GENERALIZED SPIKES WITH CIRCUITS AND COCIRCUITS OF DIFFERENT CARDINALITIES

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ABSTRACT. We consider matroids with the property that every subset of the ground set of size s is contained in a 2s-element circuit and every subset of size t is contained in a 2t-element cocircuit. We say that such a matroid has the $(s, 2s, t, 2t)$ -property. A matroid is an (s, t) -spike if there is a partition of the ground set into pairs such that the union of any s pairs is a circuit and the union of any t pairs is a cocircuit. Our main result is that all sufficiently large matroids with the $(s, 2s, t, 2t)$ property are (s, t) -spikes, generalizing a 2019 result that proved the case where $s = t$. We also present some properties of (s, t) -spikes.

1. INTRODUCTION

For integers s, u, t, and v, with $u \geq s \geq 1$ and $v \geq t \geq 1$, a matroid M has the (s, u, t, v) -property if every s-element subset of $E(M)$ is contained in a circuit of size u, and every t-element subset of $E(M)$ is contained in a cocircuit of size v . Matroids with this property appear regularly in the matroid theory literature: for example, wheels and whirls have the $(1,3,1,3)$ property, and (tipless) spikes have the $(2, 4, 2, 4)$ -property. Note that M has the (s, u, t, v) -property if and only if M^* has the (t, v, s, u) -property. Brettell, Campbell, Chun, Grace, and Whittle [\[2\]](#page-16-0) studied such matroids, and showed that if $u < 2s$ or $v < 2t$, then there are only finitely many matroids with the (s, u, t, v) -property [\[2,](#page-16-0) Theorem 3.3]. On the other hand, in the case that $s = t$ and $u = v = 2t$, any sufficiently large matroid with the (s, u, t, v) property is a member of a class of structured matroids referred to as t-spikes. In particular, when $t = 2$, this is the class typically known simply as *(tipless)* spikes.

Our focus in this paper is also on the case where $u = 2s$ and $v = 2t$, but we drop the requirement that $s = t$. For positive integers s and t, an (s, t) -spike is a matroid on at least $2 \max\{s, t\}$ elements whose ground set has a partition (S_1, S_2, \ldots, S_n) into pairs such that the union of every set of s pairs is a circuit and the union of every set of t pairs is a cocircuit. The following is our main result:

Theorem 1.1. There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that, if M is a matroid with the $(s, 2s, t, 2t)$ -property and $|E(M)| \ge f(s, t)$, then M is an (s, t) -spike.

This proves the conjecture of Brettell et al. [\[2,](#page-16-0) Conjecture 1.2].

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Our approach is essentially the same as in [\[2\]](#page-16-0), but some care is required to generalize the argument. We note also that Lemma [5.10](#page-14-0) corrects an erroneous lemma [\[2,](#page-16-0) Lemma 6.6].

This paper is one in a developing series on matroids with the (s, u, t, v) property. First, Miller [\[5\]](#page-16-1) studied matroids with the (2, 4, 2, 4)-property, proving the specialization of Theorem [1.1](#page-0-0) to the case where $s = t = 2$. As previously mentioned, Brettell et al. [\[2\]](#page-16-0) considered the more general case where $s = t$ and $u = v = 2t$, for any $t > 1$. Oxley, Pfeil, Semple, and Whittle considered the case where $s = 2$, $u = 4$, $t = 1$, and $v \in \{3, 4\}$, showing that a sufficiently large v-connected matroid with the $(2, 4, 1, v)$ property is isomorphic to $M(K_{v,n})$ for some n [\[7\]](#page-16-2). A "cyclic" analogue of the (s, u, t, v) -property has also been considered, where a cyclic ordering σ is imposed on $E(M)$, and only sets that appear consecutively with respect to σ and have size s (or size t) need appear in a circuit of size u (or a cocircuit of size v, respectively). The case where $s = u-1$ and $t = v-1$ and $s = t$ was considered by Brettell, Chun, Fife, and Semple [\[3\]](#page-16-3); whereas Brettell, Semple, and Toft dropped the requirement that $s = t$ [\[4\]](#page-16-4).

This series of papers has been motivated by problems involving matroid connectivity. The well-known Wheels-and-Whirls Theorem of Tutte [\[9\]](#page-16-5) states that wheels and whirls (which have the $(1, 3, 1, 3)$ -property) are the only 3connected matroids with no elements that can be either deleted or contracted to retain a 3-connected matroid. Similarly, spikes (which have the (2, 4, 2, 4) property) are the only 3-connected matroids on at least 13 elements that have no triangles, no triads, and no pairs of elements that can be either deleted or contracted to preserve 3-connectivity [\[10\]](#page-16-6).

The following conjecture was stated as [\[2,](#page-16-0) Conjecture 1.3]. The case where $t = 2$ was proved by Williams [\[10\]](#page-16-6).

Conjecture 1.2. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if M is a $(2t-1)$ -connected matroid with no circuits or cocircuits of size $2t-1$, and $|E(M)| \ge f(t)$, then either

- (i) there exists a t-element set $X \subseteq E(M)$ such that either M/X or $M\backslash X$ is $(t+1)$ -connected, or
- (ii) M is a (t, t) -spike.

Indeed, sufficiently large (t, t) -spikes are $(2t - 1)$ -connected matroids [\[2,](#page-16-0) Lemma 6.5], they have no circuits or cocircuits of size $(2t-1)$ [\[2,](#page-16-0) Lemma 6.3], and for every t-element subset $X \subseteq E(M)$, neither M/X nor $M\backslash X$ is $(t+1)$ connected. Optimistically, we offer the following generalization of Conjecture [1.2.](#page-1-0)

Conjecture 1.3. There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that if M is a matroid with no circuits of size at most $2s - 1$, no cocircuits of size at most $2t-1$, the matroid M is $(2\min\{s,t\}-1)$ -connected, and $|E(M)| \ge f(s,t)$, then either

- (i) there exists an s-element set $X \subseteq E(M)$ such that M/X is $(s + 1)$ connected,
- (ii) there exists a t-element set $X \subseteq E(M)$ such that $M \backslash X$ is $(t + 1)$ connected, or
- (iii) M is an (s, t) -spike.

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Section [2](#page-2-0) recalls some terminology and a Ramsey-theoretic result used later in the paper. In Section [3,](#page-2-1) we recall the definition of echidnas from [\[2\]](#page-16-0) and show that every matroid with the $(s, 2s, t, 2t)$ -property and having a sufficiently large s-echidna is an (s, t) -spike. In Section [4,](#page-4-0) we prove The-orem [1.1.](#page-0-0) Finally, Section [5](#page-9-0) describes some properties of (s, t) -spikes, as well as a construction that allows us to build an $(s, t + 1)$ -spike from an (s, t) -spike.

2. Preliminaries

Our notation and terminology follows Oxley [\[6\]](#page-16-7). We refer to the fact that a circuit and a cocircuit cannot intersect in exactly one element as "orthogonality". A set S_1 meets a set S_2 if $S_1 \cap S_2 \neq \emptyset$. We denote $\{1, 2, \ldots, n\}$ by [n], and, for positive integers $i < j$, we denote $\{i, i+1, \ldots, j\}$ by [i, j]. We denote the set of positive integers by N.

In order to prove Theorem [1.1,](#page-0-0) we will use some hypergraph Ramsey Theory $[8]$. Recall that a hypergraph is *k*-uniform if every hyperedge has size k.

Theorem 2.1 (Ramsey's Theorem for k-uniform hypergraphs). For positive integers k and n, there exists an integer $r_k(n)$ such that if H is a k-uniform hypergraph on $r_k(n)$ vertices, then H has either a clique on n vertices, or a stable set on n vertices.

3. ECHIDNAS AND (s, t) -SPIKES

Recall that M is an (s, t) -spike if there is a partition of $E(M)$ into pairs such that the union of any s pairs is a circuit and the union of any t pairs is a cocircuit. In this section, we prove a sufficient condition for M to be an (s, t) -spike. Namely, we prove as Lemma [3.3](#page-3-0) that if M has the $(s, 2s, t, 2t)$ property, and a subset of $E(M)$ can be partitioned into u pairs such that the union of any t pairs is a circuit, then, when u is sufficiently large, M is an (s, t) -spike. Conforming with [\[2\]](#page-16-0), we call such a partition a t-echidna, as defined below.

