A proof of topological completeness for S4 in (0,1)

Grigori Mints[1] and Ting Zhang[2]

[1] Philosophy Department, Stanford University mints@csli.stanford.edu
[2] Computer Science Department, Stanford University tingz@cs.stanford.edu

Summary. The completeness of the modal logic $S4$ for all topological spaces as well as for the real line $\mathbb{R}$, the n-dimensional Euclidean space $\mathbb{R}^n$ and the segment $(0,1)$ etc. (with $\Box$ interpreted as interior) was proved by McKinsey and Tarski in 1944. Several simplified proofs contain gaps. A new proof presented here combines the ideas published later by G. Mints and M. Aiello, J. van Benthem, G. Bezhanishvili with a further simplification. The proof strategy is to embed a finite rooted Kripke structure $\boldsymbol{K}$ for $S4$ into a subspace of the Cantor space which in turn encodes $(0,1)$. This provides an open and continuous map from $(0,1)$ onto the topological space corresponding to $\boldsymbol{K}$. The completeness follows as S4 is complete with respect to the class of all finite rooted Kripke structures.

1 Introduction

The correspondence between elementary topology and the modal logic $S4$ was first established by McKinsey. In [1] McKinsey introduced the topological interpretation of $S4$ where the necessitation connective $\Box$ is interpreted as the topological interior. McKinsey showed that $S4$ is complete for the class of all topological spaces. Later more mathematically interesting results were obtained by McKinsey and Tarski [2], [3]. McKinsey and Tarski showed that $S4$ is complete for any dense-in-itself separable metric space. As a consequence, $S4$ is complete for the real line $\mathbb{R}$, the n-dimensional Euclidean space $\mathbb{R}^n$, the Cantor set and the real segment $(0,1)$ etc. Recently several attempts were made to simplify the proof by McKinsey and Tarski. Mints gave a completeness proof of $S4$ for the Cantor set [4] and a completeness proof of the intuitionistic propositional logic for the real segment $(0,1)$ ([5], Chapter 9). Aiello, van Benthem and Bezhanishvili gave a completeness proof of $S4$ for $(0,1)$ ([6], Section 5). However, simplified proofs in [6], Section 5 and [5], Chapter 9 contain gaps. We present here a new proof, which combines the ideas in [4], [5] and [6], and provides a further simplification. It goes by (1) encoding reals in $(0,1)$ using a Cantor set $\mathscr{B}$, (2) unwinding a finite rooted Kripke structure $\boldsymbol{K}$ for $S4$ to cover $\mathscr{B}$. Step 1 gives a one-to-one correspondence between elements of $\mathscr{B}$ (infinite paths in the full binary tree) and real numbers in $(0,1)$. Step 2 generates a labeling of the full binary tree by worlds in $\boldsymbol{K}$ and hence establishes a one-to-one correspondence between infinite paths in $\mathscr{B}$ and infinite sequences of worlds in $\boldsymbol{K}$. Hence we have a one-to-one correspondence between reals in $(0,1)$ and infinite sequences of worlds in $\boldsymbol{K}$. Since $\boldsymbol{K}$ is finite, every infinite sequence of worlds must eventually enter a stable loop which consists of equivalent worlds with respect to the frame relation. For each such sequence we pick the label at the stabilization point where the sequence enters the loop. We map each real in $(0,1)$ to the label of its corresponding sequence. This provides an open and continuous map from $(0,1)$ onto the topological space corresponding to $\boldsymbol{K}$. The completeness follows as S4 is complete with respect to the class of finite rooted Kripke structures.

We assume basic topology terminology. In particular, we use $\boldsymbol{Int}$ and $\boldsymbol{Cl}$ to denote the interior and closure operators respectively.

Definition 1.1 (Topological Model) *A topological model is an ordered pair $M = \langle X, V\rangle$, where X is a topological space and V is a function assigning a subset of X to each propositional variable. The valuation V is extended to all $S4$ formulas as follows:*

$$V(\alpha \vee \beta) = V(\alpha) \cup V(\beta), \qquad V(\neg\alpha) = X \setminus V(\alpha),$$
$$V(\alpha \,\&\, \beta) = V(\alpha) \cap V(\beta), \qquad V(\Box\alpha) = \boldsymbol{Int}(V(\alpha)).$$

We say that α is valid in a topological model M and write $M \models \alpha$ if and only if $V(\alpha) = X$.

Definition 1.2 (Kripke Model) *A* ***Kripke frame*** *(for $S4$) is an ordered pair $F = \langle W, R\rangle$ where W is a non-empty set and R is a reflexive and transitive relation on W. The elements in W are called* ***worlds****. We say that a world w is an R-**successor** of a world w' if Rww', and w is R-**equivalent** to w' (written $w \equiv_R w'$) if both Rww' and $Rw'w$. A Kripke frame is **rooted** if there exists a world w_0 such that any world w in W is an R-successor of w_0.*

*A **Kripke model** is a tuple $M = \langle W, R, V\rangle$ with $\langle W, R\rangle$ a Kripke frame and V a valuation function, which assigns a subset of worlds in W to every propositional variable. Validity relation $\models$ is defined recursively in the standard way. In particular,*

$$(M, w) \models \Box\alpha \text{ iff } (M, w') \models \alpha \text{ for every } w' \text{ such that } Rww'.$$

*We say that a formula α is **valid in M** if and only if $(M, w) \models \alpha$ for every $w \in W$. A formula α is **valid** (written $\models \alpha$) if α is valid in every Kripke model.*

We can think of a Kripke frame as being a topological space by imposing a topology on it.

Definition 1.3 (Kripke Space) *Let $\boldsymbol{K} = \langle W, R\rangle$ be a Kripke frame. A **Kripke space** on $\boldsymbol{K}$ is a topological space $\mathcal{T} = \langle W, \mathcal{O}\rangle$ where W is the carrier and $\mathcal{O}$ is the collection of all subsets of W **closed** under R:*

$$M \in \mathcal{O} \quad \text{iff} \quad (w \in M \text{ and } Rww' \text{ implies } w' \in M) \text{ for all } w, w' \in W.$$

It is well-known that $S4$ is complete for finite rooted Kripke models [7].

Theorem 1.1 *For any $S4$ formula α, $S4 \vdash \alpha$ if and only if α is valid in all finite rooted Kripke frames.*

2 A correspondence between $(0, 1)$ and finite Kripke structures

2.1 Binary encoding of real numbers

Let $\Sigma = \{0, 1\}$, and let Σ^ω be the full infinite binary tree where each node in the tree is identified by a finite path (a finite Σ-word) from the root Λ to it. We use $\bar{b}$ and $\boldsymbol{b}$ to denote finite paths and infinite paths respectively. Let $\mathscr{C}$ be the standard Cantor set represented by Σ^ω, where each element of $\mathscr{C}$ is identified with an infinite path (an infinite Σ-word). For each $\boldsymbol{b} \in \mathscr{C}$, $\boldsymbol{b}\restriction n$ denotes the prefix of length n, i.e., the finite sequence $\boldsymbol{b}\restriction n = \boldsymbol{b}(1)\boldsymbol{b}(2)\ldots\boldsymbol{b}(n)$. We write $\boldsymbol{b}_1 \equiv_n \boldsymbol{b}_2$ if $\boldsymbol{b}_1\restriction n = \boldsymbol{b}_2\restriction n$. One can imagine adding the component $\boldsymbol{b}(0) = 0$ to account for the root Λ, but we do not do that. (See Figure 1.)

