On developing fourth-order optimal families of methods for multiple roots and their dynamics *

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Abstract

There are few optimal fourth-order methods for solving nonlinear equations when the multiplicity m of the required root is known in advance. Therefore, the first focus of this paper is on developing new fourth-order optimal families of iterative methods by a simple and elegant way. Computational and theoretical properties are fully studied along with a main theorem describing the convergence analysis. Another main focus of this paper is the dynamical analysis of the rational map associated with our proposed class for multiple roots; as far as we know, there are no deep study of this kind on iterative methods for multiple roots. Further, using Mathematica with its high precision compatibility, a variety of concrete numerical experiments and relevant results are extensively treated to confirm the theoretical development.

Keywords: Nonlinear equations. Multiple roots. Halley's method. Schröder method. Complex dynamics. Basin of attraction.

Mathematics Subject Classification (2010) 65H05.

1 Introduction

With the advancement of computer algebra, finding higher-order multi-point methods, not requiring the computation of second-order derivative for multiple roots become very important

*This research was supported by Ministerio de Economía y Competitividad MTM2014-52016-C02-02

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and interesting task from the practical point of view. These multi-point methods are of great practical importance since they overcome theoretical limits of one-point methods concerning the order and computational efficiency. Further, these multi-point iterative methods are also capable to generate root approximations of high accuracy.

Let us consider a nonlinear function $f : D \subset \mathbb{R} \to \mathbb{R}$, where D is an open interval such that $r_m \in D$ is a root of equation $f(x) = 0$ with multiplicity m.

In the last years, some optimal iterative methods (in the sense of Kung-Traub conjecture [1]) have appeared. In 2009, Li et al. [2] proposed the following fourth-order optimal two-point method which requires one function and two first-order derivative evaluations per iteration

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{2f'(x_n)}\left[\dfrac{m(m-2)\left(\frac{m}{m+2}\right)^m f'(y_n) - m^2 f'(x_n)}{f'(x_n) - \left(\frac{m}{m+2}\right)^m f'(y_n)}\right]. \end{cases} \tag{1.1}$$

Sharma and Sharma [3] proposed the following optimal variant of Jarratt's method for obtaining multiple roots

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{m}{8}\left\{(m^3 - 4m + 8) - (m+2)^2\left(\dfrac{m}{m+2}\right)^m \dfrac{f'(x_n)}{f'(y_n)}\right. \\ \qquad \left. \times \left(2(m-1) - (m+2)\left(\dfrac{m}{m+2}\right)^m \dfrac{f'(x_n)}{f'(y_n)}\right)\right\}\dfrac{f(x_n)}{f'(x_n)}. \end{cases} \tag{1.2}$$

It has fourth-order of convergence and requires one-function and two-derivative evaluation per iteration.

Again in 2010, Li et al. [4] proposed six fourth-order two-point methods with closed formulas for finding multiple zeros of nonlinear functions. Among them, the following one is optimal:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - a_3\dfrac{f(x_n)}{f'(y_n)} - \dfrac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} a_3 &= -\frac{m(m+2)(m+2)^3}{2(m^3 - 4m + 8)}\left(\frac{m}{m+2}\right)^m, \\ b_1 &= -\frac{(m^3 - 4m + 8)^2}{m(m^2 + 2m - 4)^3}, \\ b_2 &= \frac{m^2(m^3 - 4m + 8)}{(m^2 + 2m - 4)^3}\left(\frac{m+2}{m}\right)^m. \end{aligned}$$

Zhou el al. [5] in 2011 constructed a more general iteration scheme for multiple roots, requiring one function and two derivative evaluation per iteration as follows:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} Q\Big(\dfrac{f'(y_n)}{f'(x_n)}\Big), \end{cases} \tag{1.4}$$

where $Q(.) \in C^2(\mathbb{R})$ is a weight function and discussed the conditions on Q to obtain fourth-order optimal methods from it. Zhou et al. have also proved that the above methods namely, (1.2) and (1.3) are special cases of his scheme.

In 2012, Sharifi et al. [6], proposed an optimal family of fourth-order methods as below

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \left(\dfrac{m(m^2+2m-4)}{4}\dfrac{f(x_n)}{f'(x_n)} - \dfrac{m(m+2)^2}{4}\left(\dfrac{m}{m+2}\right)^m \dfrac{f(x_n)}{f'(y_n)}\right)\left[G\left(\dfrac{f'(y_n)}{f'(x_n)}\right) + H\left(\dfrac{f(x_n)}{f'(y_n)}\right)\right], \end{cases} \tag{1.5}$$

where $G(.)$ and $H(.)$ are two real valued weight functions.

On the other hand, Soleymani and Babajee [7] in 2013, developed following fourth-order optimal family of methods

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{4m\left(\frac{m}{m+2}\right)^m f(x_n)}{\left(\frac{m}{m+2}\right)^m (m^2+2m-4)f'(x_n) - m^2 f'(y_n)} H\Big(\dfrac{f'(y_n)}{f'(x_n)}\Big), \end{cases} \tag{1.6}$$

where $H(.)$ is a real valued weight function.

Zhou et al. [8] in 2013, constructed another family of fourth-order methods, requiring two-function and one-derivative evaluation per iteration as follows:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - mG(v)\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.7}$$

where $v = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $G(.) \in C^2(\mathbb{R})$ is a weight function.

Also recently, Soleymani et al. [9] proposed the following optimal method for finding multiple roots

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{16 f'(y_n) f(x_n)}{m^2(m-2)^2\left(\frac{m}{m+2}\right)^{m-3}[f'(x_n)]^2 + 2(4+\frac{8}{m}+2m-2m^2-m^3)f'(x_n)f'(y_n) + m^3\left(\frac{m}{m+2}\right)^{-m}[f'(y_n)]^2}. \end{cases} \tag{1.8}$$

Motivated and inspired by the recent activities in this direction, researchers proposed optimal fourth-order two-point methods which require either one f and two first-order derivative f' or two f and one first-order derivative f' per iteration. In this manuscript, we present a simple and elegant way to achieve as high as possible convergence order consuming as small as possible function evaluations. Therefore, to develop an optimal general class of fourth-order methods, we have taken the arithmetic mean of Halley's method [10, 11] and Schröder's method [10, 11, 12] with five disposable parameters.

In order to study the complex dynamical behavior of the operator associated to these iterative methods, it is necessary to recall some basic concepts. For a more extensive review of these tools, see [13, 14].

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be an operator that results from applying an iterative method on a particular function, where $\hat{\mathbb{C}}$ is the Riemann sphere. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as the set of successive images of z_0 by the operator, $\{z_0, R(z_0), \ldots, R^n(z_0), \ldots\}$.

