $\rho(t,z)$ be a single-qubit state with the form Eq. (6), and correspond to a point $(C_{l_1(\rho(t,z))}, C_2(\rho(t,z))) = (t, C_2(\rho(t,z)))$. We can easily find all states which violate the condition of Eq. (1). If $\rho(t,z)$ correspond to point O, we can see that $\rho(t,z)$ and any state corresponding to a point in region OAB will violate the condition of Eq. (1). However, C_2 and C_{l_1} will give the same ordering for $\rho(t,z)$ and any state corresponding to a point outside region OAB. If a point Z replaces point O, then region OAB will be replaced by regions ZXY and ZMN. An explicit example will be given as follow. We give two states:

$$
\rho_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \rho_2 = \begin{bmatrix} \frac{5+\sqrt{21}}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{5-\sqrt{21}}{10} \end{bmatrix}.
$$

Substituting ρ_1, ρ_2 into Eq.(7), Eq.(9) and Eq.(10), with an easy calculation, we have $C_{l_1}(\rho_1) = \frac{1}{2}$,$C_{l_1}(\rho_2) = \frac{2}{5}$, $C_2(\rho_1) = \frac{1}{4}$, $C_2(\rho_2) = \frac{2}{5}$, $C_1(\rho_1) \approx 0.13081$, $C_1(\rho_2) \approx 0.17344$, $C_{\frac{1}{2}}(\rho_1) \approx 0.0681$, $C_{\frac{1}{2}}(\rho_2) \approx 0.0817$. It is clear that $C_{l_1}(\rho_1) \geq C_{l_1}(\rho_2)$ but $C_\alpha(\rho_1) \leq C_\alpha(\rho_2)$, $\alpha = 2, 1, \frac{1}{2}$. It means that C_{l_1} and C_α generate the different ordering for ρ_1 and ρ_2, for any $\alpha = 2, 1, \frac{1}{2}$.

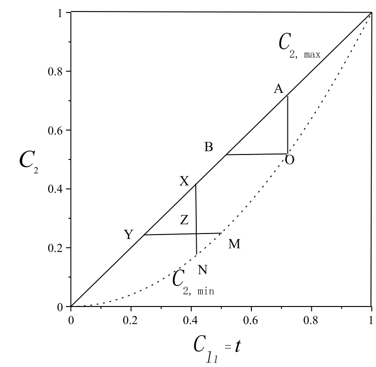

Fig. 2. A special Tsallis relative α-entropies of coherence C_2 versus l_1 norm of coherence C_{l_1}.

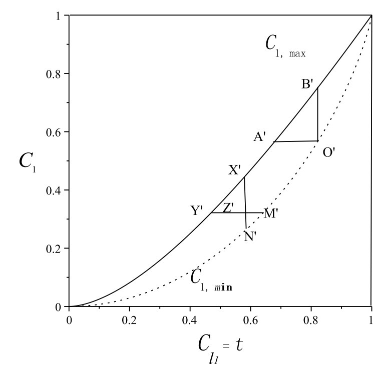

Fig. 3. A special Tsallis relative α-entropies of coherence C_1 versus l_1 norm of coherence C_{l_1}.

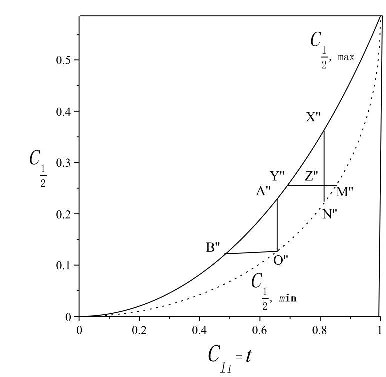

Fig. 4. A special Tsallis relative α-entropies of coherence $C_{\frac{1}{2}}$ versus l_1 norm of coherence C_{l_1}.

In the above, we have shown Tsallis relative α-entropies of coherence C_α and l_1 norm of coherence generate different ordering for some single-qubit states when α as some special values, such as 2,1,$\frac{1}{2}$. We conjecture these results remain valid for any $\alpha \in (0, 2]$.