Let M be a matroid. A *t*-echidna of order n is a partition (S_1, \ldots, S_n) of a subset of $E(M)$ such that

(i) $|S_i| = 2$ for all $i \in [n]$, and

(ii) $\bigcup_{i \in I} S_i$ is a circuit for all $I \subseteq [n]$ with $|I| = t$.

For $i \in [n]$, we say S_i is a spine. We say (S_1, \ldots, S_n) is a t-coechidna of M if (S_1, \ldots, S_n) is a t-echidna of M^* .

Let (S_1, \ldots, S_n) be a t-echidna of a matroid M. If (S_1, \ldots, S_m) is a t-echidna of M, for some $m \geq n$, we say that (S_1, \ldots, S_n) extends to (S_1, \ldots, S_m) . We say that $\pi = (S_1, \ldots, S_n)$ is maximal if π extends only to π .

Note that a matroid M is an (s, t) -spike if there exists a partition $\pi =$ (A_1, \ldots, A_m) of $E(M)$ such that π is an s-echidna and a t-coechidna, for some $m \geq \max\{s, t\}.$ In this case, we say that the (s, t) -spike M has order m, we call π the *associated partition* of the (s, t) -spike M, and we say that A_i is an arm of the (s, t) -spike for each $i \in [m]$. An (s, t) -spike with $s = t$ is also called a *t-spike*. Note that if M is an (s, t) -spike, then M^* is a (t, s) -spike.

Throughout this section, we assume that s and t are positive integers.

Lemma 3.1. Let M be a matroid with the $(s, 2s, t, 2t)$ -property. If M has an s-echidna (S_1, \ldots, S_n) , where $n \geq s + 2t - 1$, then (S_1, \ldots, S_n) is also a t-coechidna of M.

Proof. Suppose M has an s-echidna (S_1, \ldots, S_n) with $n \geq s+2t-1$, and let $S_i = \{x_i, y_i\}$ for each $i \in [n]$. We show, for every t-element subset J of $[n]$, that $\bigcup_{j\in J} S_j$ is a cocircuit. Without loss of generality, let $J = [t]$. By the $(s, 2s, t, 2t)$ -property, $\{x_1, \ldots, x_t\}$ is contained in a 2t-element cocircuit C^* . Suppose for a contradiction that $C^* \neq \bigcup_{j \in J} S_j$. Then there is some $i \in [t]$ such that $y_i \notin C^*$. Without loss of generality, say $y_1 \notin C^*$.

Let I be an $(s - 1)$ -element subset of $[t + 1, n]$. For any such I, the set $S_1 \cup \bigcup_{i \in I} S_i$ is a circuit that meets C^* . By orthogonality, $\bigcup_{i \in I} S_i$ meets C^* . Thus, \tilde{C}^* avoids at most $s-2$ of the S_i 's for $i \in [t+1, n]$. In fact, as C^* meets each S_i with $i \in [t]$, the cocircuit C^* avoids at most $s - 2$ of the S_i 's for $i \in [n]$. Thus $|C^*| \ge n - (s - 2) \ge (s + 2t - 1) - (s - 2) = 2t + 1 > 2t$, a contradiction.

Therefore, we conclude that $C^* = \bigcup_{j \in J} S_j$, and the result follows.

Lemma 3.2. Let M be a matroid with the $(s, 2s, t, 2t)$ -property, and let (S_1, \ldots, S_n) be an s-echidna of M with $n \geq \max\{s + 2t, 2s + t\} - 1$.

- (i) Let I be an $(s-1)$ -subset of $[n]$. For $z \in E(M) \bigcup_{i \in I} S_i$, there is a 2s-element circuit containing $\{z\} \cup \bigcup_{i \in I} S_i$.
- (ii) Let I be a $(t-1)$ -subset of $[n]$. For $z \in E(M) \bigcup_{i \in I} S_i$, there is a 2t-element cocircuit containing $\{z\} \cup \bigcup_{i \in I} S_i$.

Proof. First we prove (i). For $i \in [n]$, let $S_i = \{x_i, y_i\}$. By the $(s, 2s, t, 2t)$ property, there is a 2s-element circuit C containing $\{z\} \cup \{x_i : i \in I\}$. Let J be a $(t-1)$ -element subset of $[n]$ such that C and $\bigcup_{j\in J} S_j$ are disjoint (such a set exists since $|C| = 2s$ and $n \geq 2s + t - 1$). For $i \in I$, let $C_i^* = S_i \cup \bigcup_{j \in J} S_j$, and observe that $x_i \in C_i^* \cap C$, and $C_i^* \cap C \subseteq S_i$. By Lemma [3.1,](#page-3-1) (S_1, \ldots, S_n) is a *t*-coechidna as well as an *s*-echidna; therefore, C_i^* is a cocircuit. Now, for each $i \in I$, orthogonality implies that $|C_i^* \cap C| \geq 2$, and hence $y_i \in C$. So C contains $\{z\} \cup \bigcup_{i \in I} S_i$, as required.

Now, to prove (ii), recall that (S_1, \ldots, S_n) is a *t*-coechidna by Lemma Lemma [3.1.](#page-3-1) Therefore, (ii) follows by (i) and duality.

Lemma 3.3. Let M be a matroid with the $(s, 2s, t, 2t)$ -property. If M has an s-echidna $\pi = (S_1, \ldots, S_n)$, where $n \ge \max\{s + 2t - 1, 2s + t - 1, 3s + t - 3\}$, then (S_1, \ldots, S_n) extends to a partition of $E(M)$ that is both an s-echidna and a t-coechidna.

Proof. Let $\pi' = (S_1, \ldots, S_m)$ be a maximal s-echidna with $X = \bigcup_{i=1}^m S_i \subseteq$ $E(M)$. Suppose for a contradiction that $X \neq E(M)$. Since π' is maximal, $m \geq n \geq s + 2t - 1$. Therefore, by Lemma [3.1,](#page-3-1) π' is a *t*-coechidna.

Let $z \in E(M) - X$. By Lemma [3.2,](#page-3-2) there is a 2s-element circuit $C =$ $(\bigcup_{i\in [s-1]} S_i) \cup \{z, z'\}$ for some $z' \in E(M)$. We claim that $z' \notin X$. Towards a contradiction, suppose that $z' \in S_k$ for some $k \in [s, m]$. Let J be a t-element subset of $[s, m]$ containing k. Then, since (S_1, \ldots, S_m) is a t-coechidna, $\bigcup_{j\in J} S_j$ is a cocircuit that contains z'. Now, this cocircuit intersects the

circuit C in a single element z' , contradicting orthogonality. Thus, $z' \notin X$, as claimed.

We next show that $({z, z'}, S_s, S_{s+1}, \ldots, S_m)$ is a t-coechidna. Since π' is a t-coechidna, it suffices to show that $\{z, z'\} \cup \bigcup_{i \in I} S_i$ is a cocircuit for each $(t-1)$ -element subset I of [s, m]. Let I be such a set. Lemma [3.2](#page-3-2) implies that there is a 2*t*-element cocircuit C^* of M containing $\{z\} \cup \bigcup_{i \in I} S_i$. By orthogonality, $|C \cap C^*| > 1$. Therefore, $z' \in C^*$. Thus, $({z, z'}, S_s, S_{s+1}, \ldots, S_m)$ is a t-coechidna. Since this t-coechidna has order $1 + m - (s-1) \geq n - s + 2 \geq$ $2s + t - 1$, the dual of Lemma [3.1](#page-3-1) implies that $({z, z'}, S_s, S_{s+1}, \ldots, S_m)$ is also an s-echidna.

Next we show that $({z, z', S_1, S_2, \ldots, S_m})$ is a t-coechidna. Let I be a $(t-1)$ -element subset of $[m]$. We claim that $\{z, z'\} \cup \bigcup_{i \in I} S_i$ is a cocircuit. Let J be an $(s-1)$ -element subset of $[s,m]-I$. Then $C = \{z, z'\} \cup \bigcup_{j \in J} S_j$ is a circuit since $({z, z', S_s, S_{s+1}, \ldots, S_m})$ is an s-echidna. By Lemma [3.2,](#page-3-2) there is a 2t-element cocircuit C^* containing $\{z\} \cup \bigcup_{i \in I} S_i$. By orthogonality between C and C^* , we have $z' \in C^*$. Since I was arbitrarily chosen, $({z, z', S_1, S_2, \ldots, S_m})$ is a t-coechidna. By the dual of Lemma [3.1,](#page-3-1) it is also an s-echidna, contradicting the maximality of (S_1, \ldots, S_m) .

