EIGENVALUE LOCATION IN COGRAPHS

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ABSTRACT. We give an $O(n)$ time and space algorithm for constructing a diagonal matrix congruent to $A + xI$, where A is the adjacency matrix of a cograph and $x \in \mathbb{R}$. Applications include determining the number of eigenvalues of a cograph's adjacency matrix that lie in any interval, obtaining a formula for the inertia of a cograph, and exhibiting infinitely many pairs of equienergetic cographs with integer energy.

1. INTRODUCTION

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. For $v \in V$, $N(v)$ denotes the *open neighborhood* of v, that is, $\{w | \{v, w\} \in E\}$. The *closed neighborhood* $N[v] = N(v) \cup \{v\}$. If $|V| = n$, the *adjacency matrix* $A = [a_{ij}]$ is the $n \times n$ matrix of zeros and ones such that $a_{ij} = 1$ if and only if v_i is adjacent to v_j (that is, there is an edge between v_i and v_j). A value λ is an *eigenvalue* if $\det(A - \lambda I) = 0$, and since A is real symmetric its eigenvalues are real. In this paper, a graph's *eigenvalues* are the eigenvalues of its adjacency matrix.

This paper is concerned with *cographs*. This class of graphs has been discovered independently by several authors in many equivalent ways since the 1970's. Corneil, Lerchs and Burlingham [6] define cographs recursively:

  (1) A graph on a single vertex is a cograph;
  (2) A finite union of cographs is a cograph;
  (3) The complement of a cograph is a cograph.

A graph is a cograph if and only it has no induced path of length four [6]. They are often simply called P_4 *free* graphs in the literature. Linear time algorithms for recognizing cographs are given in [7] and more recently in [10].

While recognition algorithms for cographs is an interesting problem, our motivation for considering cographs comes from *spectral graph theory* [4, 8]. Spectral properties of cographs were studied by Royle in [16] where the surprising result was obtained that the rank of a cograph is the number of non-zero rows in the adjacency matrix. An elementary proof of this property was later given in [5]. More recently, in [2] Bıyıkoğlu, Simić and Stanić obtained the multiplicity of -1 and 0 for cographs.

The purpose of this paper is to extend to cographs eigenvalue location algorithms that exist for trees [11], threshold graphs [12] and generalized lollipop graphs [9]. Recall that two real symmetric matrices R and S are *congruent* if there exists a nonsingular matrix P for which $R = P^T S P$. Our main focus is an algorithm that uses $O(n)$ time and space for constructing a diagonal matrix congruent to $A + xI$, where A is adjacency matrix of a cograph, and $x \in \mathbb{R}$. Our paper is similar in spirit to the papers [11, 12] which describe $O(n)$ diagonalization algorithms for trees and for threshold graphs. Threshold

graphs are P_4, C_4, and $2K_2$ free, and therefore are a subclass of cographs. Hence our algorithm is an extension of the algorithm in [12].

Several points are worth noting. First, while one might expect linear time algorithms for graphs with *sparse* adjacency matrices such as trees, the adjacency matrix of a cograph can be *dense*. Next, while our algorithm's correctness is based on elementary matrix operations, its implementation operates directly on the cotree and uses only $O(n)$ space. Finally, the *analysis* of algorithms for trees and threshold graphs has led to interesting theoretical results. For example, in [15] conditions were determined for the index (largest eigenvalue) in trees to be integer. In [12] the authors showed that all eigenvalues of threshold graphs, except -1 and 0, are simple. In [13] the algorithm was used to show that no threshold graphs have eigenvalues in $(-1, 0)$.

If G is a graph having eigenvalues $\lambda_1, \ldots, \lambda_n$, its *energy*, denoted $E(G)$ is defined to be $\sum_{i=1}^{n} |\lambda_i|$. Two non-cospectral graphs with the same energy are called *equienergetic*. Finding non-cospectral equienergetic graphs is a relevant problem. In [13] the authors presented infinite sequences of connected, equienergetic pairs of non-cospectral threshold graphs with integer energy. In this paper, we continue this investigation.

