# GENERALIZED SPIKES WITH CIRCUITS AND COCIRCUITS OF DIFFERENT CARDINALITIES

NICK BRETTELL AND KEVIN GRACE

ABSTRACT. We consider matroids with the property that every subset of the ground set of size s is contained in a 2s-element circuit and every subset of size t is contained in a 2t-element cocircuit. We say that such a matroid has the (s, 2s, t, 2t)-property. A matroid is an (s, t)-spike if there is a partition of the ground set into pairs such that the union of any s pairs is a circuit and the union of any t pairs is a cocircuit. Our main result is that all sufficiently large matroids with the (s, 2s, t, 2t)property are (s, t)-spikes, generalizing a 2019 result that proved the case where s = t. We also present some properties of (s, t)-spikes.

## 1. INTRODUCTION

For integers s, u, t, and v, with  $u \ge s \ge 1$  and  $v \ge t \ge 1$ , a matroid Mhas the (s, u, t, v)-property if every s-element subset of E(M) is contained in a circuit of size u, and every t-element subset of E(M) is contained in a cocircuit of size v. Matroids with this property appear regularly in the matroid theory literature: for example, wheels and whirls have the (1, 3, 1, 3)property, and (tipless) spikes have the (2, 4, 2, 4)-property. Note that M has the (s, u, t, v)-property if and only if  $M^*$  has the (t, v, s, u)-property. Brettell, Campbell, Chun, Grace, and Whittle [2] studied such matroids, and showed that if u < 2s or v < 2t, then there are only finitely many matroids with the (s, u, t, v)-property [2, Theorem 3.3]. On the other hand, in the case that s = t and u = v = 2t, any sufficiently large matroid with the (s, u, t, v)property is a member of a class of structured matroids referred to as t-spikes. In particular, when t = 2, this is the class typically known simply as (tipless) spikes.

Our focus in this paper is also on the case where u = 2s and v = 2t, but we drop the requirement that s = t. For positive integers s and t, an (s,t)-spike is a matroid on at least  $2 \max\{s,t\}$  elements whose ground set has a partition  $(S_1, S_2, \ldots, S_n)$  into pairs such that the union of every set of s pairs is a circuit and the union of every set of t pairs is a cocircuit. The following is our main result:

**Theorem 1.1.** There exists a function  $f : \mathbb{N}^2 \to \mathbb{N}$  such that, if M is a matroid with the (s, 2s, t, 2t)-property and  $|E(M)| \ge f(s, t)$ , then M is an (s, t)-spike.

This proves the conjecture of Brettell et al. [2, Conjecture 1.2].

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Our approach is essentially the same as in [2], but some care is required to generalize the argument. We note also that Lemma 5.10 corrects an erroneous lemma [2, Lemma 6.6].

This paper is one in a developing series on matroids with the (s, u, t, v)property. First, Miller [5] studied matroids with the (2, 4, 2, 4)-property, proving the specialization of Theorem 1.1 to the case where s = t = 2. As previously mentioned, Brettell et al. [2] considered the more general case where s = t and u = v = 2t, for any  $t \ge 1$ . Oxley, Pfeil, Semple, and Whittle considered the case where s = 2, u = 4, t = 1, and  $v \in \{3, 4\}$ , showing that a sufficiently large v-connected matroid with the (2, 4, 1, v)property is isomorphic to  $M(K_{v,n})$  for some n [7]. A "cyclic" analogue of the (s, u, t, v)-property has also been considered, where a cyclic ordering  $\sigma$  is imposed on E(M), and only sets that appear consecutively with respect to  $\sigma$  and have size s (or size t) need appear in a circuit of size u (or a cocircuit of size v, respectively). The case where s = u - 1 and t = v - 1 and s = t was considered by Brettell, Chun, Fife, and Semple [3]; whereas Brettell, Semple, and Toft dropped the requirement that s = t [4].

This series of papers has been motivated by problems involving matroid connectivity. The well-known Wheels-and-Whirls Theorem of Tutte [9] states that wheels and whirls (which have the (1, 3, 1, 3)-property) are the only 3-connected matroids with no elements that can be either deleted or contracted to retain a 3-connected matroid. Similarly, spikes (which have the (2, 4, 2, 4)-property) are the only 3-connected matroids on at least 13 elements that have no triangles, no triads, and no pairs of elements that can be either deleted or contracted to routracted to preserve 3-connectivity [10].

The following conjecture was stated as [2, Conjecture 1.3]. The case where t = 2 was proved by Williams [10].

**Conjecture 1.2.** There exists a function  $f : \mathbb{N} \to \mathbb{N}$  such that if M is a (2t-1)-connected matroid with no circuits or cocircuits of size 2t-1, and  $|E(M)| \ge f(t)$ , then either

- (i) there exists a t-element set  $X \subseteq E(M)$  such that either M/X or  $M \setminus X$  is (t+1)-connected, or
- (ii) M is a (t, t)-spike.

Indeed, sufficiently large (t, t)-spikes are (2t - 1)-connected matroids [2, Lemma 6.5], they have no circuits or cocircuits of size (2t-1) [2, Lemma 6.3], and for every t-element subset  $X \subseteq E(M)$ , neither M/X nor  $M \setminus X$  is (t+1)-connected. Optimistically, we offer the following generalization of Conjecture 1.2.

**Conjecture 1.3.** There exists a function  $f : \mathbb{N}^2 \to \mathbb{N}$  such that if M is a matroid with no circuits of size at most 2s - 1, no cocircuits of size at most 2t - 1, the matroid M is  $(2\min\{s,t\} - 1)$ -connected, and  $|E(M)| \ge f(s,t)$ , then either

- (i) there exists an s-element set  $X \subseteq E(M)$  such that M/X is (s+1)-connected,
- (ii) there exists a t-element set  $X \subseteq E(M)$  such that  $M \setminus X$  is (t + 1)-connected, or
- (iii) M is an (s,t)-spike.

#### GENERALIZED SPIKES

Section 2 recalls some terminology and a Ramsey-theoretic result used later in the paper. In Section 3, we recall the definition of echidnas from [2] and show that every matroid with the (s, 2s, t, 2t)-property and having a sufficiently large *s*-echidna is an (s, t)-spike. In Section 4, we prove Theorem 1.1. Finally, Section 5 describes some properties of (s, t)-spikes, as well as a construction that allows us to build an (s, t + 1)-spike from an (s, t)-spike.

## 2. Preliminaries

Our notation and terminology follows Oxley [6]. We refer to the fact that a circuit and a cocircuit cannot intersect in exactly one element as "orthogonality". A set  $S_1$  meets a set  $S_2$  if  $S_1 \cap S_2 \neq \emptyset$ . We denote  $\{1, 2, \ldots, n\}$  by [n], and, for positive integers i < j, we denote  $\{i, i + 1, \ldots, j\}$  by [i, j]. We denote the set of positive integers by  $\mathbb{N}$ .

In order to prove Theorem 1.1, we will use some hypergraph Ramsey Theory [8]. Recall that a hypergraph is k-uniform if every hyperedge has size k.

**Theorem 2.1** (Ramsey's Theorem for k-uniform hypergraphs). For positive integers k and n, there exists an integer  $r_k(n)$  such that if H is a k-uniform hypergraph on  $r_k(n)$  vertices, then H has either a clique on n vertices, or a stable set on n vertices.

### 3. Echidnas and (s, t)-spikes

Recall that M is an (s, t)-spike if there is a partition of E(M) into pairs such that the union of any s pairs is a circuit and the union of any t pairs is a cocircuit. In this section, we prove a sufficient condition for M to be an (s, t)-spike. Namely, we prove as Lemma 3.3 that if M has the (s, 2s, t, 2t)property, and a subset of E(M) can be partitioned into u pairs such that the union of any t pairs is a circuit, then, when u is sufficiently large, M is an (s, t)-spike. Conforming with [2], we call such a partition a t-echidna, as defined below.

Let M be a matroid. A *t*-echidna of order n is a partition  $(S_1, \ldots, S_n)$  of a subset of E(M) such that

(i)  $|S_i| = 2$  for all  $i \in [n]$ , and

(ii)  $\bigcup_{i \in I} S_i$  is a circuit for all  $I \subseteq [n]$  with |I| = t.

