High order family of multivariate iterative methods: convergence and stability⋆

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Abstract

In this manuscript, we design an efficient sixth-order scheme for solving nonlinear systems of equations, with only two steps in its iterative expression. Moreover, it belongs to a new parametric class of methods whose order of convergence is, at least, four. In this family, the most stable members have been selected by using techniques of real multidimensional dynamics; also, some members with undesirable chaotic behavior have been found and rejected for practical purposes. Finally, all these high-order schemes have been numerically checked and compared with other existing procedures of the same order of convergence, showing good and stable performance.

Keywords: Nonlinear systems of equations; iterative methods; acceleration of convergence; real multidimensional dynamics.

1. Introduction

New and efficient solution techniques are needed for obtaining the roots of a system of nonlinear equations of the form

$$F(x) = 0, \tag{1}$$

where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, which partake of scientific, engineering and various other models. One of the best known one-point optimal method is the classical Newton's method [17]. Many other schemes have been developed which improve the convergence rate of the Newton method at the expense of additional evaluations of the nonlinear function F, its associate Jacobian matrix F' and more steps in the iterative expression. All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices.

With the advances of computer algebra, a number of sixth-order methods are also appearing as extensions of Newton's method or Newton-like schemes (see for example [11, 16, 10, 12, 1, 13, 18]). In 2014, Sharma and Arora [16], suggested a new efficient Jarratt-like method for solving nonlinear systems, which is defined as follows:

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \left[\frac{23}{8}I - F'(x^{(k)})^{-1}F'(y^{(k)})\Big(3I - \frac{9}{8}F'(x^{(k)})^{-1}F'(y^{(k)})\Big)\right]F'(x^{(k)})^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left(\frac{5}{2}I - \frac{3}{2}F'(x^{(k)})^{-1}F'(y^{(k)})\right)F'(x^{(k)})^{-1}F(z^{(k)}).
\end{aligned} \tag{2}$$

On the other hand, in 2015 Hueso et al. [10] and Lotfi [12] proposed three point sixth-order methods, which are given by

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - G_s(x^{(k)}, y^{(k)})F'(x^{(k)})^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - G_t(x^{(k)}, y^{(k)})F'(y^{(k)})^{-1}F(z^{(k)}),
\end{aligned} \tag{3}$$

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and

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - 2\Big(F'(x^{(k)}) + F'(y^{(k)})\Big)^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left(\frac{7}{2}I - 4F'(x^{(k)})^{-1}F'(y^{(k)}) + \frac{3}{2}\big(F'(x^{(k)})^{-1}F'(y^{(k)})\big)^2\right)F'(x^{(k)})^{-1}F(z^{(k)}),
\end{aligned}
\tag{4}
$$

where $H(x^{(k)}, y^{(k)}) = F'(x^{(k)})^{-1}F'(y^{(k)})$, $G_s(x^{(k)}, y^{(k)}) = s_1 I + s_2 H(y^{(k)}, x^{(k)}) + s_3 H(x^{(k)}, y^{(k)}) + s_4 H(y^{(k)}, x^{(k)})^2$, $G_t(x^{(k)}, y^{(k)}) = t_1 I + t_2 H(x^{(k)}, y^{(k)}) + t_3 H(y^{(k)}, x^{(k)}) + t_4 H(x^{(k)}, y^{(k)})^2$ and the values of disposable parameters given in [10].

Recently in 2016, Abbasbandy et al. [1] and Narang et al. [13], presented new efficient three-points methods for solving nonlinear systems

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \Big[I + \frac{21}{8}\mu - \frac{9}{2}\mu^2 + \frac{15}{8}\mu^3\Big]F'(x^{(k)})^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \Big(3I - \frac{5}{2}\mu + \frac{1}{2}\mu^2\Big)F'(x^{(k)})^{-1}F(z^{(k)}),
\end{aligned}
\tag{5}
$$

being $\mu = F'(x^{(k)})^{-1}F'(y^{(k)})$ and

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \left[I + \frac{1}{2\beta}\left(I - \frac{\lambda}{\beta}G(x^{(k)})\right)^{-1}G(x^{(k)})\right]H\big(G(x^{(k)})\big)u(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \Big(pI + qG(x^{(k)})\Big)F'(x^{(k)})^{-1}F(z^{(k)}),
\end{aligned}
\tag{6}
$$

where $G(x^{(k)}) = I - F'(x^{(k)})^{-1}F'(y^{(k)})$, $H\big(G(x^{(k)})\big) = I + \frac{3\beta-2}{4\beta}G(x^{(k)}) + \frac{9\beta^2-3\beta-4\lambda+2}{8\beta^2}G(x^{(k)})^2$, $u(x^{(k)}) = F'(x^{(k)})^{-1}F(x^{(k)})$, $p = 1$ and $q = \frac{3}{2}$. Scheme (6) has sixth-order of convergence for all $\lambda, \beta \in \mathbb{R}$ except $\beta = 0$.

More recently, in 2017, Wang and Li in [18] have proposed an efficient three-point sixth-order method, which is defined as follows:

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - \Big(2I - F'(x^{(k)})^{-1}F'(y^{(k)})\Big)F'(x^{(k)})^{-1}F(y^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \Big(2I - F'(x^{(k)})^{-1}F'(y^{(k)})\Big)F'(x^{(k)})^{-1}F(z^{(k)}).
\end{aligned}
\tag{7}
$$

Obtaining new two-point methods of order six is a very important and interesting task from the practical point of view. In this way, we can attain sixth-order convergence with two steps per iteration, instead of three ones. Motivated and inspired in this way, we propose a two-point sixth-order family of schemes having simpler body structure than the existing three-point families of sixth-order methods. Further, our class of iterative schemes is more competent in all the tested examples than the existing three-point sixth-order methods available in the literature. The dynamic study of these methods supports the theoretical aspects and show the dependence on initial estimations of the different members of the new family.

The following section of this manuscript is devoted to the design of the class of iterative methods for solving nonlinear systems and shows the analysis of its convergence. Section 3 deals with the study of the stability of the proposed family by using techniques of real multidimensional discrete dynamics. The class is found to be very stable and some elements with particular good performance are obtained. They are used in the numerical Section 4 and their performance is compared with those of existing methods.

2. Development of the scheme for multi-dimensional case

In this section, we propose a new two-point high-order family, which is given by

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}),\\
x^{(k+1)} &= y^{(k)} - \Big(\alpha I + \beta\mu^{(k)} + \gamma{\mu^{(k)}}^2 + \mu^{(k)}(\mu^{(k)} - I)(I - \eta^{(k)})\Big)F'(y^{(k)})^{-1}F(y^{(k)}),
\end{aligned}
\tag{8}
$$

where α, β and γ are free disposable parameters with $\mu^{(k)} = F'(x^{(k)})^{-1}F'(y^{(k)})$ and $\eta^{(k)} = F'(x^{(k)})^{-1}[y^{(k)}, x^{(k)}; F]$, being I the identity matrix of size $n \times n$. In addition, $[y^{(k)}, x^{(k)}; F]$ is a finite difference of order one.

In Theorem (1), we demonstrate the fourth-convergence order of the class of iterative schemes (8), which has a member of order six. In the proof of this result, we use the tools and procedure introduced in [3] which we recall briefly.

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently differentiable in D. The qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \ldots, v_q) \in \mathbb{R}^n$. It is easy to observe that

1. $F^{(q)}(u)(v_1, \ldots, v_{q-1}, \cdot) \in \mathcal{L}(\mathbb{R}^n)$,
2. $F^{(q)}(u)(v_{\sigma(1)}, \ldots, v_{\sigma(q)}) = F^{(q)}(u)(v_1, \ldots, v_q)$, for all permutation σ of $\{1, 2, \ldots, q\}$.

From the above properties we can use the following notation, where $v^q = (v, v, \ldots, v)$, q-times :

(a) $F^{(q)}(u)(v_1, \ldots, v_q) = F^{(q)}(u)v_1 \ldots v_q$,

(b) The element of $\mathcal{L}(\mathbb{R}^n)$, $F^{(q)}(u)v^{q-1}$ acting on $F^{(p)}v^p$ are denoted by

$$F^{(q)}(u)v^{q-1}F^{(p)}v^p = F^{(q)}(u)F^{(p)}(u)v^{q+p-1}.$$

On the other hand, for $\xi^* + h \in \mathbb{R}^n$ lying in a neighborhood of a solution ξ^* of $F(x) = 0$, we can apply Taylor's expansion and assuming that the Jacobian matrix $F'(\xi^*)$ is nonsingular, we have

$$F(\xi^* + h) = F'(\xi^*)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p), \tag{9}$$

where $C_q = (1/q!)[F'(\xi^*)]^{-1}F^{(q)}(\xi^*)$, $q \geq 2$. We observe that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\xi^*) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\xi^*)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$.

In addition, we can express F' as

$$F'(\xi^* + h) = F'(\xi^*)\left[I + \sum_{q=2}^{p-2} qC_q h^{q-1}\right] + O(h^{p-1}), \tag{10}$$

where I is the identity matrix. Therefore, $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. From (10), we obtain

$$[F'(\xi^* + h)]^{-1} = \left[I + X_2 h + X_3 h^2 + X_4 h^4 + \cdots\right][F'(\xi^*)]^{-1} + O(h^p), \tag{11}$$

where

$$\begin{aligned} X_2 &= -2C_2, \\ X_3 &= 4C_2^2 - 3C_3, \\ X_4 &= -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4, \\ &\vdots \end{aligned}$$

We denote $e^{(k)} = x^{(k)} - \xi^*$ the error in the kth-iteration in the multidimensional case. The equation

$$e^{(k+1)} = Me^{(k)^p} + O(e^{(k)^{p+1}})$$

where M is a p-linear function $M \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$, is called the *error equation* and p is the *order of convergence*. Let us observe that $e^{(k)^p}$ is $(e^{(k)}, e^{(k)}, \cdots, e^{(k)})$.

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an open neighborhood D of its zero ξ^*. Let us consider that $F'(x)$ is continuous and nonsingular in ξ^*. In addition, we assume the initial guess $x^{(0)}$ is close enough to ξ^* for the guaranteed convergence. Then, the parametric iterative schemes of class (8) satisfying*

$$\beta = 2 - 2\alpha \ \text{ and } \ \gamma = \alpha - 1, \tag{12}$$

have fourth-order of convergence. In particular, the method resulting for $\alpha = \frac{7}{4}$ has sixth-order of convergence, being its error equation

$$e^{(k+1)} = \left(14C_2^5 - 7C_3C_2^3 + C_4C_2^2 - \frac{3}{4}C_3^2C_2\right)e^{(k)^6} + O(e^{(k)^7}).$$

Proof. Let $e^{(k)} = x^{(k)} - \xi^*$ be the error of the kth-iteration. Developing $F(x^{(k)})$ and $F'(x^{(k)})$ in a neighborhood of ξ^*, we write

$$F(x^{(k)}) = F'(\xi^*)\left[e^{(k)} + C_2{e^{(k)}}^2 + C_3{e^{(k)}}^3 + C_4{e^{(k)}}^4 + C_5{e^{(k)}}^5 + C_6{e^{(k)}}^6\right] + O({e^{(k)}}^7) \tag{13}$$

and

$$F'(x^{(k)}) = F'(\xi^*)\left[I + 2C_2e^{(k)} + 3C_3{e^{(k)}}^2 + 4C_4{e^{(k)}}^3 + 5C_5{e^{(k)}}^4 + 6C_6{e^{(k)}}^5\right] + O({e^{(k)}}^6), \tag{14}$$

where I is the identity matrix of size $n \times n$ and $C_k = \frac{1}{k!}F'(\xi^*)^{-1}F^{(k)}(\xi^*)$, $k \geq 2$.