Here is an outline of the remainder of this paper. In Section 2 we describe cotrees, and present some known facts. In Section 3 we give the elementary matrix operations used in our algorithm. In Section 4 we give the complete diagonalization algorithm. In Section 5, using Sylvester's Law of Inertia, we show how to efficiently determine how many eigenvalues of a cograph lie in a given interval. The *inertia* of a graph G is the triple (n_+, n_0, n_-) giving the number of eigenvalues of G that are positive, zero, and negative, and in Section 6 we give a formula for cograph inertia. Finally in Section 7 we exhibit infinitely many non-threshold cographs equienergetic to a complete graph.

2. COTREES AND ADJACENCY MATRIX

Cographs have been represented in various ways, and it is useful to recall the representation given in [6]. The unique *normalizend form* of a cograph G is defined recursively: If G is connected, then it is in normalized form if it is expressed as a single vertex, or the *complemented union* of $k \geq 2$

$$G = \overline{G_1 \cup G_2 \cup \ldots \cup G_k}$$

connected cographs G_i in normalized form. If G disconnected its normalized form is the complement of a connected cograph in normalized form. The unique rooted tree T_G representing the parse structure of the cograph's normalized form is called the *cotree*. The leaves or terminal vertices of T_G correspond to vertices in the cograph. The interior nodes represent $\overline{\cup}$ operations.

It is not difficult to show that the class of cographs is also the smallest class of graphs containing K_1, and closed under the union $\cup$ and join $\otimes$ operators. In fact one can transform the cotree of Corneil, Lerchs and Burlingham into an equivalent tree T_G using $\cup$ and $\otimes$. In the connected case, we simply place a $\otimes$ at the tree's root, placing $\cup$ on interior nodes with odd depth, and placing $\otimes$ on interior nodes with even depth. To build a cotree for a disconnected cograph, we place $\cup$ at the root, and place $\otimes$'s at odd depths, and $\cup$'s at even depths. It will be convenient for us to use this unique alternating representation. In [2] this structure is called a *minimal cotree*, but throughout this paper we call it simply a *cotree*. All interior nodes of cotrees have at least two children. Figure 1 shows a cograph and cotree. The following is well known.

Lemma 1. *If G is a cograph with cotree T_G, vertices u and v are adjacent in G if and only if their least common ancestor in T_G is $\otimes$.*
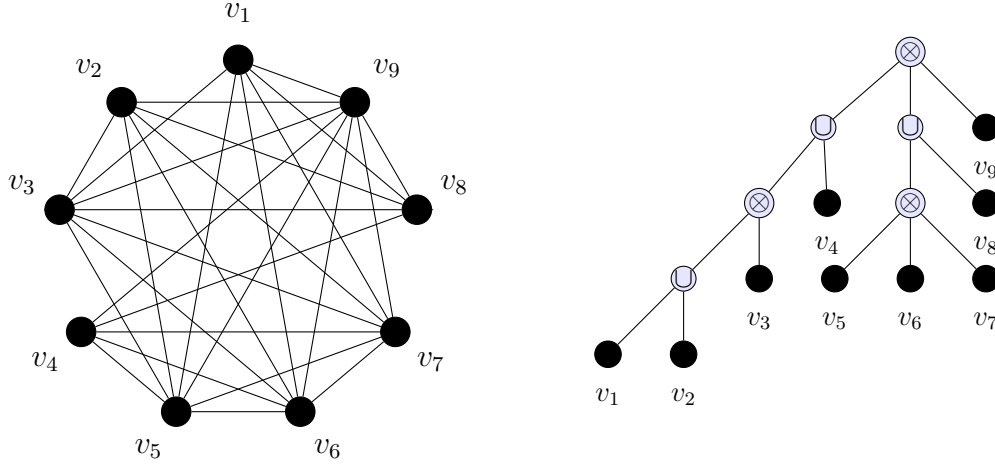

FIGURE 1. A cograph G and its cotree.

Two vertices u and v are *duplicates* if $N(u) = N(v)$ and *coduplicates* if $N[u] = N[v]$. We call u and v *siblings* if they are either duplicates or coduplicates. Siblings play an important role in the structure of cographs, as well as in this paper.