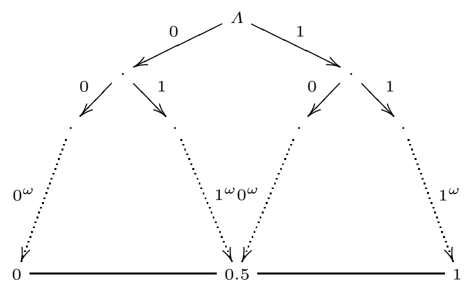

Fig. 1. The full infinite binary tree

Let

$$\mathscr{B} = \mathscr{C} \setminus (\{0^\omega\} \cup \Sigma^* 1^\omega),$$

i.e., $\mathscr{B}$ is obtained by deleting from $\mathscr{C}$ the leftmost path which corresponds to the word 0^ω, as well as paths going right from some point on, which correspond to sequences with the infinite tail of 1's. So for each path $\boldsymbol{b} \in \mathscr{B}$, $\boldsymbol{b}$ either always goes left from some point on, or goes both left and right infinitely often. In the former case $\boldsymbol{b}$ ends with 0^ω and in the latter case $\boldsymbol{b}$ contains infinitely many 0's as well as infinitely many 1's. Formally let

$$\mathscr{B}_1 = \{\boldsymbol{b} \in \mathscr{B} \mid \boldsymbol{b} = b_1 b_2 \ldots b_i 0^\omega \text{ for some } i > 0\}, \quad \mathscr{B}_2 = \mathscr{B} \setminus \mathscr{B}_1.$$

We view a sequence in $\mathscr{B}$ as a binary encoding of a real number in $(0,1)$. A one-to-one correspondence between $\mathscr{B}$ and $(0,1)$ is given by

$$\boldsymbol{real}(\boldsymbol{b}) = \sum_{i=1}^{\infty} \boldsymbol{b}(i) 2^{-i},$$

$$B(x) = \text{ the unique } \boldsymbol{b} \in \mathscr{B} \text{ such that } \boldsymbol{real}(\boldsymbol{b}) = x.$$

The sequences in $\mathscr{B}_1$ represent binary rational numbers in $(0,1)$; the sequences in $\mathscr{B}_2$ represent all other real numbers in $(0,1)$. For example, 0.375, in binary 0.011, is represented by 0110^ω. Now it should be clear why $\mathscr{B}$ excludes some binary sequences; sequence 0^ω represents 0, and numbers represented by sequences of the form $b_1 b_2 \ldots b_n 01^\omega$ can also be represented by sequences of the form $b_1 b_2 \ldots b_n 10^\omega$.

Proposition 2.1 *Let $x, y \in (0,1)$, $B(x)\restriction(n+1) = b_1 b_2 \ldots b_n 0$ and $B(y)\restriction(n+1) = b_1 b_2 \ldots b_n 1$. Then for any $z \in (0,1)$, if $x < z < y$, then $B(z)\restriction n = b_1 b_2 \ldots b_n$.*

Proof. It follows immediately from basic properties of the binary representation. ☐

Proposition 2.2 *Let* $x \in (0,1)$ *and* $B(x) = b_1 b_2 \dots b_n 10^\omega$. *Then for any* $y \in (0,1)$,

1. *if* $0 < x - y < 2^{-(n+2)}$, *then* $B(y) \upharpoonright (n+2) = b_1 b_2 \dots b_n 01$, *and*
2. *if* $0 < y - x < 2^{-(n+2)}$, *then* $B(y) \upharpoonright (n+2) = b_1 b_2 \dots b_n 10$.

Proof. (See Figure 2.) Let $\boldsymbol{b} = b_1 b_2 \dots b_n 10^\omega$, $\boldsymbol{b}' = b_1 b_2 \dots b_n 01^\omega$, $\boldsymbol{l} = b_1 b_2 \dots b_n 010^\omega$ and $\boldsymbol{u} = b_1 b_2 \dots b_n 101^\omega$. We know that $B(x) = \boldsymbol{real}(\boldsymbol{b}) = \boldsymbol{real}(\boldsymbol{b}')$. Let $l = \boldsymbol{real}(\boldsymbol{l})$ and $u = \boldsymbol{real}(\boldsymbol{u})$. Since $l + 2^{-(n+2)} = x$, for any y such that $0 < x - y < 2^{-(n+2)}$, $l < y < x$, and so by Proposition 2.1 $B(y)$ has the prefix $b_1 b_2 \dots b_n 01$. Similarly, since $x + 2^{-(n+2)} = u$, for any y such that $0 < y - x < 2^{-(n+2)}$, $x < y < l$, and so $B(y)$ has the prefix $b_1 b_2 \dots b_n 10$. □

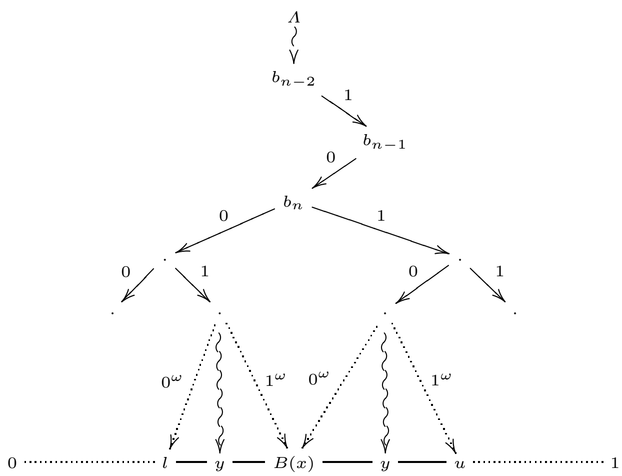

Fig. 2. Proposition 2.2

Proposition 2.3 *Let* $x, y \in (0,1)$. *If* $B(x) \equiv_n B(y)$, *then* $|x - y| < 2^{-n}$.