The dynamical behavior of the orbit of a point on the complex plane can be classified depending on its asymptotic behavior. In this way, a point $z_0 \in \mathbb{C}$ is a fixed point of R if $R(z_0) = z_0$. A fixed point is attracting, repelling or neutral if $|R'(z_0)|$ is less than, greater than or equal to 1, respectively. Moreover, if $|R'(z_0)| = 0$, the fixed point is superattracting.

If z_f^* is an attracting fixed point of the operator R, its basin of attraction $\mathcal{A}(z_f^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(z_f^*) = \left\{ z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z_f^*, n \to \infty \right\}.$$

A point z_c is defined as a critical point if $R'(z_c) = 0$. A classical result establishes that there is at least one critical point associated with each invariant Fatou component [14].

The set of points whose orbits tends to an attracting fixed point z_f^* is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set, the Julia set $\mathcal{J}(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the boundaries between the basins of attraction.

The aim of this paper is to highlight the advantages of this new approach over the traditional approaches like weight function approach, quadrature approach, functional approach, etc. It is also observed that the body structures of our proposed families of methods can be simpler than the existing families of fourth-order methods for multiple roots. Further, these families of iterative methods are more effective and equally competent in all the tested examples to the existing methods available in the literature. The dynamic study of these methods not only supports the theoretical aspects, but also shows those elements of the parametric families more stable and which ones must be avoided because of their dangerous numerical behavior.

These tools have been widely used in the last years for studying the stability of many iterative methods for solving nonlinear equations with simple roots (see, for example, [15, 16, 17, 18]).

Recently, these studies have been extended to iterative methods for multiple roots as in [19, 20]. Let us remark that, in [20], the authors analyzed the dynamics of the methods under study applied on polynomial $p(z) = ((z-a)(z-b))^m$. In this manuscript, we will analyze the dynamics of the proposed family on polynomial $p(z) = (z-a)^m(z-b)$, where $m = 2$ or $m = 3$.

The rest of the paper is organized as follows: in Section 2 we describe the new optimal fourth-order families and analyze their convergence; in Section 3 the complex dynamics of the families is studied, showing curious behavior in terms of stability and convergence to the roots. Section 4 is devoted to numerical performances of the methods, comparing the described schemes with other known ones. Finally, some conclusions and remarks are presented.

2 An optimal general class of iterative methods

The well-known Halley's method [10] for simple zeros and Schröder's method [12] for multiple roots are given by

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{2.1}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}, \tag{2.2}$$

respectively. We now intend to develop new optimal families of Jarratt-type method [11]. For this, we take the arithmetic mean of (2.1) and (2.2) to get

$$x_{n+1} = x_n - \frac{1}{2}\left[\frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} + \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}\right]. \tag{2.3}$$

Now, consider a Newton-type iterative method for multiple roots as

$$y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \tag{2.4}$$

where $m \geq 1$ is the multiplicity of the multiple root $x = r_m$.

Now, expanding the function $f'(y_n) = f'\left(x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}\right)$ about the point $x = x_n$ by Taylor's series expansion, we have

$$f'\left(x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}\right) \approx f'(x_n) - \frac{2m}{m+2}\frac{f(x_n)f''(x_n)}{f'(x_n)},$$

therefore, we obtain

$$f''(x_n) \approx \frac{(m+2)f'(x_n)[f'(x_n) - f'(y_n)]}{2mf(x_n)}. \tag{2.5}$$

Using this approximate value of $f''(x_n)$ in the method (2.3) and after some simplification, we get

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \left[\dfrac{2mf(x_n)}{(3m-2)f'(x_n) + (m+2)f'(y_n)} + \dfrac{mf(x_n)}{(m-2)f'(x_n) + (m+2)f'(y_n)}\right]. \end{cases} \tag{2.6}$$

This method satisfies the following error equation

$$e_{n+1}=\left[1-m\left(\frac{1}{(-2+m)m+u(2+m)^2}+\frac{2}{u(2+m)^2+m(-2+3m)}\right)\right]e_n+O(e_n^2),$$

where $u=\left(\dfrac{m}{m+2}\right)^m$. It is clear that, according to the Kung-Traub conjecture, the above method (2.6) is not an optimal method because it has linear-order convergence and requires three evaluations of function per full iteration. Therefore, to build our optimal families of methods, we shall take five free disposable parameters. Therefore, we consider

$$\begin{cases} y_n=x_n-\dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)},\\ x_{n+1}=x_n-\left[\dfrac{2ma_1f(x_n)}{(3m-2)f'(x_n)+(m+2)a_2f'(y_n)}+\dfrac{ma_3f(x_n)}{(m-2)a_4f'(x_n)+(m+2)a_5f'(y_n)}\right],\end{cases}\tag{2.7}$$

where a_1, a_2, a_3, a_4, a_5 are real parameters such that the order of convergence reaches at the optimal level four without using any more functional evaluations. Theorem 2.1 indicates that under what choices on the disposable parameters in (2.7), the order of convergence will reach at the optimal level four.

Theorem 2.1 *Let $f:D\subseteq\mathbb{R}\to\mathbb{R}$ be a sufficiently smooth function defined on an open interval D, enclosing a multiple zero of $f(x)$, say $x=r_m$ with multiplicity $m\geq 1$. Then the family of iterative methods defined by* (2.7) *has fourth-order convergence when*

$$\begin{cases} a_1=-\dfrac{(3m-2)(m^2(m-2)a_4+u(m^3+4m^2-8)a_5)^3}{2(m^3(m-2)a_4+u(m^4+2m^3-4m^2+16)a_5)((m-2)^2m^3a_4^2+2m^3u(m^2-4)a_4a_5+u^2(m+2)^2(m^3-4m+8)a_5^2)},\\ a_2=-\dfrac{m^2\left(\frac{m}{m+2}\right)^{1-m}(3m-2)((m-2)a_4+(m+2)ua_5)}{m^3(m-2)a_4+u(m^4+2m^3-4m^2+16)a_5},\\ a_3=-\dfrac{(m-2)(m(m-2)a_4+u(m+2)^2a_5)^3}{2((m-2)^2m^3a_4^2+2m^3u(m^2-4)a_4a_5+u^2(m+2)^2(m^3-4m+8)a_5^2)},\end{cases}\tag{2.8}$$

where $u=\left(\frac{m}{m+2}\right)^m$ and a_4 and a_5 are free parameters. The family (2.7) satisfies the following error equation

$$e_{n+1}=\left(\frac{\alpha_1}{\alpha_2}c_1^3-\frac{1}{m}c_1c_2+\frac{m}{(m+2)^2}c_3\right)e_n^4+O(e_n^5),\tag{2.9}$$

where $\alpha_1=(u^2(m+2)^2(m^6+6m^5+10m^4-2m^3-24m^2+8m-32)a_5^2+2u(m^8+4m^7-18m^5-24m^4+24m^3+8m^2-64m+96)a_4a_5+m^3(m-2)^2(m^3+2m^2+2m-2)a_4^2)$,
$\alpha_2=3m^4(m+2)^2(u(m+2)^2a_5+m(m-2)a_4)(m^2(m-2)a_4+u(m^3+4m^2-8))$ and $c_k=\dfrac{m!}{(m+k)!}\dfrac{f^{(m+k)}(r_m)}{f^{(m)}(r_m)}$, $k=1,2,3,\ldots$