4. MATROIDS WITH THE $(s, 2s, t, 2t)$ -PROPERTY

In this section, we prove that every sufficiently large matroid with the $(s, 2s, t, 2t)$ -property is an (s, t) -spike. We will show that a sufficiently large matroid with the $(s, 2s, t, 2t)$ -property has a large s-echidna or t-coechidna; it then follows, by Lemma [3.3,](#page-3-0) that the matroid is an (s, t) -spike. As in the previous section, we assume that s and t are positive integers.

Lemma 4.1. Let M be a matroid with the $(s, 2s, t, 2t)$ -property, and let $X \subseteq E(M)$.

- (i) If $r(X) < s$, then X is independent.
- (ii) If $r(X) = s$, then $M|X \cong U_{s,|X|}$ and $|X| < s + 2t$.

Proof. Every subset of $E(M)$ of size at most s is independent since it is contained in a circuit of size 2s. In particular, [\(i\)](#page-4-1) holds.

Now let $r(X) = s$. Then every $(s + 1)$ -element subset of X is a circuit, so $M|X \cong U_{s,|X|}$. Suppose for a contradiction that $|X| \geq s + 2t$. Let C^* be a 2t-element cocircuit such that there is some $x \in X \cap C^*$. Then $X - C^*$ is contained in the hyperplane $E(M) - C^*$. Since $x \in X \cap C^*$, we have $r(X - C^*) < r(X) = s$. Therefore, $X - C^*$ is an independent set, so $|X - C^*| < s$. Since $|X| \geq s + 2t$, we have $|C^*| > 2t$, a contradiction. Thus, [\(ii\)](#page-4-2) holds.

Lemma 4.2. Let M be a matroid with the $(s, 2s, t, 2t)$ -property, and let $C_1^*, C_2^*, \ldots, C_{s-1}^*$ be a collection of pairwise disjoint cocircuits of M. Let $Y = E(M) - \bigcup_{i \in [s-1]} C_i^*$. For all $y \in Y$, there is a 2s-element circuit C_y $containing y such that either$

- (i) $|C_y \cap C_i^*| = 2$ for all $i \in [s-1]$, or
- (ii) $|C_y \cap C_j^*| = 3$ for some $j \in [s-1]$, and $|C_y \cap C_i^*| = 2$ for all $i \in [s-1]-\{j\}.$

Moreover, if C_y satisfies [\(ii\)](#page-4-3), then there are at most s+2t-1 elements $w \in Y$ such that $(C_y - y) \cup \{w\}$ is a circuit.

Proof. Choose an element $c_i \in C_i^*$ for each $i \in [s-1]$. By the $(s, 2s, t, 2t)$ property, there is a 2s-element circuit C_y containing $\{c_1, c_2, \ldots, c_{s-1}, y\}$, for each $y \in Y$. By orthogonality, C_y satisfies [\(i\)](#page-4-4) or [\(ii\).](#page-4-3)

Suppose C_y satisfies [\(ii\),](#page-4-3) and let $S = C_y - Y = C_y - \{y\}$. Let $W = \{w \in$ $Y : S \cup \{w\}$ is a circuit}. It remains to prove that $|W| < s + 2t$. Observe that $W \subseteq cl(S) \cap Y$, and, since S contains s – 1 elements in pairwise disjoint cocircuits that avoid Y, we have $r(cl(S) \cup Y) \geq r(Y) + (s-1)$. Thus,

$$
r(W) \le r(\text{cl}(S) \cap Y)
$$

\n
$$
\le r(\text{cl}(S)) + r(Y) - r(\text{cl}(S) \cup Y)
$$

\n
$$
\le (2s - 1) + r(Y) - (r(Y) + (s - 1))
$$

\n
$$
= s,
$$

using submodularity of the rank function at the second line.

Now, by Lemma [4.1](#page-4-5)[\(i\),](#page-4-1) if $r(W) < s$, then W is independent, so $|W| =$ $r(W) < s < s+2t$. On the other hand, by Lemma [4.1](#page-4-5)[\(ii\),](#page-4-2) if $r(W) = s$, then $M|W \cong U_{t,|W|}$ and $|W| < s + 2t$, as required.

Lemma 4.3. There exists a function h such that if M is a matroid with at least $h(k, d, t)$ k-element circuits, and the property that every t-element set is contained in a 2t-element cocircuit for some positive integer t, then M has a collection of d pairwise disjoint 2t-element cocircuits.

Proof. By [\[2,](#page-16-0) Lemma 3.2], there is a function q such that if M has at least $g(k, d)$ k-element circuits, then M has a collection of d pairwise disjoint circuits. We define $h(k, d, t) = g(k, dt)$, and claim that a matroid with at least $h(k, d, t)$ k-element circuits, and the property that every t-element set is contained in a 2t-element cocircuit, has a collection of d pairwise disjoint 2t-element cocircuits.

Let M be such a matroid. Then M has a collection of dt pairwise disjoint circuits. We partition these into d groups of size t : call this partition $(\mathcal{C}_1, \ldots, \mathcal{C}_d)$. Since the t circuits in any cell of this partition are pairwise disjoint, it now suffices to show that, for each $i \in [d]$, there is a 2t-element cocircuit contained in the union of the members of \mathcal{C}_i . Let $\mathcal{C}_i = \{C_1, \ldots, C_t\}$ for some $i \in [d]$. Pick some $c_j \in C_j$ for each $j \in [t]$. Then, since $\{c_1, c_2, \ldots, c_t\}$ is a t -element set, it is contained in a $2t$ -element cocircuit, which, by orthogonality, is contained in $\bigcup_{j\in[t]} C_j$.

Lemma 4.4. Let M be a matroid with the $(s, 2s, t, 2t)$ -property such that $r(M) \geq r^*(M)$. There exists a function g such that, if $|E(M)| \geq g(s,t,q)$, then M has $s-1$ pairwise disjoint 2t-element cocircuits $C_1^*, C_2^*, \ldots, C_{s-1}^*$, and there is some $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$ such that

- (i) $r_M(Z) \geq q$, and
- (ii) for each $z \in Z$, there exists an element $z' \in Z \{z\}$ such that $\{z, z'\}$ is contained in a 2s-element circuit C with $|C \cap C_i^*| = 2$ for each $i \in [s-1]$.

Proof. By Lemma [4.3,](#page-5-0) there is a function h such that if M has at least $h(k, d, t)$ k-element circuits, then M has d pairwise disjoint 2t-element cocircuits.

Suppose $|E(M)| \geq 2s \cdot h(2s, s-1, t)$. By the $(s, 2s, t, 2t)$ -property, M has at least $h(2s, s-1, t)$ distinct 2s-element circuits. Therefore, by Lemma [4.3,](#page-5-0) M has a collection of $s-1$ pairwise disjoint 2t-element cocircuits C_1^*, \ldots, C_{s-1}^* .

Let $X = \bigcup_{i \in [s-1]} C_i^*$ and $Y = E(M) - X$. By Lemma [4.2,](#page-4-6) for each $y \in Y$ there is a 2s-element circuit C_y containing y such that $|C_y \cap C_j^*| = 3$ for at most one $j \in [s-1]$ and $|C_y \cap C_i^*| = 2$ otherwise. Let W be the set of all $w \in Y$ such that w is in a 2s-element circuit C with $|C \cap C_j^*| = 3$ for some $j \in [s-1]$, and $|C \cap C_i^*| = 2$ for all $i \in [s-1] - \{j\}$. Now, letting $Z = Y - W$, we see that [\(ii\)](#page-5-1) is satisfied. It remains to show that [\(i\)](#page-5-2) holds.