For  $i \in [n]$ , we say  $S_i$  is a *spine*. We say  $(S_1, \ldots, S_n)$  is a *t*-coechidna of M if  $(S_1, \ldots, S_n)$  is a *t*-echidna of  $M^*$ .

Let  $(S_1, \ldots, S_n)$  be a *t*-echidna of a matroid M. If  $(S_1, \ldots, S_m)$  is a *t*-echidna of M, for some  $m \ge n$ , we say that  $(S_1, \ldots, S_n)$  extends to  $(S_1, \ldots, S_m)$ . We say that  $\pi = (S_1, \ldots, S_n)$  is maximal if  $\pi$  extends only to  $\pi$ .

Note that a matroid M is an (s,t)-spike if there exists a partition  $\pi = (A_1, \ldots, A_m)$  of E(M) such that  $\pi$  is an s-echidna and a t-coechidna, for some  $m \ge \max\{s,t\}$ . In this case, we say that the (s,t)-spike M has order m, we call  $\pi$  the associated partition of the (s,t)-spike M, and we say that  $A_i$  is an arm of the (s,t)-spike for each  $i \in [m]$ . An (s,t)-spike with s = t is also called a t-spike. Note that if M is an (s,t)-spike, then  $M^*$  is a (t,s)-spike.

Throughout this section, we assume that s and t are positive integers.

**Lemma 3.1.** Let M be a matroid with the (s, 2s, t, 2t)-property. If M has an s-echidna  $(S_1, \ldots, S_n)$ , where  $n \ge s + 2t - 1$ , then  $(S_1, \ldots, S_n)$  is also a t-coechidna of M.

Proof. Suppose M has an s-echidna  $(S_1, \ldots, S_n)$  with  $n \ge s + 2t - 1$ , and let  $S_i = \{x_i, y_i\}$  for each  $i \in [n]$ . We show, for every t-element subset J of [n], that  $\bigcup_{j \in J} S_j$  is a cocircuit. Without loss of generality, let J = [t]. By the (s, 2s, t, 2t)-property,  $\{x_1, \ldots, x_t\}$  is contained in a 2t-element cocircuit  $C^*$ . Suppose for a contradiction that  $C^* \neq \bigcup_{j \in J} S_j$ . Then there is some  $i \in [t]$  such that  $y_i \notin C^*$ . Without loss of generality, say  $y_1 \notin C^*$ .

Let I be an (s-1)-element subset of [t+1, n]. For any such I, the set  $S_1 \cup \bigcup_{i \in I} S_i$  is a circuit that meets  $C^*$ . By orthogonality,  $\bigcup_{i \in I} S_i$  meets  $C^*$ . Thus,  $C^*$  avoids at most s-2 of the  $S_i$ 's for  $i \in [t+1, n]$ . In fact, as  $C^*$  meets each  $S_i$  with  $i \in [t]$ , the cocircuit  $C^*$  avoids at most s-2 of the  $S_i$ 's for  $i \in [n]$ . Thus  $|C^*| \ge n - (s-2) \ge (s+2t-1) - (s-2) = 2t+1 > 2t$ , a contradiction.

Therefore, we conclude that  $C^* = \bigcup_{i \in J} S_i$ , and the result follows.

**Lemma 3.2.** Let M be a matroid with the (s, 2s, t, 2t)-property, and let  $(S_1, \ldots, S_n)$  be an s-echidna of M with  $n \ge \max\{s + 2t, 2s + t\} - 1$ .

(i) Let I be an (s-1)-subset of [n]. For  $z \in E(M) - \bigcup_{i \in I} S_i$ , there is a 2s-element circuit containing  $\{z\} \cup \bigcup_{i \in I} S_i$ .

(ii) Let I be a (t-1)-subset of [n]. For  $z \in E(M) - \bigcup_{i \in I} S_i$ , there is a 2t-element cocircuit containing  $\{z\} \cup \bigcup_{i \in I} S_i$ .

Proof. First we prove (i). For  $i \in [n]$ , let  $S_i = \{x_i, y_i\}$ . By the (s, 2s, t, 2t)property, there is a 2s-element circuit C containing  $\{z\} \cup \{x_i : i \in I\}$ . Let Jbe a (t-1)-element subset of [n] such that C and  $\bigcup_{j \in J} S_j$  are disjoint (such a
set exists since |C| = 2s and  $n \geq 2s+t-1$ ). For  $i \in I$ , let  $C_i^* = S_i \cup \bigcup_{j \in J} S_j$ ,
and observe that  $x_i \in C_i^* \cap C$ , and  $C_i^* \cap C \subseteq S_i$ . By Lemma 3.1,  $(S_1, \ldots, S_n)$ is a t-coechidna as well as an s-echidna; therefore,  $C_i^*$  is a cocircuit. Now,
for each  $i \in I$ , orthogonality implies that  $|C_i^* \cap C| \geq 2$ , and hence  $y_i \in C$ .
So C contains  $\{z\} \cup \bigcup_{i \in I} S_i$ , as required.

Now, to prove (ii), recall that  $(S_1, \ldots, S_n)$  is a *t*-coechidna by Lemma Lemma 3.1. Therefore, (ii) follows by (i) and duality.

**Lemma 3.3.** Let M be a matroid with the (s, 2s, t, 2t)-property. If M has an s-echidna  $\pi = (S_1, \ldots, S_n)$ , where  $n \ge \max\{s + 2t - 1, 2s + t - 1, 3s + t - 3\}$ , then  $(S_1, \ldots, S_n)$  extends to a partition of E(M) that is both an s-echidna and a t-coechidna.

Proof. Let  $\pi' = (S_1, \ldots, S_m)$  be a maximal s-echidna with  $X = \bigcup_{i=1}^m S_i \subseteq E(M)$ . Suppose for a contradiction that  $X \neq E(M)$ . Since  $\pi'$  is maximal,  $m \ge n \ge s + 2t - 1$ . Therefore, by Lemma 3.1,  $\pi'$  is a t-coechidna.

Let  $z \in E(M) - X$ . By Lemma 3.2, there is a 2*s*-element circuit  $C = (\bigcup_{i \in [s-1]} S_i) \cup \{z, z'\}$  for some  $z' \in E(M)$ . We claim that  $z' \notin X$ . Towards a contradiction, suppose that  $z' \in S_k$  for some  $k \in [s, m]$ . Let J be a *t*-element subset of [s, m] containing k. Then, since  $(S_1, \ldots, S_m)$  is a *t*-coechidna,  $\bigcup_{i \in J} S_j$  is a cocircuit that contains z'. Now, this cocircuit intersects the

circuit C in a single element z', contradicting orthogonality. Thus,  $z' \notin X$ , as claimed.

We next show that  $(\{z, z'\}, S_s, S_{s+1}, \ldots, S_m)$  is a *t*-coechidna. Since  $\pi'$  is a *t*-coechidna, it suffices to show that  $\{z, z'\} \cup \bigcup_{i \in I} S_i$  is a cocircuit for each (t-1)-element subset I of [s, m]. Let I be such a set. Lemma 3.2 implies that there is a 2*t*-element cocircuit  $C^*$  of M containing  $\{z\} \cup \bigcup_{i \in I} S_i$ . By orthogonality,  $|C \cap C^*| > 1$ . Therefore,  $z' \in C^*$ . Thus,  $(\{z, z'\}, S_s, S_{s+1}, \ldots, S_m)$  is a *t*-coechidna. Since this *t*-coechidna has order  $1 + m - (s-1) \ge n - s + 2 \ge 2s + t - 1$ , the dual of Lemma 3.1 implies that  $(\{z, z'\}, S_s, S_{s+1}, \ldots, S_m)$  is also an *s*-echidna.