Proof. If $B(x) \equiv_n B(y)$, then obviously $|x-y| \le 2^{-n}$. To have $|x-y| = 2^{-n}$, one of $B(x)$ and $B(y)$ must end with 0^ω and the other must end with 1^ω. Since paths ending with 1^ω has been excluded from $\mathscr{B}$, we have $|x-y| < 2^{-n}$. □

Proposition 2.4 *Let* $x \in (0,1)$ *and* $B(x) \upharpoonright (n+2) = b_1 b_2 \dots b_n 01$. *If* $y \in (0,1)$, $|y - x| < 2^{-(n+2)}$, *then* $B(y) \upharpoonright n = b_1 \dots b_n$ *and* $B(y) \neq b_1 b_2 \dots b_n 0^\omega$.

Proof. (See Figure 3.) Let $\boldsymbol{u} = b_1 b_2 \dots b_n 01^\omega$, $\boldsymbol{l} = b_1 b_2 \dots b_n 010^\omega$, $\boldsymbol{u}_1 = b_1 b_2 \dots b_n 001^\omega$, $\boldsymbol{l}_1 = b_1 b_2 \dots b_n 0^\omega$, $\boldsymbol{u}_2 = b_1 b_2 \dots b_n 101^\omega$, and $\boldsymbol{l}_2 = b_1 b_2 \dots b_n 10^\omega$. Let $u = \boldsymbol{real}(\boldsymbol{u})$, $l = \boldsymbol{real}(\boldsymbol{l})$, $u_1 = \boldsymbol{real}(\boldsymbol{u}_1)$, $l_1 =$

$\boldsymbol{real}(\boldsymbol{l}_1)$, $u_2 = \boldsymbol{real}(\boldsymbol{u}_2)$ and $l_2 = \boldsymbol{real}(\boldsymbol{l}_2)$. Obviously, we have $l_1 + 2^{-(n+2)} = u_1 = l$, $l + 2^{-(n+2)} = u = l_2$ and $l_2 + 2^{-(n+2)} = u_2$. Since $B(x) \upharpoonright (n+2) = b_1 b_2 \dots b_n 01$, $l \leq x < u$ and so $l_1 < y < u_2$ as $|y - x| < 2^{-(n+2)}$. Hence by Proposition 2.1 $B(y) \upharpoonright n = b_1 \dots b_n$ and $B(y) \neq b_1 b_2 \dots b_n 0^\omega$. □

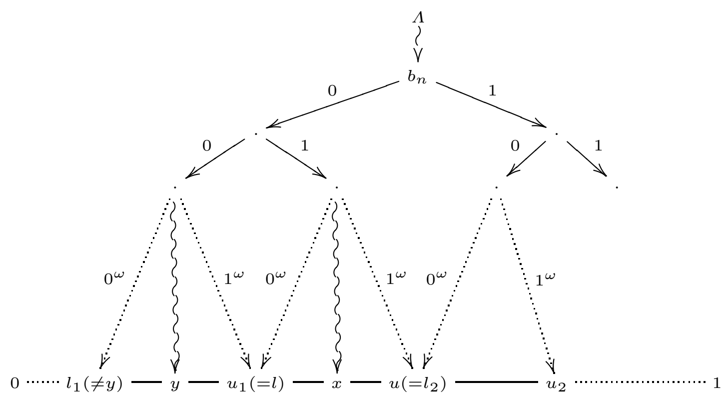

Fig. 3. Proposition 2.4

Proposition 2.5 *For any $x, y \in (0,1)$ if $B(x) = b_1 b_2 \dots b_n 10^\omega$, $B(y) \upharpoonright m = b_1 b_2 \dots b_n 011^{m-(n+2)}$, then $B(y) \neq b_1 b_2 \dots b_n 011^{m-(n+2)} 0^\omega$ if and only if $|x - y| < 2^{-m}$.*

Proof. Let $\boldsymbol{u} = b_1 b_2 \dots b_n 01^\omega$, $\boldsymbol{l} = b_1 b_2 \dots b_n 011^{m-(n+2)} 0^\omega$ and $u = \boldsymbol{real}(\boldsymbol{u})$, $l = \boldsymbol{real}(\boldsymbol{l})$. Obviously, $u = x$ and $u - l = 2^{-m}$. Since $B(y) \upharpoonright m = b_1 b_2 \dots b_n 011^{m-(n+2)}$, $y < u = x$. If $|x - y| < 2^{-m}$, then $y > l$ and so $B(y) \neq \boldsymbol{l}$. On the other hand, if $B(y) \neq \boldsymbol{l}$ and $B(y) \upharpoonright m = b_1 b_2 \dots b_n 011^{m-(n+2)}$, then $l < y < u$ and so $|x - y| < 2^{-m}$. □

2.2 Unwinding a finite rooted model into the Cantor space

Let $\boldsymbol{K} = \langle W, R \rangle$ be a finite Kripke model with root w_0 and $\mathcal{K}$ be the corresponding Kripke space. In the following sections by Kripke model we always mean a finite rooted one.

Definition 2.1 (Unwinding and Labeling) *The labeling function $\mathcal{W} : \Sigma^* \to W$ is defined recursively as follows. (See Figure 4.)*

1. *$\mathcal{W}(\Lambda) = w_0$.*
2. *Let $\bar{b} \in \Sigma^*$ be a node in $\mathscr{B}$. Suppose $\bar{b}$ is already labeled by a world w (i.e., $\mathcal{W}(\bar{b}) = w$) , while none of its children has yet been labeled. Let $w, w_1, \dots, w_m$ be all R-successors of w. Then*

$$\mathcal{W}(\bar{b}0^i) = w \textit{ for } 0 < i \leq 2m,$$
$$\mathcal{W}(\bar{b}0^{2i-1}1) = w_i \textit{ for } 0 < i \leq m,$$
$$\mathcal{W}(\bar{b}0^{2i}1) = w \textit{ for } 0 \leq i < m.$$

Note that in placing R-successors of w at right branches $\bar{b}0^{2i-1}1$ $(i > 0)$, we interleave w with each of its other successors. This is the main distinction from the construction in [6]. (See Figure 4.)

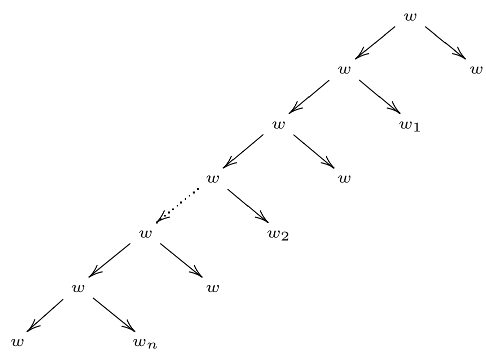

Fig. 4. Unwinding and labeling

Definition 2.2 (Monotonic Sequences) *Let $\boldsymbol{K} = \langle W, R\rangle$ be a Kripke model. An infinite sequence $\boldsymbol{b}$ of worlds in W is **monotonic** (with respect to R) if $R\boldsymbol{b}(i)\boldsymbol{b}(j)$ holds for any $i < j$. We write W^* for the set of all monotonic sequences in W^ω.*