Since each C_i^* has size $2t$, there are $(s-1)\binom{2t}{3}$ $\binom{2t}{3}\binom{2t}{2}^{s-2}$ sets $X' \subseteq X$ with $|X' \cap C_j^*| = 3$ for some $j \in [s-1]$ and $|X' \cap C_i^*| = 2$ for all $i \in [s-1] - \{j\}$. It follows, by Lemma [4.2,](#page-4-6) that $|W| \le f(s,t)$ where

$$
f(s,t) = (s + 2t - 1) \left[(s - 1) {2t \choose 3} {2t \choose 2}^{s-2} \right].
$$

We define

$$
g(s,t,q) = \max \{2s \cdot h(2s, s-1,t), 2(2t(s-1) + f(s,t) + q)\}.
$$

Suppose that $|E(M)| \ge g(s,t,q)$. Since $r(M) \ge r^*(M)$ and $|E(M)| \ge$ $2(2t(s-1) + f(s,t) + q)$, we have $r(M) \ge 2t(s-1) + f(s,t) + q$. Then,

$$
r_M(Z) \ge r_M(Y) - |W|
$$

\n
$$
\ge (r(M) - 2t(s - 1)) - f(s, t)
$$

\n
$$
\ge q,
$$

so [\(i\)](#page-5-2) holds as well.

Lemma 4.5. Let M be a matroid with the $(s, 2s, t, 2t)$ -property. Suppose M has $s-1$ pairwise disjoint 2t-element cocircuits $C_1^*, C_2^*, \ldots, C_{s-1}^*$ and, for some positive integer p, there is a set $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$ such that

- (a) $r(Z) \geq \binom{2t}{2}$ $\binom{2t}{2}^{s-1}(p+2(s-1)),$ and
- (b) for each $z \in Z$, there exists an element $z' \in Z \{z\}$ such that $\{z, z'\}$ is contained in a 2s-element circuit C of M with $|C \cap C_i^*| = 2$ for each $i \in [s-1]$.

There exists a subset $Z' \subseteq Z$ and a partition $\pi = (Z'_1, \ldots, Z'_p)$ of Z' into pairs such that

- (i) each circuit of M/Z' is a union of pairs in π , and
- (ii) the union of any s pairs in π contains a circuit.

Proof. We first prove the following:

4.5.1. There exists a $(2s-2)$ -element set X such that $|X \cap C_i^*| = 2$ for every $i \in [s-1]$ and a set $Z' \subseteq Z$ with a partition $\pi = \{Z'_1, \ldots, Z'_p\}$ of Z' into pairs such that

- (I) $X \cup Z'_i$ is a circuit, for each $i \in [p]$ and
- (II) π partitions the ground set of $(M/X)|Z'$ into parallel classes such that $r_{M/X}(\bigcup_{i\in[p]}Z'_i)=p$.

Proof. By (b), for each $z \in Z$, there exists an element $z' \in Z - \{z\}$ and a set X' such that $\{z, z'\} \cup X'$ is a circuit of M and X' is the union of pairs Y_i for $i \in [s-1]$, with $Y_i \subseteq C_i^*$. Since $|C_i^*| = 2t$ for each $i \in [s-1]$, there are $\binom{2t}{2}$ $\left(\frac{2t}{2}\right)^{s-1}$ choices for $(Y_1, Y_2, \ldots, Y_{s-1})$. Therefore, for some $m \leq \binom{2t}{2}$ $\binom{2t}{2}^{s-1}$, there are $(2s-2)$ -element sets X_1, X_2, \ldots, X_m , and sets Z_1, Z_2, \ldots, Z_m whose union is Z, such that each of X_1, X_2, \ldots, X_m intersects C_i^* in two elements for each $i \in [s-1]$, and such that, for each $j \in [m]$ and each $z_j \in Z_j$, there is an element z'_j such that $\{z_j, z'_j\} \cup X_j$ is a circuit. Since $Z = \bigcup_{i \in [m]} Z_i$, we have $\sum_{i\in[m]} r(Z_i) \ge r(Z)$. Thus, the pigeonhole principle implies that there is some $j \in [m]$ such that

$$
r(Z_j) \ge \frac{r(Z)}{\binom{2t}{2}^{s-1}} \ge p + 2(s-1),
$$

by (a) .

We define $Z' = Z_j$ and $X = X_j$. Observe that $X \cup \{z, z'\}$ is a circuit, for some pair $\{z, z'\} \subseteq \overline{Z'}$, if and only if $\{z, z'\}$ is a parallel pair in M/X . Therefore, there is a partition of the ground set of $(M/X)|Z'$ into parallel classes, where every parallel class has size at least two. Let $\{\{z_1, z'_1\}, \ldots, \{z_n, z'_n\}\}\$ be a collection of pairs from each parallel class such that $\{z_1, z_2, \ldots, z_n\}$ is an independent set in $(M/X)|Z'$. Note that $n \geq r_{M/X}(Z') = r(Z' \cup X) - r(X) \geq$ $r(Z') - 2(s - 1) \ge p$. For $i \in [p]$, let $Z'_i = \{z_i, z'_i\}$. Then $\pi = \{Z'_1, \ldots, Z'_p\}$ satisfies [4.5.1.](#page-6-0) \square

Let X, π , and Z' be as described in [4.5.1,](#page-6-0) and let $\mathcal{X} = \{X_1, \ldots, X_{s-1}\},\$ where $X_i = \{x_i, x'_i\} = X \cap C_i^*$.

4.5.2. Each circuit of $M|(X \cup Z')$ is a union of pairs in $X \cup \pi$.

Proof. Let C be a circuit of $M|(X \cup Z')$. If $x_i \in C$, for some $\{x_i, x'_i\} \in \mathcal{X}$, then orthogonality with C_i^* implies that $x'_i \in C$. Assume for a contradiction that $\{z, z'\} \in \pi$ and $C \cap \{z, z'\} = \{z\}$. Let W be the union of the pairs in π containing elements of $(C - \{z\}) \cap Z'$. Then $z \in \text{cl}(X \cup W)$. Hence $z \in \text{cl}_{M/X}(W)$, contradicting [4.5.1](#page-6-0)[\(II\).](#page-7-0)

4.5.3. Every union of s pairs in $\mathcal{X} \cup \pi$ contains a circuit.

Proof. Let W be a subset of $\mathcal{X} \cup \pi$ of size s. We proceed by induction on the number of pairs in $W \cap \pi$. If there is only one pair in $W \cap \pi$, then the union of the pairs in W contains a circuit (indeed, is a circuit) by $4.5.1(I)$. Suppose the result holds for any subset containing k pairs in π , and let W be a subset containing $k + 1$ pairs in π . Let $\{x, x'\}$ be a pair in $\mathcal{X} - \mathcal{W}$, and let $W = \bigcup_{W' \in \mathcal{W}} W'$. Then $W \cup \{x, x'\}$ is the union of $s + 1$ pairs of $\mathcal{X} \cup \pi$, of which $k+1$ are in π , so, by the induction hypothesis, $W \cup \{x, x'\}$ properly contains a circuit C_1 . If $\{x, x'\} \subseteq E(M) - C_1$, then $C_1 \subseteq W$, in which case the union of the pairs in W contains a circuit, as desired. Therefore, we may assume, by [4.5.2,](#page-7-2) that $\{x, x'\} \subseteq C_1$. Since X is independent, there is a pair $\{z, z'\} \subseteq Z' \cap C_1$. By the induction hypothesis, there is a circuit C_2

contained in $(W - \{z, z'\}) \cup \{x, x'\}.$ Observe that C_1 and C_2 are distinct, and $\{x, x'\} \subseteq C_1 \cap C_2$. Circuit elimination on C_1 and C_2 , and [4.5.2,](#page-7-2) imply that there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{x, x'\} \subseteq W$, as desired. The claim now follows by induction.

Now, [4.5.3](#page-7-3) implies that the union of any s pairs in π contains a circuit, and the result follows.

Lemma 4.6. If M is a matroid with the $(1, 2, t, 2t)$ -property and at least t elements, then M is a $(1,t)$ -spike. Dually, if M is a matroid with the $(s, 2s, 1, 2)$ -property and at least s elements, then M is an $(s, 1)$ -spike.