With the help of above expression (14), we have

$$F'(x^{(k)})^{-1} = \left[I - 2C_2e^{(k)} + \Omega_0{e^{(k)}}^2 + \Omega_1{e^{(k)}}^3 + \Omega_2{e^{(k)}}^4 + \Omega_3{e^{(k)}}^5 + \Omega_4{e^{(k)}}^6 + O({e^{(k)}}^7)\right]F'(\xi^*)^{-1} \tag{15}$$

where $\Omega_i = \Omega_i(C_2, C_3, ..., C_6)$, for example $\Omega_0 = 4C_2^2 - 3C_3,\ \Omega_1 = -(8C_2^3 - 6C_2C_3 - 6C_3C2 + 4C_4),\ \Omega_2 = 8C_2C_4 + 9C_3^2 + 8C_4C_2 - 12C_2^2C_3 - 12C_2C_3C_2 - 12C_3C_2^2 + 16C_2^4 - 5C_5,\ \Omega_3 = 10C_2C_5 + 12C_3C_4 + 12C_4C_3 + 10C_5C_2 - 16C_2^2C_4 - 18C_2C_3^2 - 16C_2C_4C_2 - 18C_3C_2C_3 - 18C_3^2C_2 - 16C_4C_2^2 + 24C_2^3C_3 + 24C_2^2C_3C_2 + 24C_2C_3C_2^2 + 24C_3C_2^3 - 32C_2^5 - 6C_6$, etc.

From expressions (13) and (15), we further yields

$$F'(x^{(k)})^{-1}F(x^{(k)}) = e^{(k)} + \Delta_0{e^{(k)}}^2 + \Delta_1{e^{(k)}}^3 + \Delta_2{e^{(k)}}^4 + \Delta_3{e^{(k)}}^5 + \Delta_4{e^{(k)}}^6 + O({e^{(k)}}^7), \tag{16}$$

where $\Delta_j = \Delta_j(C_2, C_3, ..., C_6)$, for example $\Delta_0 = -C_2,\ \Delta_1 = 2C_2^2 - 2C_3,\ \Delta_2 = -(4C_2^3 - 4C_2C_3 - 3C_3C_2 + 3C_4),\ \Delta_3 = 6C_2C_4 + 6C_3^2 + 4C_4C_2 - 8C_2^2C_3 - 6C_2C_3C_2 - 6C_3C_2^2 + 8C_2^4 - 4C_5,\ \Delta_4 = 8C_2C_5 + 9C_3C_4 + 8C_4C_3 + 5C_5C_2 - 12C_2^2C_4 - 12C_2C_3^2 - 8C_2C_4C_2 - 12C_3C_2C_3 - 9C_3^2C_2 - 8C_4C_2^2 + 16C_2^3C_3 + 12C_2^2C_3C_2 + 12C_2C_3C_2^2 + 12C_3C_2^3 - 16C_2^5 - 5C_6$, etc.

By inserting the above expression (16) in the first substep of (8), we obtain

$$y^{(k)} - \xi^* = -\Delta_0{e^{(k)}}^2 - \Delta_1{e^{(k)}}^3 - \Delta_2{e^{(k)}}^4 - \Delta_3{e^{(k)}}^5 - \Delta_4{e^{(k)}}^6 + O({e^{(k)}}^7), \tag{17}$$

which further produce

$$\begin{aligned} F(y^{(k)}) =& F'(\xi^*)\Big[-\Delta_0{e^{(k)}}^2 - \Delta_1{e^{(k)}}^3 + (C_2\Delta_0^2 - \Delta_2){e^{(k)}}^4 + (2C_2\Delta_0\Delta_1 - \Delta_3){e^{(k)}}^5 + \\ &- \left(C_3\Delta_0^3 - C_2\left(\Delta_1^2 + 2\Delta_0\Delta_2\right) + \Delta_4\right){e^{(k)}}^6 + O({e^{(k)}}^7)\Big] \end{aligned} \tag{18}$$

and

$$\begin{aligned} F'(y^{(k)}) =& F'(\xi^*)\Big[1 - 2C_2\Delta_0 - 2C_2\Delta_1{e^{(k)}}^3 + \left(3C_3\Delta_0^2 - 2C_2\Delta_2\right){e^{(k)}}^4 + \left(6C_3\Delta_0\Delta_1 - 2C_2\Delta_3\right){e^{(k)}}^5 \\ &+ \left(-4C_4\Delta_0^3 + 3C_3\left(\Delta_1^2 + 2\Delta_0\Delta_2\right) - 2C_2\Delta_4\right){e^{(k)}}^6 + O({e^{(k)}}^7)\Big]. \end{aligned} \tag{19}$$

From expressions (18) and (19), we further obtain

$$F'(y^{(k)})^{-1}F(y^{(k)}) = -\Delta_0{e^{(k)}}^2 - \Delta_1{e^{(k)}}^3 - \left(C_2\Delta_0^2 + \Delta_2\right){e^{(k)}}^4 - \left(2C_2\Delta_0\Delta_1 + \Delta_3\right){e^{(k)}}^5 + O({e^{(k)}}^6), \tag{20}$$

$$\begin{aligned} \left[y^{(k)}, x^{(k)}; F\right] =& 1 + C_2e^{(k)} + \left(C_3 - C_2\Delta_0\right){e^{(k)}}^2 - \left(C_3\Delta_0 + C_2\Delta_1 - C_4\right){e^{(k)}}^3 \\ &+ \left(C_3\Delta_0^2 - C_4\Delta_0 - C_3\Delta_1 - C_2\Delta_2 + C_5\right){e^{(k)}}^4 + O({e^{(k)}}^5). \end{aligned} \tag{21}$$

By using the expressions (13) – (21) in the scheme (8), we have

$$e^{(k+1)} - \xi^* = \Theta_0{e^{(k)}}^2 + \Theta_1{e^{(k)}}^3 + \Theta_2{e^{(k)}}^4 + \Theta_3{e^{(k)}}^5 + \Theta_4{e^{(k)}}^6 + O({e^{(k)}}^7). \tag{22}$$

where $\Theta_i = \Theta_i(\Delta_0, \Delta_1, \ldots, \Delta_4, C_2, C_3, \ldots, C_6, \alpha, \beta, \gamma)$, for example $\Theta_0 = (\alpha + \beta + \gamma - 1)\Delta_0$, $\Theta_1 = (\alpha + \beta + \gamma - 1)\Delta_1 - 2(\beta + 2\gamma)C_2\Delta_0$, etc.

From equation (22), it is clear that by substituting the following value

$$\gamma = 1 - \alpha - \beta, \tag{23}$$

we get at least third-order convergence.

Further, by using (23) into $\Theta_1 = 0$, we have

$$2\alpha + \beta - 2 = 0,$$

which further yields

$$\beta = 2 - 2\alpha. \tag{24}$$

By using the values of Δ_0, Δ_1 and equations (23) and (24) into (22), we get the fourth-order of the family,

$$e^{(k+1)} = (7-4\alpha)C_2^3 {e^{(k)}}^4 + (4\alpha - 7)C_2^2\left(8C_2^2 - 5C_3\right){e^{(k)}}^5 + \bar{\Theta}_4 {e^{(k)}}^6 + O({e^{(k)}}^7). \tag{25}$$

It is clear from the above expression (25) that we have sixth-order convergence when we choose

$$\alpha = \frac{7}{4}. \tag{26}$$

Finally, by adopting the expressions (23), (24) and (26) into (25), we obtain the following error equation

$$e^{(k+1)} = \frac{1}{4}C_2\left(-4C_2\left(2C_4 + \Delta_2\right) + 40C_2^4 - 3C_3^2\right){e^{(k)}}^6 + O({e^{(k)}}^7). \tag{27}$$

Therefore, the particular scheme of (8) has sixth-order convergence with two substeps. ■

3. Stability analysis

Now, let us recall some concepts about real multidimensional discrete dynamics that we use along this manuscript.

Let us denote with $G(x)$ the vectorial fixed-point rational function associated to the iterative method applied to n-variable polynomial system $p(x) = 0$, $p : \mathbb{R}^n \to \mathbb{R}^n$. Most of the following concepts are direct extensions of those considered in complex dynamics (see, for example, [4, 15] for more explanations).

The orbit of $x^{(0)} \in \mathbb{R}^n$ is defined as $\left\{x^{(0)}, G(x^{(0)}), \ldots, G^m(x^{(0)}), \ldots\right\}$. A point $x^* \in \mathbb{R}^n$ is a fixed point of G if $G(x^*) = x^*$, and it is called strange fixed point when it is not a root of $p(x) = 0$. The stability of the fixed points is characterized in a result by Robinson ([15], page 558), stating that the character of a k-periodic point x^* depends on the eigenvalues of $G'(x^*)$, $\lambda_1, \lambda_2, \ldots, \lambda_n$. It is attracting if all $|\lambda_j| < 1$, repelling if all $|\lambda_j| > 1$ ($j = 1, 2, \ldots, n$)), and unstable or saddle if at least one λ_{j_0} exists such that $|\lambda_{j_0}| > 1$. In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$.

In the following, we denote by $G(x, \alpha) = (g_1(x, \alpha), g_2(x, \alpha), \ldots, g_n(x, \alpha))$, the fixed point function associated to (8) class, applied on n-dimensional quadratic polynomial $p(x) = 0$, where:

$$p_i(x) = x_i^2 - 1,\ i = 1, 2, \ldots, n. \tag{28}$$

As the polynomial system has separated variables, all the coordinate functions $g_j(x, \alpha)$ have the same expression, with the only difference of the sub-index $j = 1, 2, \ldots, n$. These coordinate functions of the multidimensional rational operator can be expressed as:

$$g_j(x, \alpha) \;=\; \frac{(21-4\alpha)x_j^{10} + (16\alpha + 77)x_j^8 - 6(4\alpha - 7)x_j^6 + 2(8\alpha - 7)x_j^4 + (1 - 4\alpha)x_j^2 + 1}{64x_j^7\left(x_j^2 + 1\right)}, \tag{29}$$

for $j = 1, 2, \ldots, n$.