By Definition 2.1 each path in $\mathscr{B}$ is labeled by a monotonic sequence in W^*. We write $\boldsymbol{W}$ for the **induced map** from $\mathscr{B}$ to W^*, i.e.,

$$\boldsymbol{W}(\boldsymbol{b}) = \lambda n : \omega.\mathcal{W}(\boldsymbol{b}\restriction n) = \mathcal{W}(\boldsymbol{b}\restriction 1)\mathcal{W}(\boldsymbol{b}\restriction 2)\dots.$$

Proposition 2.6 *Let $\mathcal{W}(\bar{b}) = w_1$, $\mathcal{W}(\bar{b}1) = w_2$. If $w_1 \neq w_2$, then $\mathcal{W}(\bar{b}01) = w_1$.*

Proof. If $\mathcal{W}(\bar{b}) = w_1$, then $\mathcal{W}(\bar{b}0) = w_1$. In addition, if $\mathcal{W}(\bar{b}1) = w_2$, then Rw_1w_2. But since $w_1 \neq w_2$, and w_1 is interleaved with any other proper successor of w_1 during the unwinding process, $\bar{b}01$ must be labeled by w_1, that is, $\mathcal{W}(\bar{b}01) = w_1$. □

Proposition 2.7 *Let $\mathcal{W}(\bar{b}) = w$. Then for any $w' \in W$ with Rww' there exist infinitely many $i > 0$ such that $\mathcal{W}(\bar{b}0^i1) = w'$.*

Proof. Let $w, w_1, \ldots, w_m$ be all R-successors of w. By Definition 2.1

$$\mathcal{W}(\bar{b}0^{2i-1}1) = w_i \quad \textit{for} \quad 0 < i \leq m \quad \text{and} \quad \mathcal{W}(\bar{b}0^{2m}0) = w.$$

By our definition, we have for all $k \geq 0$

$$\mathcal{W}(\bar{b}0^{2mk+(2i-1)}1) = w_i \quad \textit{for} \quad 0 < i \leq m.$$

□

Proposition 2.8 *Let $\mathcal{W}(\bar{b}) = w$. If $\mathcal{W}(\bar{b}1) = w$, then for any $i \geq 1$, $\mathcal{W}(\bar{b}1^i) = w$.*

Proof. Note that $\bar{b}11$ gets labeled only after $\bar{b}1$ has been labeled. By Definition 2.1, $\mathcal{W}(\bar{b}11) = w$. Repeating this argument we have $\mathcal{W}(\bar{b}1^i) = w$ for any $i \geq 1$. □

Definition 2.3 (Stabilization Point) *We say a point i is a stabilization point for a monotonic sequence $\boldsymbol{b}$ if $\boldsymbol{b}(i) \equiv_R \boldsymbol{b}(j)$ for any $j > i$.*

If $\boldsymbol{K}$ is a finite model, each sequence in W^* must eventually enter a stable loop consisting of R-equivalent worlds. (Note that the loop may consist of a single world.) We define function $\lambda : \mathscr{B} \to \mathbb{N}$ by

$$\lambda(\boldsymbol{b}) = \mu n[n \geq 1 \;\&\; (\forall i, j \geq n \; R\boldsymbol{W}(\boldsymbol{b})(i)\boldsymbol{W}(\boldsymbol{b})(j))].$$

In other words, the function λ returns the non-root "R-stabilization point" of $\boldsymbol{W}(\boldsymbol{b})$ for each $\boldsymbol{b} \in \mathscr{B}$.

Definition 2.4 *We define a map $\delta : \mathscr{B} \to \mathbb{N}$ as follows:*

$$\delta(\boldsymbol{b}) = \begin{cases} \delta_1(\boldsymbol{b}) & \textit{if } \boldsymbol{b} \in \mathscr{B}_1 \\ \delta_2(\boldsymbol{b}) & \textit{if } \boldsymbol{b} \in \mathscr{B}_2 \end{cases}$$

where $\delta_1 : \mathscr{B}_1 \to \mathbb{N}$ is defined by

$$\delta_1(\boldsymbol{b}) = max(1, n), \quad \textit{if} \quad \boldsymbol{b} = b_1 b_2 \ldots b_n 10^\omega,$$

and $\delta_2 : \mathscr{B}_2 \to \mathbb{N}$ is defined by

$$\delta_2(\boldsymbol{b}) = \mu n(n > \lambda(\boldsymbol{b}) \;\&\; \boldsymbol{b}(n) = 1 \;\&\; \boldsymbol{b}(n-1) = 0).$$

The map δ will serve as the "modulus of continuity" for the map $\pi : \mathscr{B} \to W$ introduced below.

Definition 2.5 (Selection Function) *We define a selection function $\rho : \mathscr{B} \to \mathbb{N}$ and a map $\pi : \mathscr{B} \to W$ as follows: (See Figure 5.)*

$$\rho(\boldsymbol{b}) = \begin{cases} \delta_1(\boldsymbol{b}) & \textit{if } \boldsymbol{b} \in \mathscr{B}_1, \\ \lambda(\boldsymbol{b}) & \textit{if } \boldsymbol{b} \in \mathscr{B}_2. \end{cases}$$

$$\pi(\boldsymbol{b}) = \boldsymbol{W}(\boldsymbol{b})(\rho(\boldsymbol{b})).$$

For notation simplicity we identify function $f : \mathscr{B} \to X$ with the corresponding function $B \circ f : (0,1) \to X$. For example, $\rho(x)$ ($x \in (0,1)$) should be understood as $\rho(B(x))$. In particular,

$$\pi(x) = \boldsymbol{W}(B(x))(\rho(B(x))).$$

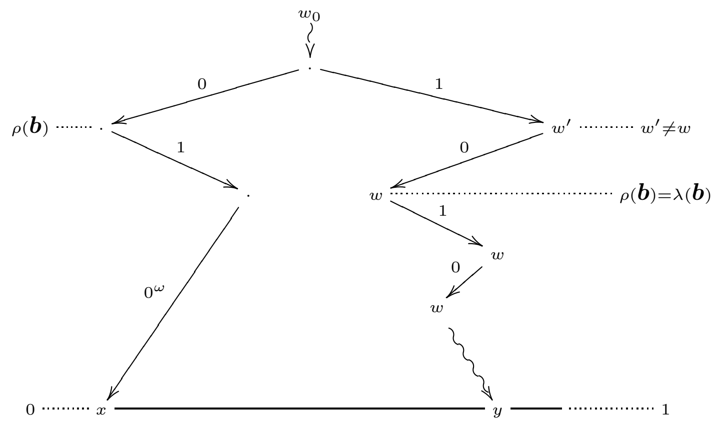

Fig. 5. The selection function ρ: Case $\boldsymbol{b} \in \mathscr{B}_1$ (left) and Case $\boldsymbol{b} \in \mathscr{B}_2$ (right)