Proof. By duality, it suffices to consider the case where M has the $(1, 2, t, 2t)$ property and at least t elements. Since every element of M is contained in a 2-element circuit, there is a partition of $E(M)$ into parallel classes P_1, P_2, \ldots, P_n , where $|P_i| \geq 2$ for each i. For each P_i , let $x_i \in P_i$.

First, we consider the case where $n \geq t$. Let X be a t-element subset of $\{x_1, \ldots, x_n\}$; for ease of notation, we assume $X = \{x_1, \ldots, x_t\}$. By the $(1, 2, t, 2t)$ -property, $X \subseteq C^*$ for some 2t-element cocircuit C^* . Since P_i is a parallel class, $\{x_i, y_i\}$ is a circuit for each $y_i \in P_i - \{x_i\}$. By orthogonality, $y_i \in C^*$ for each such y_i , so $P_i \subseteq C^*$. Since $|C^*| = 2t$, and X is an arbitrary t-element subset of $\{x_1, \ldots, x_n\}$, it follows that $|P_i| = 2$ for each $i \in [n]$, and that the union of any t of the P_i 's is a cocircuit. Thus M is a $(1, t)$ -spike.

It remains to consider the case where $n < t$. Since M has at least t elements, let X be any t-element set containing $\{x_1, \ldots, x_n\}$. By the $(1, 2, t, 2t)$ property, there is a 2t-element cocircuit C^* containing X. For $i \in [n]$ and each $y_i \in P_i - \{x_i\}$, orthogonality implies $y_i \in C^*$. Thus, $E(M) = C^*$. It follows that $M \cong U_{1,2t}$, which is a $(1, t)$ -spike. ■

We now prove Theorem [1.1,](#page-0-0) restated below.

Theorem 4.7. There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that, if M is a matroid with the $(s, 2s, t, 2t)$ -property and $|E(M)| > f(s, t)$, then M is an (s, t) -spike.

Proof. If $s = 1$ or $t = 1$, then, by Lemma [4.6,](#page-8-0) the theorem holds with $f(s,t) = \max\{s,t\}$. So we may assume that $\min\{s,t\} \geq 2$. A matroid is an (s, t) -spike if and only if its dual is a (t, s) -spike; moreover, a matroid has the $(s, 2s, t, 2t)$ -property if and only if its dual has the $(t, 2t, s, 2s)$ -property. Therefore, by duality, we may also assume that $r(M) \geq r^*(M)$.

Let $r_k(n)$ be the Ramsey number described in Theorem [2.1.](#page-2-2) For $k \in [s]$, we define the function $h_k : \mathbb{N}^2 \to \mathbb{N}$ such that

$$
h_s(s,t) = \max\{s+2t-1, 2s+t-1, 3s+t-3, s+3t-3\}
$$

and such that $h_k(s,t) = r_k(h_{k+1}(s,t))$ for $k \in [s-1]$. Note that $h_k(s,t) \geq$ $h_{k+1}(s,t) \geq h_s(s,t)$, for each $k \in [s-1]$.

Let $p = h_1(s, t)$ and let $q(s, t) = \binom{2t}{2}$ $\frac{2t}{2}$ ^{s-1} $(p+2(s-1))$. By Lemma [4.4,](#page-5-3) there exists a function g such that if $|E(M)| \ge g(s,t,q(s,t))$, then M has s – 1 pairwise disjoint 2t-element cocircuits $C_1^*, C_2^*, \ldots, C_{s-1}^*$, and there is some $Z \subseteq E(M) - \bigcup_{i \in [s-1]} C_i^*$ such that $r_M(Z) \ge q(s,t)$, and, for each $z \in \mathbb{Z}$, there exists an element $z' \in \mathbb{Z}' - \{z\}$ such that $\{z, z'\}$ is contained in a 2s-element circuit C with $|C \cap C_i^*| = 2$ for each $i \in [s-1]$.

Let $f(s,t) = g(s,t,q(s,t))$, and suppose that $|E(M)| \ge f(s,t)$. Then, by Lemma [4.5,](#page-6-1) there exists a subset $Z \subseteq Z'$ such that Z has a partition into pairs $\pi = (Z_1, \ldots, Z_p)$ such that

- (I) each circuit of M/Z is a union of pairs in π , and
- (II) the union of any s pairs in π contains a circuit.

Let $m = h_s(s, t)$. By Lemma [3.3](#page-3-0) and its dual, it suffices to show that M has either an *s*-echidna or a *t*-coechidna of order m. If the smallest circuit in M/Z has size 2s, then, by [\(II\),](#page-9-1) π is an s-echidna of order $p \geq m$. So we may assume that the smallest circuit in M/Z has size 2j for some $j \in [s-1]$.

4.7.1. If the smallest circuit in M|Z has size 2j, for $j \in [s-1]$, and $|\pi| \ge$ $h_i(s, t)$, then either

- (i) M has a t-coechidna of order m, or
- (ii) there exists some $Z' \subseteq Z$ that is the union of $h_{j+1}(s,t)$ pairs in π for which the smallest circuit in $M|Z'$ has size at least $2(j + 1)$.

Proof. We define H to be the j-uniform hypergraph with vertex set π whose hyperedges are the j-subsets of π that are partitions of circuits in M/Z . By Theorem [2.1,](#page-2-2) and the definition of h_k , as H has at least $h_i(s, t)$ vertices, it has either a clique or a stable set, on $h_{j+1}(s,t)$ vertices. If H has a stable set π' on $h_{j+1}(s,t)$ vertices, then clearly [\(ii\)](#page-9-2) holds, with $Z' = \bigcup_{P \in \pi'} P$.

Therefore, we may assume that there are $h_{i+1}(s,t)$ pairs in π such that the union of any j of these pairs is a circuit. Let Z'' be the union of these $h_{j+1}(s,t)$ pairs. We claim that the union of any set of t pairs contained in Z'' is a cocircuit. Let T be a transversal of t pairs in π contained in Z'' , and let C^* be the 2t-element cocircuit containing T. Suppose, for a contradiction, that there exists some pair $P \in \pi$ with $P \subseteq Z''$ such that $|C^* \cap P| = 1$. Select j − 1 pairs Z''_1, \ldots, Z''_{j-1} in π that are each contained in $Z'' - C^*$ (these exist since $h_{j+1}(s,t) \geq s+2t-1 \geq 2t+j-1$). Then $P \cup (\bigcup_{i \in [j-1]} Z_i'')$ is a circuit intersecting C^* in a single element, contradicting orthogonality. We deduce that the union of any t pairs in π that are contained in Z'' is a cocircuit. Thus, M has a t-coechidna of order $h_{j+1}(t) \geq m$, satisfying [\(i\).](#page-9-3)

We now apply [4.7.1](#page-9-4) iteratively, for a maximum of $s - j$ iterations. If [\(i\)](#page-9-3) holds, at any iteration, then M has a t-coechidna of order m , as required. Otherwise, we let π' be the partition of Z' induced by π ; then, at the next iteration, we relabel $Z = Z'$ and $\pi = \pi'$. If [\(ii\)](#page-9-2) holds for each of $s - j$ iterations, then we obtain a subset Z' of Z such that the smallest circuit in $M|Z'$ has size 2s. Then, by [\(II\),](#page-9-1) M has an s-echidna of order $h_s(s,t) = m$, completing the proof.

5. PROPERTIES OF (s, t) -SPIKES

In this section, we prove some properties of (s, t) -spikes. In particular, we show that an (s, t) -spike has order at least $s+t-1$; an (s, t) -spike of order m has 2m elements and rank $m + s - t$; and the circuits of an (s, t) -spike that are not a union of s arms meet all but at most $t-2$ of the arms. We also give some results about the connectivity of (s, t) -spikes of sufficiently large order.

We also show that an appropriate concatenation of the associated partition of a t-spike is a $(2t-1)$ -anemone, following the terminology of [\[1\]](#page-16-9). Finally, we describe a construction that can be used to obtain an $(s, t+1)$ -spike from an (s, t) -spike of sufficiently large order, and we show that every $(s, t + 1)$ -spike can be constructed from some (s, t) -spike in this way.

We again assume that s and t are positive integers.

Basic properties.

Lemma 5.1. Let M be an (s, t) -spike with associated partition (A_1, \ldots, A_m) . Then $m \geq s + t - 1$.