All the information about the stability of these fixed points appears in the next result.

Theorem 2. *The rational function $G(x, \alpha)$ associated to the family of iterative methods (8) has 2^n superattracting fixed points whose components are roots of $p(x)$. This operator also has a different number of real strange fixed points whose components are found combining the roots of polynomial $q(t) = t^8(4\alpha+43) + t^6(30-12\alpha) + t^4(12\alpha-12) + t^2(2-4\alpha) + 1$ depending on α, and the roots of $p(x)$:*

(a) If $\alpha < -\frac{43}{4}$, two roots of polynomial $q(t)$, denoted by $q_i(\alpha)$, $i = 1, 2$ are real, being their respective eigenvalues of the associate Jacobian matrix greater than one (in absolute value). So, the strange fixed point expressed as $(q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_n})$ being $\sigma_i \in \{1, 2\}$, are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) are equal to 1 or -1, it will be a saddle fixed point.

(b) If $-\frac{43}{4} \leq \alpha < t^*$*, being* $t^* \approx 2.99238$ *the biggest real root of polynomial* $1225 - 3390t + 2964t^2 - 1304t^3 + 216t^4$*, then the roots of polynomial* $q(t)$ *are complex and there not exist any real strange fixed point.*

(c) If $\alpha = t^*$*, then* $q_1(\alpha)$ *and* $q_3(\alpha)$ *are real. Moreover, the strange fixed point* $(q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_n})$ *being* $\sigma_i \in \{1, 3\}$*, are non-hyperbolic as all the eigenvalues of the respective Jacobian matrices are equal to one in absolute value. Indeed, if at least one of the components of the strange fixed point (but not all) are equal to* ± 1*, it will be a saddle fixed point.*

(d) Finally, if $\alpha \geq t^*$*, then four roots of* $q(t)$ *are real, denoted by* $q_i(\alpha)$*,* $i = 1, 2, 3, 4$*, and the strange fixed points* $(q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_n})$ *being* $\sigma_i \in \{1, 2, 3, 4\}$ *are all repulsive. If any of the components (but not all) of the strange fixed point is equal to* ± 1*, then it would be a saddle point.*

Proof. The fixed points of the multidimensional rational function are obtained by solving $g_j(x, \alpha) = x_j$, :

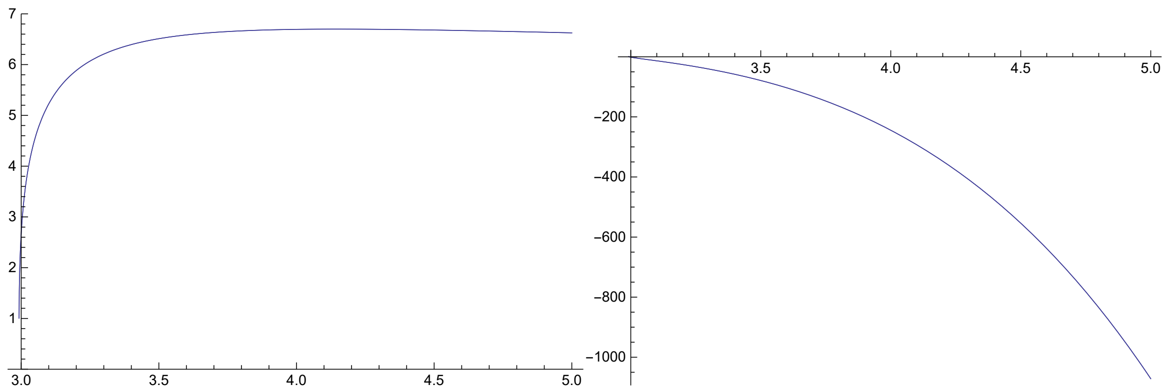
$$-(x_j^2 - 1)(1 + x_j^2(2 - 4\alpha) + 12x_j^4(-1 + \alpha) - 6x_j^6(-5 + 2\alpha) + x_j^8(43 + 4\alpha)) = 0, \;\; j = 1, 2, \ldots, n,$$

i.e., the components of the fixed points are those of the roots of $p(x)$, $x_j = \pm 1$ and also the roots of the polynomial $q(t) = t^8(4\alpha + 43) + t^6(30 - 12\alpha) + t^4(12\alpha - 12) + t^2(2 - 4\alpha) + 1$, provided that $t \neq 0$.

Let us denote the roots of $q(t)$ as $q_i(\alpha)$, $i = 1, 2, \ldots, 8$. As most four of the roots of $q(t)$ are real, depending on the value of α. The stability of the fixed points of $G(x, \alpha)$ is given by the absolute value of the eigenvalues of the associated Jacobian matrix evaluated at the fixed points. Due to the nature of the polynomial system, these eigenvalues coincide with the coordinate function of the rational operator:

$$Eig_j(q_j(\alpha)) = -\frac{\left(q_j(\alpha)^2 - 1\right)^3 \left((4\alpha - 21)q_j(\alpha)^6 + (40\alpha - 49)q_j(\alpha)^4 + 5(4\alpha - 7)q_j(\alpha)^2 - 7\right)}{64q_j(\alpha)^8 \left(q_j(\alpha)^2 + 1\right)^2}.$$

By taking into account the absolute value of this eigenvalues in the intervals where different fixed points are real, we state that those fixed points whose components are ± 1 are superattracting. As the roots of $q(t)$ are real for α not in the interval $]-\frac{43}{4}, t^*[$, being $t^* \approx 2.99238$ the biggest real root of polynomial $1225 - 3390t + 2964t^2 - 1304t^3 + 216t^4$, then it can be checked that the respective eigenvalues are bigger than one in absolute value in all cases (see Figure 1), except in case $\alpha = t^*$, where they are equal to one. Therefore, all combinations among these roots of $q(t)$ give rise to repulsive strange fixed points (non-hyperbolic in case $\alpha = t^*$). Moreover, all the fixed points whose components are ± 1 and real $q_j(\alpha)$ are classified as saddle.



(a) $Eig_j(q_1(\alpha), q_4(\alpha), \alpha)$ (b) $Eig_j(q_2(\alpha), q_3(\alpha), \alpha)$

Figure 1: Stability of the fixed points for $\alpha > t^*$ for $j = 1, 2, \ldots, n$

■

3.1. Critical points and bifurcation diagrams

Firstly, we analyze the Jacobian matrix $G'(x, \alpha)$ of the rational function $G(x, \alpha)$ and its critical points. Let us recall that, in this context, the critical points are the solutions of $\det G'(x, \alpha) = 0$. When the critical point is not a solution of $p(x) = 0$, it is called free critical point.

Theorem 3. *The free critical points of operator $G(x,\alpha)$, $(c_{\sigma_1}(\alpha), c_{\sigma_n}(\alpha), \ldots, c_{\sigma_n}(\alpha))$, $\sigma_i \in \{1, 2, \ldots, m\}$ $m \leq 4$, are the only ones making null all the partial derivatives of $g_j(x,\alpha)$, for $j = 1, 2, \ldots, n$, whose components are different from those of the roots of $p(x)$, that is:*

(a) If $\alpha \leq \frac{7}{4}$, there are no real free critical point.

(b) If $\frac{7}{4} < \alpha < \frac{21}{4}$, $c_1(\alpha) = -\sqrt{r^*}$, $c_2(\alpha) = \sqrt{r^*}$, $c_3(\alpha) = -\sqrt{r^{**}}$,$c_4(\alpha) = \sqrt{r^{**}}$ are the different components of the free critical points, being r^* and r^{**} the two positive roots of polynomial $-7 + (-35 + 20\alpha)t + (-49 + 40\alpha)t^2 + (-21 + 4\alpha)t^3$.

(c) If $\alpha = \frac{21}{4}$, the free critical points have as only different components $-\sqrt{\frac{1}{23}\left(4\sqrt{3} - 5\right)}$ and $\sqrt{\frac{1}{23}\left(4\sqrt{3} - 5\right)}$.

(d) Finally, if $\alpha > \frac{21}{4}$, $c_1(\alpha) = -\sqrt{r^\star}$, $c_2(\alpha) = \sqrt{r^\star}$ are the different components of the free critical points, being $r^\star$ the only positive root of polynomial $-7 + (-35 + 20\alpha)t + (-49 + 40\alpha)t^2 + (-21 + 4\alpha)t^3$ in this interval.

Proof. The no null entries of the Jacobian matrix $G'(x,\alpha)$ are

$$\frac{\partial g_j(x,\alpha)}{\partial x_j} = -\frac{\left(x_j^2 - 1\right)^3\left((4\alpha - 21)x_j^6 + (40\alpha - 49)x_j^4 + 5(4\alpha - 7)x_j^2 - 7\right)}{64x_j^8\left(x_j^2 + 1\right)^2}, \quad j = 1, 2.$$

It is straightforward that the roots of polynomial $(4\alpha - 21)x_j^6 + (40\alpha - 49)x_j^4 + 5(4\alpha - 7)x_j^2 - 7$, when they are real, are the components of the critical points. ■

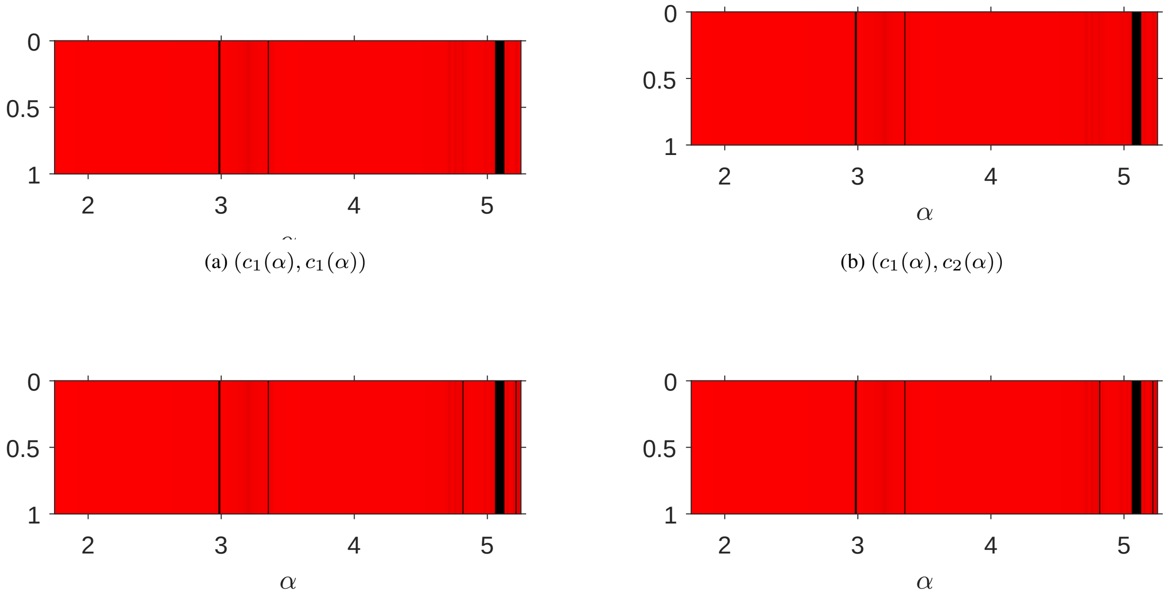
The importance of knowing the critical points is in a classical result from Julia and Fatou (see, for example, [7]), stating that in the immediate basin of attraction of any attracting point (fixed or periodic) there exist at least one critical point. So, the existence of these free critical points states the possibility of another attracting behavior different from that of the roots.