Proof: Let $x=r_m$ be a multiple zero of $f(x)$. Expanding $f(x_n)$ and $f'(x_n)$ about $x=r_m$ by the Taylor's series expansion, we have

$$f(x_n)=\frac{f^{(m)}(r_m)}{m!}e_n^m\left(1+c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4\right)+O(e_n^5),\tag{2.10}$$

and

$$f'(x_n) = \frac{f^m(r_m)}{(m-1)!}e_n^{m-1}\left(1 + \frac{m+1}{m}c_1e_n + \frac{m+2}{m}c_2e_n^2 + \frac{m+3}{m}c_3e_n^3 + \frac{(m+4)}{m}c_4e_n^4\right) + O(e_n^5), \tag{2.11}$$

respectively.

From equations (2.10) and (2.11), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}e_n - \frac{c_1}{m^2}e_n^2 + \left(\frac{(m+1)c_1^2 - 2mc_2}{m^3}\right)e_n^3 + O(e_n^4). \tag{2.12}$$

Furthermore, we have

$$\begin{aligned}\frac{2ma_1f(x_n)}{(3m-2)f'(x_n) + (m+2)a_2f'(y_n)} =& \left(\frac{2a_1}{3m-2+a_2m^{m-1}(m+2)^{2-m}}\right)e_n \\ &- \left(\frac{2a_1(m+2)^m((3m-2)m^2(m+2)^m + a_2m^m(m^3+4m^2-8))c_1}{m[a_2m^m(m+2)^2 + m(m+2)^m(3m-2)]^2}\right)e_n^2 \\ &+ Ae_n^3 + O(e_n^4),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}A =& \frac{2a_1}{m^2[u(m+2)^2a_2 + m(3m-2)]^3}\left[(m^3(3m-2)^2(m+1) + 2u(3m^6 + 13m^5 + 8m^4 - 18m^3 - 20m^2\right. \\ &- 8m + 16)a_2 + u^2(m+2)^2(m^4 + 5m^3 + 4m^2 - 8m - 16)a_2^2)c_1^2 - 2m((3m-2)^2m^3 \\ &\left.+2mu(3m^4 + 10m^3 - 2m^2 - 16m + 8)a_2 + u^2(m+2)^3(m^2 + 2m - 4)a_2^2)c_2\right].\end{aligned}$$

Similarly, we can also get

$$\begin{aligned}\frac{ma_3f(x_n)}{(m-2)a_4f'(x_n) + (m+2)a_5f'(y_n)} =& \left(\frac{ma_3}{m(m-2)a_4 + (m+2)^2ua_5}\right)e_n \\ &- \left(\frac{a_3(m^2(m-2)a_4 + u(m^3+4m^2-8)a_5)c_1}{m(m(m-2)a_4 + (m+2)^2ua_5)^2}\right)e_n^2 \\ &+ Be_n^3 + O(e_n^4),\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}B =& \frac{a_3}{m^2(m(m-2)a_4 + (m+2)^2ua_5)^3}\left[(m-2)^2m^3a_4^2((m+1)c_1^2 - 2mc_2) + 2u(m^2-4)a_4a_5((m^4 + 3m^3 - 2m - 4)c_1^2\right. \\ &\left.-2m^2(m^2 + 2m - 2)c_2) + u^2(m+2)^2a_5^2((m^4 + 5m^3 + 4m^2 - 8m - 16)c_1^2 - 2m(m^3 + 4m^2 - 8m)c_2)\right].\end{aligned}$$

Using equations (2.13) and (2.14) in scheme (2.7), we get the following error equation

$$\begin{aligned}e_{n+1} =& e_n - \left[\frac{2ma_1f(x_n)}{(3m-2)f'(x_n) + (m+2)a_2f'(y_n)} + \frac{ma_3f(x_n)}{(m-2)a_4f'(x_n) + (m+2)a_5f'(y_n)}\right], \\ =& \left(1 - \frac{2a_1}{3m-2+a_2m^{m-1}(m+2)^{2-m}} - \frac{ma_3}{m(m-2)a_4 + (m+2)^2ua_5}\right)e_n \\ &+ \left(\frac{2a_1(m+2)^m((3m-2)m^2(m+2)^m + a_2m^m(m^3+4m^2-8))c_1}{m[a_2m^m(m+2)^2 + m(m+2)^m(3m-2)]^2} + \frac{a_3(m^2(m-2)a_4 + u(m^3+4m^2-8)a_5)c_1}{m(m(m-2)a_4 + (m+2)^2ua_5)^2}\right)c_1e_n^2 \\ &- (A+B)e_n^3 + O(e_n^4).\end{aligned} \tag{2.15}$$

For obtaining an iterative method of order four, the coefficients of e_n, e_n^2, and e_n^3 in the error equation (2.15) must be zero simultaneously. After simplifying the equation (2.15), we have the following values of $a_1,\ a_2,\ a_3$ involving two free parameters a_4 and a_5

$$\begin{cases} a_1 = -\dfrac{(3m-2)(m^2(m-2)a_4+u(m^3+4m^2-8)a_5)^3}{2(m^3(m-2)a_4+u(m^4+2m^3-4m^2+16)a_5)((m-2)^2m^3a_4^2+2m^3u(m^2-4)a_4a_5+u^2(m+2)^2(m^3-4m+8)a_5^2)}, \\ a_2 = -\dfrac{m^2\left(\frac{m}{m+2}\right)^{1-m}(3m-2)((m-2)a_4+(m+2)ua_5)}{m^3(m-2)a_4+u(m^4+2m^3-4m^2+16)a_5}, \\ a_3 = -\dfrac{(m-2)(m(m-2)a_4+u(m+2)^2a_5)^3}{2((m-2)^2m^3a_4^2+2m^3u(m^2-4)a_4a_5+u^2(m+2)^2(m^3-4m+8)a_5^2)}, \end{cases} \tag{2.16}$$

where $u = \left(\frac{m}{m+2}\right)^m$.