Lemma 2. *Two vertices v and u in a cograph are siblings if and only if they share the same parent w node in the cotree. Moreover, if $w = \cup$, they are duplicates. If $w = \otimes$ they are coduplicates.*

Lemma 3. *A cograph G of order $n \geq 2$ has a pair of siblings.*

Proof. The cotree of G must have an interior vertex adjacent to two leaves. $\qquad\square$

Let G be a cograph with cotree T_G. Let $G - v$ denote the subgraph obtained by removing v. It is known that $G - v$ is a cograph, so we shall use $T - v$ to denote the cotree of $G - v$. There is a general method for constructing $T - v$ [6, Lem. 1]. However, it somewhat simplifies the process if v has maximum depth. The following lemma can be proved with Lemma 1.

Lemma 4. *Let T_G be a cotree, and let $\{v, u\}$ be siblings of greatest depth, whose parent w has k children. If $k > 2$ we obtain $T - v$ by removing v. If $k = 2$ and w is not the root, we obtain $T - v$ by moving u to the parent of w, and removing v and w. If $k = 2$ and w is the root, the cotree is u.*

We end this section by making an important observation.

Lemma 5. *Let G be a cograph with adjacency matrix A and cotree T_G. Let $\{v_k, v_l\}$ be siblings with parent w. If $w = \cup$ (they are duplicates), then rows(columns) k and l in A are equal. If $w = \otimes$ (they are coduplicates), then the rows (columns) are equal except in two positions, namely $A[k, k] = A[l, l] = 0$ and $A[k, l] = A[l, k] = 1$.*

3. Diagonalizing a row and column

Given a cograph G with adjacency matrix A, and $x \in \mathbb{R}$, we will transform $B = A + xI$ into a congruent diagonal matrix in stages. In this section, we illustrate a stage, given

a partially transformed matrix. Recall that matrices are congruent if one can obtain the other by a sequence of *pairs* of elementary operations, each pair consisting of a row operation followed by the *same* column operation.

Let T_G be the cotree of G, and let $\{v_k, v_l\}$ be a pair of siblings. We assume the diagonal values d_k and d_l of rows k and l, respectively, may have been modified by previous computations, but *only* diagonal values. The goal is to annihilate off-diagonal 1's in the row and column corresponding to v_k, maintaining congruence to B. Let w be the parent of the siblings in T_G. There are two cases.

Case 1: $w = \otimes$. By Lemma 2 we know $\{v_k, v_l\}$ are coduplicates. By Lemma 5 rows (columns) l and k of the matrix B have the form

$$
\begin{bmatrix}
 & & & a_1 & a_1 & & \\
 & & & \vdots & \vdots & & \\
 & & & a_i & a_i & & \\
 & & & \vdots & \vdots & & \\
a_1 & \dots & a_i & \dots & d_l & 1 & \dots & a_n \\
 & & & & & & \\
a_1 & \dots & a_i & \dots & 1 & d_k & \dots & a_n \\
 & & & \vdots & \vdots & & \\
 & & & a_n & a_n & & \\
\end{bmatrix},
$$

where $a_i \in \{0, 1\}$. The row and column operations

$$R_k \leftarrow R_k - R_l$$
$$C_k \leftarrow C_k - C_l$$

give:

$$
\begin{bmatrix}
 & & & a_1 & 0 & & \\
 & & & \vdots & \vdots & & \\
 & & & a_i & 0 & & \\
 & & & \vdots & \vdots & & \\
a_1 & \dots & a_i & \dots & d_l & 1 - d_l & \dots & a_n \\
 & & & & & & \\
0 & \dots & 0 & \dots & 1 - d_l & d_k + d_l - 2 & \dots & 0 \\
 & & & \vdots & \vdots & & \\
 & & & a_n & 0 & & \\
\end{bmatrix}.
$$

Most of the non-zero elements in row and column k have been removed. But we must now remove the two entries $1 - d_l$. There are three subcases, depending on whether $d_k + d_l - 2 \neq 0$ and whether $d_l = 1$.