Proof. By the definition of an (s, t) -spike, we have $m \ge \max\{s, t\}$. Let $Y = \bigcup_{j \in [t]} A_j$, and let $y \in Y$. Since Y is a cocircuit, $Z = (E(M) - Y) \cup \{y\}$ spans M. Therefore, $r(M) \leq |Z| = 2m - 2t + 1$. Similarly, by duality, $r^*(M) \leq 2m - 2s + 1$. Therefore,

$$
2m = |E(M)| = r(M) + r^*(M) \le (2m - 2t + 1) + (2m - 2s + 1).
$$

The result follows.

Lemma 5.2. Let M be an (s,t) -spike of order m. Then $r(M) = m + s - t$ and $r^*(M) = m - s + t$.

Proof. Let (A_1, \ldots, A_m) be the associated partition of M, and let $A_i =$ $\{x_i, y_i\}$ for each $i \in [m]$. Choose $I \subseteq J \subseteq [m]$ such that $|I| = s - 1$ and $|J| = m-t$. (This is possible by Lemma [5.1.](#page-10-0)) Let $X = \{y_i : i \in I\} \cup \{x_i : j \in I\}$ J}. Note that $\bigcup_{i\in I\cup J} A_i \subseteq \text{cl}(X)$. Since $E(M) - \bigcup_{i\in I\cup J} A_i$ is a cocircuit, $\bigcup_{i\in I\cup J} A_i$ is a hyperplane. Therefore, $\bigcup_{i\in I\cup J} A_i = \text{cl}(X)$, and we have $r(M)-1 = r(X) \le |X| = |I| + |J| = m + s - t - 1$. Thus, $r(M) \le m + s - t$. Similarly, by duality, $r^*(M) \leq m - s + t$.

Therefore, we have

$$
2m = |E(M)| = r(M) + r^*(M) \le (m + s - t) + (m - s + t) = 2m.
$$

Thus, we must have equality, and the result holds.

Lemma 5.3. Let M be an (s, t) -spike of order m with associated partition (A_1, \ldots, A_m) , and let C be a circuit of M.

- (i) $C = \bigcup_{j \in J} A_j$ for some s-element set $J \subseteq [m]$, or
- (ii) $|\{i \in [m] : A_i \cap C \neq \emptyset\}| \geq m (t 2) \text{ and } |\{i \in [m] : A_i \subseteq C\}| < s.$

Proof. Let $S = \{i \in [m] : A_i \cap C \neq \emptyset\}$. Thus, S is the minimal subset of $[m]$ such that $C \subseteq \bigcup_{i \in S} A_i$. We have $|S| \geq s$ since C is independent otherwise. If $|S| = s$, then C satisfies [\(i\).](#page-10-1) Therefore, we may assume $|S| > s$. We must have $|\{i \in [m] : A_i \subseteq C\}| < s$; otherwise C properly contains a circuit. Thus, there is some $j \in S$ such that $A_i - C \neq \emptyset$. If $|S| \geq m-(t-2)$, then C satisfies [\(ii\).](#page-10-2) Therefore, we may assume $|S| \leq m - (t - 1)$. Let $T = (m - S) \cup \{j\}$. Then $|T| \geq t$, implying that $\bigcup_{i \in T} A_i$ contains a cocircuit intersecting C in one element. This contradicts orthogonality.

In the remainder of the paper, if (A_1, \ldots, A_m) is the associated partition of an (s, t) -spike and $J \subseteq [m]$, then we define

$$
A_J = \bigcup_{j \in J} A_j.
$$

Proposition 5.4. Let $\pi = (A_1, \ldots, A_m)$ be the associated partition of an (s, t) -spike. If $J \subseteq [m]$, then

$$
r(A_J) = \begin{cases} 2|J| & \text{if } |J| < s, \\ s + |J| - 1 & \text{if } s \le |J| \le m - t + 1, \\ m + s - t & \text{if } |J| \ge m - t + 1. \end{cases}
$$

Proof. If $|J| < s$, then A_J is properly contained in a circuit and is therefore independent. Thus, $r(A_J) = |A_J| = 2|J|$.

We now prove that $r(A_J) = s + |J| - 1$ if $s \leq |J| \leq m - t + 1$. We proceed by induction on |J|. As a base case, if $|J| = s$, then A_J is a circuit. Therefore, $r(A_J) = |A_J| - 1 = s + |J| - 1$. Now, for the inductive step, let $s < |J| \leq m - t + 1$, and let $J' \subseteq J$ with $|J'| = |J| - 1$. By induction, $r(A_{J}) = s + |J| - 2$. Let $\{x_i, y_i\} = A_J - A_{J'}$. By Lemma [5.3,](#page-10-3) since $|J| < m-t+2$, there is no circuit C such that $x_i \in C \subseteq A_{J'} \cup \{x_i\}.$ Therefore, $x_i \notin \text{cl}(A_{J'})$, and $r(A_{J'} \cup \{x_i\}) = r(A_{J'}) + 1$. On the other hand, since $|J| > s$, there is a circuit C such that $y_i \in C \subseteq A_J$. Therefore, $y_i \in \text{cl}(A_{J'} \cup \{x_i\}), \text{ and } r(A_J) = r(A_{J'}) + 1 = s + |J| - 1.$

Note that the preceding argument, along with Lemma [5.2](#page-10-4) implies that, if $|J| = m - t + 1$, then A_J is spanning. Thus, if $|J| \geq m - t + 1$, then $r(A_J) = r(M) = m + s - t.$

Connectivity. Let M be a matroid with ground set E . Recall that the connectivity function of M, denoted by λ , is defined as

$$
\lambda(X) = r(X) + r(E - X) - r(M),
$$

for all subsets X of E. In the case where M is an (s, t) -spike of order m and $X = A_J$ for some set $J \subseteq [m]$, this implies

$$
\lambda(A_J) = r(A_J) + r(A_{[m]-J}) - r(M).
$$

Therefore, Proposition [5.4](#page-10-5) allows us to easily compute $\lambda(A_J)$.

Lemma 5.5. Let $\pi = (A_1, \ldots, A_m)$ be the associated partition of an (s, t) spike, and let (J, K) be a partition of $[m]$ with $|J| \leq |K|$.

- (i) If $|J| \leq t 1$, then $\lambda(A_J) = r(A_J)$.
- (ii) If $t-1 \leq |J| \leq m-s$, then

$$
\lambda(A_J) = \begin{cases} t + |J| - 1 & \text{if } |J| < s, \\ s + t - 2 & \text{if } s \le |J| \le m - t + 1. \end{cases}
$$

(iii) If $|J| > m - s$, then $\lambda(A_J) = m - s + t$.

Proof. If $|J| \le t - 1$, then $|K| \ge m - t + 1$. Therefore, A_K is spanning, and $\lambda(A_J) = r(A_J) + r(A_K) - r(M) = r(A_J)$. Statement (i) follows.

If $t-1 \leq |J| \leq m-s$, then $s \leq |K| \leq m-t+1$. Therefore, $\lambda(A_J)$ $r(A_J) + r(A_K) - r(M) = r(A_J) + s + m - |J| - 1 - (m + s - t)$. Statement (ii) follows. (Note that we cannot have $|J| > m - t + 1$ because otherwise $|K| < t - 1 < |J|$.)

If $|J| > m - s$, then $s > |K| \ge |J|$. Therefore, $\lambda(A_J) = r(A_J) + r(A_K) - s$ $r(M) = 2|J| + 2(m-|J|) - (m+s-t) = m-s+t$. Statement (iii) follows. \blacksquare

Using the terminology of [\[1\]](#page-16-9), Lemma [5.5](#page-11-0) implies the following.

Proposition 5.6. Let (A_1, \ldots, A_m) be the associated partition of an (s, t) spike M, and suppose that (P_1, \ldots, P_k) is a partition of $E(M)$ such that, for $each i \in [k], P_i = \bigcup_{i \in I} A_i$ for some subset I of $[m],$ with $|I| \ge \max\{s-1, t-1\}$ 1}. Then $(P_1, ..., P_k)$ is an $(s + t - 1)$ -anemone.

We now continue our study of the connectivity of (s, t) -spikes.