The family (2.7) satisfies the following error equation

$$e_{n+1} = \left(\frac{\alpha_1}{\alpha_2}c_1^3 - \frac{1}{m}c_1c_2 + \frac{m}{(m+2)^2}c_3\right)e_n^4 + O(e_n^5), \tag{2.17}$$

where $\alpha_1 = (u^2(m+2)^2(m^6+6m^5+10m^4-2m^3-24m^2+8m-32)a_5^2+2u(m^8+4m^7-18m^5-24m^4+24m^3+8m^2-64m+96)a_4a_5+m^3(m-2)^2(m^3+2m^2+2m-2)a_4^2)$,
$\alpha_2 = 3m^4(m+2)^2(u(m+2)^2a_5+m(m-2)a_4)(m^2(m-2)a_4+u(m^3+4m^2-8))$.

This reveals that the general two-step class of methods (2.7) reaches the optimal order of convergence four by using only three functional evaluations per full iteration. This completes the proof of Theorem 2.1. □

2.1 Some special cases

It is straight forward to see from the above mentioned general class (2.7) that for different specific values of a_4 and a_5, the following various optimal families of methods can be derived by fixing one of the free disposable parameters. Some of the important families of methods are given below as:

(i) For $a_5 = -1$, family (2.7) reads as

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{mf(x_n)\alpha_5}{2((m+2)f'(y_n)-(m-2)a_4f'(x_n))\alpha_6}, \end{cases} \tag{2.18}$$

where

$\alpha_5 = 2u(m^2-4)(u(m^4+2m^3-2m^2-4m-8)f'(x_n)-m^2(m^2-6)f'(y_n))a_4+m^3(m-2)^2((m-2)f'(y_n) - muf'(x_n))a_4^2) - u^2(m+2)^2(u(m-2)(m+2)^3f'(x_n)-(m^4+2m^3-8m^2-16m-16)f'(y_n))$, and
$\alpha_6 = (u(m+2)(u(m^3-4m+8)f'(x_n)-m^3f'(y_n))+m^3(m-2)(f'(y_n)-uf'(x_n))a_4)$.

The above scheme is a new fourth-order multi-point family of iterative methods.

Some specific subcases of above family (2.18) are:

(a) For $a_4 = 0$, family (2.18) reads as a fourth-order method (1.3) proposed by Li et al. [4].

(b) For $a_4 = -\frac{u(m+2)^2}{m^2}$, family (2.18) reads as

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{muf(x_n)[m(m^5-9m^3+20m-8)f'(y_n) - u(m^6+2m^5-5m^4-10m^3+8m+16)f'(x_n)]}{\left(m^2f'(y_n)+u(m^2-4)f'(x_n)\right)\left(u(m^3-4m+4)f'(x_n)-m(m^2-2)f'(y_n)\right)}. \end{cases} \tag{2.19}$$

This is a new fourth-order multi-point iterative method.

(c) For $a_4 = -u(m+2)^2$, family (2.18) corresponds to

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)mu\left[f'(y_n)h_1 + f'(x_n)u(48+16m-20m^3+5m^4+4m^5-6m^6+m^8)\right]}{2\left(f'(y_n)+f'(x_n)u\left(-4+m^2\right)\right)\left(-f'(y_n)m^3\left(-3+m^2\right)+f'(x_n)u\left(8-4m-3m^3+m^5\right)\right)}, \end{cases} \tag{2.20}$$

where $h_1 = \left(-16+16m-40m^2+30m^3+3m^4-16m^5+6m^6+2m^7-m^8\right)$.

It is another new fourth-order multi-point iterative method.

(d) For

$$a_4 = -\frac{2(2+m)\sqrt{(16+16m+12m^2-4m^3-3m^4)} + (16+16m+8m^2-2m^3-4m^4-m^5)}{u^{-1}(-2+m)m^4},$$

general class (2.7) reads as a fourth-order method (1.8) proposed by Soleymani et al. [9].

(ii) For $a_5 = 0$ and $a_4 \neq 0$, general class (2.7) is a fourth-order method (1.1) proposed by Li et al. [2].

3 Dynamical analysis in the complex plane

Under the point of view of complex dynamics, we will study the general convergence of family (2.7) on polynomials with multiple roots of multiplicity 2 and 3. It is known that the roots of a polynomial can be transformed by an affine map with no qualitative changes on the dynamics of the family. So, we can use the polynomials $p(z) = (z-a)^2(z-b)$ and $q(z) = (z-a)^3(z-b)$. Let us remark that the operator of the family on $p(z)$ and $q(z)$ is a rational function that depends, not only on a_4 and a_5, but also on parameters a and b.

Blanchard in [21] considered the conjugacy map $h(z) = \dfrac{z-a}{z-b}$, (a Möbius transformation) with the following properties:

$$\text{i) } h(\infty) = 1, \quad \text{ii) } h(a) = 0, \quad \text{iii) } h(b) = \infty,$$

and proved that, for quadratic polynomials, Newton's operator is conjugate to the rational map z^2. In what follows, we will use this transformation in order to avoid the appearance of parameters a and b in the rational functions resulting from applying the fixed point operator of the iterative method on polynomials $p(z)$ and $q(z)$.

In the next sections we are going to analyze, under the dynamical point of view, the stability and reliability of the members of the proposed family. Firstly, we will study the fixed points of the rational function that are not related with the original roots of the polynomial (called *strange fixed points*), and the *free critical points*, that is, the critical points of the associated rational function different from 0 and ∞.

3.1 Complex dynamics on double roots

For $p(z)$, the operator associated to family (2.7) is the rational function:

$$M_p(z,a,b) = \frac{-3z^4 + a^3(2b+z) + a^2\left(8b^2 - 20bz + 3z^2\right) + a\left(8b^3 - 32b^2z + 42bz^2 - 9z^3\right)}{3a^3 + 8b^3 + a^2(2b-11z) - 32b^2z + 42bz^2 - 21z^3 + a\left(8b^2 - 20bz + 21z^2\right)},$$

depending on the parameters a and b (although does not depend on a_4 and a_5); moreover, in this case $a_3 = 0$ since $m = 2$ and the iterative expression of the method would be simpler. On the other hand, operator $M_p(z,a,b)$ on $p(z)$ is conjugated to operator $O_p(z)$,

$$O_p(z) = \left(h \circ M_p \circ h^{-1}\right)(z) = \frac{3z^4}{8 - 6z^2 + z^3}. \tag{3.1}$$

Let us observe that the parameters a and b have been obviated in $O_p(z)$.