3 Proof of completeness

Lemma 3.1 *If $\boldsymbol{b} \in \mathscr{B}_1$ with $\boldsymbol{b}(n+1) = 1$ $(n > 0)$, then $\rho(\boldsymbol{b}) \geq n$.*

Proof. Suppose that $\boldsymbol{b} = b_1 \dots b_m 10^\omega$. Since $\boldsymbol{b}(n+1) = 1$ and $n > 0$, we have $m > 0$ and $m = \rho(\boldsymbol{b})$. It follows immediately that $n \leq m$ as $\boldsymbol{b}$ has the prefix $b_1 \dots b_n 1$. □

Lemma 3.2 *If $\boldsymbol{b} \in \mathscr{B}_2$, then either $\rho(\boldsymbol{b}) = \lambda(\boldsymbol{b}) = 1$ or $\boldsymbol{b}(\lambda(\boldsymbol{b})) = \boldsymbol{b}(\rho(\boldsymbol{b})) = 1$.*

Proof. For $\boldsymbol{b} \in \mathscr{B}_2$ either $\boldsymbol{W}(\boldsymbol{b})$ stabilizes at the root, or $\boldsymbol{W}(\boldsymbol{b})$ stabilizes at point n for $n > 0$. In the former case, $\rho(\boldsymbol{b}) = \lambda(\boldsymbol{b}) = 1$. In the latter case, we must have $\boldsymbol{b}(n) = 1$. Otherwise $\boldsymbol{W}(\boldsymbol{b})$ stabilizes at point $n-1$ as $\mathcal{W}(\boldsymbol{b})(n-1) = \mathcal{W}(\boldsymbol{b})(n)$. So $\boldsymbol{b}(\lambda(\boldsymbol{b})) = \boldsymbol{b}(\rho(\boldsymbol{b})) = 1$. □

Lemma 3.3 *Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathscr{B}$ with $\rho(\boldsymbol{b}_1) = n_1$, $\rho(\boldsymbol{b}_2) = n_2$. If $n_1 \leq n_2$ and $\boldsymbol{b}_1 \equiv_{n_1} \boldsymbol{b}_2$, then $R\pi(\boldsymbol{b}_1)\pi(\boldsymbol{b}_2)$.*

Proof. Since $n_1 \leq n_2$ and $\boldsymbol{b}_1 \equiv_{n_1} \boldsymbol{b}_2$, the node $\boldsymbol{b}_2 \restriction n_2$ (labeled by $\boldsymbol{W}(\boldsymbol{b}_2)(n_2)$) is in the subtree with root $\boldsymbol{b}_1 \restriction n_1$ (labeled by $\boldsymbol{W}(\boldsymbol{b}_1)(n_1)$). So $R\boldsymbol{W}(\boldsymbol{b}_1)(n_1)\boldsymbol{W}(\boldsymbol{b}_2)(n_2)$, that is, $R\pi(\boldsymbol{b}_1)\pi(\boldsymbol{b}_2)$. □

Lemma 3.4 *Let $\boldsymbol{b}_1 \in \mathscr{B}$ with $\rho(\boldsymbol{b}_1) = n_1$, $\boldsymbol{b}_2 \in \mathscr{B}_2$. If $\boldsymbol{b}_1 \equiv_{n_1} \boldsymbol{b}_2$, then $R\pi(\boldsymbol{b}_1)\pi(\boldsymbol{b}_2)$.*

Proof. Let $\rho(\boldsymbol{b}_2) = \lambda(\boldsymbol{b}_2) = n_2$. If $n_1 \leq n_2$, then $R\pi(\boldsymbol{b}_1)\pi(\boldsymbol{b}_2)$ by Lemma 3.3. Suppose that $n_1 > n_2$. Since $\boldsymbol{b}_1 \equiv_{n_1} \boldsymbol{b}_2$ and n_2 is the stabilization point of $\boldsymbol{W}(\boldsymbol{b}_2)$, $\boldsymbol{W}(\boldsymbol{b}_1)(n_1)$ is in the final stabilization loop where $\boldsymbol{W}(\boldsymbol{b}_2)(n_2)$ belongs. So $R\boldsymbol{w}_1(n_1)\boldsymbol{w}_2(n_2)$, i.e., $R\pi(\boldsymbol{b}_1)\pi(\boldsymbol{b}_2)$. □

Lemma 3.5 *For any $x, y \in (0,1)$, if $|y - x| < 2^{-(\delta(x)+2)}$, then $R\pi(x)\pi(y)$.*

Proof. Consider all possible cases.

1. Case $B(x) \in \mathscr{B}_1$. (See Figure 6.)

   If $B(x) = 10^\omega$, then $\pi(x) = w_0$, the root of $\boldsymbol{K}$, and trivially $R\pi(x)\pi(y)$ for any $y \in (0,1)$. Suppose that $B(x) = b_1 b_2 \dots b_n 10^\omega$ for $n \geq 1$. Then $\rho(x) = \delta(x) = \delta_1(x) = n$, $|y - x| < 2^{-(n+2)}$ and $B(x)(n+1) = 1$.

   a) Case $y < x$.

      By Proposition 2.2, $B(y)$ has $b_1 b_2 \dots b_n 01$ as a prefix. So $B(x) \equiv_n B(y)$ and $B(y)(n+2) = 1$. If $B(y) \in \mathscr{B}_1$, then by Lemma 3.1 $\rho(y) = \delta_1(y) \geq n+1 > n = \rho(x)$. By Lemma 3.3 $R\pi(x)\pi(y)$. If $B(y) \in \mathscr{B}_2$, then $\rho(y) = \lambda(y)$. Since $B(x) \equiv_n B(y)$, by Lemma 3.4 $R\pi(x)\pi(y)$.

   b) Case $y > x$.

      By Proposition 2.2, $B(y)$ has $b_1 b_2 \dots b_n 10$ as a prefix. So $B(x) \equiv_{n+2} B(y)$ and $B(y)(n+1) = 1$. If $B(y) \in \mathscr{B}_1$, then again by Lemma 3.1 $\rho(y) = \delta_1(y) \geq n$ and by Lemma 3.3 $R\pi(x)\pi(y)$. If $B(y) \in \mathscr{B}_2$, then $\rho(y) = \lambda(y)$. Since $B(x) \equiv_n B(y)$, by Lemma 3.4 $R\pi(x)\pi(y)$.