Lemma 5.7. Let M be an (s, t) -spike of order $m \geq 3 \max\{s, t\} - 2$, and let $X \subseteq E(M)$ such that $|X| \leq 2 \min\{s,t\} - 1$. Then $\lambda(X) = |X|$.

Proof. By Lemma [5.3,](#page-10-3) if X is dependent, then either $|X| = 2s$ or $|X| \ge$ $m - t + 2 \geq 3 \max\{s, t\} - 2 - t + 2 = 3 \max\{s, t\} - t \geq 2 \max\{s, t\} \geq 2s.$ However, $|X| \leq 2 \min\{s,t\} - 1 < 2s$. Therefore, X is independent, which implies that $r(X) = |X|$.

By a similar argument, using the dual of Lemma 5.3 , X is coindependent, implying that $r(E(M) - X) = r(M)$. Therefore,

$$
\lambda(X) = r(X) + r(E(M) - X) - r(M)
$$

= |X| + r(M) - r(M)
= |X|,

proving the lemma.

Theorem 5.8. Let M be an (s, t) -spike of order

 $m > \max\{3s + t, s + 3t\} - 4,$

where $\min\{s,t\} \geq 2$. Then M is $(2\min\{s,t\}-1)$ -connected.

Proof. Because M^* is a (t, s) -spike and because $\lambda_{M^*} = \lambda_M$, we may assume without loss of generality that $t \leq s$. Note that $\max\{3s + t, s + 3t\}$ $3 \max\{s, t\} + \min\{s, t\}.$ Therefore, $m \geq 3s + t - 4$, and we must show that M is $(2t-1)$ -connected.

Now, suppose for a contradiction that M is not $(2t-1)$ -connected. Then there is a k-separation (P,Q) of M, with $|P| \ge |Q|$, for some $k < 2t - 1$. Therefore, $\lambda(P) = \lambda(Q) < k \leq 2t - 2$.

First, we consider the case where $A_I \subseteq P$, for some $(t-1)$ -element set $I \subseteq [m]$. Let $U = \{u \in [m] : |P \cap A_u| = 1\}$. Then $A_i \subseteq \text{cl}_{M^*}(P)$ for each $j \in U$. For such a j, it follows, by the definition of λ_{M^*} (which is equal to $\lambda_M = \lambda$, that $\lambda(P \cup A_i) \leq \lambda(P)$. We use this repeatedly below; in particular, we see that $\lambda(P \cup A_U) \leq \lambda(P)$.

Let $P' = P \cup A_U$, and let $Q' = E(M) - P'$. Then there is a partition (J, K) of $[m]$, with $|J| \leq |K|$, such that $Q' = A_J$ and $P' = A_K$. Moreover, $\lambda(Q') = \lambda(P') \leq \lambda(P).$

Suppose $|J| \geq t - 1$. Note that $m \geq 3s + t - 4 \geq 2s$ since $\min\{s,t\} \geq 2$. Therefore, $|J| \leq \frac{1}{2}m = m - \frac{1}{2}m \leq m - \frac{1}{2}(2s) = m - s$. Thus, to determine $\lambda(Q')$, we need only consider Lemma [5.5\(](#page-11-0)ii). If $|J| \geq s$, then by Lemma [5.5\(](#page-11-0)ii),

$$
\lambda(P) \ge \lambda(P') = \lambda(Q') = s + t - 2 \ge 2t - 2,
$$

a contradiction. Otherwise, $|J| < s$, implying by Lemma [5.5\(](#page-11-0)ii) that

$$
\lambda(P) \ge \lambda(P') = \lambda(Q') = t + |J| - 1 \ge t + t - 1 - 1 = 2t - 2,
$$

another contradiction.

Therefore, $|J| < t - 1$. Let $U' \subseteq U$ such that $|U'| = |Q| - (2t - 2)$. Then $\lambda(P) \geq \lambda(P \cup A_{U'}) = \lambda(Q - A_{U'})$. Since $|Q - A_{U'}| = 2t - 2$ and $m \geq 3s + t - 4 \geq 3s - 2$, Lemma [5.7](#page-12-0) implies that $\lambda (Q - A_{U'}) = 2t - 2$, so $\lambda(P) \geq 2t - 2$, a contradiction.

Now we consider the case that $|\{i \in [m] : A_i \subseteq P\}| < t - 1$. Since $|Q| \le$ |P|, it follows that $|\{i \in [m] : A_i \subseteq Q\}| \leq |\{i \in [m] : A_i \subseteq P\}| < t - 1 < s$. Now, since $|\{i \in [m] : A_i \subseteq P\}| < t-1$, we have $|\{i \in [m] : A_i \cap Q \neq \emptyset\}|$

 $\lvert \emptyset \rvert > m - (t - 1)$. Therefore, $r(Q) \geq m - (t - 1)$ by Lemma [5.3.](#page-10-3) Similarly, $r(P) \geq m - (t - 1)$. Thus,

$$
\lambda(P) = r(P) + r(Q) - r(M)
$$

\n
$$
\geq (m - (t - 1)) + (m - (t - 1)) - (m + s - t)
$$

\n
$$
= m - s - t + 2
$$

\n
$$
\geq 3s + t - 4 - s - t + 2
$$

\n
$$
= 2s - 2
$$

\n
$$
\geq 2t - 2,
$$

a contradiction. This completes the proof.

Constructions. In [\[2\]](#page-16-0), a construction is described that, starting from a (t, t) -spike M_0 , obtains a $(t + 1, t + 1)$ -spike M_1 . This construction consists of a certain elementary quotient M'_0 of M_0 , followed by a certain elementary lift M_1 of M'_0 . It is shown in [\[2\]](#page-16-0) that M_1 is a $(t + 1, t + 1)$ -spike as long as the order of M_0 is sufficiently large.

In the process of constructing M_1 in this way, the intermediary matroid M'_0 is a $(t, t + 1)$ -spike. For the sake of completeness, we will review this construction in the more general case where M_0 is an (s, t) -spike, in which case M'_0 is an $(s, t + 1)$ -spike. To construct an $(s + 1, t)$ -spike, we perform the construction on M^* and dualize. Since $(2, 2)$ -spikes (and indeed, $(1, 1)$ spikes) are well known to exist, this means that (s, t) -spikes exist for all positive integers s and t.

It is also shown in [\[2\]](#page-16-0) that all (t, t) -spikes can be constructed in this manner. We also extend this to the general case of (s, t) -spikes below.

Recall that M_1 is an *elementary quotient* of M_0 if there is a single-element extension M_0^+ of M_0 by an element e such that $M_1 = M_0^+/e$. If M_1 is an elementary quotient of M_0 , then M_0 is an *elementary lift* of M_1 . Also, note that if M_1 is an elementary lift of M_0 , then M_1^* is an elementary quotient of M_0^* .

Construction 5.9. Let M be an (s, t) -spike of order $m \geq s + t$, with associated partition π . Let $M + e$ be a single-element extension of M by an element e such that e blocks each 2t-element cocircuit that is a union of t arms of M. Then let $M' = (M + e)/e$.

In other words, $M + e$ has the property that $e \notin cl_{M+e}(E(M) - C^*)$ for every 2t-element cocircuit C^* that is the union of t arms. Note that one possibility is that $M + e$ is the free extension of M by an element e. Since $m-t \geq s$, we have $e \notin \mathrm{cl}_{M+e}(C)$ for each 2s-element circuit C. Thus, in M', the union of any s arms of the (s, t) -spike M is still a circuit of M'. However,

since $r(M') = r(M) - 1$, the union of any $t + 1$ arms is a $2(t + 1)$ -element cocircuit. Therefore, M' is an $(s, t + 1)$ -spike.

Note that M' is not unique; more than one $(s, t + 1)$ -spike can be constructed from a given (s, t) -spike M using Construction [5.9.](#page-13-0) Given an $(s + 1, t)$ -spike M', we will describe how to obtain an (s, t) -spike M from M' by a specific elementary quotient. This process reverses the dual of Construction [5.9.](#page-13-0) This will then imply that every (s, t) -spike can be constructed from a (1, 1)-spike by repeated use of Construction [5.9](#page-13-0) and its dual. Lemma [5.10](#page-14-0) describes the single-element extension that gives rise to the elementary quotient we desire. Intuitively, the extension adds a "tip" to the (s, t) -spike. In the proof of this lemma, we assume knowledge of the theory of modular cuts (see [\[6,](#page-16-7) Section 7.2]).