In order to analyze the stability of the fixed points of $O_p(z)$, it is necessary to calculate its derivative:

$$O_p'(z) = \frac{3z^3\left(32 - 12z^2 + z^3\right)}{\left(8 - 6z^2 + z^3\right)^2}.$$

Lemma 3.1 *The fixed points of $O_p(z)$ and their stability are described in the following statements:*

- *Fixed points of $O_p(z)$ are the roots of the equation $O_p(z) = z$, that is, $z = 0$, $z = \infty$ (that corresponds to the roots of $p(z)$) and the strange fixed points $z_1 = -2$ and $z_2 = 1$.*
- *By analyzing the derivative of the fixed point operator on each of the fixed points, their character is:*
  - *$z = 0$ is superattracting, as $O_p'(0) = 0$.*
  - *$z = \infty$ is attracting, being $O_p'(\infty) = \frac{1}{3}$.*
  - *z_1 is a parabolic fixed point, as $O_p'(z_1) = 1$.*

– z_2 *is repulsive, as* $O'_p(z_2) = 7$.

From this result we would like to remark that the methods guaranties the high-order convergence on the multiple root (this is not the case of the simple one). Moreover, the existence of a parabolic fixed point could imply a dangerous numerical behavior if there exist any free critical point close to it.

Lemma 3.2 *The critical points of* $O_p(z)$ *are* $z = 0$ *and the roots of polynomial* $32 - 12z + z^3$*, that is, the free critical points are approximately* $cr_1 \approx 11.769$*,* $cr_2 \approx 1.7685$ *and* $cr_3 \approx -1.53747$*.*

Let us remark that cr_3 is close to fixed point z_1 and it yields to the existence of its own basin of attraction, as can be observed in the Figure 1. This basin appears in black because the convergence to this fixed point is very slow. The same happens between the attracting fixed point $z = \infty$ and cr_1.

For the representation of the dynamical planes we have used the software described in [22]. We draw a mesh with four hundred points per axis; each point of the mesh is a different initial estimation which we introduce in each procedure. When the method reaches one fixed point in less than eighty iterations, this point is drawn in a different color for each attracting fixed point (with a tolerance of 10^{-3}). The color will be more intense when the number of iterations is lower. These attracting fixed points will be represented by white stars in the different pictures, while fixed points (independently from their character) are plotted as white circles and critical points are represented by white squares. Finally, if there is no convergence to any attracting fixed point, after a maximum of 80 iterations, then the point of the mesh used as initial estimation is painted in black.

Let us remark that two different dynamical behaviors have been observed, depending on the multiplicity of the roots: if the root is multiple ($m = 2$ in this case), a wide basin of attraction with fourth-order convergence is found. However, the second order of convergence is not reached for the simple root; the result is that the respective fixed point is attracting, but not superattracting.

3.2 Complex dynamics on roots of multiplicity three

When $m = 3$ and the behavior of the method on $q(z)$ is considered, the operator of the family is a rational function $M_p(z, a_4, a_5, a, b)$ that depends on all the parameters. By means of Möbius transformation, this rational function is conjugated to operator $O_q(z, a_4, a_5) =$

$\left(h \circ M_p \circ h^{-1}\right)(z)$, which numerator is the polynomial

$$z^4 \left(50 \left(467289 + 788859z + 518130z^2 + 173934z^3 + 34341z^4 + 3751z^5\right) a_4 a_5 + \right.$$
$$\left. 625(3+z)^3 \left(441 + 318z + 49z^2\right) a_4^2 + 81 \left(220239 + 368793z + 239022z^2 + 80490z^3 + 17075z^4 + 2125z^5\right) a_5^2\right)$$

and its denominator is the polynomial

$$625(3+z)^3 \left(2187 + 972z - 1431z^2 - 1008z^3 + 6z^4 + 82z^5\right) a_4^2 +$$
$$50 \left(2302911 + 3247695z + 196830z^2 - 2071818z^3 - 1428759z^4 - 320949z^5 + 28620z^6 + 27456z^7 + 4318z^8\right) a_4 a_5 +$$
$$243 \left(360855 + 503739z + 25110z^2 - 320058z^3 - 217935z^4 - 50553z^5 + 3140z^6 + 4200z^7 + 750z^8\right) a_5^2.$$

Let us observe that the parameters a and b have been obviated in $O_q(z, a_4, a_5)$.

Lemma 3.3 *The fixed points of $O_q(z, a_4, a_5)$ and their stability are described in the following statements:*

(i) Fixed points of $O_q(z, a_4, a_5)$ are the roots of the equation $O_q(z, a_4, a_5) = z$, that is, $z = 0$, $z = \infty$ (that corresponds to the roots of $q(z)$) and the strange fixed points $z_1 = -3$ (except for $a_5 = -\frac{25}{27} a_4$ or $a_5 = 0$) and $z_2 = 1$ and the roots of the polynomial

$$\begin{aligned} r(z) &= -455625a_4^2 - 1421550a_4a_5 - 1082565a_5^2 + z\left(-658125a_4^2 - 2004750a_4a_5 - 1511217a_5^2\right) \\ &\quad + z^2\left(-360000a_4^2 - 1069200a_4a_5 - 797040a_5^2\right) + z^3\left(-58125a_4^2 - 190350a_4a_5 - 147825a_5^2\right) \\ &\quad + z^4\left(6875a_4^2 + 9450a_4a_5 + 3375a_5^2\right). \end{aligned}$$

(ii) By analyzing the derivative of the fixed point operator on each of the fixed points, their character is:

- *$z = 0$ is superattracting, as $O_p'(0) = 0$.*
- *$z_1 = -3$ is always parabolic, as $O_p'(z_1) = 1$.*
- *$z_2 = 1$ can be attracting in the following cases:*
  - *If $a_5 < 0$, then $\frac{-584550-300\sqrt{52701}}{381250} a_5 > a_4 > \frac{-584550+300\sqrt{52701}}{381250} a_5$.*
  - *If $a_5 > 0$, then $\frac{-584550-300\sqrt{52701}}{381250} a_5 < a_4 < \frac{-584550+300\sqrt{52701}}{381250} a_5$.*
- *$z = \infty$ can be attracting in the following cases:*
  - *If $a_5 < 0$, then $-\frac{1377a_5}{1225} < a_4 < -5a_5$.*
  - *If $a_5 > 0$, then $-5a_5 < a_4 < -\frac{1377a_5}{1225}$.*

Proof:

(i) Let us start with the fixed points of operator $O_q(z, a_4, a_5)$. In order to get them we factorize $O_q(z, a_4, a_5) - z$ which numerator is

$$\begin{aligned} n(z) \;=\; & (3(-1+z)z(3+z)^3(-455625a_4^2 - 1421550a_4a_5 - 1082565a_5^2 - 658125a_4^2z - 2004750a_4a_5z \\ & -1511217a_5^2z - 360000a_4^2z^2 - 1069200a_4a_5z^2 - 797040a_5^2z^2 - 58125a_4^2z^3 \\ & -190350a_4a_5z^3 - 147825a_5^2z^3 + 6875a_4^2z^4 + 9450a_4a_5z^4 + 3375a_5^2z^4)) \end{aligned}$$

From this factorization the thesis of this item is obtained.