By using the information that gives us the iteration of the operator $G(x,\alpha)$ on the different free critical points, we can deduce the global behavior of the family depending on the value of the parameter. When the complex dynamics of iterative methods on polynomials is analyzed, the parameter plane plays an important role (see, for example, [6] or [2]). We propose in Figures 2 to 3 a real parametric plot to show the orbits of each one of the free critical points (see Theorem 3) for $n = 2$. In each one of these pictures, a different free critical point is used as starting point of each member of the family of iterative schemes, taking values of the parameter α in $I_1 = \left]\frac{7}{4}, \frac{21}{4}\right[$ and also in $I_2 = \left]\frac{21}{4}, 10\right[$ (where the components of the critical points are real, see Theorem 3). To plot these parameter lines, a mesh of 500×500 points is made in the region $[0, 1] \times I_i$, $i = 1, 2$. The unit interval correspond to the intention of fatten the interval where parameter α is defined, allowing a better visualization. So, the color corresponding to each value of α is red if the corresponding critical point converges to one of the roots of the polynomial system, blue in case of divergence and black in other cases. This color is also assigned to all the values of $[0, 1]$ with the same value of the parameter. The maximum number of iterations used is 200 and the tolerance for the error estimation is 10^{-3}, when the iterates tend to a fixed point.

We consider that this way to visualize the behavior of the critical points is more similar to that used in complex dynamics and provides useful information in a more friendly way as that used in [4]. Both techniques are the only ones defined until now (as far as the authors know) to manage the information provided by critical points in real multidimensional dynamics.

In Figures 2 and 3, the parameter lines of all the pairs of free critical points with different or equal components are presented, where $\frac{7}{4} < \alpha < \frac{21}{4}$ (due to the symmetry of the polynomial system). For values of parameter $\alpha > \frac{21}{4}$, there only exist two possible different components for free critical points (see Theorem 3). So, other parameter plots can be obtained by combining them in pairs; they are shown in Figure 4. Let us notice that in all cases the most usual behavior of the free critical points is the convergence to any of the four roots. No divergence behavior is observed (blue color). However, there exist in all the parameter lines black bands corresponding to subintervals of I_1 and I_2 where the corresponding critical points converge to attracting elements that are not roots of the polynomial system. The biggest black areas appear for $\alpha \in [2.996, 3.003]$ and $\alpha \in [5.074, 5.144]$, although some other smaller intervals appear with unstable performance. The values of α with change of stability are known as bifurcation values.

We use Feigenbaum diagrams to analyze the bifurcations of $G(x,\alpha)$ by using each one of the free critical points of the map as a starting point and observing their behavior for different ranges of α. When the rational function is iterated on these critical points, different behavior can be found after 1000 iterations of the method corresponding to each value of α in a mesh of 3000 subintervals. The resulting behavior is from convergence to the roots to periodic orbits or even other chaotic attractors.



(a) $(c_1(\alpha), c_1(\alpha))$
(b) $(c_1(\alpha), c_2(\alpha))$
(c) $(c_1(\alpha), c_3(\alpha))$
(d) $(c_1(\alpha), c_4(\alpha))$

Figure 2: Parameter lines of $G(x, \alpha)$ involving $c_1(\alpha)$ for $\frac{7}{4} < \alpha < \frac{21}{4}$

Figure 5 corresponds to the bifurcation diagrams in the common black areas of the parameter lines for $\alpha > \frac{21}{4}$ (see Figure 4). In Figures 6a and 6b, the blue curves is the non-hyperbolic strange fixed point that bifurcates for bigger values of α by a period-doubling cascade. However, in Figures 6c and 6d, when the critical point does not converge to the roots, it converges to a periodic orbit of period 6, that for bigger values of α bifurcates in others of doubling periods.

Further, some blue regions appear in the bifurcation diagrams associated with chaotic behavior. In them, strange attractors can be found (visualized in the bidimensional case, $n = 2$). To do it, we plotted in the (x_1, x_2)-space the iteration of $M((x_1, x_2), \alpha)$, for fixed values of α in the blue area. For each fixed value of parameter α, 1000 different initial estimations have been used and, for each of them, the iterates have been plotted following this code color: The first 100 iterations are not plotted, while the following 400 are plotted in green color, the next 400 in blue color, and the last hundred in magenta color. This can be observed in Figure 7 as a non-hyperbolic strange fixed point, after bifurcating in periodic orbits of increasing periods, falls in a chaotic behavior where the orbits are dense in a small rectangular region of the (x_1, x_2)-space.

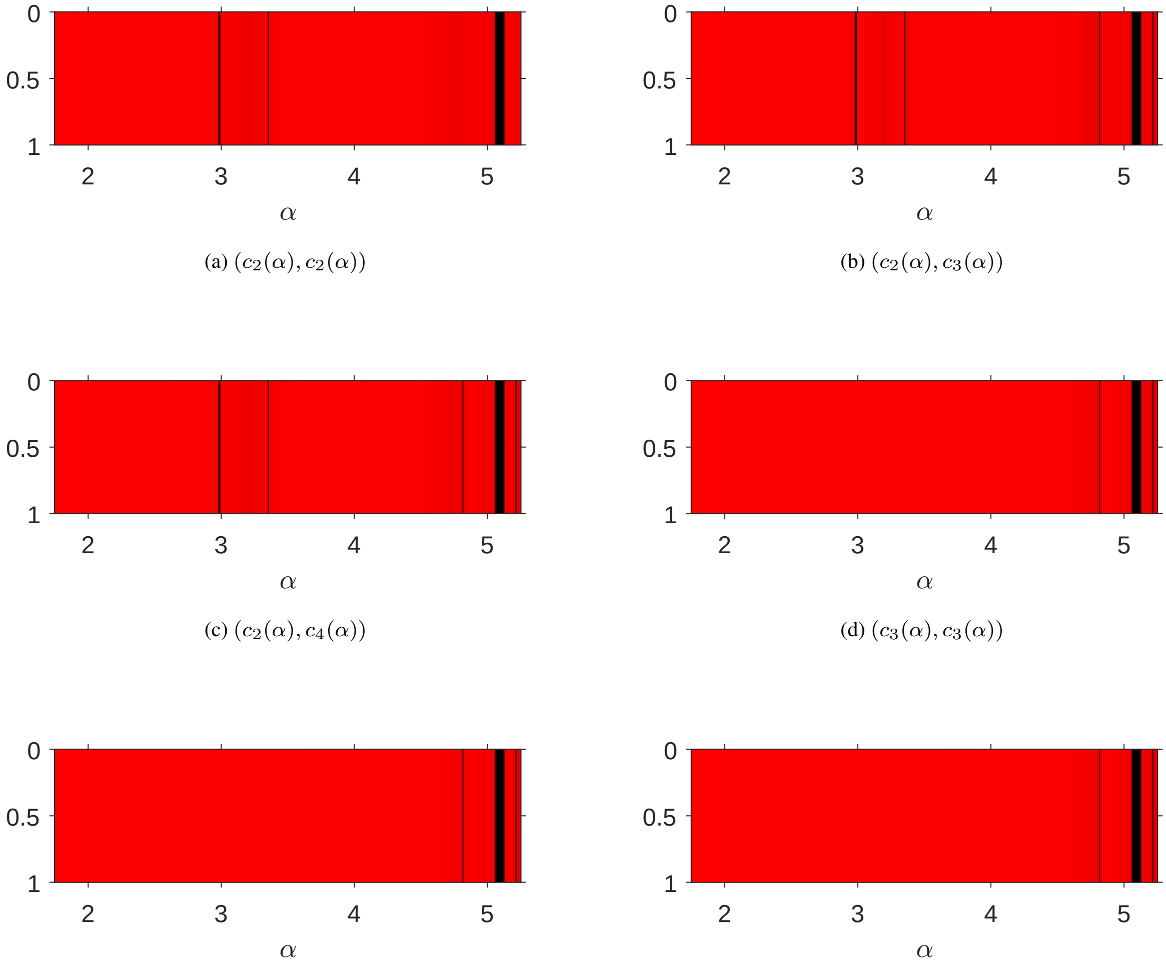
4. Dynamical planes for $n = 2$

Now, the sensitivity of the members of class M4 to the initial estimation is checked, depending on some values of the parameter α that have resulted to provide stable or unstable behavior in the previous dynamical analysis. This is achieved by plotting the dynamical planes of $G(x, \alpha)$ for different values of α. They have been obtained using the routines appearing in [2] in the following way: a mesh of 400×400 points was used, the maximum number of iterations employed was 80, and the stopping criterium has a tolerance of 10^{-3}. We have painted a point of this mesh with different colors depending on the root they converge to. This color is brighter when the number of iterations used is lower; moreover, it is colored in black if it reaches the maximum number of iterations without converging to any of the roots.

The first dynamical plane corresponds to stable behavior, that is, there exist only convergence to the roots (see Figure 8). The unstable performance can be observed in Figure 9. For the particular value of $\alpha = 5.08$, the dynamical plane in Figure 9b shows the different basins of attraction of the roots and also a black area that is the basin of attraction of the existing 4-periodic orbit that is attractive, appearing in yellow color.