Next we show that  $(\{z, z'\}, S_1, S_2, \ldots, S_m)$  is a *t*-coechidna. Let *I* be a (t-1)-element subset of [m]. We claim that  $\{z, z'\} \cup \bigcup_{i \in I} S_i$  is a cocircuit. Let *J* be an (s-1)-element subset of [s,m] - I. Then  $C = \{z, z'\} \cup \bigcup_{j \in J} S_j$  is a circuit since  $(\{z, z'\}, S_s, S_{s+1}, \ldots, S_m)$  is an *s*-echidna. By Lemma 3.2, there is a 2*t*-element cocircuit  $C^*$  containing  $\{z\} \cup \bigcup_{i \in I} S_i$ . By orthogonality between *C* and  $C^*$ , we have  $z' \in C^*$ . Since *I* was arbitrarily chosen,  $(\{z, z'\}, S_1, S_2, \ldots, S_m)$  is a *t*-coechidna. By the dual of Lemma 3.1, it is also an *s*-echidna, contradicting the maximality of  $(S_1, \ldots, S_m)$ .

### 4. MATROIDS WITH THE (s, 2s, t, 2t)-property

In this section, we prove that every sufficiently large matroid with the (s, 2s, t, 2t)-property is an (s, t)-spike. We will show that a sufficiently large matroid with the (s, 2s, t, 2t)-property has a large s-echidna or t-coechidna; it then follows, by Lemma 3.3, that the matroid is an (s, t)-spike. As in the previous section, we assume that s and t are positive integers.

**Lemma 4.1.** Let M be a matroid with the (s, 2s, t, 2t)-property, and let  $X \subseteq E(M)$ .

- (i) If r(X) < s, then X is independent.
- (ii) If r(X) = s, then  $M|X \cong U_{s,|X|}$  and |X| < s + 2t.

*Proof.* Every subset of E(M) of size at most s is independent since it is contained in a circuit of size 2s. In particular, (i) holds.

Now let r(X) = s. Then every (s + 1)-element subset of X is a circuit, so  $M|X \cong U_{s,|X|}$ . Suppose for a contradiction that  $|X| \ge s + 2t$ . Let  $C^*$  be a 2t-element cocircuit such that there is some  $x \in X \cap C^*$ . Then  $X - C^*$  is contained in the hyperplane  $E(M) - C^*$ . Since  $x \in X \cap C^*$ , we have  $r(X - C^*) < r(X) = s$ . Therefore,  $X - C^*$  is an independent set, so  $|X - C^*| < s$ . Since  $|X| \ge s + 2t$ , we have  $|C^*| > 2t$ , a contradiction. Thus, (ii) holds.

**Lemma 4.2.** Let M be a matroid with the (s, 2s, t, 2t)-property, and let  $C_1^*, C_2^*, \ldots, C_{s-1}^*$  be a collection of pairwise disjoint cocircuits of M. Let  $Y = E(M) - \bigcup_{i \in [s-1]} C_i^*$ . For all  $y \in Y$ , there is a 2s-element circuit  $C_y$  containing y such that either

- (i)  $|C_y \cap C_i^*| = 2$  for all  $i \in [s-1]$ , or
- (ii)  $|C_y \cap C_j^*| = 3$  for some  $j \in [s-1]$ , and  $|C_y \cap C_i^*| = 2$  for all  $i \in [s-1] \{j\}$ .

Moreover, if  $C_y$  satisfies (ii), then there are at most s+2t-1 elements  $w \in Y$  such that  $(C_y - y) \cup \{w\}$  is a circuit.

*Proof.* Choose an element  $c_i \in C_i^*$  for each  $i \in [s-1]$ . By the (s, 2s, t, 2t)-property, there is a 2s-element circuit  $C_y$  containing  $\{c_1, c_2, \ldots, c_{s-1}, y\}$ , for each  $y \in Y$ . By orthogonality,  $C_y$  satisfies (i) or (ii).

Suppose  $C_y$  satisfies (ii), and let  $S = C_y - Y = C_y - \{y\}$ . Let  $W = \{w \in Y : S \cup \{w\}$  is a circuit $\}$ . It remains to prove that |W| < s + 2t. Observe that  $W \subseteq \operatorname{cl}(S) \cap Y$ , and, since S contains s - 1 elements in pairwise disjoint cocircuits that avoid Y, we have  $r(\operatorname{cl}(S) \cup Y) \ge r(Y) + (s - 1)$ . Thus,

$$\begin{aligned} r(W) &\leq r(\operatorname{cl}(S) \cap Y) \\ &\leq r(\operatorname{cl}(S)) + r(Y) - r(\operatorname{cl}(S) \cup Y) \\ &\leq (2s-1) + r(Y) - (r(Y) + (s-1)) \\ &= s, \end{aligned}$$

using submodularity of the rank function at the second line.

Now, by Lemma 4.1(i), if r(W) < s, then W is independent, so |W| = r(W) < s < s + 2t. On the other hand, by Lemma 4.1(ii), if r(W) = s, then  $M|W \cong U_{t,|W|}$  and |W| < s + 2t, as required.

**Lemma 4.3.** There exists a function h such that if M is a matroid with at least h(k, d, t) k-element circuits, and the property that every t-element set is contained in a 2t-element cocircuit for some positive integer t, then M has a collection of d pairwise disjoint 2t-element cocircuits.

*Proof.* By [2, Lemma 3.2], there is a function g such that if M has at least g(k,d) k-element circuits, then M has a collection of d pairwise disjoint circuits. We define h(k, d, t) = g(k, dt), and claim that a matroid with at least h(k, d, t) k-element circuits, and the property that every t-element set is contained in a 2t-element cocircuit, has a collection of d pairwise disjoint 2t-element cocircuits.

Let M be such a matroid. Then M has a collection of dt pairwise disjoint circuits. We partition these into d groups of size t: call this partition  $(\mathcal{C}_1, \ldots, \mathcal{C}_d)$ . Since the t circuits in any cell of this partition are pairwise disjoint, it now suffices to show that, for each  $i \in [d]$ , there is a 2t-element cocircuit contained in the union of the members of  $\mathcal{C}_i$ . Let  $\mathcal{C}_i = \{C_1, \ldots, C_t\}$  for some  $i \in [d]$ . Pick some  $c_j \in C_j$  for each  $j \in [t]$ . Then, since  $\{c_1, c_2, \ldots, c_t\}$  is a t-element set, it is contained in a 2t-element cocircuit, which, by orthogonality, is contained in  $\bigcup_{j \in [t]} C_j$ .

**Lemma 4.4.** Let M be a matroid with the (s, 2s, t, 2t)-property such that  $r(M) \ge r^*(M)$ . There exists a function g such that, if  $|E(M)| \ge g(s, t, q)$ , then M has s - 1 pairwise disjoint 2t-element cocircuits  $C_1^*, C_2^*, \ldots, C_{s-1}^*$ , and there is some  $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$  such that

- (i)  $r_M(Z) \ge q$ , and
- (ii) for each z ∈ Z, there exists an element z' ∈ Z {z} such that {z, z'} is contained in a 2s-element circuit C with |C ∩ C<sub>i</sub><sup>\*</sup>| = 2 for each i ∈ [s 1].

*Proof.* By Lemma 4.3, there is a function h such that if M has at least h(k, d, t) k-element circuits, then M has d pairwise disjoint 2t-element cocircuits.

Suppose  $|E(M)| \ge 2s \cdot h(2s, s-1, t)$ . By the (s, 2s, t, 2t)-property, M has at least h(2s, s-1, t) distinct 2s-element circuits. Therefore, by Lemma 4.3, M has a collection of s-1 pairwise disjoint 2t-element cocircuits  $C_1^*, \ldots, C_{s-1}^*$ .

Let  $X = \bigcup_{i \in [s-1]} C_i^*$  and Y = E(M) - X. By Lemma 4.2, for each  $y \in Y$ there is a 2*s*-element circuit  $C_y$  containing y such that  $|C_y \cap C_j^*| = 3$  for at most one  $j \in [s-1]$  and  $|C_y \cap C_i^*| = 2$  otherwise. Let W be the set of all  $w \in Y$  such that w is in a 2*s*-element circuit C with  $|C \cap C_j^*| = 3$  for some  $j \in [s-1]$ , and  $|C \cap C_i^*| = 2$  for all  $i \in [s-1] - \{j\}$ . Now, letting Z = Y - W, we see that (ii) is satisfied. It remains to show that (i) holds.