(ii) In order to analyze the stability of the fixed points of $O_q(z, a_4, a_5)$, it is necessary to calculate its derivative:

$$O_q'(z, a_4, a_5) = \frac{3z^3\left(32 - 12z^2 + z^3\right)}{\left(8 - 6z^2 + z^3\right)^2}.$$

- Obviously, $O_q'(0, a_4, a_5) = 0$ and $z = 0$ is superattracting.
- It is also clear that $O_q'(-3, a_4, a_5) = 1$, being this point parabolic.
- As
$$O_q'(1, a_4, a_5) = \frac{635000a_4^2 + 1947625a_4a_5 + 1472499a_5^2}{63125a_4^2 + 193975a_4a_5 + 146772a_5^2},$$
and $O_q(1, a_4, a_5) < 1$ is equivalent to $571875a_4^2 + 1753650a_4a_5 + 1325727a_5^2 < 0$, the stated results can be deduced (by analyzing the previous parabola in a_4).
- When $z = \infty$ is considered,
$$O_q'(\infty, a_4, a_5) = \frac{2\left(784765625a_4^4 + 8111937500a_4^3a_5 + 27447428750a_4^2a_5^2 + 35671387500a_4a_5^3 + 15684890625a_5^4\right)}{\left(30625a_4^2 + 187550a_4a_5 + 172125a_5^2\right)^2}$$
and by using a similar reasoning than in case $z_2 = 1$, the thesis can be stated.

□

Let us remark that the strange fixed points that are roots of the fourth-order polynomial stated in the Lemma 3.3 have very complicated expressions and to study their stability analytically is not possible. However, we can analyze numerically their stability functions, whose graphics appear in Figure 2.

It is interesting to notice that $z = \infty$ (the simple root of the original polynomial $q(z)$) is not always an attracting fixed point. This is the reason why it is not a critical point, meanwhile $z = 0$ (the multiple root of $q(z)$) is always a critical point. As we have also seen in the previous section, the behavior of the roots is different depending on the multiplicity: meanwhile the multiple root shows a very stable behavior, the simple one can be even a repulsive fixed point.

Some interesting cases can be extracted from the previous results: it is easy to check that if $z_2 = 1$ is an attracting strange fixed point, then $z = \infty$ is also attracting. In Figure 3a, the dynamical plane corresponding to $a_4 = 0.857$ and $a_5 = -0.5$ is showed. It can be observed that four basins of attraction appear, corresponding to $z = 0$, $z = \infty$, $z_2 = 1$ and $z_1 = -3$. In Figure 3b the dynamical plane of $a_3 = 0$ is presented; the areas of convergence of $z = 0$, $z_1 = -3$ (in black because of the slow convergence) and $z = \infty$ appear.

In Figure 4b the dynamical plane associated to $a_4 = 0$ and any value of a_5 is presented. In it, the basin of attraction of $z = \infty$ does not exist, the one of $z = 0$ appears in orange and a big black area that corresponds to $z_1 = -3$ and $z = 48.8272$, parabolic and attracting fixed points respectively, with a very slow convergence. This is the reason why these dynamical planes have been drawn with 800×800 points and 2000 iterations. Even under this circumstances, the basin of $z_1 = -3$ appears in black.

Now, we are going to analyze the cases in which $z_1 = -3$ is not a parabolic point. It has been stated that it happens when $a_5 = 0$ or $a_5 = -\frac{25}{27}a_4$ (see Lemma 3.3). Both cases have the same associated rational function,

$$O_q(z) = \frac{z^4\left(441 + 318z + 49z^2\right)}{2187 + 972z - 1431z^2 - 1008z^3 + 6z^4 + 82z^5}.$$

This rational function has as fixed points $z = 0$ and $z = \infty$, being $z = 1$ and the roots of polynomial $s(z) = -729 - 1053z - 576z^2 - 93z^3 + 11z^4$ the strange fixed points. respect to the stability of these fixed points, let us remark that $z = 0$ is superattracting, $z = \infty$ and $z = 1$ are repulsive and three of the roots of $s(z)$ are repulsive, being the last one attractive.

As in case of parametric families of iterative methods for finding simple roots, the dynamical behavior, when multiple roots are searched, depends on the value of the parameter and stable or unstable cases can be found and the regions of the parameter plane where these behaviors happen are delimited. Nevertheless, the differences between the simple and the multiple root remain, due to the different order of convergence of the methods on them.

4 Numerical results

In this section, we shall check the effectiveness of the new optimal methods. We employ the present methods namely, method (2.19) and (2.20) denoted by OM_4^1 and OM_4^2 respectively to solve the following nonlinear equations in table 1. We compare them with the method of Zhou et al., iterative expression (11) of [5], denoted by ZM_4; method (1.2) of Sharma and Sharma [3] (called SSM_4); method (1.3) designed by Li et al. in [4] and denoted by LM_4. In addition, Soleymani and Babajee constructed in [7] several fourth-order methods for multiple roots, between them we will use expression (27) and expression (29), denoted by SBM_4^1 and

SBM_4^2, respectively. Finally, we will also compare our schemes with the method of Sharifi et al. SHM_4, expression (35) of [6]. For better comparisons of our proposed methods, we have given three types of comparison tables in each example: one is corresponding to absolute error value of given nonlinear functions, other is with respect to number of iterations and third one is regarding the computational order of convergence (obtained from the expression (see [23])) in tables 2-4, respectively

$$\rho \approx \frac{\ln |(x_{k+1} - r_m)/(x_k - r_m)|}{\ln |(x_k - r_m)/(x_{k-1} - r_m)|}.$$

All computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. We use $\epsilon = 10^{-34}$ as a tolerance error. The following stopping criterium are used for computer programs: $|x_{n+1} - x_n| < \epsilon$ and $|f(x_{n+1})| < \epsilon$.