One of the eight attracting 2-period orbits existing for $\alpha = 3$ is shown in Figure 9a, whose elements are

$$\{(-1.0000, -0.4216), (-1.0000, -0.4308)\}.$$



(a) $(c_2(\alpha), c_2(\alpha))$ (b) $(c_2(\alpha), c_3(\alpha))$

(c) $(c_2(\alpha), c_4(\alpha))$ (d) $(c_3(\alpha), c_3(\alpha))$

(e) $(c_3(\alpha), c_4(\alpha))$ (f) $(c_4(\alpha), c_4(\alpha))$

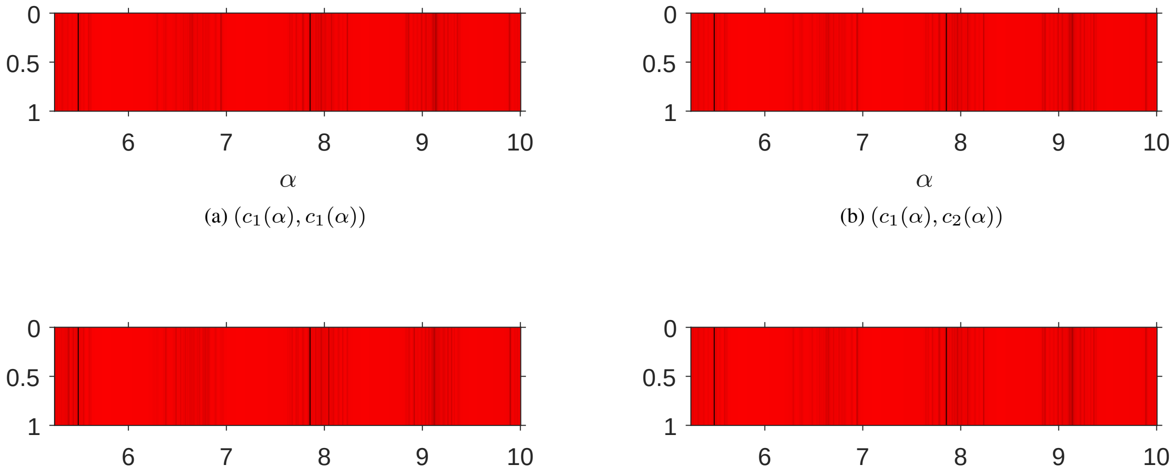
Figure 3: Parameter lines of $G(x, \alpha)$ not involving $c_1(\alpha)$ for $\frac{7}{4} < \alpha < \frac{21}{4}$

The rest of periodic orbits are $\{(-1, 0.4216), (-1, 0.4308)\}$, $\{(1, -0.4216), (1, -0.4308)\}$, $\{(1, 0.4216), (1, 0.4308)\}$, $\{(-0.4216, -1), (-0.4308, -1)\}$, $\{(0.4216, -1), (0.4308, -1)\}$, $\{(-0.4216, 1), (-0.4308, 1)\}$ and $\{(0.4216, 1), (0.4308, 1)\}$.

In Figure 9c, a part of the strange attractor found for $\alpha = 5.14$ can be observed; this dynamical plane have been obtained by avoiding the lines in the orbit of the initial estimation selected in the black area and plotting by yellow circles the elements of the orbit. It can be seen as the most this elements lay in the small area of the strange attractor.

Finally, in Figure 9d, the phase space for $\alpha = 7.7805$ is represented. In it, the 8-period orbit $\{(0.2271, 0.2271), (-139.7847, -139.7847), (22.8703, 22.8703), (-3.6004, -3.6004), (-0.2271, -0.2271), (139.7847, 139.7847), (-22.8703, -22.8703), (3.6004, 3.6004)\}$ appears in yellow.

To sum up, the main performance of the members of M4 class of iterative methods on this kind of polynomial systems is stable. There are no attracting strange fixed points and the only attracting behavior different from the roots lay in very small intervals of α. The most stable member of the class is exactly that providing sixth-order of convergence; many other members of the class (all satisfying $\alpha < \frac{7}{4}$) remain totally stable as there not exist free critical points giving rise to other basins of attraction different from the roots, but their order of convergence is four. However, when the unstable behavior appears, it yields to undesirable convergence performance that is interesting under the dynamical point of view.



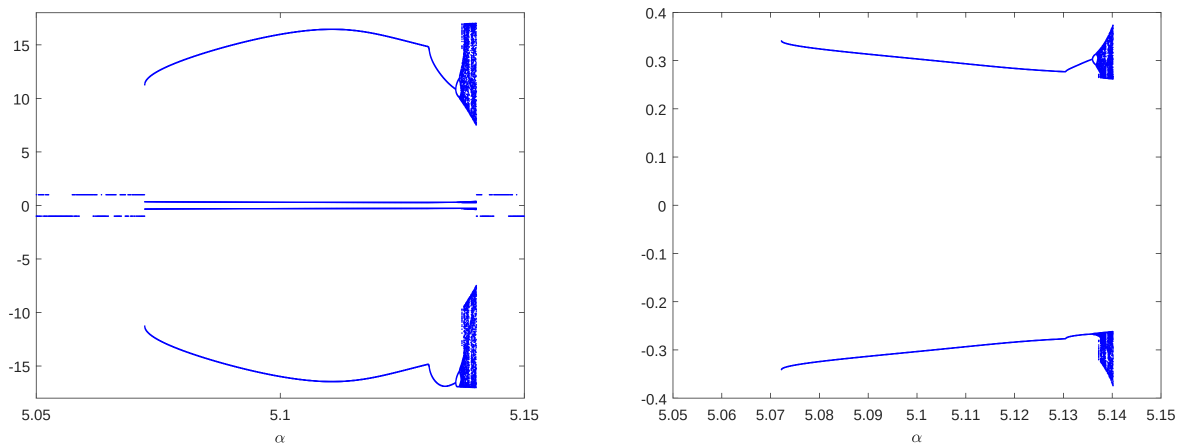
(a) $(c_1(\alpha), c_1(\alpha))$

(b) $(c_1(\alpha), c_2(\alpha))$

(c) $(c_2(\alpha), c_1(\alpha))$

(d) $(c_2(\alpha), c_2(\alpha))$

Figure 4: Parameter lines of $G(x, \alpha)$ for $\alpha > \dfrac{21}{4}$



(a) $(c_1(\alpha), c_1(\alpha))$

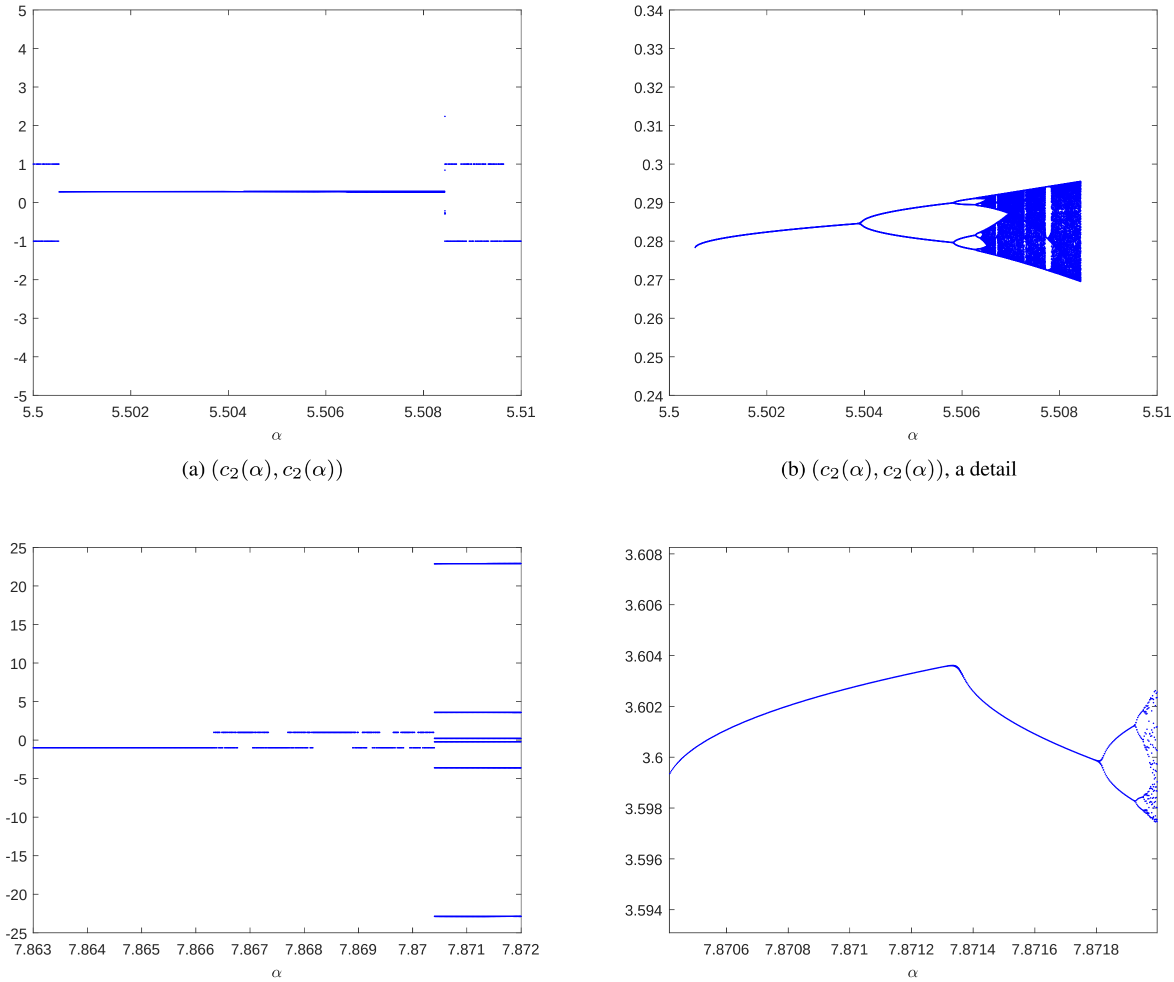
(b) $(c_1(\alpha), c_1(\alpha))$, a detail

Figure 5: Feigenbaum diagrams of $G(x, \alpha)$ for $\dfrac{7}{4} < \alpha < \dfrac{21}{4}$

5. Numerical experiments

Here, we check the efficiency and effectiveness of our scheme on real life and standard academic test problems. Therefore, we consider five examples, whose details can be seen in Examples (1) – (5). Further, we also depict the starting points and zeros of the considered nonlinear system in the Examples (1)–(5). Now, we employ our sixth-order scheme (8) with $\alpha = \dfrac{7}{4}$, called (OM) to verify the computational performance of them with existing methods considered in the Introduction.

Now, we compare (8) with two sixth-order families of iterative methods that have been recently designed by Abbasbandy et al. [1] and Hueso et al. [10]; out of them we choose their best expressions (8) and (14–15) $\left(\text{for } t_1 = -\frac{9}{4} \text{ and } s_2 = \frac{9}{8}\right)$, respectively denoted by (AM) and (HM). Moreover, we also compare our scheme with sixth-order families of iterative methods proposed by Sharma and Arora [16] and Wang and Li [18]; out of them we choose methods namely (13) and (6), respectively, denoted by (SM) and (WM). Finally, we compare (8) with sixth-order methods suggested by Narang



(a) $(c_2(\alpha), c_2(\alpha))$

(b) $(c_2(\alpha), c_2(\alpha))$, a detail

(c) $(c_2(\alpha), c_2(\alpha))$, a detail

(d) $(c_2(\alpha), c_2(\alpha))$, a detail

Figure 6: Feigenbaum diagrams $G(x, \alpha)$ for $\alpha > \dfrac{21}{4}$

et al. [13] and Lotfi et al. [12], we consider methods (3.1) $\left(\text{for } \lambda = 1,\ \beta = 2,\ p = 1 \text{ and } q = \frac{3}{2}\right)$ and (5), respectively, denoted by (MM) and (LM).