Since each  $C_i^*$  has size 2t, there are  $(s-1)\binom{2t}{3}\binom{2t}{2}^{s-2}$  sets  $X' \subseteq X$  with  $|X' \cap C_j^*| = 3$  for some  $j \in [s-1]$  and  $|X' \cap C_i^*| = 2$  for all  $i \in [s-1] - \{j\}$ . It follows, by Lemma 4.2, that  $|W| \leq f(s,t)$  where

$$f(s,t) = (s+2t-1) \left[ (s-1) \binom{2t}{3} \binom{2t}{2}^{s-2} \right].$$

We define

$$g(s,t,q) = \max\left\{2s \cdot h(2s,s-1,t), 2(2t(s-1) + f(s,t) + q)\right\}.$$

Suppose that  $|E(M)| \ge g(s,t,q)$ . Since  $r(M) \ge r^*(M)$  and  $|E(M)| \ge 2(2t(s-1) + f(s,t) + q)$ , we have  $r(M) \ge 2t(s-1) + f(s,t) + q$ . Then,

$$r_M(Z) \ge r_M(Y) - |W|$$
  
$$\ge (r(M) - 2t(s-1)) - f(s,t)$$
  
$$\ge q,$$

so (i) holds as well.

**Lemma 4.5.** Let M be a matroid with the (s, 2s, t, 2t)-property. Suppose M has s - 1 pairwise disjoint 2t-element cocircuits  $C_1^*, C_2^*, \ldots, C_{s-1}^*$  and, for some positive integer p, there is a set  $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$  such that

- (a)  $r(Z) \ge {\binom{2t}{2}}^{s-1}(p+2(s-1))$ , and
- (b) for each z ∈ Z, there exists an element z' ∈ Z {z} such that {z, z'} is contained in a 2s-element circuit C of M with |C ∩ C<sub>i</sub><sup>\*</sup>| = 2 for each i ∈ [s − 1].

There exists a subset  $Z' \subseteq Z$  and a partition  $\pi = (Z'_1, \ldots, Z'_p)$  of Z' into pairs such that

- (i) each circuit of M|Z' is a union of pairs in  $\pi$ , and
- (ii) the union of any s pairs in  $\pi$  contains a circuit.

*Proof.* We first prove the following:

**4.5.1.** There exists a (2s-2)-element set X such that  $|X \cap C_i^*| = 2$  for every  $i \in [s-1]$  and a set  $Z' \subseteq Z$  with a partition  $\pi = \{Z'_1, \ldots, Z'_p\}$  of Z' into pairs such that

- (I)  $X \cup Z'_i$  is a circuit, for each  $i \in [p]$  and
- (II)  $\pi$  partitions the ground set of (M/X)|Z' into parallel classes such that  $r_{M/X}(\bigcup_{i \in [p]} Z'_i) = p$ .

*Proof.* By (b), for each  $z \in Z$ , there exists an element  $z' \in Z - \{z\}$  and a set X' such that  $\{z, z'\} \cup X'$  is a circuit of M and X' is the union of pairs  $Y_i$  for  $i \in [s-1]$ , with  $Y_i \subseteq C_i^*$ . Since  $|C_i^*| = 2t$  for each  $i \in [s-1]$ , there are  $\binom{2t}{2}^{s-1}$  choices for  $(Y_1, Y_2, \ldots, Y_{s-1})$ . Therefore, for some  $m \leq \binom{2t}{2}^{s-1}$ , there are (2s-2)-element sets  $X_1, X_2, \ldots, X_m$ , and sets  $Z_1, Z_2, \ldots, Z_m$  whose union is Z, such that each of  $X_1, X_2, \ldots, X_m$  intersects  $C_i^*$  in two elements for each  $i \in [s-1]$ , and such that, for each  $j \in [m]$  and each  $z_j \in Z_j$ , there is an element  $z'_j$  such that  $\{z_j, z'_j\} \cup X_j$  is a circuit. Since  $Z = \bigcup_{i \in [m]} Z_i$ , we have  $\sum_{i \in [m]} r(Z_i) \ge r(Z)$ . Thus, the pigeonhole principle implies that there is some  $j \in [m]$  such that

$$r(Z_j) \ge \frac{r(Z)}{\binom{2t}{2}^{s-1}} \ge p + 2(s-1),$$

by (a).

We define  $Z' = Z_j$  and  $X = X_j$ . Observe that  $X \cup \{z, z'\}$  is a circuit, for some pair  $\{z, z'\} \subseteq Z'$ , if and only if  $\{z, z'\}$  is a parallel pair in M/X. Therefore, there is a partition of the ground set of (M/X)|Z' into parallel classes, where every parallel class has size at least two. Let  $\{\{z_1, z'_1\}, \ldots, \{z_n, z'_n\}\}$ be a collection of pairs from each parallel class such that  $\{z_1, z_2, \ldots, z_n\}$  is an independent set in (M/X)|Z'. Note that  $n \ge r_{M/X}(Z') = r(Z' \cup X) - r(X) \ge$  $r(Z') - 2(s - 1) \ge p$ . For  $i \in [p]$ , let  $Z'_i = \{z_i, z'_i\}$ . Then  $\pi = \{Z'_1, \ldots, Z'_p\}$ satisfies 4.5.1.

Let X,  $\pi$ , and Z' be as described in 4.5.1, and let  $\mathcal{X} = \{X_1, \ldots, X_{s-1}\}$ , where  $X_i = \{x_i, x'_i\} = X \cap C_i^*$ .

**4.5.2.** Each circuit of  $M|(X \cup Z')$  is a union of pairs in  $\mathcal{X} \cup \pi$ .

Proof. Let C be a circuit of  $M|(X \cup Z')$ . If  $x_i \in C$ , for some  $\{x_i, x'_i\} \in \mathcal{X}$ , then orthogonality with  $C_i^*$  implies that  $x'_i \in C$ . Assume for a contradiction that  $\{z, z'\} \in \pi$  and  $C \cap \{z, z'\} = \{z\}$ . Let W be the union of the pairs in  $\pi$  containing elements of  $(C - \{z\}) \cap Z'$ . Then  $z \in cl(X \cup W)$ . Hence  $z \in cl_{M/X}(W)$ , contradicting 4.5.1(II).

**4.5.3.** Every union of s pairs in  $\mathcal{X} \cup \pi$  contains a circuit.

Proof. Let  $\mathcal{W}$  be a subset of  $\mathcal{X} \cup \pi$  of size s. We proceed by induction on the number of pairs in  $\mathcal{W} \cap \pi$ . If there is only one pair in  $\mathcal{W} \cap \pi$ , then the union of the pairs in  $\mathcal{W}$  contains a circuit (indeed, is a circuit) by 4.5.1(I). Suppose the result holds for any subset containing k pairs in  $\pi$ , and let  $\mathcal{W}$ be a subset containing k + 1 pairs in  $\pi$ . Let  $\{x, x'\}$  be a pair in  $\mathcal{X} - \mathcal{W}$ , and let  $W = \bigcup_{W' \in \mathcal{W}} W'$ . Then  $W \cup \{x, x'\}$  is the union of s + 1 pairs of  $\mathcal{X} \cup \pi$ , of which k + 1 are in  $\pi$ , so, by the induction hypothesis,  $W \cup \{x, x'\}$  properly contains a circuit  $C_1$ . If  $\{x, x'\} \subseteq E(M) - C_1$ , then  $C_1 \subseteq W$ , in which case the union of the pairs in  $\mathcal{W}$  contains a circuit, as desired. Therefore, we may assume, by 4.5.2, that  $\{x, x'\} \subseteq C_1$ . Since X is independent, there is a pair  $\{z, z'\} \subseteq Z' \cap C_1$ . By the induction hypothesis, there is a circuit  $C_2$ 

contained in  $(W - \{z, z'\}) \cup \{x, x'\}$ . Observe that  $C_1$  and  $C_2$  are distinct, and  $\{x, x'\} \subseteq C_1 \cap C_2$ . Circuit elimination on  $C_1$  and  $C_2$ , and 4.5.2, imply that there is a circuit  $C_3 \subseteq (C_1 \cup C_2) - \{x, x'\} \subseteq W$ , as desired. The claim now follows by induction.