5 Concluding remarks

In this paper, we have proposed a simple and different technique to construct higher-order optimal methods for computing multiple roots of nonlinear equations numerically. The family of methods requires one function and two of its first-order derivative evaluations per iteration step. Proposed methods are free from second or higher order derivatives. Further, one can easily generate many new optimal families and some existing methods by fixing one of the free disposable parameters in our proposed schemes (2.7). It is observed from Tables 2-4 that the proposed methods namely, method (2.19) and (2.20) have at least equal or better performance as compared with other similar robust methods available in literature. On the other hand, investigation has been made on the complex plane for such methods to reveal their dynamical behavior on polynomials with double and triple roots. The dynamic study of our families of iterative methods allows us to select iterative schemes with good stability and reliability properties and detect iterative methods with dangerous numerical behavior. It has been also observed that the difference between simple and multiple roots in terms of dynamical behavior is clear, when a method designed for multiple roots is applied, as it only hold the order of convergence for the multiple zeros, and not for the simple ones.

Acknowledgments

The authors thank to the anonymous referees for their valuable comments and suggestions.

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Table 1: Test problems

$f(x)$	r_m	multiplicity of r_m
$f_1(x) = (\sin^2 x + x)^5$	0.0000000000000000000000000000000000	5
$f_2(x) = (xe^x + \cos x)^5$	−1.2010606007342119750993085470974788	5
$f_3(x) = x^2 \sin(4x)$	0.0000000000000000000000000000000000	3
$f_4(x) = (x^2 - e^x - 3x + 2)^6$	0.25753028543986076045536730493724178	6
$f_5(x) = \left(5\tan^{-1} x - 4x\right)^8$	−0.94913461128828951372581521479848875	8
$f_6(x) = \left(1 + \cos\left(\frac{x\pi}{2}\right) - \sqrt{1-x^2}\right)^4$	−0.72858404644482671671233310242278337	4

Table 2: Comparison of different fourth-order methods with the same total number of functional evaluations (TNFE=12)

$f(x)$	x_0	ZM_4	SSM_4	LM_4	SBM_4^1	SBM_4^2	SHM_4	OM_4^1	OM_4^2
$f_1(x)$	−3.4	2.0e−100	2.5e−102	3.4e−109	D	D	D	1.1e−110	5.7e−114
	1.2	3.7e−46	1.6e−97	1.5e−97	1.6e−97	1.6e−97	D	1.5e−97	1.5e−97
$f_2(x)$	−2.4	9.0e−79	1.7e−76	1.8e−70	C	C	2.5e+13	1.8e−69	1.5e−67
	2.0	2.1e−113	3.5e−127	2.1e−206	C	C	5.4e−17	1.1e−172	1.8e−145
$f_3(x)$	−3.5	1.2e−314	2.5e−314	9.3e−315	6.9e−315	2.7e−315	4.7e−284	9.2e−315	8.9e−315
	3.5	1.2e−314	2.5e−314	9.3e−315	6.9e−315	2.7e−315	4.7e−284	9.2e−315	8.9e−315
$f_4(x)$	−7.0	8.6e−317	4.0e−325	7.0e−356	1.0e−291	1.5e−290	D	1.8e−361	1.8e−361
	1.4	5.6e−743	1.5e−742	1.6e−741	8.6e−736	1.3e−735	1.2e−320	2.0e−741	2.0e−741
$f_5(x)$	−2.3	3.3e−410	2.1e−410	4.2e−411	8.2e−416	7.7e−416	8.1e−410	3.6e−411	2.8e−411
	0.3	C	C	C	C	C	C	7.4e−9	5.5−10
$f_6(x)$	−0.88	1.2e−158	8.8e−161	3.0e−167	2.1e−94	2.1e−74	4.3e−143	1.0e−168	1.9e−172
	−0.1	4.3e−133	3.4e−133	2.4e−133	7.3e−133	4.3e−133	D	2.3e−133	2.1e−133

C: converges to undesired root, D: divergent.

Table 3: Comparison of different fourth-order methods with respect to number of iteration

$f(x)$	x_0	ZM_4	SSM_4	LM_4	SBM_4^1	SBM_4^2	SHM_4	OM_4^1	OM_4^2
$f_1(x)$	−3.4	6	6	6	D	D	D	6	6
	1.2	6	6	6	6	6	D	6	6
$f_2(x)$	−2.4	6	6	5	C	C	12	6	6
	2.0	6	6	5	C	C	7	5	6
$f_3(x)$	−3.5	5	5	5	5	5	5	5	5
	3.5	5	5	5	5	5	5	5	5
$f_4(x)$	−7.0	5	5	5	5	6	D	5	5
	1.4	5	5	5	5	5	5	5	5
$f_5(x)$	−2.3	5	5	5	5	5	5	5	5
	0.3	C	C	C	C	C	C	8	8
$f_6(x)$	−0.88	5	5	5	6	6	5	5	5
	−0.1	5	5	5	5	5	D	5	5

Table 4: Computational order of convergence of different fourth-order methods

$f(x)$	x_0	ZM_4	SSM_4	LM_4	SBM_4^1	SBM_4^2	SHM_4	OM_4^1	OM_4^2
$f_1(x)$	−3.4	4.000	4.000	4.000	D	D	D	4.000	4.000
	1.2	3.999	3.999	4.000	4.000	4.000	D	4.000	4.000
$f_2(x)$	−2.4	4.000	4.000	4.000	C	C	4.000	4.000	4.000
	2.0	4.000	4.000	4.000	C	C	4.001	3.997	4.000
$f_3(x)$	−3.5	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
	3.5	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
$f_4(x)$	−7.0	4.000	4.000	4.000	4.000	4.000	D	4.000	4.000
	1.4	4.000	4.000	4.000	4.000	4.000	3.999	4.000	4.000
$f_5(x)$	−2.3	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000
	0.3	C	C	C	C	C	C	4.000	4.000
$f_6(x)$	−0.88	3.998	3.999	3.999	4.000	4.000	4.003	4.000	3.999
	−0.1	3.995	3.995	3.995	3.995	3.995	3.995	3.995	3.995