In Tables 2–6, we mention the number of iteration indexes (n), residual error of the corresponding function $(\|F(x^{(k)})\|)$, error between the two consecutive iterations $\|x^{(k+1)} - x^{(k)}\|$, computational order of convergence

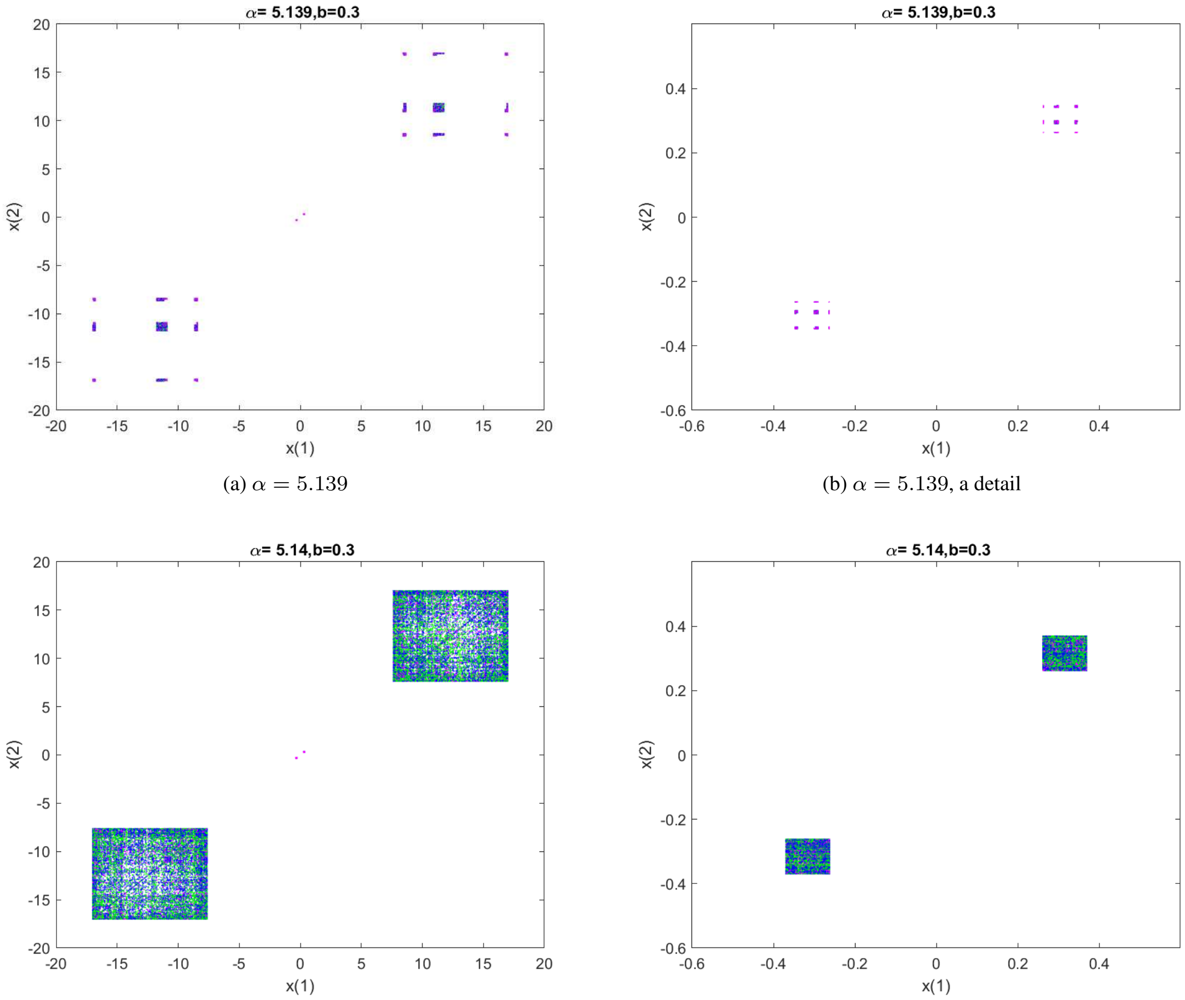
$$\rho = \frac{\log\left[\|x^{(k+1)} - x^{(k)}\| / \|x^{(k)} - x^{(k-1)}\|\right]}{\log\left[\|x^{(k)} - x^{(k-1)}\| / \|x^{(k-1)} - x^{(k-2)}\|\right]},$$

introduced in [5], $\frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k)} - x^{(k-1)}\|^6}$ and η. Further, the value of η is the last calculated value of $\frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k)} - x^{(k-1)}\|^6}$.

During the current numerical experiments with programming language Mathematica (Version 9), all the computations have been done with multiple precision arithmetics with 1000 digits of mantissa, which minimize round-off errors. The meaning of $a\ (\pm b)$ is $a \times 10^{(\pm b)}$ in all the tables.

Example 1. *In this example, we consider the well-known Hammerstein integral equation (see [14, pp. 19-20] to check the effectiveness and applicability of our method as compared to the other existing ones, is given as follows:*

$$x(s) = 1 + \frac{1}{5}\int_0^1 F(s,t)x(t)^3 dt, \tag{30}$$



(a) $\alpha = 5.139$ (b) $\alpha = 5.139$, a detail

(c) $\alpha = 5.14$ (d) $\alpha = 5.14$, a detail

Figure 7: Strange attractors of $G(x, \alpha)$ for α values in some period-doubling cascade regions

where $x \in C[0, 1]$ $s, t \in [0, 1]$ *and the kernel* F *is*

$$F(s,t) = \begin{cases} (1-s)t, t \leq s, \\ s(1-t), s \leq t. \end{cases}$$

We transform the above equation into a finite-dimensional problem by using Gauss-Legendre quadrature formula given as $\int_0^1 f(t)dt \simeq \sum_{j=1}^{2} w_j f(t_j)$, *where the abscissas* t_j *and the weights* w_j *are determined for* $n = 2$ *by Gauss-Legendre quadrature formula. By denoting the approximations of* $x(t_i)$ *by* x_i $(i = 1, 2)$, *one gets the system of nonlinear equations* $5x_i - 5 - \sum_{j=1}^{2} a_{ij} x_j^3 = 0$, *where* $i = 1, 2$ *and*

$$a_{ij} = \begin{cases} w_j t_j (1-t_i), j \leq i, \\ w_j t_i (1-t_j), i < j. \end{cases}$$

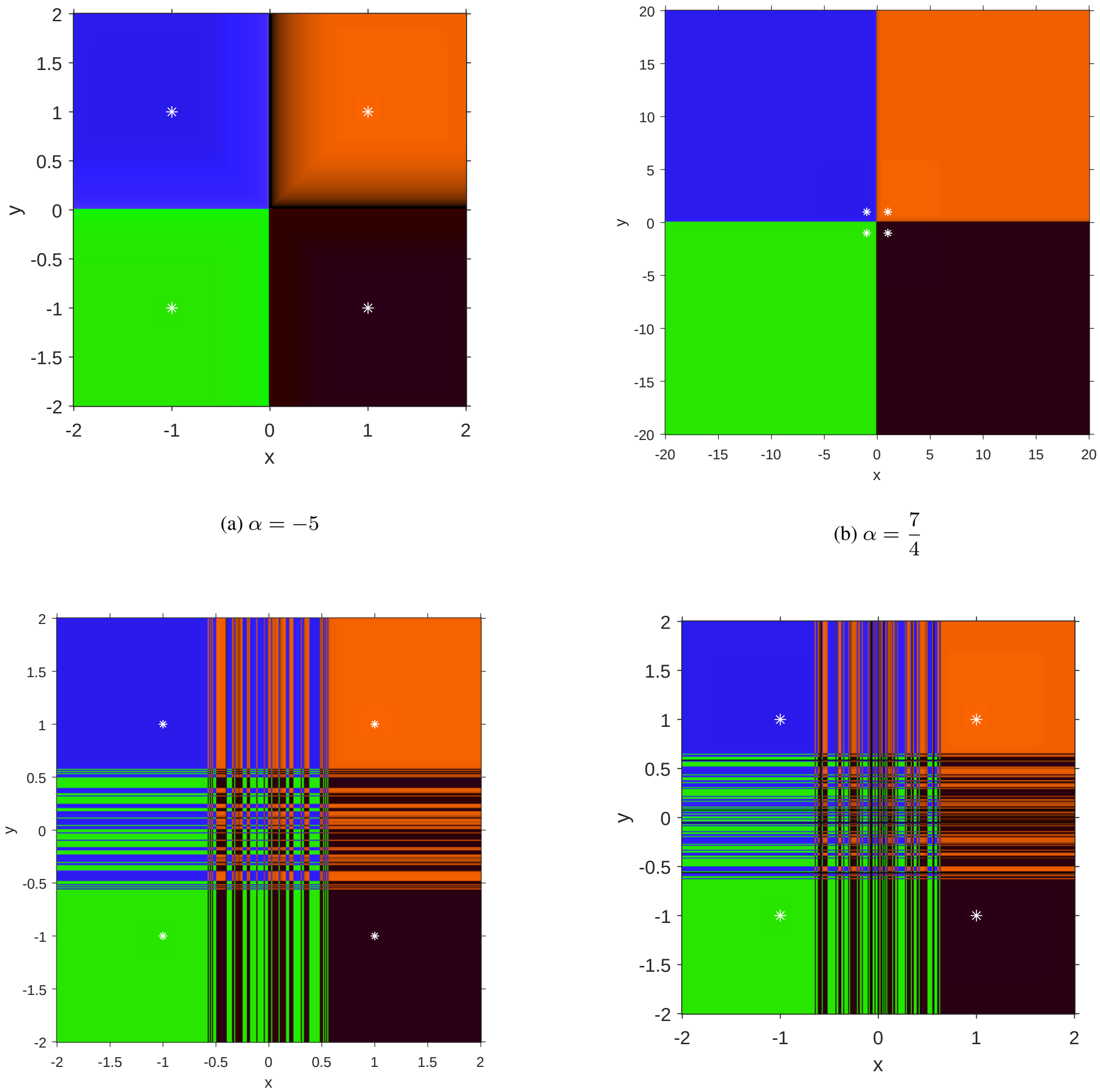
Here the abscissas t_j *and the weights* w_j *are known and depicted in Table 2 when* $n = 2$. *The convergence of the methods towards the root* $\xi^* \approx (1.022598\ldots,\ 1.022598\ldots)^T$, *is tested in Table 2, on the basis of the initial guess* $x^{(0)} = (-1,\ -1)^T$.

Table 1: Abscissas and weights of Gauss-Legendre quadrature formula for $n = 2$

j	t_j	w_j
1	0.21132486540518711774542560...	0.50000000000000000000000000...
2	0.78867513459481288225457440...	0.50000000000000000000000000...