subcase 1a: $d_k + d_l - 2 \neq 0$. Then we may perform the operations

$$R_l \leftarrow R_l - \frac{1 - d_l}{d_k + d_l - 2} R_k$$
$$C_l \leftarrow C_l - \frac{1 - d_l}{d_k + d_l - 2} C_k$$

obtaining:

$$
\begin{bmatrix}
& & & & a_1 & & 0 & & \\
& & & & \vdots & & \vdots & & \\
& & & & a_i & & 0 & & \\
& & & & \vdots & & \vdots & & \\
a_1 & \dots & a_i & \dots & \gamma & & 0 & \dots & a_n \\
& & & & & & & & \\
0 & \dots & 0 & \dots & 0 & & d_k + d_l - 2 & \dots & 0 \\
& & & & \vdots & & \vdots & & \\
& & & & a_n & & 0 & &
\end{bmatrix},
$$

where

$$
\gamma = d_l - \frac{(1 - d_l)^2}{d_k + d_l - 2} = \frac{d_k d_l - 1}{d_k + d_l - 2}.
$$

The following assignments are made

$$
d_k \leftarrow d_k + d_l - 2 \qquad d_l \leftarrow \frac{d_k d_l - 1}{d_k + d_l - 2}. \tag{1}
$$

Since row (and column) k is diagonalized, the value d_k becomes permanent value and we remove v_k from the cotree:

$$
T_G \leftarrow T_G - v_k.
$$

The assignments in (1) are technically incorrect since d_k is modified in the first assignment and used in the second. In our algorithm's pseudo-code, we will use temporary variables and assign $\alpha \leftarrow d_k$ and $\beta \leftarrow d_l$, and then assign $d_k \leftarrow \alpha + \beta - 2$ and $d_l \leftarrow \frac{\alpha\beta - 1}{\alpha + \beta - 2}$. To keep notation simple, in the remainder of this section, we will not do this.

subcase 1b: $d_k + d_l = 2$ and $d_l = 1$. Then the matrix looks like

$$
\begin{bmatrix}
& & & & a_1 & & 0 & & \\
& & & & \vdots & & \vdots & & \\
& & & & a_i & & 0 & & \\
& & & & \vdots & & \vdots & & \\
a_1 & \dots & a_i & \dots & 1 & & 0 & \dots & a_n \\
& & & & & & & & \\
0 & \dots & 0 & \dots & 0 & & 0 & \dots & 0 \\
& & & & \vdots & & \vdots & & \\
& & & & a_n & & 0 & &
\end{bmatrix},
$$

and we are done. We make the assignments

$$
d_k \leftarrow 0 \qquad d_l \leftarrow 1 \qquad T_G \leftarrow T_G - v_k
$$

as d_k becomes permanent, and v_k is removed.

subcase 1c: $d_k + d_l - 2 = 0$ and $d_l \neq 1$. Then our matrix looks like

$$\begin{bmatrix} & & & & a_1 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ & & & & a_i & & 0 & & \\ & & & & \vdots & & \vdots & & \\ a_1 & \dots & a_i & \dots & d_l & & 1-d_l & \dots & a_n \\ 0 & \dots & 0 & \dots & 1-d_l & & 0 & \dots & 0 \\ & & & & \vdots & & \vdots & & \\ & & & & a_n & & 0 & & \end{bmatrix} .$$

We note that $a_1, \dots, a_n \in \{0, 1\}$ and since $1 - d_l \neq 0$, for each $i \neq k, l$ such that $a_{il} = 1$, we perform:

$$\begin{aligned} R_i &\leftarrow R_i - \frac{1}{1 - d_l} R_k \\ C_i &\leftarrow C_i - \frac{1}{1 - d_l} C_k \end{aligned} \tag{2}$$

This annihilates most of row (column) l, without changing any other values:

$$\begin{bmatrix} & & & & 0 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ & & & & 0 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ 0 & \dots & 0 & \dots & d_l & & 1-d_l & \dots & 0 \\ 0 & \dots & 0 & \dots & 1-d_l & & 0 & \dots & 0 \\ & & & & \vdots & & \vdots & & \\ & & & & 0 & & 0 & & \end{bmatrix} .$$