Conjecture 1: For any $\alpha \in (0,2]$, C_α and C_{l_1} don't generate the same ordering for some single-qubit states with the form Eq. (6).

In Sec. III, we find any two Tsallis relative α-entropies of coherence C_{α_1} and C_{α_2}, $\alpha_1, \alpha_2 \in (0,2]$, give the same ordering for any single-qubit pure states. Now we consider whether this result is still valid for any single-qubit mixed states. We give a counterexample to show that it is not true. We give three mixed states with the form Eq. (6).

$$\rho_1 = \rho(0.5, 0.5) = \begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.25 \end{bmatrix},$$

$$\rho_2 = \rho(0.48, 0.58) = \begin{bmatrix} 0.79 & 0.24 \\ 0.24 & 0.21 \end{bmatrix},$$

$$\rho_3 = \rho(0.48, 0.64) = \begin{bmatrix} 0.82 & 0.24 \\ 0.24 & 0.18 \end{bmatrix}.$$

By using of Eq. (4), we can have $C_1(\rho_1) = 0.1458$, $C_1(\rho_2) = 0.1400$, $C_1(\rho_3) = 0.1463$, $C_2(\rho_1) = 0.3090$, $C_2(\rho_2) = 0.3100$, $C_2(\rho_3) = 0.3326$, $C_{\frac{1}{2}}(\rho_1) = 0.0746$, $C_{\frac{1}{2}}(\rho_2) = 0.0707$, $C_{\frac{1}{2}}(\rho_3) = 0.0733$. It is clear that

(1) $C_1(\rho_1) > C_1(\rho_2)$ but $C_2(\rho_1) < C_2(\rho_2)$,

(2) $C_{\frac{1}{2}}(\rho_1) > C_{\frac{1}{2}}(\rho_2)$ but $C_2(\rho_1) < C_2(\rho_2)$,

(3) $C_{\frac{1}{2}}(\rho_1) > C_{\frac{1}{2}}(\rho_3)$ but $C_1(\rho_1) < C_1(\rho_3)$.

So, we can know that any two of these three Tsallis relative α-entropies of coherence don't give the same ordering for some single-qubit mixed states. We conjecture this result is also effective for all Tsallis relative α-entropies of coherence.

Conjecture 2: For any $\alpha_1, \alpha_2 \in (0,2]$, C_{α_1} and C_{α_2} don't give the same ordering for some single-qubit mixed states with form Eq. (6).

V. CONCLUSION

In this paper, we studied the ordering with l_1 norm of coherence measures and Tsallis relative α-entropies of coherence for single-qubit states. First, we showed that any Tsallis α-entropies of coherence and l_1 norm of coherence give the same ordering for single-qubit pure states, but this result is not true for high dimensional pure states, even though these states are pure. Second, we investigated some special Tsallis, α-entropies of coherence, such as C_1, C_2 and $C_{\frac{1}{2}}$. We found any one of these three measures and C_{l_1} don't generate the same ordering for single-qubit mixed states. For any single-qubit state, as presented in Fig. 2, Fig. 3, Fig. 4, we could find all states which will violate the condition (1). We conjectured that above results remain valid for any Tsallis relative α-entropies of coherence. Finally, we considered that any two of these three special measures don't generate the same ordering for single-qubit mixed states by a counter-example. Furthermore, we conjectured that it is also true for any two Tsallis relative α-entropies of coherence.

VI. ACKNOWLEDGMENTS

This paper is supported by NSFC(Grants No. 11271237, No.61228305), the Higher School Doctoral Subject Foundation of Ministry of Education of China(Grant No.20130202110001).

VII. APPENDIX

We provide a proof of $\frac{\partial r}{\partial z} \geq 0$. The first equation comes from the derivation of $r_{\frac{1}{2}}$ with respect z. In the second equation, we use distributive law and then unite like terms. The last inequality comes from the fact $2 - z^2 - t^2 \geq 2\sqrt{1-\sqrt{z^2+t^2}}$.

[1] M. O. Scully and M. S. Zubairy, Quantum Optics (Canbrudge University Press, Cambridge, 1997).

[2] M. A. Nielsen and I. L. Chuang *Quantum Computation and Quantum Information* (Cambridge: Cambridge Univ. Press) (2000).