Table 2: Convergence behavior of different methods on Example 1

Methods	k	$\|F(x^{(k)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	ρ	$\frac{\|x^{(k+1)}-x^{(k)}\|}{\|x^{(k)}-x^{(k-1)}\|^6}$	η
	1	$7.2(-10)$	$7.7(-10)$			
AM	2	$7.8(-60)$	$8.3(-60)$		$3.955178954(-5)$	$3.955178960(-5)$
	3	$1.2(-359)$	$1.3(-359)$	6.0000	$3.955178960(-5)$	
	1	$6.3(-9)$	$6.7(-9)$			
HM	2	$7.1(-46)$	$7.6(-46)$		$8.068403312(+4)$	$7.145579086(+40)$
	3	$1.3(-230)$	$1.4(-230)$	5.0000	$7.145579086(+40)$	
	1	$5.5(-10)$	$5.9(-10)$			
SM	2	$1.2(-60)$	$1.3(-60)$		$2.957756070(-5)$	$2.957756073(-5)$
	3	$1.2(-364)$	$1.3(-364)$	6.0000	$2.957756073(-5)$	
	1	$1.4(-10)$	$1.6(-10)$			
WM	2	$1.1(-64)$	$1.2(-64)$		$8.558182383(-6)$	$8.558182386(-6)$
	3	$2.4(-389)$	$2.5(-389)$	6.0000	$8.558182386(-6)$	
	1	$5.1(-10)$	$5.5(-10)$			
MM	2	$6.9(-61)$	$7.4(-61)$		$2.709163941(-5)$	$2.709163944(-5)$
	3	$4.2(-366)$	$4.5(-366)$	6.0000	$2.709163944(-5)$	
	1	$2.6(-10)$	$2.8(-10)$			
LM	2	$5.7(-63)$	$6.1(-63)$		$1.181398901(-5)$	$1.181398901(-5)$
	3	$5.4(-379)$	$5.8(-379)$	6.0000	$1.181398901(-5)$	
	1	$1.4(-10)$	$1.5(-10)$			
OM	2	$9.6(-65)$	$1.0(-64)$		$8.653105286(-6)$	$8.653105289(-6)$
	3	$9.3(-390)$	$9.9(-390)$	5.0000	$8.653105289(-6)$	



(a) $\alpha = -5$

(b) $\alpha = \frac{7}{4}$

(c) $\alpha = 5$

(d) $\alpha = 10$

Figure 8: Stable dynamical planes of M4 on $p(x)$

Example 2. *We assume the following nonlinear system of equations:*

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + \exp(x_2) - \cos x_2 = 0 \\ f_2(x_1, x_2) &= 3x_1 - x_2 - \sin x_2 = 0. \end{aligned} \tag{31}$$

The required solution of the system is $\xi^* = (0,\ 0)^T$ and we consider the initial guess $x^{(0)} = (0.5,\ 0.5)^T$. The obtained results can be observed in Table 3.

Example 3. *We assume a nonlinear system (selected from [8]), defined by the nonlinear function:*

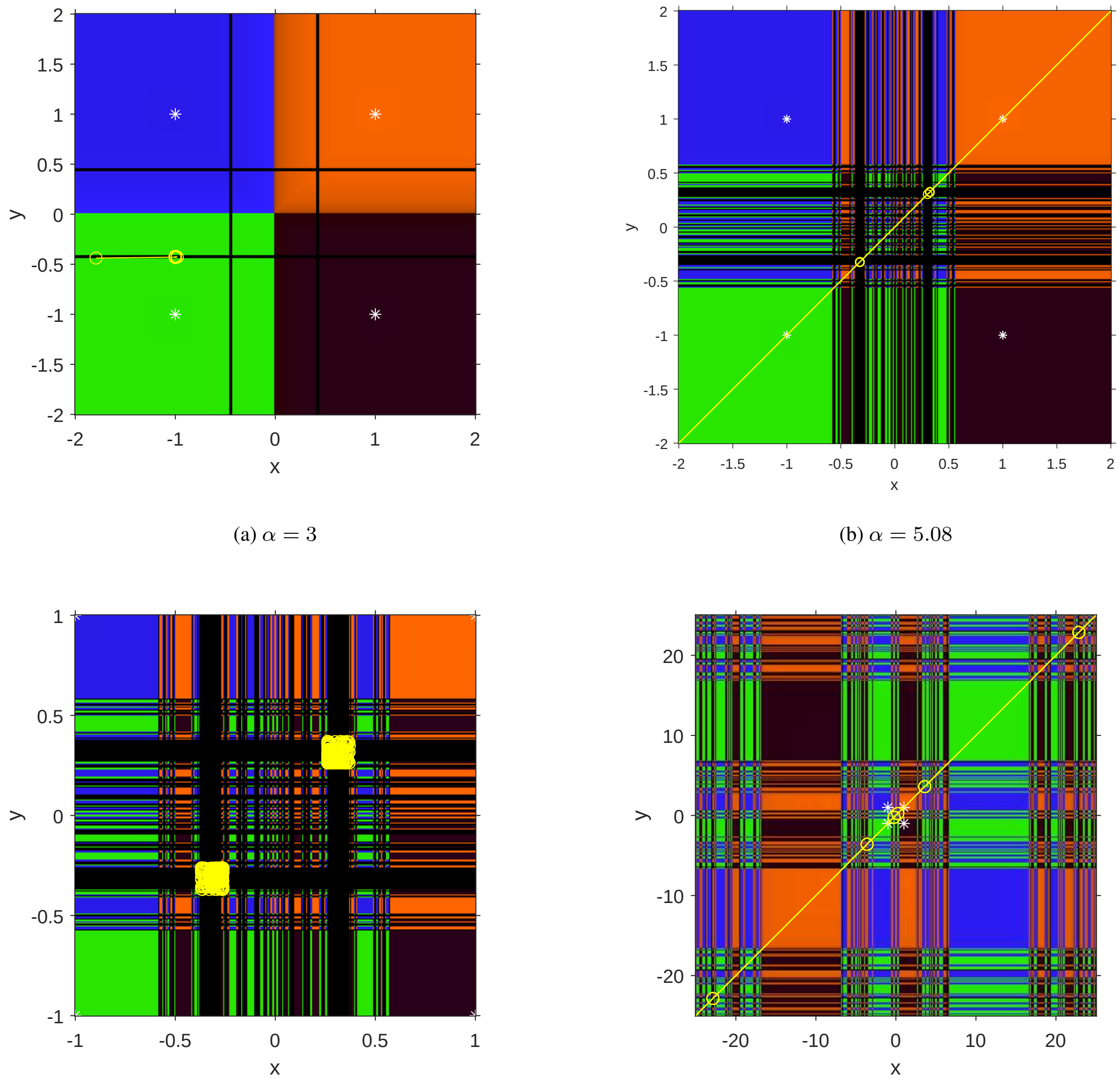
$$F(x) = x_k - \cos\left(2x_k - \sum_{i=1}^{n} x_i\right), \quad 1 \leq k \leq n. \tag{32}$$

We choose $n = 6$ and the initial guess $x^{(0)} = (0.3, 0.3, 0.3, 0.3, 0.3, 0.3)^T$ for this problem. The required solution is

$$\xi^* \approx (0.307733\ldots, 0.307733\ldots, 0.307733\ldots, 0.307733\ldots, 0.307733\ldots, 0.307733\ldots)^T.$$

Table 3: Convergence behavior of different methods on Example 2

Methods	k	$\\|F(x^{(k)})\\|$	$\\|x^{(k+1)} - x^{(k)}\\|$	ρ	$\frac{\\|x^{(k+1)}-x^{(k)}\\|}{\\|x^{(k)}-x^{(k-1)}\\|^6}$	η
	1	6.6(−3)	4.7(−3)			
AM	2	2.0(−14)	1.4(−14)		1.305023650	1.517979331
	3	1.8(−83)	1.3(−83)	5.9926	1.517979331	
	1	3.7(−3)	2.7(−3)			
HM	2	3.1(−14)	2.2(−14)		6.021334705(+1)	7.984925036(+12)
	3	1.4(−69)	9.7(−70)	4.9946	7.984925036(+12)	
	1	4.8(−3)	3.5(−3)			
SM	2	1.9(−15)	1.3(−15)		7.945105897(−1)	9.219151003(−1)
	3	7.6(−90)	5.5(−90)	5.9931	9.219151003(−1)	
	1	5.1(−3)	3.7(−3)			
WM	2	2.7(−15)	2.0(−15)		8.015635635(−1)	9.295921854(−1)
	3	7.4(−89)	5.4(−89)	5.9925	9.295921854(−1)	
	1	4.3(−3)	3.1(−3)			
MM	2	8.1(−16)	5.9(−16)		6.618188308(−1)	7.659557287(−1)
	3	4.3(−92)	3.1(−92)	5.9934	7.659557287(−1)	
	1	4.1(−3)	2.9(−3)			
LM	2	5.1(−16)	3.7(−16)		5.884542177(−1)	5.884542177(−1)
	3	2.3(−93)	1.7(−93)	5.9946	6.647289210(−1)	
	1	2.7(−3)	1.9(−3)			
OM	2	1.2(−15)	3.8(−16)		8.035052667	4.635630014(+7)
	3	4.3(−85)	1.4(−85)	5.6309	4.635630014(+7)	



(a) $\alpha = 3$

(b) $\alpha = 5.08$

(c) $\alpha = 5.14$

(d) $\alpha = 7.7805$

Figure 9: Unstable dynamical planes of M4 on $p(x)$

The numerical results are shown in Table 4.

Example 4. *Let us consider the following system of nonlinear equations (chosen from Grau-Sánchez et al. [9]), defined by the function*

$$F(x) = \sum_{j=1, j \neq i}^{n} x_j - e^{-xi}, \quad 1 \leq i \leq n. \tag{33}$$

We choose $n = 150$ *and the initial guess* $x^{(0)} = (0.5,\ 0.5,\ 0.5,\ \cdots,\ 0.5)^T$ *for this problem. The required solution of this problem is*

$$\xi^* \approx (0.5671433\ldots, 0.5671433\ldots, 0.5671433\ldots, \cdots,\ 0.5671433\ldots)^T.$$

Table 5 shows the numerical results obtained in this case.