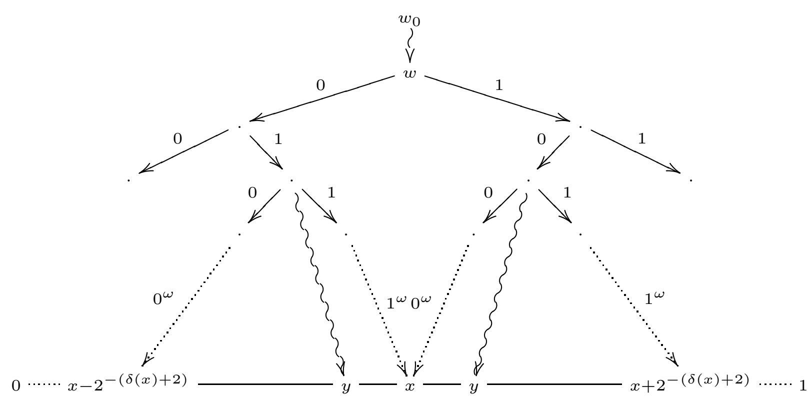

Fig. 6. Lemma 3.5 Case $B(x) \in \mathscr{B}_1$

2. Case $B(x) \in \mathscr{B}_2$. (See Figures 7, 8.)

   Let $\rho(x) = \lambda(x) = m$ and $\delta(x) = n$. If $m = 1$, then $\pi(x) = w_0$, the root of $\boldsymbol{K}$, and so $R\pi(x)\pi(y)$ for any $y \in (0,1)$. Suppose that $m > 1$. By Lemma 3.2 $B(x)(m) = 1$. By Definition 2.4 $n = \delta(x) > m = \lambda(x)$, $n \geq m+2$. Assume that

   $$B(x) \upharpoonright n = b_1 b_2 \dots b_{m-1} 1 b_{m+1} \dots b_{n-2} 01$$

   Since $|y - x| < 2^{-(n+2)}$, by Proposition 2.4

   $$B(y) \upharpoonright (n-2) = b_1 b_2 \dots b_{m-1} 1 b_{m+1} \dots b_{n-2} \quad \text{and} \quad B(y) \neq b_1 b_2 \dots b_{n-2} 0^\omega$$

a) Case $B(y) \in \mathscr{B}_1$. (See Figure 7.)

It immediately follows from the above condition that for any $i \le n-2$

$$B(y) \neq b_1 b_2 \ldots b_i 0^\omega$$

Then $n-1$ is the least value k for which it is possible to have

$$B(y) = b_1 b_2 \ldots b_k 10^\omega$$

By Proposition 3.1 $\rho(y) \ge n-1 > m$. Since $B(y) \equiv_m B(x)$ and $\rho(x) = m$, by Lemma 3.3 $R\pi(x)\pi(y)$.

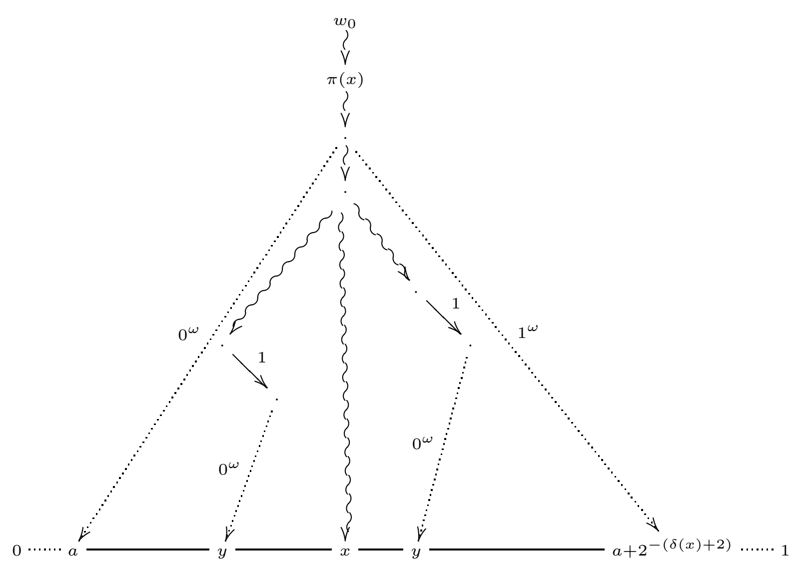

Fig. 7. Lemma 3.5 Case $B(x) \in \mathscr{B}_2$ & $B(y) \in \mathscr{B}_1$

b) Case $B(y) \in \mathscr{B}_2$. (See Figure 8.)

As $n-2 \ge m$, $B(y) \equiv_m B(x)$. Since $\rho(x) = m$, by Lemma 3.4 $R\pi(x)\pi(y)$.

□

Lemma 3.6 *For any $x \in (0,1)$, $\epsilon > 0$, $w \in W$ with $R\pi(x)w$, there exists $y \in (0,1)$ such that $|y - x| < \epsilon$ and $\pi(y) = w$.*

Proof. 1. Case $B(x) \in \mathscr{B}_2$. (See Figure 9.)

Let $m = \lambda(x)$ and take $n > m$ such that $2^{-n} < \epsilon$. Assume that

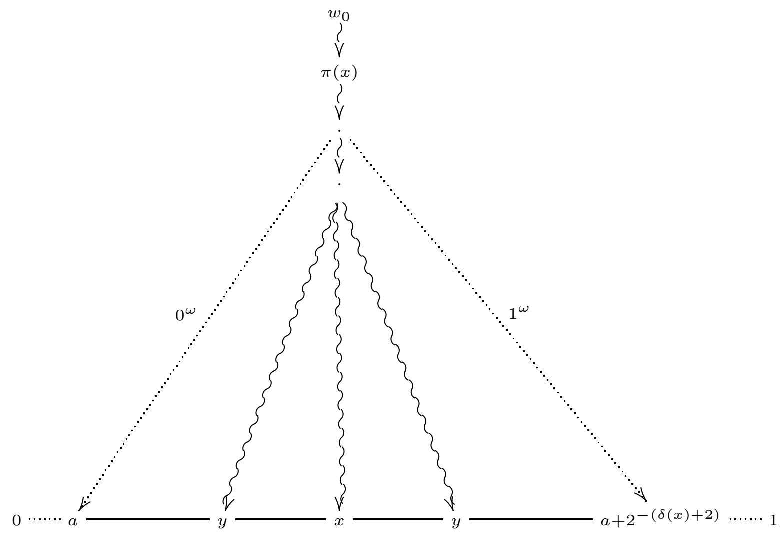

Fig. 8. Lemma 3.5 Case $B(x) \in \mathscr{B}_2$ & $B(y) \in \mathscr{B}_2$

$$B(x)\upharpoonright n = b_1 b_2 \ldots b_m \ldots b_n,\quad \mathcal{W}(b_1 \ldots b_m) = w_1,\quad \mathcal{W}(b_1 \ldots b_m \ldots b_n) = w_2$$

By the assumption $\pi(x) = w_1$ and w_1, w_2 are R-equivalent. By Proposition 2.7 for any R-successor w of w_1 (and hence of w_2) there exists $i \geq 0$ such that

$$\mathcal{W}(b_1 \ldots b_m \ldots b_n 0^i 1) = w$$

Let

$$\boldsymbol{b} = b_1 \ldots b_m \ldots b_n 0^i 110^\omega$$

Then $\boldsymbol{b} \in \mathscr{B}_1$ and $\pi(\boldsymbol{b}) = w$. By Proposition 2.3, for any $\boldsymbol{b}_1, \boldsymbol{b}_2, \in \mathscr{B}$, if $\boldsymbol{b}_1 \equiv_n \boldsymbol{b}_2$, then $|B^{-1}(\boldsymbol{b}_1) - B^{-1}(\boldsymbol{b}_2)| < 2^{-n}$. Take $y = B^{-1}(\boldsymbol{b})$, then we have $|y - x| < 2^{-n} < \epsilon$ and $\pi(y) = w$.