[3] C. A. Rodríguez-Rosario, T. Frauenheim, and A. Aspuru-Guzik. arXiv:1308.1245.

[4] J Å berg. Phys. Rev. Lett. **113**, 150402 (2014).

[5] M. Horodecki and J. Oppenheim, Nat. Commun. **4**, 2059 (2013).

[6] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).

[7] V. Narasimhachar and G. Gour, Nat. Commun. **6**, 7689 (2015).

[8] J. Å berg, arXiv:0612146.

[9] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).

[10] S. Rana, P. Parashar, and M. Lewenstein. Phys. Rev. A **93**, 012110 (2016).

[11] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92**, 022124 (2015).

[12] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91**, 042120 (2015).

[13] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).

[14] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso. Phys. Rev. Lett. **116**,150502 2016.

[15] Y.-R. Zhang, L.-H. Shao, Y. Li, and H. Fan. Phys. Rev. A **93**, 012334 (2016).

[16] X.-D. Yu, D.-J. Zhang, G. F. Xu, D. M. Tong. arXiv:1606.03181.

$$
\begin{aligned}
\frac{\partial r_{\frac{1}{2}}}{\partial z} =& \frac{1}{\sqrt{2}}[\frac{\sqrt{1+\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}+z)}{\sqrt{z^2+t^2}} + \frac{\sqrt{1-\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}-z)}{\sqrt{z^2+t^2}}] \\
& [\frac{\sqrt{2}}{8}\frac{z(\sqrt{z^2+t^2}+z)}{\sqrt{1+\sqrt{z^2+t^2}(z^2+t^2)}} - \frac{\sqrt{2}}{8}\frac{z(\sqrt{z^2+t^2}-z)}{\sqrt{1-\sqrt{z^2+t^2}(z^2+t^2)}} - \frac{1}{2\sqrt{2}}\frac{\sqrt{1+\sqrt{z^2+t^2}}z(\sqrt{z^2+t^2}+z)}{(z^2+t^2)^{\frac{3}{2}}} \\
& + \frac{1}{2\sqrt{2}}\frac{\sqrt{1+\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}+z)}{(z^2+t^2)} - \frac{1}{2\sqrt{2}}\frac{\sqrt{1-\sqrt{z^2+t^2}}z(\sqrt{z^2+t^2}-z)}{(z^2+t^2)^{\frac{3}{2}}} - \frac{1}{2\sqrt{2}}\frac{\sqrt{1-\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}-z)}{(z^2+t^2)}] \\
& + \frac{1}{\sqrt{2}}[\frac{\sqrt{1+\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}-z)}{\sqrt{z^2+t^2}} + \frac{\sqrt{1-\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}+z)}{\sqrt{z^2+t^2}}] \\
& [\frac{\sqrt{2}}{8}\frac{z(\sqrt{z^2+t^2}-z)}{\sqrt{1+\sqrt{z^2+t^2}(z^2+t^2)}} - \frac{\sqrt{2}}{8}\frac{z(\sqrt{z^2+t^2}+z)}{\sqrt{1-\sqrt{z^2+t^2}(z^2+t^2)}} - \frac{1}{2\sqrt{2}}\frac{\sqrt{1+\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}-z)}{(z^2+t^2)} \\
& - \frac{1}{2\sqrt{2}}\frac{\sqrt{1+\sqrt{z^2+t^2}}z(\sqrt{z^2+t^2}-z)}{(z^2+t^2)^{\frac{3}{2}}} + \frac{1}{2\sqrt{2}}\frac{\sqrt{1-\sqrt{z^2+t^2}}(\sqrt{z^2+t^2}+z)}{(z^2+t^2)} - \frac{1}{2\sqrt{2}}\frac{\sqrt{1-\sqrt{z^2+t^2}}z(\sqrt{z^2+t^2}+z)}{(z^2+t^2)^{\frac{3}{2}}}]. \\
=& \frac{1}{2}\frac{(\sqrt{z^2+t^2}+z)^2}{(z^2+t^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{(\sqrt{z^2+t^2}-z)^2}{(z^2+t^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{z(\sqrt{z^2+t^2}-z)^2}{(z^2+t^2)^2} - \frac{1}{2}\frac{z(\sqrt{z^2+t^2}+z)^2}{(z^2+t^2)^2} \\
& - \frac{zt^2\sqrt{1-(z^2+t^2)}}{(z^2+t^2)^2} + \frac{1}{4}\frac{zt^2\sqrt{1-\sqrt{z^2+t^2}}}{\sqrt{(1+\sqrt{z^2+t^2})}(z^2+t^2)^{\frac{3}{2}}} - \frac{1}{4}\frac{zt^2\sqrt{1+\sqrt{z^2+t^2}}}{\sqrt{1-\sqrt{z^2+t^2}}(z^2+t^2)^{\frac{3}{2}}} \\
=& \frac{2z\sqrt{z^2+t^2}}{(z^2+t^2)^{\frac{3}{2}}} - \frac{z(2z^2+t^2)}{(z^2+t^2)^2} - \frac{zt^2\sqrt{1-(z^2+t^2)}}{(z^2+t^2)^2} - \frac{1}{2}\frac{zt^2\sqrt{z^2+t^2}}{\sqrt{1-(z^2+t^2)}(z^2+t^2)^{\frac{3}{2}}} \\
=& \frac{z^2}{2\sqrt{1-(z^2+t^2)}(z^2+t^2)}[-2+z^2+t^2+2\sqrt{1-\sqrt{z^2+t^2}}] \leq 0.
\end{aligned}
$$