The proof of Lemma [5.10](#page-14-0) will be very similar to the proof of [\[2,](#page-16-0) Lemma 6.6]. However, we note that [\[2,](#page-16-0) Lemma 6.6] is falsely stated; what is proven in [\[2\]](#page-16-0) is essentially the specialisation of Lemma [5.10,](#page-14-0) below, in the case that $s =$ t. The statement of [\[2,](#page-16-0) Lemma 6.6] replaces the condition that M is a (t, t) spike with the weaker condition that M has a t-echidna. To demonstrate that this is overly general, consider the rank-3 matroid consisting of two disjoint lines with four points. Let these lines be $\{a, b, c, d\}$ and $\{w, x, y, z\}$. Then $({a, b}, {w, x})$ is a 2-echidna of order 2. For [\[2,](#page-16-0) Lemma 6.6] to be true, we would need a single-element extension M^+ by an element e such that $e \in \text{cl}_{M^+}(\{a, b\})$ but $e \notin \text{cl}_{M^+}(\{c, d\})$. This is impossible since $\text{cl}_{M}(\{a, b\})$ = $\text{cl}_M(\{c,d\}).$

Lemma 5.10. Let M be an (s, t) -spike. There is a single-element extension M^+ of M by an element e having the property that, for every $X \subseteq E(M)$, $e \in cl_{M^+}(X)$ if and only if X contains at least $s - 1$ arms of M.

Proof. Since M is an (s, t) -spike, there is a partition $\pi = (S_1, \ldots, S_m)$ of $E(M)$ that is both an s-echidna and a t-coechidna. Let

$$
\mathcal{F} = \left\{ \bigcup_{i \in I} S_i : I \subseteq [m] \text{ and } |I| = s - 1 \right\}.
$$

By the definition of an s-echidna, $\mathcal F$ is a collection of flats of M. Let M be the set of all flats of M containing some flat $F \in \mathcal{F}$. We claim that M is a modular cut. Recall that, for distinct $F_1, F_2 \in \mathcal{M}$, the pair (F_1, F_2) is modular if $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$. To show that M is a modular cut, it suffices to prove that, for any $F_1, F_2 \in \mathcal{M}$ such that (F_1, F_2) is a modular pair, $F_1 \cap F_2 \in \mathcal{M}$.

For any $F \in \mathcal{M}$, since F contains at least $s-1$ arms of M, and the union of any s arms is a circuit, it follows that F is a union of arms of M . Thus, let $F_1, F_2 \in \mathcal{M}$ be such that $F_1 = \bigcup_{i \in I_1} S_i$ and $F_2 = \bigcup_{i \in I_2} S_i$, where I_1 and I₂ are distinct subsets of $[m]$ with $u_1 = |I_1| \geq s - 1$ and $u_2 = |I_2| \geq s - 1$.

Let $q = |I_1 \cap I_2|$. Then $F_1 \cup F_2$ is the union of $u_1 + u_2 - q \geq s - 1$ arms, and $F_1 \cap F_2$ is the union of q arms. We show that if $q < s - 1$, then (F_1, F_2) is not a modular pair.

We consider several cases. First, suppose $u_1, u_2 \leq m - t + 1$. By Proposition [5.4,](#page-10-5)

$$
r(F_1) + r(F_2) = (s + u_1 - 1) + (s + u_2 - 1)
$$

> $(s - 1 + u_1 + u_2 - q) + 2q$
= $s + |I_1 \cup I_2| - 1 + 2|I_1 \cap I_2|$
 $\ge r(F_1 \cup F_2) + r(F_1 \cap F_2).$

Next, consider the case where $u_2 \leq m - t + 1 \leq u_1$. (By symmetry, the argument is the same if u_1 and u_2 are swapped.) One can check that $u_1 + u_2 - q > m - t + 1$. By Proposition [5.4,](#page-10-5)

$$
r(F_1) + r(F_2) = (m + s - t) + (s + u_2 - 1)
$$

>
$$
(m + s - t) + 2q
$$

=
$$
r(F_1 \cup F_2) + r(F_1 \cap F_2).
$$

Finally, consider the case where $u_1, u_2 > m - t - 1$. We have

$$
r(F_1) + r(F_2) = 2m + 2s - 2t,
$$

which by Lemma [5.1,](#page-10-0) is at least

$$
m+3s-t-1 > m+s-t+2q
$$

= $r(F_1 \cup F_2) + r(F_1 \cap F_2).$

Thus, in all cases, (F_1, F_2) is not a modular pair. Therefore, we have shown that M is a modular cut. Now, there is a single-element extension corresponding to the modular cut \mathcal{M} , and this extension satisfies the requirements of the lemma (see, for example, [\[6,](#page-16-7) Theorem 7.2.3]).

Theorem 5.11. Let M be an (s,t) -spike of order $m \geq s+t$. Then M can be constructed from a $(1,1)$ -spike of order m by applying Construction [5.9](#page-13-0) $t-1$ times, followed by the dual of Construction [5.9](#page-13-0) s – 1 times.

Proof. For $s = t = 1$, the result is clear. Otherwise, by duality, we may assume without loss of generality that $t > 1$. By induction and duality, it suffices to show that M can be constructed from an $(s - 1, t)$ -spike of order m by applying the dual of Construction [5.9](#page-13-0) once.

Let $\pi = (A_1, \ldots, A_m)$ be the associated partition of M. Let M^+ be the single-element extension of M by an element e described in Lemma [5.10.](#page-14-0)

Let $M' = M^+/e$. We claim that π is an $(s-1)$ -echidna and a t-coechidna that partitions the ground set of M' .

Let X be the union of any $s-1$ spines of π . Then X is independent in M, and $X \cup \{e\}$ is a circuit in M^+ , so X is a circuit in M'. Thus, π is an $(s-1)$ -echidna of M'. Now let C^* be the union of any t spines of π , and let $H = E(M) - C^*$. Then H is the union of at least $s - 1$ spines, so $e \in \text{cl}_{M^+}(H)$. Now $H \cup \{e\}$ is a hyperplane in M^+ , so C^* is a cocircuit in M^+ and therefore in M'. Hence π is a t-coechidna of M'.

Note that M' is an elementary quotient of M , so M is an elementary lift of M' where none of the $2(s-1)$ -element circuits of M' are preserved in M. So the (s, t) -spike M can be obtained from the $(s - 1, t)$ -spike M' using the dual of Construction [5.9.](#page-13-0)

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REFERENCES

- [1] J. Aikin and J. Oxley. The structure of crossing separations in matroids. Advances in Applied Mathematics, 41(1):10–26, 2008.
- [2] N. Brettell, R. Campbell, D. Chun, K. Grace, and G. Whittle. On a generalization of spikes. SIAM Journal on Discrete Mathematics, 33(1):358–372, 2019.
- [3] N. Brettell, D. Chun, T. Fife, and C. Semple. Matroids with a cyclic arrangement of circuits and cocircuits. European Journal of Combinatorics 81:195–209, 2019.
- [4] N. Brettell, C. Semple, and G. Toft. Cyclic matroids. arXiv 2112.14914.
- [5] J. Miller. Matroids in which every pair of elements belongs to both a 4-circuit and a 4-cocircuit. M.Sc. thesis, Victoria University of Wellington, 2014.
- [6] J. Oxley. Matroid Theory, volume 21 of Oxford Graduate Texts in Mathematics. Oxford University Press, New York, second edition, 2011.
- [7] J. Oxley, S. Pfeil, C. Semple, and G. Whittle. Matroids with many small circuits and cocircuits. Advances in Applied Mathematics 105:1–24, 2019.
- [8] F. P. Ramsey. On a problem of formal logic. Proceedings of the London Mathematical Society, 2(1):264–286, 1930.
- [9] W. T. Tutte. Connectivity in matroids. Canadian Journal of Mathematics, 18:1301– 1324, 1966.
- [10] A. Williams. Detachable Pairs in 3-Connected Matroids. Ph.D. thesis, Victoria University of Wellington, 2015.

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