The operations

$$\begin{aligned} R_l &\leftarrow R_l + \frac{1}{2} R_k \\ C_l &\leftarrow C_l + \frac{1}{2} C_k \end{aligned}$$

replace the diagonal d_l with one, while the operations

$$\begin{aligned} R_k &\leftarrow R_k - (1 - d_l) R_l \\ C_k &\leftarrow C_k - (1 - d_l) C_l \end{aligned}$$

eliminate the two off-diagonal elements, finally giving:

$$\begin{bmatrix} & & & & 0 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ & & & & 0 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ 0 & \dots & 0 & \dots & 1 & & 0 & \dots & 0 \\ & & & & & & & & \\ 0 & \dots & 0 & \dots & 0 & & -(1-d_l)^2 & \dots & 0 \\ & & & & \vdots & & \vdots & & \\ & & & & 0 & & 0 & & \end{bmatrix}.$$

What is different about this subcase is that both rows k and l have been diagonalized. We make the following assignments

$$d_k \leftarrow -(1-d_l)^2 \qquad d_l \leftarrow 1 \qquad T_G \leftarrow T_G - v_k \qquad T_G \leftarrow T_G - v_l$$

removing both vertices from T_G and making both variables permanent.

Case 2: $w = \cup$. By Lemma 5, row k and l of the matrix B look like:

$$\begin{bmatrix} & & & & a_1 & & a_1 & & \\ & & & & \vdots & & \vdots & & \\ & & & & a_i & & a_i & & \\ & & & & \vdots & & \vdots & & \\ a_1 & \dots & a_i & \dots & d_l & & 0 & \dots & a_n \\ & & & & & & & & \\ a_1 & \dots & a_i & \dots & 0 & & d_k & \dots & a_n \\ & & & & \vdots & & \vdots & & \\ & & & & a_n & & a_n & & \end{bmatrix},$$

Similar to Case 1, the row and column operations

$$R_k \leftarrow R_k - R_l$$
$$C_k \leftarrow C_k - C_l$$

give:

$$\begin{bmatrix} & & & & a_1 & & 0 & & \\ & & & & \vdots & & \vdots & & \\ & & & & a_i & & 0 & & \\ & & & & \vdots & & \vdots & & \\ a_1 & \dots & a_i & \dots & d_l & & -d_l & \dots & a_n \\ & & & & & & & & \\ 0 & \dots & 0 & \dots & -d_l & & d_k + d_l & \dots & 0 \\ & & & & \vdots & & \vdots & & \\ & & & & a_n & & 0 & & \end{bmatrix},$$

subcase 2a: $d_k + d_l \neq 0$. Then the matrix operations

$$R_l \leftarrow R_l + \frac{d_l}{d_k + d_l} R_k$$
$$C_l \leftarrow C_l + \frac{d_l}{d_k + d_l} C_k$$

diagonalize the matrix and the following assignments are made:

$$d_k \leftarrow d_k + d_l \qquad T_G \leftarrow T_G - v_k \qquad d_l \leftarrow \frac{d_k d_l}{d_k + d_l}$$

subcase 2b: $d_k + d_l = 0$, and $d_l = 0$. Similar to **subcase 1b**, the matrix is in diagonal form and the following assignments are made:

$$d_k \leftarrow 0 \qquad T_G \leftarrow T_G - v_k \qquad d_l \leftarrow 0$$

subcase 2c: $d_k + d_l = 0$, and $d_l \neq 0$. Since $d_l \neq 0$, we use operations similar to (2) to annihilate most of rows and columns l:

$$
\begin{bmatrix}
 & & 0 & 0 & & \\
 & & \vdots & \vdots & & \\
 & & 0 & 0 & & \\
 & & \vdots & \vdots & & \\
0 & \cdots & 0 & \cdots & d_l & -d_l & \cdots & 0 \\
0 & \cdots & 0 & \cdots & -d_l & 0 & \cdots & 0 \\
 & & \vdots & \vdots & & \\
 & & 0 & 0 & &
\end{bmatrix} .
$$

The operations

$$
\begin{aligned}
R_k &\leftarrow R_k + R_l \\
C_k &\leftarrow C_k + C_l
\end{aligned}
$$

complete the diagonlization. The following assignments are made:

$$d_k \leftarrow -d_l \qquad d_l \leftarrow d_l \qquad T_G \leftarrow T_G - v_k \qquad T_G \leftarrow T_G - v_l$$

4. Diagonalizing $A + xI$

The algorithm in Figure 2 constructs a diagonal matrix D, congruent to $B = A + xI$, where A is the adjacency matrix of a cograph G, and $x \in \mathbb{R}$. The algorithm's input is the cotree T_G and x. Note the algorithm does not store the matrix, but rather only records changes on the diagonal values d_i of B, allowing only $O(n)$ space.