Now, 4.5.3 implies that the union of any s pairs in  $\pi$  contains a circuit, and the result follows.

**Lemma 4.6.** If M is a matroid with the (1, 2, t, 2t)-property and at least t elements, then M is a (1, t)-spike. Dually, if M is a matroid with the (s, 2s, 1, 2)-property and at least s elements, then M is an (s, 1)-spike.

*Proof.* By duality, it suffices to consider the case where M has the (1, 2, t, 2t)property and at least t elements. Since every element of M is contained
in a 2-element circuit, there is a partition of E(M) into parallel classes  $P_1, P_2, \ldots, P_n$ , where  $|P_i| \ge 2$  for each i. For each  $P_i$ , let  $x_i \in P_i$ .

First, we consider the case where  $n \geq t$ . Let X be a t-element subset of  $\{x_1, \ldots, x_n\}$ ; for ease of notation, we assume  $X = \{x_1, \ldots, x_t\}$ . By the (1, 2, t, 2t)-property,  $X \subseteq C^*$  for some 2t-element cocircuit  $C^*$ . Since  $P_i$  is a parallel class,  $\{x_i, y_i\}$  is a circuit for each  $y_i \in P_i - \{x_i\}$ . By orthogonality,  $y_i \in C^*$  for each such  $y_i$ , so  $P_i \subseteq C^*$ . Since  $|C^*| = 2t$ , and X is an arbitrary t-element subset of  $\{x_1, \ldots, x_n\}$ , it follows that  $|P_i| = 2$  for each  $i \in [n]$ , and that the union of any t of the  $P_i$ 's is a cocircuit. Thus M is a (1, t)-spike.

It remains to consider the case where n < t. Since M has at least t elements, let X be any t-element set containing  $\{x_1, \ldots, x_n\}$ . By the (1, 2, t, 2t)-property, there is a 2t-element cocircuit  $C^*$  containing X. For  $i \in [n]$  and each  $y_i \in P_i - \{x_i\}$ , orthogonality implies  $y_i \in C^*$ . Thus,  $E(M) = C^*$ . It follows that  $M \cong U_{1,2t}$ , which is a (1, t)-spike.

We now prove Theorem 1.1, restated below.

**Theorem 4.7.** There exists a function  $f : \mathbb{N}^2 \to \mathbb{N}$  such that, if M is a matroid with the (s, 2s, t, 2t)-property and  $|E(M)| \ge f(s, t)$ , then M is an (s, t)-spike.

*Proof.* If s = 1 or t = 1, then, by Lemma 4.6, the theorem holds with  $f(s,t) = \max\{s,t\}$ . So we may assume that  $\min\{s,t\} \ge 2$ . A matroid is an (s,t)-spike if and only if its dual is a (t,s)-spike; moreover, a matroid has the (s, 2s, t, 2t)-property if and only if its dual has the (t, 2t, s, 2s)-property. Therefore, by duality, we may also assume that  $r(M) \ge r^*(M)$ .

Let  $r_k(n)$  be the Ramsey number described in Theorem 2.1. For  $k \in [s]$ , we define the function  $h_k : \mathbb{N}^2 \to \mathbb{N}$  such that

$$h_s(s,t) = \max\{s + 2t - 1, 2s + t - 1, 3s + t - 3, s + 3t - 3\}$$

and such that  $h_k(s,t) = r_k(h_{k+1}(s,t))$  for  $k \in [s-1]$ . Note that  $h_k(s,t) \ge h_{k+1}(s,t) \ge h_s(s,t)$ , for each  $k \in [s-1]$ .

Let  $p = h_1(s,t)$  and let  $q(s,t) = \binom{2t}{2}^{s-1}(p+2(s-1))$ . By Lemma 4.4, there exists a function g such that if  $|E(M)| \ge g(s,t,q(s,t))$ , then M has s-1 pairwise disjoint 2t-element cocircuits  $C_1^*, C_2^*, \ldots, C_{s-1}^*$ , and there is some  $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$  such that  $r_M(Z) \ge q(s,t)$ , and, for each  $z \in Z$ , there exists an element  $z' \in Z' - \{z\}$  such that  $\{z, z'\}$  is contained in a 2s-element circuit C with  $|C \cap C_i^*| = 2$  for each  $i \in [s-1]$ . Let f(s,t) = g(s,t,q(s,t)), and suppose that  $|E(M)| \ge f(s,t)$ . Then, by Lemma 4.5, there exists a subset  $Z \subseteq Z'$  such that Z has a partition into pairs  $\pi = (Z_1, \ldots, Z_p)$  such that

- (I) each circuit of M|Z is a union of pairs in  $\pi$ , and
- (II) the union of any s pairs in  $\pi$  contains a circuit.

Let  $m = h_s(s, t)$ . By Lemma 3.3 and its dual, it suffices to show that M has either an s-echidna or a t-coechidna of order m. If the smallest circuit in M|Z has size 2s, then, by (II),  $\pi$  is an s-echidna of order  $p \ge m$ . So we may assume that the smallest circuit in M|Z has size 2j for some  $j \in [s-1]$ .

**4.7.1.** If the smallest circuit in M|Z has size 2j, for  $j \in [s-1]$ , and  $|\pi| \ge h_j(s,t)$ , then either

- (i) M has a t-coechidna of order m, or
- (ii) there exists some Z' ⊆ Z that is the union of h<sub>j+1</sub>(s,t) pairs in π for which the smallest circuit in M|Z' has size at least 2(j + 1).

Proof. We define H to be the *j*-uniform hypergraph with vertex set  $\pi$  whose hyperedges are the *j*-subsets of  $\pi$  that are partitions of circuits in M|Z. By Theorem 2.1, and the definition of  $h_k$ , as H has at least  $h_j(s,t)$  vertices, it has either a clique or a stable set, on  $h_{j+1}(s,t)$  vertices. If H has a stable set  $\pi'$  on  $h_{j+1}(s,t)$  vertices, then clearly (ii) holds, with  $Z' = \bigcup_{P \in \pi'} P$ .

Therefore, we may assume that there are  $h_{j+1}(s,t)$  pairs in  $\pi$  such that the union of any j of these pairs is a circuit. Let Z'' be the union of these  $h_{j+1}(s,t)$  pairs. We claim that the union of any set of t pairs contained in Z''is a cocircuit. Let T be a transversal of t pairs in  $\pi$  contained in Z'', and let  $C^*$  be the 2t-element cocircuit containing T. Suppose, for a contradiction, that there exists some pair  $P \in \pi$  with  $P \subseteq Z''$  such that  $|C^* \cap P| = 1$ . Select j-1 pairs  $Z''_1, \ldots, Z''_{j-1}$  in  $\pi$  that are each contained in  $Z'' - C^*$  (these exist since  $h_{j+1}(s,t) \ge s+2t-1 \ge 2t+j-1$ ). Then  $P \cup (\bigcup_{i \in [j-1]} Z''_i)$  is a circuit intersecting  $C^*$  in a single element, contradicting orthogonality. We deduce that the union of any t pairs in  $\pi$  that are contained in Z'' is a cocircuit. Thus, M has a t-coechidna of order  $h_{j+1}(t) \ge m$ , satisfying (i).

We now apply 4.7.1 iteratively, for a maximum of s - j iterations. If (i) holds, at any iteration, then M has a *t*-coechidna of order m, as required. Otherwise, we let  $\pi'$  be the partition of Z' induced by  $\pi$ ; then, at the next iteration, we relabel Z = Z' and  $\pi = \pi'$ . If (ii) holds for each of s - j iterations, then we obtain a subset Z' of Z such that the smallest circuit in M|Z' has size 2s. Then, by (II), M has an s-echidna of order  $h_s(s,t) = m$ , completing the proof.