Example 5. *Let us consider the following nonlinear system*

$$F(x) = \begin{cases} x_j^2 x_{j+1} - 1 = 0,\ 1 \leq j \leq n, \\ x_n^2 x_1 - 1 = 0, \quad otherwise. \end{cases} \tag{34}$$

Table 4: Convergence behavior of different methods on Example 3

Methods	k	$\|F(x^{(k)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	ρ	$\frac{\|x^{(k+1)}-x^{(k)}\|}{\|x^{(k)}-x^{(k-1)}\|^6}$	η
	1	$6.9(-10)$	$1.4(-10)$			
AM	2	$4.0(-60)$	$8.4(-61)$		$9.404958387(-2)$	$9.404958401(-2)$
	3	$1.6(-361)$	$3.3(-362)$	6.0000	$9.404958401(-2)$	
	1	$5.5(-9)$	$1.1(-9)$			
HM	2	$2.5(-46)$	$5.2(-47)$		$2.286554826(+7)$	$5.046419320(+44)$
	3	$4.8(-233)$	$9.9(-234)$	5.0000	$5.046419320(+44)$	
	1	$5.2(-10)$	$1.1(-10)$			
SM	2	$6.0(-61)$	$1.3(-61)$		$7.370958415(-2)$	$7.370958423(-2)$
	3	$1.4(-366)$	$2.9(-367)$	6.0000	$7.370958423(-2)$	
	1	$1.1(-10)$	$2.3(-10)$			
WM	2	$8.8(-66)$	$1.8(-66)$		$1.317712273(-2)$	$1.317712273(-2)$
	3	$2.4(-396)$	$4.9(-397)$	6.0000	$1.317712273(-2)$	
	1	$4.8(-10)$	$1.0(-10)$			
MM	2	$3.5(-61)$	$7.2(-62)$		$6.878880403(-2)$	$6.878880410(-2)$
	3	$4.2(-368)$	$9.6(-369)$	6.0000	$6.878880410(-2)$	
	1	$2.9(-10)$	$6.1(-11)$			
LM	2	$1.2(-62)$	$2.5(-63)$		$4.845034257(-2)$	$4.845034256(-2)$
	3	$5.7(-377)$	$1.2(-377)$	6.0000	$4.845034256(-2)$	
	1	$9.0(-11)$	$1.9(-11)$			
OM	2	$1.7(-66)$	$3.5(-67)$		$8.043571822(-3)$	$8.043571826(-3)$
	3	$7.0(-401)$	$1.5(-401)$	6.0000	$8.043571826(-3)$	

Table 5: Convergence behavior of different methods on Example 4

Methods	k	$\|F(x^{(k)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	ρ	$\frac{\|x^{(k+1)}-x^{(k)}\|}{\|x^{(k)}-x^{(k-1)}\|^6}$	η
	1	$1.4(-6)$	$5.9(-9)$			
AM	2	$1.8(-55)$	$7.8(-58)$		$1.814507829(-8)$	$1.814507829(-8)$
	3	$9.7(-349)$	$4.1(-351)$	6.0000	$1.814507829(-8)$	
	1	$4.6(-6)$	$2.0(-8)$			
HM	2	$3.5(-44)$	$1.5(-46)$		2.485682422	$3.301595366(+38)$
	3	$8.4(-235)$	$3.6(-237)$	5.0000	$3.301595366(+38)$	
	1	$7.7(-7)$	$3.3(-9)$			
SM	2	$3.1(-57)$	$1.3(-59)$		$1.007050020(-8)$	$1.007050020(-8)$
	3	$1.3(-359)$	$5.4(-362)$	6.0000	$1.007050020(-8)$	
	1	$1.6(-6)$	$6.7(-9)$			
WM	2	$4.3(-55)$	$1.9(-57)$		$2.111852506(-8)$	$2.111852506(-8)$
	3	$2.1(-346)$	$8.8(-349)$	6.0000	$2.111852506(-8)$	
	1	$5.9(-7)$	$2.5(-9)$			
MM	2	$4.8(-58)$	$2.1(-60)$		$7.685599982(-9)$	$7.685599980(-9)$
	3	$1.4(-364)$	$6.1(-367)$	6.0000	$7.685599980(-9)$	
	1	$1.4(-6)$	$6.1(-9)$			
LM	2	$2.3(-55)$	$1.0(-57)$		$1.946645858(-8)$	$1.946645858(-8)$
	3	$4.5(-348)$	$1.9(-350)$	6.0000	$1.946645858(-8)$	
	1	$6.9(-8)$	$3.0(-10)$			
OM	2	$1.3(-64)$	$5.4(-67)$		$7.778375898(-10)$	$7.778375897(-10)$
	3	$4.3(-405)$	$1.8(-407)$	6.0000	$7.778375897(-10)$	

Table 6: Convergence behavior of different methods on Example 5

Methods	k	$\|F(x^{(k)})\|$	$\|x^{(k+1)} - x^{(k)}\|$	ρ	$\frac{\|x^{(k+1)}-x^{(k)}\|}{\|x^{(k)}-x^{(k-1)}\|^6}$	η
	1	$5.5(-2)$	$1.8(-2)$			
AM	2	$8.2(-15)$	$2.7(-15)$		$7.599151595(-5)$	$7.695242316(-5)$
	3	$9.5(-92)$	$3.2(-92)$	5.9993	$7.695242316(-5)$	
	1	$3.0(-2)$	$1.0(-2)$			
HM	2	$2.0(-14)$	$6.7(-15)$		$6.625126396(-3)$	$9.962407479(+9)$
	3	$2.7(-75)$	$8.9(-76)$	4.9997	$9.962407479(+9)$	
	1	$3.9(-2)$	$1.3(-2)$			
SM	2	$6.4(-16)$	$2.1(-16)$		$4.636076578(-5)$	$4.674761498(-5)$
	3	$1.3(-98)$	$4.3(-99)$	6.0000	$4.674761498(-5)$	
	1	$4.2(-2)$	$1.4(-2)$			
WM	2	$1.2(-15)$	$3.9(-16)$		$5.254428593(-5)$	$5.303300859(-5)$
	3	$5.7(-97)$	$1.9(-97)$	5.9995	$5.303300859(-5)$	
	1	$3.4(-2)$	$1.1(-2)$			
MM	2	$2.6(-16)$	$8.5(-17)$		$3.831428538(-5)$	$3.858533300(-5)$
	3	$4.4(-101)$	$1.5(-101)$	5.9996	$3.858533300(-5)$	
	1	$3.4(-2)$	$1.1(-2)$			
LM	2	$2.3(-16)$	$7.7(-17)$		$3.721130070(-5)$	$3.746683848(-5)$
	3	$2.3(-101)$	$7.6(-102)$	5.9996	$3.746683848(-5)$	
	1	$2.1(-1)$	$7.1(-3)$			
OM	2	$7.6(-18)$	$2.5(-18)$		$2.039428975(-5)$	$2.047663387(-5)$
	3	$1.6(-110)$	$5.3(-111)$	5.9998	$2.047663387(-5)$	

In order to obtain a large system of nonlinear equations, we choose $n = 200$ *and the initial approximation* $x^{(0)} = (1.25,\ 1.25,\ 1.25,\ \cdots,\ 1.25)^T$ *for this problem. The required solution of this problem is* $\xi^* = (1,\ 1,\ 1,\ \cdots,\ 1)^T$. *The obtained results can be observed in Table 6.*

The numerical performance of the proposed and existing methods are similar in all examples. Our proposed scheme improves slightly the results of other schemes in the third iteration. Moreover, the behavior is the same with independence of the size of the system.

6. Concluding remarks

A parametric family of iterative methods with two steps for solving nonlinear systems has been designed. This class, under standard conditions, converges to a solution of the system with order, at least, four. Moreover, one element of the class reaches sixth-order of convergence. This element improves the efficiency regarding other known methods with the same order, as it has only two steps in its iterative expression. As far as we know, all the sixth-order vectorial schemes have three steps per iteration. A deep multidimensional dynamical analysis has been also developed on the fourth-order family; one of the conclusions is that one of the most stable elements is that providing order six. Indeed, some values of the parameter with chaotic behavior have been detected in order to avoid its use in practice. Finally, the numerical tests confirm the theoretical results, showing that the proposed sixth-order scheme is robust, efficient and stable.
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References

References

[1] S. Abbasbandy, P. Bakhtiari, A. Cordero, J.R. Torregrosa, T.Lotfi, New efficient methods for solving nonlinear systems of equations with arbitrary even order, Appl. Math. Comput. 287–288, (2016) 94–103.

[2] F.I. Chicharro, A. Cordero, J.R. Torregrosa, Drawing dynamical and parameters planes of iterative families and methods, The Scientific World Journal Volume 2013, Article ID 780153, 11 pages.

[3] A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa, A modified Newton-Jarratt's composition, Numer. Algor. 55, (2010) 87–99.

[4] A. Cordero, F. Soleymani, J.R. Torregrosa, Dynamical analysis of iterative methods for nonlinear systems or how to deal with the dimension?, Appl. Math. Comput. 244, (2014) 398–412.

[5] A. Cordero, J.R. Torregrosa, Variants of Newton's method using fifth-order quadrature formulas, Appl. Math. Comput. 190, (2007) 686–698.

[6] A. Cordero, J.R. Torregrosa, P. Vindel, Dynamics of a family of Chebyshev-Halley type methods, Appl. Math. Comput. 219, (2013) 8568–8583.

[7] R. L. Devaney, An Introduction to Chaotic Dynamical Systems, Advances in Mathematics and Engineering, CRC Press, 2003.

[8] M. Grau-Sánchez, Á. Grau, M. Noguera, On the computational efficiency index and some iterative methods for solving systems of nonlinear equations, J. Comput. Appl. Math. 236, (2011) 1259–1266.

[9] M. Grau-Sánchez, Á. Noguera, S. Amat, On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods, J. Comput. Appl. Math. 237, (2013) 363–372.

[10] J.L. Hueso, E. Martínez, C. Teruel, Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems, J. Comput. Appl. Math. 275, (2015) 412–420.

[11] Y. Khan, M. Fardi, K. Sayevand, A new general eighth-order family of uterative methods for solving nonlinear equations, Appl. Math. Letters 25(12), (2012) 2262–2266.

[12] T. Lotfi, P. Bakhtiari, A. Cordero, K. Mahdiani, J.R. Torregrosa, Some new efficient multipoint iterative methods for solving nonlinear systems of equations, Int. J. Comput. Math. 92(9), (2015) 1921–1934.

[13] M. Narang, S. Bhatia, V. Kanwar, New two parameter Chebyshev-Halley like family of fourth and sixth-order methods for systems of nonlinear equations, Appl. Math. Comput. 275, (2016) 394–403.

[14] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New-York, 1970.

[15] R.C. Robinson, An Introduction to Dynamical Systems, Continous and Discrete, Americal Mathematical Society, Providence, RI, USA, 2012.

[16] J.R. Sharma, H. Arora, Efficient Jarratt-like methods for solving systems of nonlinear equations, Calcolo 51(1), (2014) 193–210.

[17] J.F. Traub, Iterative methods for the solution of equations, Prentice-Hall, Englewood Cliffs, 1964.

[18] X. Wang, Y. Li, An Efficient Sixth Order Newton Type Method for Solving Nonlinear Systems, Algorithms 10(45), (2017) 1–9.