2. Case $B(x) \in \mathscr{B}_1$. (See Figures 10, 11.)

   Suppose that

$$B(x) = b_1 b_2 \ldots b_n 10^\omega,\quad \mathcal{W}(b_1 b_2 \ldots b_n) = w_1,\quad \mathcal{W}(b_1 b_2 \ldots b_n 1) = w_2.$$

   We have $\rho(x) = \delta_1(x) = n$, $\pi(x) = w_1$.

   a) Case $w_1 = w_2$. (See Figure 10.)

      Let $w \in W$ be an R-successor of w_2. By Proposition 2.7 there exists $m > n + 1$ such that

$$2^{-m} < \epsilon \quad \text{and} \quad \mathcal{W}(b_1 \ldots b_n 10^{m-(n+1)} 1) = w.$$

      Let

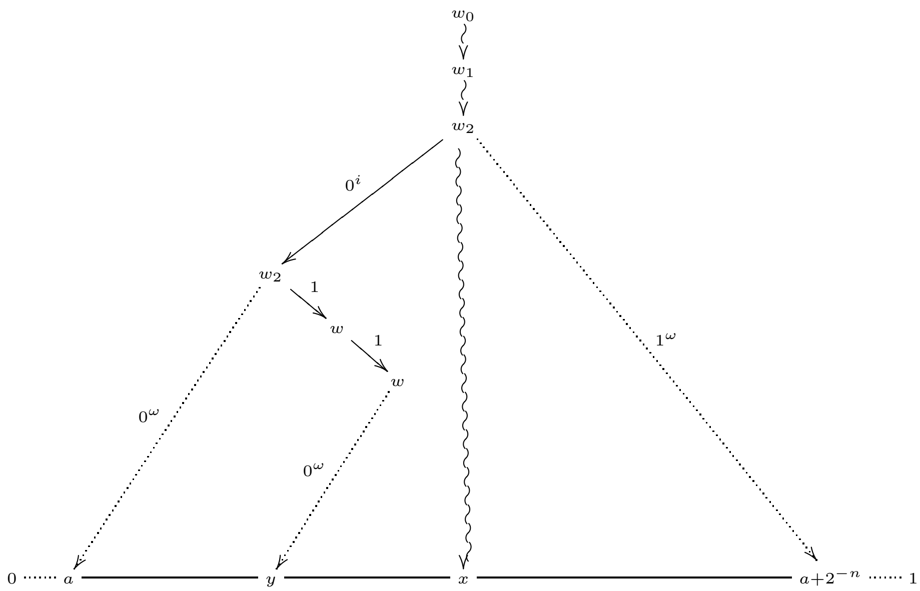

Fig. 9. Lemma 3.6 Case $B(x) \in \mathscr{B}_2$

$$\boldsymbol{b} = b_1 \ldots b_n 10^{m-(n+1)} 110^\omega$$

Then $\boldsymbol{b} \in \mathscr{B}_1$ and $\pi(\boldsymbol{b}) = w$. Now let $y = B^{-1}(\boldsymbol{b})$, so $\pi(y) = w$. Since $B(x) \equiv_m B(y)$, by Proposition 2.3, $|y - x| < 2^{-m} < \epsilon$ as desired.

b) Case $w_1 \neq w_2$. (See Figure 11.)

Let $w \in W$ with Rw_1w. By Proposition 2.6

$$\mathcal{W}(b_1 b_2 \ldots b_n 01) = w_1$$

By Proposition 2.8 we can take $m > n + 2$ such that

$$\mathcal{W}(b_1 b_2 \ldots b_n 011^{m-(n+2)}) = w_1$$

By Proposition 2.7 there exists $k > m$ such that

$$\mathcal{W}(b_1 b_2 \ldots b_n 011^{m-(n+2)} 0^{k-m} 1) = w$$

Let

$$\boldsymbol{b} = b_1 b_2 \ldots b_n 011^{m-(n+2)} 0^{k-m} 110^\omega$$

Then $\boldsymbol{b} \in \mathscr{B}_1$ and $\pi(\boldsymbol{b}) = w$. Now let $y = B^{-1}(\boldsymbol{b})$, so $\pi(y) = w$. By Proposition 2.5, $|y - x| < 2^{-m} < \epsilon$ as desired.

$\square$

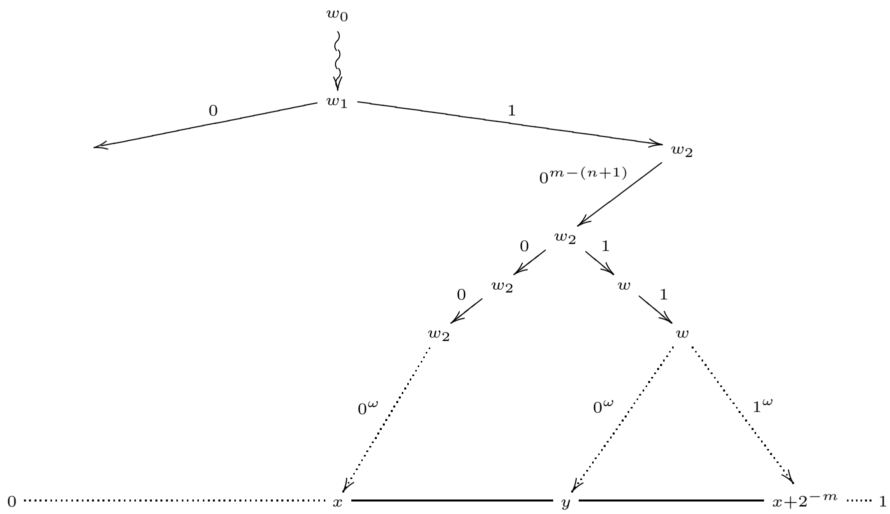

Fig. 10. Lemma 3.6 Case $B(x) \in \mathscr{B}_1$ & $w_1 = w_2$

Theorem 3.1 *The function π is an open and continuous from $\mathscr{B}$ onto the Kripke space $\mathcal{K}$.*

Proof. 1. Continuity.