#### 5. Properties of (s, t)-spikes

In this section, we prove some properties of (s, t)-spikes. In particular, we show that an (s, t)-spike has order at least s + t - 1; an (s, t)-spike of order m has 2m elements and rank m + s - t; and the circuits of an (s, t)-spike that are not a union of s arms meet all but at most t - 2 of the arms. We also give some results about the connectivity of (s, t)-spikes of sufficiently large order.

We also show that an appropriate concatenation of the associated partition of a t-spike is a (2t-1)-anemone, following the terminology of [1]. Finally, we describe a construction that can be used to obtain an (s, t+1)-spike from an (s,t)-spike of sufficiently large order, and we show that every (s,t+1)-spike can be constructed from some (s, t)-spike in this way.

We again assume that s and t are positive integers.

### Basic properties.

**Lemma 5.1.** Let M be an (s, t)-spike with associated partition  $(A_1, \ldots, A_m)$ . Then  $m \ge s + t - 1$ .

*Proof.* By the definition of an (s,t)-spike, we have  $m \geq \max\{s,t\}$ . Let  $Y = \bigcup_{j \in [t]} A_j$ , and let  $y \in Y$ . Since Y is a cocircuit,  $Z = (E(M) - Y) \cup \{y\}$ spans M. Therefore,  $r(M) \leq |Z| = 2m - 2t + 1$ . Similarly, by duality,  $r^*(M) \leq 2m - 2s + 1$ . Therefore,

$$2m = |E(M)| = r(M) + r^*(M) \le (2m - 2t + 1) + (2m - 2s + 1)$$

The result follows.

**Lemma 5.2.** Let M be an (s,t)-spike of order m. Then r(M) = m + s - tand  $r^*(M) = m - s + t$ .

*Proof.* Let  $(A_1, \ldots, A_m)$  be the associated partition of M, and let  $A_i =$  $\{x_i, y_i\}$  for each  $i \in [m]$ . Choose  $I \subseteq J \subseteq [m]$  such that |I| = s - 1 and |J| = m - t. (This is possible by Lemma 5.1.) Let  $X = \{y_j : i \in I\} \cup \{x_j : j \in I\}$ J}. Note that  $\bigcup_{i \in I \cup J} A_i \subseteq cl(X)$ . Since  $E(M) - \bigcup_{i \in I \cup J} A_i$  is a cocircuit,  $\bigcup_{i \in I \cup J} A_i$  is a hyperplane. Therefore,  $\bigcup_{i \in I \cup J} A_i = \operatorname{cl}(X)$ , and we have  $r(M) - 1 = r(X) \le |X| = |I| + |J| = m + s - t - 1$ . Thus,  $r(M) \le m + s - t$ . Similarly, by duality,  $r^*(M) \leq m - s + t$ .

Therefore, we have

$$2m = |E(M)| = r(M) + r^*(M) \le (m + s - t) + (m - s + t) = 2m.$$

Thus, we must have equality, and the result holds.

**Lemma 5.3.** Let M be an (s,t)-spike of order m with associated partition  $(A_1, \ldots, A_m)$ , and let C be a circuit of M.

- (i)  $C = \bigcup_{j \in J} A_j$  for some s-element set  $J \subseteq [m]$ , or (ii)  $|\{i \in [m] : A_i \cap C \neq \emptyset\}| \ge m (t-2)$  and  $|\{i \in [m] : A_i \subseteq C\}| < s$ .

*Proof.* Let  $S = \{i \in [m] : A_i \cap C \neq \emptyset\}$ . Thus, S is the minimal subset of [m]such that  $C \subseteq \bigcup_{i \in S} A_i$ . We have  $|S| \ge s$  since C is independent otherwise. If |S| = s, then C satisfies (i). Therefore, we may assume |S| > s. We must have  $|\{i \in [m] : A_i \subseteq C\}| < s$ ; otherwise C properly contains a circuit. Thus, there is some  $j \in S$  such that  $A_j - C \neq \emptyset$ . If  $|S| \ge m - (t-2)$ , then C satisfies (ii). Therefore, we may assume  $|S| \le m - (t-1)$ . Let  $T = ([m] - S) \cup \{j\}$ . Then  $|T| \ge t$ , implying that  $\bigcup_{i \in T} A_i$  contains a cocircuit intersecting C in one element. This contradicts orthogonality.

In the remainder of the paper, if  $(A_1, \ldots, A_m)$  is the associated partition of an (s, t)-spike and  $J \subseteq [m]$ , then we define

$$A_J = \bigcup_{j \in J} A_j.$$

**Proposition 5.4.** Let  $\pi = (A_1, \ldots, A_m)$  be the associated partition of an (s,t)-spike. If  $J \subseteq [m]$ , then

$$r(A_J) = \begin{cases} 2|J| & \text{if } |J| < s, \\ s + |J| - 1 & \text{if } s \le |J| \le m - t + 1, \\ m + s - t & \text{if } |J| \ge m - t + 1. \end{cases}$$

*Proof.* If |J| < s, then  $A_J$  is properly contained in a circuit and is therefore independent. Thus,  $r(A_J) = |A_J| = 2|J|$ .

We now prove that  $r(A_J) = s + |J| - 1$  if  $s \leq |J| \leq m - t + 1$ . We proceed by induction on |J|. As a base case, if |J| = s, then  $A_J$  is a circuit. Therefore,  $r(A_J) = |A_J| - 1 = s + |J| - 1$ . Now, for the inductive step, let  $s < |J| \leq m - t + 1$ , and let  $J' \subseteq J$  with |J'| = |J| - 1. By induction,  $r(A_{J'}) = s + |J| - 2$ . Let  $\{x_i, y_i\} = A_J - A_{J'}$ . By Lemma 5.3, since |J| < m - t + 2, there is no circuit C such that  $x_i \in C \subseteq A_{J'} \cup \{x_i\}$ . Therefore,  $x_i \notin cl(A_{J'})$ , and  $r(A_{J'} \cup \{x_i\}) = r(A_{J'}) + 1$ . On the other hand, since |J| > s, there is a circuit C such that  $y_i \in C \subseteq A_J$ . Therefore,  $y_i \in cl(A_{J'} \cup \{x_i\})$ , and  $r(A_J) = r(A_{J'}) + 1 = s + |J| - 1$ .

Note that the preceding argument, along with Lemma 5.2 implies that, if |J| = m - t + 1, then  $A_J$  is spanning. Thus, if  $|J| \ge m - t + 1$ , then  $r(A_J) = r(M) = m + s - t$ .

**Connectivity.** Let M be a matroid with ground set E. Recall that the *connectivity function* of M, denoted by  $\lambda$ , is defined as

$$\lambda(X) = r(X) + r(E - X) - r(M),$$

for all subsets X of E. In the case where M is an (s,t)-spike of order m and  $X = A_J$  for some set  $J \subseteq [m]$ , this implies

$$\lambda(A_J) = r(A_J) + r(A_{[m]-J}) - r(M).$$

Therefore, Proposition 5.4 allows us to easily compute  $\lambda(A_J)$ .

**Lemma 5.5.** Let  $\pi = (A_1, \ldots, A_m)$  be the associated partition of an (s, t)-spike, and let (J, K) be a partition of [m] with  $|J| \leq |K|$ .

- (i) If  $|J| \leq t 1$ , then  $\lambda(A_J) = r(A_J)$ .
- (ii) If  $t 1 \le |J| \le m s$ , then

$$\lambda(A_J) = \begin{cases} t + |J| - 1 & \text{if } |J| < s, \\ s + t - 2 & \text{if } s \le |J| \le m - t + 1. \end{cases}$$

(iii) If |J| > m - s, then  $\lambda(A_J) = m - s + t$ .

*Proof.* If  $|J| \le t - 1$ , then  $|K| \ge m - t + 1$ . Therefore,  $A_K$  is spanning, and  $\lambda(A_J) = r(A_J) + r(A_K) - r(M) = r(A_J)$ . Statement (i) follows.