[17] A .E. Rastegin. Phys. Rev. A **93**, 032136 (2016).

[18] C. L. Liu , X. D. Yu , G. F. Xu , D. M. Tong. arXiv:1601.03936.

[19] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Phys. Rev. A **92**, 022112 (2015).

[20] S. Du, Z. Bai and Y. Guo, Phys. Rev. A **91**, 052120 (2015).

[21] S. Cheng and M. J. W. Hall. Phys. Rev. A **92**, 042101(2015).

[22] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati. Phys. Rev. A **92**, 012118 (2015).

[23] Z. Xi, Y. Li, and H. Fan. Sci. Rep. **5**, 10922 (2015).

[24] A. Winter and D. Yang. Phys. Rev. Lett. **116**, 120404 (2016).

[25] T. R. Bromley, M. Cianciaruso, and G. Adesso. Phys. Rev. Lett. **114**, 210401 (2015).

[26] J. Xu, Phys. Rev. A **93**, 032111 (2016).

[27] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral. arXiv:1512.02085.

[28] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery. Phys. Rev. Lett. **116**, 160406 (2016).

[29] E. Chitambar, G. Gour. arXiv:1602.06969.

[30] G. S. L. F. Brandão, G. Gour. Phys. Rev. Lett. **115**, 070503 (2015).

[31] J. Eisert and M. B. Plenio, J. Mod. Opt. **46**, 145 (1999).

[32] S. Virmani and M. B. Plenio, Phys. Lett. A **268**, 31 (2000).

[33] K. Zyczkowski and I. Bengtsson, Ann. Phys. (N.Y.) **295**, 115 (2002).

[34] A. Miranowicz and A. Grudka, Phys. Rev. A **70**, 032326 (2004).

[35] T. C. Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro, and F. Verstraete, Phys. Rev. A **67**, 022110 (2003).

[36] M. Ziman and V. Bŭzek, Phys. Rev. A **73**, 012312 (2006).

[37] A. Sen De and U. Sen, J. Mod. Opt. **50**, 981 (2003).

[38] M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. A **81**, 042105 (2010).

[39] M. D. Lang, C. M. Caves, and A. Shaji, Int. J. Quantum Inf. **09**, 1553 (2011).

[40] Galve F, Plastina F, Paris M G A, et al. Phys. Rev. lett. **110**, 010501 (2013).

[41] M. Okrasa and Z. Walczak, Europhys. Lett. **98**, 40003 (2012).

[42] S. Furuichi , K. Yanagi, K. Kuriyama. J. Math. Phys. **45**, 4868 (2004).

[43] F. Hiai, M. Mosonyi, D. Petz, C.Bény. Rev. Math. Phy. **23**, 691 (2011).