Let $W_0 \subseteq W$ be an open set of the Kripke space $\mathcal{K}$ (i.e., W_0 is closed under R). For any $w \in W_0$, let $x \in \pi^{-1}(w)$, i.e., $\pi(x) = w$. Take a set $O_x = \{y \mid |x - y| < 2^{-(\delta(x)+2)}\}$. Obviously O_x is an open subset of $(0, 1)$. By Lemma 3.5 all worlds in $\pi(O_x)$ are R-successors of w. Since $w \in W_0$ and W_0 is closed under R, we have $\pi(O_x) \subseteq W_0$. Hence π is continuous.

2. Openness.

Let $\mathcal{O}_x$ be the collection of sets $O_{x,i} = \{y \mid |x - y| < 2^{-(i+\delta(x)+2)}\}$ for $i \geq 0$. Clearly $\bigcup_x \mathcal{O}_x$ is a base of the standard topology on $(0, 1)$. By Lemma 3.5 for any $w \in \pi(O_{x,i})$ we have $R\pi(x)w$. And by Lemma 3.6 for any w with $R\pi(x)w$, there exists $y \in O_{x,i}$ such that $\pi(y) = w$, that is, $w \in \pi(O_{x,i})$. Hence $\pi(O_{x,i}) = \{w \in W \mid R\pi(x)w\}$, which is obviously closed under R. Hence π is an open map.

□

Lemma 3.7 *Let X_1, X_2 be two topological spaces and $f : X_1 \to X_2$ a continuous and open map. Let V_2 be a valuation for topological semantics on X_2 and define*

$$V_1(p) = f^{-1}(V_2(p)) \tag{1}$$

for each propositional variable p. Then

$$V_1(\alpha) = f^{-1}(V_2(\alpha))$$

for any S4-formula α.

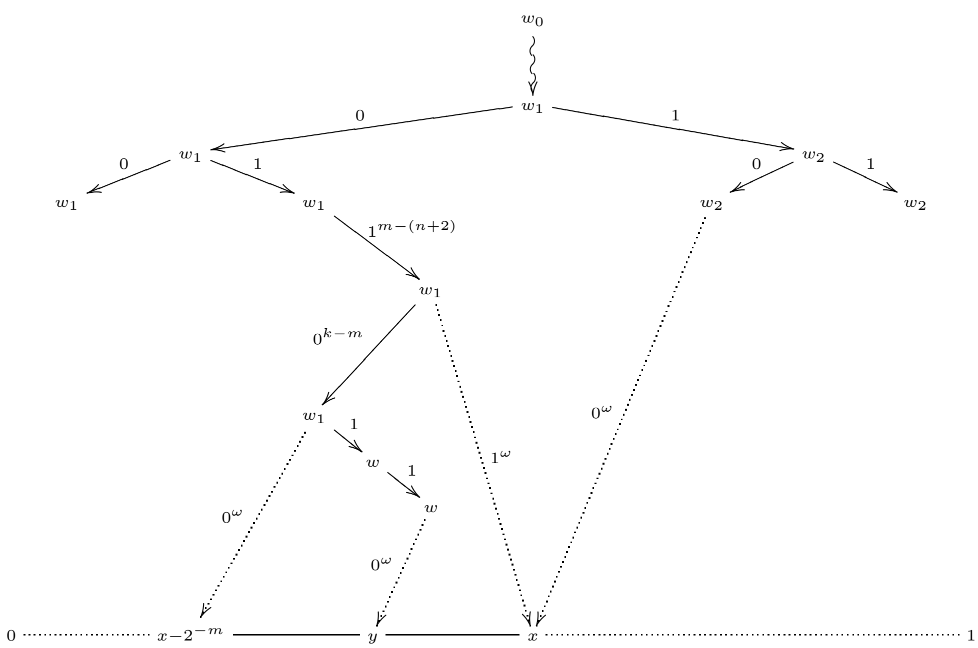

Fig. 11. Lemma 3.6 Case $B(x) \in \mathscr{B}_1$ & $w_1 \neq w_2$

Proof. The proof uses induction. The base case and induction steps for connectives $\vee, \&, \neg$ are straightforward. Now suppose $\alpha = \Box\beta$. By induction hypothesis,

$$V_1(\beta) = f^{-1}(V_2(\beta)).$$

It follows from openness and continuity that

$$Int(f^{-1}(V_2(\beta))) = f^{-1}(Int(V_2(\beta))).$$

Hence we have

$$\begin{aligned} V_1(\alpha) &= V_1(\Box\beta) = Int(V_1(\beta)) = Int(f^{-1}(V_2(\beta))) \\ &= f^{-1}(Int(V_2(\beta))) = f^{-1}(V_2(\Box\beta)) = f^{-1}(V_2(\alpha)). \end{aligned}$$

□

Lemma 3.8 *Let X_1, X_2 be two topological spaces and $f : X_1 \to X_2$ a continuous and open map. Let V_2 be a valuation for topological semantics on X_2 and define V_1 by the equation* (1). *Then for any S4-formula α,*

$$\langle X_2, V_2\rangle \models \alpha \quad \textit{implies} \quad \langle X_1, V_1\rangle \models \alpha.$$

Moreover if f is onto, then

$$\langle X_2, V_2\rangle \models \alpha \textit{ iff } \langle X_1, V_1\rangle \models \alpha.$$

Proof. Suppose $\langle X_2, V_2\rangle \models \alpha$, that is, $V_2(\alpha) = X_2$. By Lemma 3.7 $V_1(\alpha) = f^{-1}(V_2(\alpha))$, and so $V_1(\alpha) = X_1$ as required. Now suppose that f is onto and $\langle X_1, V_1\rangle \models \alpha$, but $\langle X_2, V_2\rangle \not\models \alpha$, i.e., $V_2(\alpha) \neq X_2$. Since f is onto and $V_1(\alpha) = f^{-1}(V_2(\alpha))$, we have $V_1(\alpha) \neq X_1$, that is, $\langle X_1, V_1\rangle \not\models \alpha$, a contradiction. □

Theorem 3.2 *$S4$ is complete for the standard topology on* $(0, 1)$.

Proof. It suffices to show that every non-theorem of $S4$ can be refuted on $(0, 1)$. Let α be such an non-theorem. We need to find a valuation V such that $V(\alpha) \neq (0, 1)$. By Theorem 1.1 there exists a finite rooted Kripke model $\boldsymbol{K} = \langle X, V'\rangle$ such that $\boldsymbol{K} \not\models \alpha$. By Theorem 3.1, we have a continuous and open map π from $(0, 1)$ onto $\boldsymbol{K}$. Let V be the valuation on $(0, 1)$ such that

$$V(p) = \pi^{-1}(V'(p))$$

for every propositional variable p. By Lemma 3.8 $V'(\beta) = X$ if and only if $V(\beta) = (0, 1)$ for any $S4$-formula β. In particular since $V'(\alpha) \neq X$, $V(\alpha) \neq (0, 1)$. It follows that $S4$ is complete for $(0, 1)$. □

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