If  $t-1 \leq |J| \leq m-s$ , then  $s \leq |K| \leq m-t+1$ . Therefore,  $\lambda(A_J) = r(A_J) + r(A_K) - r(M) = r(A_J) + s + m - |J| - 1 - (m+s-t)$ . Statement (ii) follows. (Note that we cannot have |J| > m-t+1 because otherwise  $|K| < t-1 \leq |J|$ .)

If |J| > m - s, then  $s > |K| \ge |J|$ . Therefore,  $\lambda(A_J) = r(A_J) + r(A_K) - r(M) = 2|J| + 2(m - |J|) - (m + s - t) = m - s + t$ . Statement (iii) follows.

Using the terminology of [1], Lemma 5.5 implies the following.

**Proposition 5.6.** Let  $(A_1, \ldots, A_m)$  be the associated partition of an (s, t)-spike M, and suppose that  $(P_1, \ldots, P_k)$  is a partition of E(M) such that, for each  $i \in [k]$ ,  $P_i = \bigcup_{i \in I} A_i$  for some subset I of [m], with  $|I| \ge \max\{s-1, t-1\}$ . Then  $(P_1, \ldots, P_k)$  is an (s + t - 1)-anemone.

We now continue our study of the connectivity of (s, t)-spikes.

**Lemma 5.7.** Let M be an (s,t)-spike of order  $m \ge 3 \max\{s,t\} - 2$ , and let  $X \subseteq E(M)$  such that  $|X| \le 2 \min\{s,t\} - 1$ . Then  $\lambda(X) = |X|$ .

*Proof.* By Lemma 5.3, if X is dependent, then either |X| = 2s or  $|X| \ge m - t + 2 \ge 3 \max\{s, t\} - 2 - t + 2 = 3 \max\{s, t\} - t \ge 2 \max\{s, t\} \ge 2s$ . However,  $|X| \le 2 \min\{s, t\} - 1 < 2s$ . Therefore, X is independent, which implies that r(X) = |X|.

By a similar argument, using the dual of Lemma 5.3, X is coindependent, implying that r(E(M) - X) = r(M). Therefore,

$$\lambda(X) = r(X) + r(E(M) - X) - r(M)$$
$$= |X| + r(M) - r(M)$$
$$= |X|,$$

proving the lemma.

**Theorem 5.8.** Let M be an (s,t)-spike of order

 $m \ge \max\{3s+t, s+3t\} - 4,$ 

where  $\min\{s,t\} \ge 2$ . Then M is  $(2\min\{s,t\}-1)$ -connected.

*Proof.* Because  $M^*$  is a (t, s)-spike and because  $\lambda_{M^*} = \lambda_M$ , we may assume without loss of generality that  $t \leq s$ . Note that  $\max\{3s + t, s + 3t\} = 3\max\{s,t\} + \min\{s,t\}$ . Therefore,  $m \geq 3s + t - 4$ , and we must show that M is (2t - 1)-connected.

Now, suppose for a contradiction that M is not (2t-1)-connected. Then there is a k-separation (P,Q) of M, with  $|P| \ge |Q|$ , for some k < 2t - 1. Therefore,  $\lambda(P) = \lambda(Q) < k \le 2t - 2$ .

First, we consider the case where  $A_I \subseteq P$ , for some (t-1)-element set  $I \subseteq [m]$ . Let  $U = \{u \in [m] : |P \cap A_u| = 1\}$ . Then  $A_j \subseteq \operatorname{cl}_{M^*}(P)$  for each  $j \in U$ . For such a j, it follows, by the definition of  $\lambda_{M^*}$  (which is equal to  $\lambda_M = \lambda$ ), that  $\lambda(P \cup A_j) \leq \lambda(P)$ . We use this repeatedly below; in particular, we see that  $\lambda(P \cup A_U) \leq \lambda(P)$ .

Let  $P' = P \cup A_U$ , and let Q' = E(M) - P'. Then there is a partition (J, K) of [m], with  $|J| \leq |K|$ , such that  $Q' = A_J$  and  $P' = A_K$ . Moreover,  $\lambda(Q') = \lambda(P') \leq \lambda(P)$ .

Suppose  $|J| \ge t - 1$ . Note that  $m \ge 3s + t - 4 \ge 2s$  since  $\min\{s, t\} \ge 2$ . Therefore,  $|J| \le \frac{1}{2}m = m - \frac{1}{2}m \le m - \frac{1}{2}(2s) = m - s$ . Thus, to determine  $\lambda(Q')$ , we need only consider Lemma 5.5(ii). If  $|J| \ge s$ , then by Lemma 5.5(ii),

$$\lambda(P) \ge \lambda(P') = \lambda(Q') = s + t - 2 \ge 2t - 2,$$

a contradiction. Otherwise, |J| < s, implying by Lemma 5.5(ii) that

$$\lambda(P) \ge \lambda(P') = \lambda(Q') = t + |J| - 1 \ge t + t - 1 - 1 = 2t - 2,$$

another contradiction.

Therefore, |J| < t - 1. Let  $U' \subseteq U$  such that |U'| = |Q| - (2t - 2). Then  $\lambda(P) \geq \lambda(P \cup A_{U'}) = \lambda(Q - A_{U'})$ . Since  $|Q - A_{U'}| = 2t - 2$  and  $m \geq 3s + t - 4 \geq 3s - 2$ , Lemma 5.7 implies that  $\lambda(Q - A_{U'}) = 2t - 2$ , so  $\lambda(P) \geq 2t - 2$ , a contradiction.

Now we consider the case that  $|\{i \in [m] : A_i \subseteq P\}| < t - 1$ . Since  $|Q| \leq |P|$ , it follows that  $|\{i \in [m] : A_i \subseteq Q\}| \leq |\{i \in [m] : A_i \subseteq P\}| < t - 1 < s$ .

Now, since  $|\{i \in [m] : A_i \subseteq P\}| < t-1$ , we have  $|\{i \in [m] : A_i \cap Q \neq \emptyset\}| > m - (t-1)$ . Therefore,  $r(Q) \ge m - (t-1)$  by Lemma 5.3. Similarly,  $r(P) \ge m - (t-1)$ . Thus,

$$\begin{split} \lambda(P) &= r(P) + r(Q) - r(M) \\ &\geq (m - (t - 1)) + (m - (t - 1)) - (m + s - t) \\ &= m - s - t + 2 \\ &\geq 3s + t - 4 - s - t + 2 \\ &= 2s - 2 \\ &\geq 2t - 2, \end{split}$$

a contradiction. This completes the proof.

**Constructions.** In [2], a construction is described that, starting from a (t,t)-spike  $M_0$ , obtains a (t+1,t+1)-spike  $M_1$ . This construction consists of a certain elementary quotient  $M'_0$  of  $M_0$ , followed by a certain elementary lift  $M_1$  of  $M'_0$ . It is shown in [2] that  $M_1$  is a (t+1,t+1)-spike as long as the order of  $M_0$  is sufficiently large.

In the process of constructing  $M_1$  in this way, the intermediary matroid  $M'_0$  is a (t, t + 1)-spike. For the sake of completeness, we will review this construction in the more general case where  $M_0$  is an (s, t)-spike, in which case  $M'_0$  is an (s, t + 1)-spike. To construct an (s + 1, t)-spike, we perform the construction on  $M^*$  and dualize. Since (2, 2)-spikes (and indeed, (1, 1)-spikes) are well known to exist, this means that (s, t)-spikes exist for all positive integers s and t.

It is also shown in [2] that all (t,t)-spikes can be constructed in this manner. We also extend this to the general case of (s,t)-spikes below.

Recall that  $M_1$  is an *elementary quotient* of  $M_0$  if there is a single-element extension  $M_0^+$  of  $M_0$  by an element e such that  $M_1 = M_0^+/e$ . If  $M_1$  is an elementary quotient of  $M_0$ , then  $M_0$  is an *elementary lift* of  $M_1$ . Also, note that if  $M_1$  is an elementary lift of  $M_0$ , then  $M_1^*$  is an elementary quotient of  $M_0^*$ .

**Construction 5.9.** Let M be an (s,t)-spike of order  $m \ge s + t$ , with associated partition  $\pi$ . Let M + e be a single-element extension of M by an element e such that e blocks each 2t-element cocircuit that is a union of t arms of M. Then let M' = (M + e)/e.