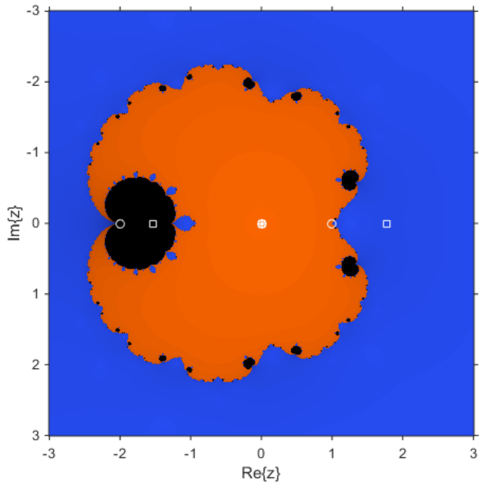
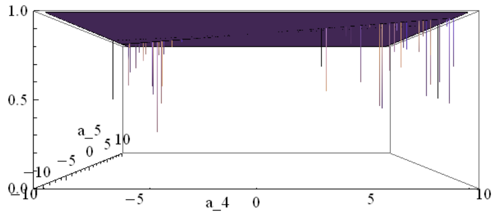
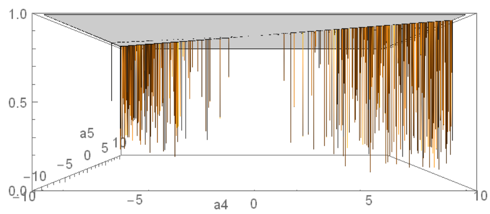


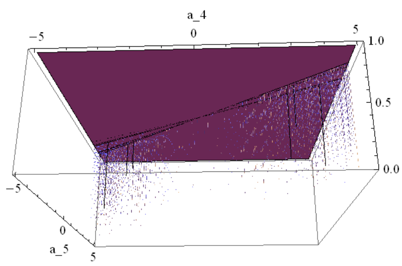
Figure 1: Dynamical plane associated to $O_p(z)$ for $m = 2$



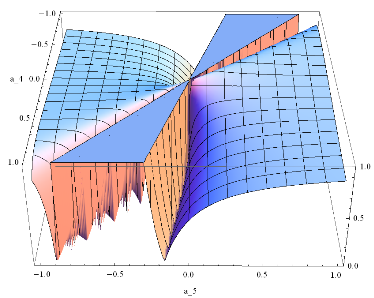
(a) $|O'_q(ex_1)|$



(b) $|O'_q(ex_2)|$

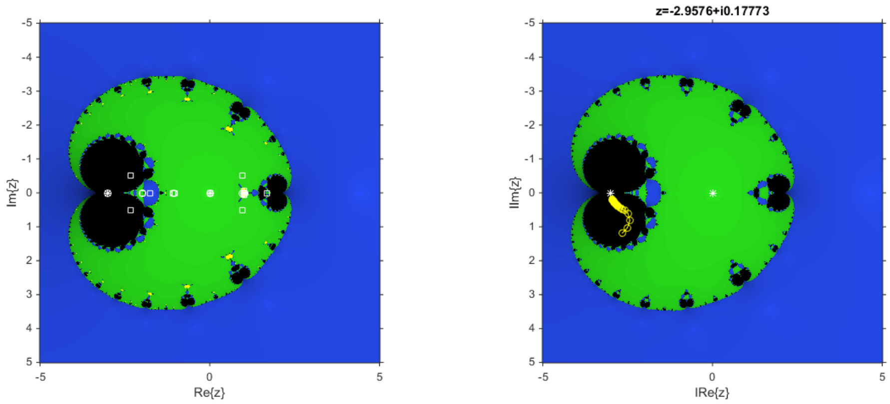


(c) $|O'_q(ex_3)|$



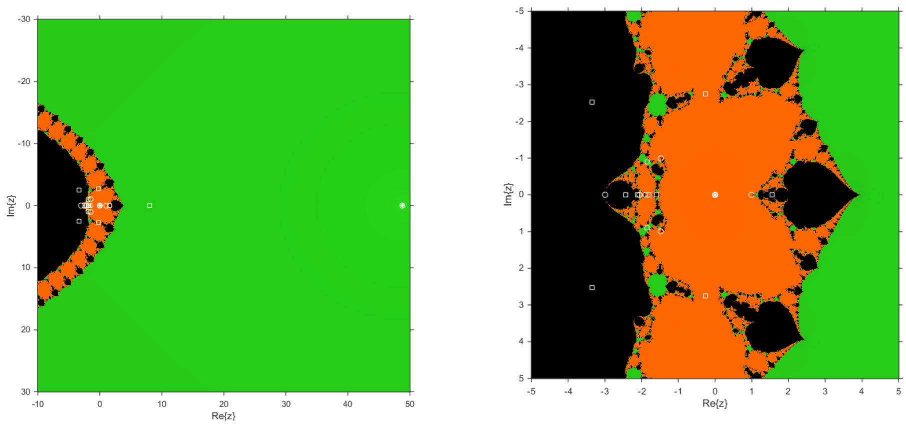
(d) $|O'_q(ex_4)|$

Figure 2: Stability functions of some strange fixed points



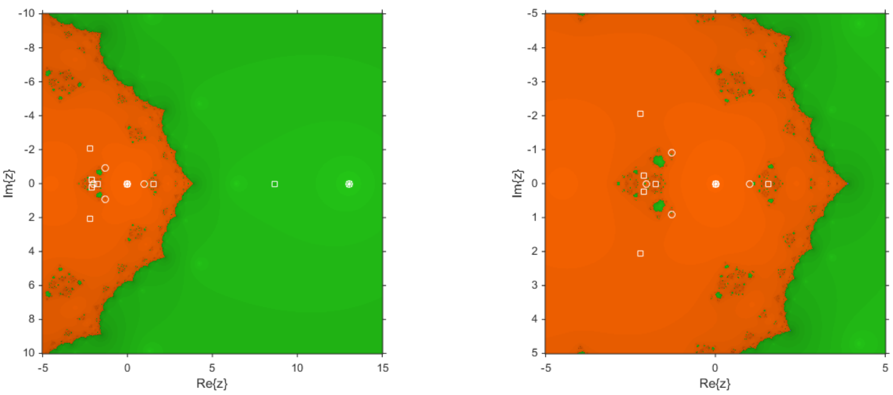
(a) $a_4 = 0.857$ and $a_5 = -0.5$ (b) $a_4 = 0$

Figure 3: Dynamical plane associated to $O_q(z, a_4, a_5)$ for $m = 3$



(a) $a_4 = 0$ (b) A detail

Figure 4: Dynamical plane associated to $O_q(z, a_4, a_5)$ for $m = 3$ and $a_4 = 0$



(a) $a_5 = 0$ (b) A detail

Figure 5: Dynamical plane associated to $O_q(z, a_4, a_5)$ for $m = 3$ and $a_5 = 0$