In other words, M + e has the property that  $e \notin cl_{M+e}(E(M) - C^*)$  for every 2t-element cocircuit  $C^*$  that is the union of t arms. Note that one possibility is that M + e is the free extension of M by an element e. Since  $m-t \geq s$ , we have  $e \notin cl_{M+e}(C)$  for each 2s-element circuit C. Thus, in M', the union of any s arms of the (s, t)-spike M is still a circuit of M'. However,

since r(M') = r(M) - 1, the union of any t + 1 arms is a 2(t + 1)-element cocircuit. Therefore, M' is an (s, t + 1)-spike.

Note that M' is not unique; more than one (s, t + 1)-spike can be constructed from a given (s, t)-spike M using Construction 5.9. Given an (s + 1, t)-spike M', we will describe how to obtain an (s, t)-spike M from M' by a specific elementary quotient. This process reverses the dual of Construction 5.9. This will then imply that every (s, t)-spike can be constructed from a (1, 1)-spike by repeated use of Construction 5.9 and its dual. Lemma 5.10 describes the single-element extension that gives rise to the elementary quotient we desire. Intuitively, the extension adds a "tip" to the (s, t)-spike. In the proof of this lemma, we assume knowledge of the theory of modular cuts (see [6, Section 7.2]).

The proof of Lemma 5.10 will be very similar to the proof of [2, Lemma 6.6]. However, we note that [2, Lemma 6.6] is falsely stated; what is proven in [2] is essentially the specialisation of Lemma 5.10, below, in the case that s = t. The statement of [2, Lemma 6.6] replaces the condition that M is a (t, t)-spike with the weaker condition that M has a t-echidna. To demonstrate that this is overly general, consider the rank-3 matroid consisting of two disjoint lines with four points. Let these lines be  $\{a, b, c, d\}$  and  $\{w, x, y, z\}$ . Then  $(\{a, b\}, \{w, x\})$  is a 2-echidna of order 2. For [2, Lemma 6.6] to be true, we would need a single-element extension  $M^+$  by an element e such that  $e \in cl_{M^+}(\{a, b\})$  but  $e \notin cl_{M^+}(\{c, d\})$ . This is impossible since  $cl_M(\{a, b\}) = cl_M(\{c, d\})$ .

**Lemma 5.10.** Let M be an (s, t)-spike. There is a single-element extension  $M^+$  of M by an element e having the property that, for every  $X \subseteq E(M)$ ,  $e \in cl_{M^+}(X)$  if and only if X contains at least s - 1 arms of M.

*Proof.* Since M is an (s,t)-spike, there is a partition  $\pi = (S_1, \ldots, S_m)$  of E(M) that is both an s-echidna and a t-coechidna. Let

$$\mathcal{F} = \left\{ \bigcup_{i \in I} S_i : I \subseteq [m] \text{ and } |I| = s - 1 \right\}.$$

By the definition of an s-echidna,  $\mathcal{F}$  is a collection of flats of M. Let  $\mathcal{M}$  be the set of all flats of M containing some flat  $F \in \mathcal{F}$ . We claim that  $\mathcal{M}$  is a modular cut. Recall that, for distinct  $F_1, F_2 \in \mathcal{M}$ , the pair  $(F_1, F_2)$  is modular if  $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$ . To show that  $\mathcal{M}$  is a modular cut, it suffices to prove that, for any  $F_1, F_2 \in \mathcal{M}$  such that  $(F_1, F_2)$  is a modular pair,  $F_1 \cap F_2 \in \mathcal{M}$ .

For any  $F \in \mathcal{M}$ , since F contains at least s - 1 arms of  $\mathcal{M}$ , and the union of any s arms is a circuit, it follows that F is a union of arms of  $\mathcal{M}$ . Thus, let  $F_1, F_2 \in \mathcal{M}$  be such that  $F_1 = \bigcup_{i \in I_1} S_i$  and  $F_2 = \bigcup_{i \in I_2} S_i$ , where  $I_1$  and  $I_2$  are distinct subsets of [m] with  $u_1 = |I_1| \ge s - 1$  and  $u_2 = |I_2| \ge s - 1$ .

Let  $q = |I_1 \cap I_2|$ . Then  $F_1 \cup F_2$  is the union of  $u_1 + u_2 - q \ge s - 1$  arms, and  $F_1 \cap F_2$  is the union of q arms. We show that if q < s - 1, then  $(F_1, F_2)$ is not a modular pair. We consider several cases. First, suppose  $u_1, u_2 \leq m - t + 1$ . By Proposition 5.4,

$$r(F_1) + r(F_2) = (s + u_1 - 1) + (s + u_2 - 1)$$
  
>  $(s - 1 + u_1 + u_2 - q) + 2q$   
=  $s + |I_1 \cup I_2| - 1 + 2|I_1 \cap I_2$   
 $\ge r(F_1 \cup F_2) + r(F_1 \cap F_2).$ 

Next, consider the case where  $u_2 \leq m - t + 1 < u_1$ . (By symmetry, the argument is the same if  $u_1$  and  $u_2$  are swapped.) One can check that  $u_1 + u_2 - q > m - t + 1$ . By Proposition 5.4,

$$r(F_1) + r(F_2) = (m + s - t) + (s + u_2 - 1)$$
  
> (m + s - t) + 2q  
= r(F\_1 \cup F\_2) + r(F\_1 \cap F\_2).

Finally, consider the case where  $u_1, u_2 > m - t - 1$ . We have

$$r(F_1) + r(F_2) = 2m + 2s - 2t,$$

which by Lemma 5.1, is at least

$$m + 3s - t - 1 > m + s - t + 2q$$
  
=  $r(F_1 \cup F_2) + r(F_1 \cap F_2).$ 

Thus, in all cases,  $(F_1, F_2)$  is not a modular pair. Therefore, we have shown that  $\mathcal{M}$  is a modular cut. Now, there is a single-element extension corresponding to the modular cut  $\mathcal{M}$ , and this extension satisfies the requirements of the lemma (see, for example, [6, Theorem 7.2.3]).

**Theorem 5.11.** Let M be an (s, t)-spike of order  $m \ge s + t$ . Then M can be constructed from a (1, 1)-spike of order m by applying Construction 5.9 t - 1 times, followed by the dual of Construction 5.9 s - 1 times.

*Proof.* For s = t = 1, the result is clear. Otherwise, by duality, we may assume without loss of generality that t > 1. By induction and duality, it suffices to show that M can be constructed from an (s - 1, t)-spike of order m by applying the dual of Construction 5.9 once.

Let  $\pi = (A_1, \ldots, A_m)$  be the associated partition of M. Let  $M^+$  be the single-element extension of M by an element e described in Lemma 5.10.

Let  $M' = M^+/e$ . We claim that  $\pi$  is an (s-1)-echidna and a *t*-coechidna that partitions the ground set of M'.

Let X be the union of any s - 1 spines of  $\pi$ . Then X is independent in M, and  $X \cup \{e\}$  is a circuit in  $M^+$ , so X is a circuit in M'. Thus,  $\pi$  is an (s - 1)-echidna of M'. Now let  $C^*$  be the union of any t spines of  $\pi$ , and let  $H = E(M) - C^*$ . Then H is the union of at least s - 1 spines, so  $e \in cl_{M^+}(H)$ . Now  $H \cup \{e\}$  is a hyperplane in  $M^+$ , so  $C^*$  is a cocircuit in  $M^+$  and therefore in M'. Hence  $\pi$  is a t-coechidna of M'.

Note that M' is an elementary quotient of M, so M is an elementary lift of M' where none of the 2(s-1)-element circuits of M' are preserved in M. So the (s,t)-spike M can be obtained from the (s-1,t)-spike M' using the dual of Construction 5.9.

#### GENERALIZED SPIKES

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School of Mathematics and Statistics, Victoria University of Wellington, New Zealand

Email address: nick.brettell@vuw.ac.nz

Department of Mathematics, Vanderbilt University, Nashville, Tennessee *Email address*: kevin.m.grace@vanderbilt.edu