It initializes all entries of D with x. At the beginning of each iteration, the cotree represents the subgraph induced by vertices whose rows and columns have not yet been diagonalized. During each iteration, a pair of siblings $\{v_k, v_l\}$ from the cotree is selected, whose existence is guaranteed by Lemma 3. Note that the algorithm can't assume that either of the diagonal elements d_k and d_l are still x.

Each iteration of the loop annihilates either one or two rows (columns), updating diagonal values wth arithmetic from Section 3. Once a row (column) is diagonalized, those entries never participate again in row and column operations, and so their values remain unchanged. When a row (column) corresponding to vertex v has been diagonalized, the subgraph induced by those vertices whose rows (columns) are undiagonalized, has been reduced. Thus v is removed from the cotree. It is slightly easier to reconstruct the cotree when leaves of maximum depth are removed, using the method of Lemma 4. Hence we always choose sibling pairs of maximum depth. The algorithm terminates when all rows and columns have been diagonalized. Each iteration of `Diagonalize` takes constant time, so its running time is $O(n)$.

```
INPUT: cotree T_G, scalar x
OUTPUT: diagonal matrix D = [d_1, d_2, ..., d_n] congruent to A(G) + xI
 Algorithm Diagonalize (T_G, x)
    initialize d_i := x, for 1 ≤ i ≤ n
    while  T_G has ≥ 2 leaves
        select siblings {v_k, v_l} of maximum depth with parent w
        α ← d_k    β ← d_l
        if w = ⊗
           if α + β ≠ 2                      //subcase 1a
               d_k ← α + β − 2;  d_l ← (αβ−1)/(α+β−2);  T_G = T_G − v_k
           else if  β = 1                    //subcase 1b
               d_k ← 0;  d_l ← 1;  T_G = T_G − v_k
           else                              //subcase 1c
               d_k ← −(1 − β)^2;  d_l ← 1;  T_G = T_G − v_k;  T_G = T_G − v_l
        else if w = ∪
           if α + β ≠ 0                      //subcase 2a
               d_k ← α + β;  d_l ← (αβ)/(α+β);  T_G = T_G − v_k
           else if  β = 0                    //subcase 2b
               d_k ← 0;  d_l ← 0;  T_G = T_G − v_k
           else                              //subcase 2c
               d_k ← −β;  d_l ← β;  T_G = T_G − v_k;  T_G = T_G − v_l
    end loop
```

FIGURE 2. Diagonalization algorithm

Theorem 1. *For inputs T_G and x, where T_G is the cotree of cograph G having adjacency matrix A, algorithm* Diagonalize *computes a diagonal matrix D, which is congruent to $A + xI$ using $O(n)$ time and space.*

It is interesting that for threshold graphs, the cotree is a *caterpillar*, as shown in [17, Cor. 3.2]. When Diagonalize is given such a cotree, it performs essentially the same arithmetic as the threshold graph algorithm in [12].

Example 1. *In the remainder of this section we apply Algorithm* Diagonalize *to the cograph in Figure 1 with $x = 0$. In our figures, diagonal values d_i appear under the vertex v_i in the cotree. Initially, all d_i will be $x = 0$. Sibling pairs appear in red. Dashed edges indicate vertices about to be removed from the cotree. To follow this example, the reader need not be concerned with labels of vertices, but simply check that each cotree produces the next one.*

In the first iteration of the algorithm, siblings $\{v_k, v_l\}$ are chosen of maximum depth. Since $d_k = \alpha = 0$ and $d_l = \beta = 0$, **subcase 2b** occurs. The assignments

$$d_k \leftarrow 0 \qquad d_l \leftarrow 0$$

are made, represented in Figure 3. Figure 4 depicts the cotree after vertex v_k is removed and v_l is relocated under its parent's parent using the rules in Lemma 4.

In the next step, a sibling pair $\{v_k, v_l\}$ of depth three is chosen. Since their parent is $\otimes$, and $d_k = \alpha = 0$ and $d_l = \beta = 0$, **subcase 1a** is executed and the following assignments are made:

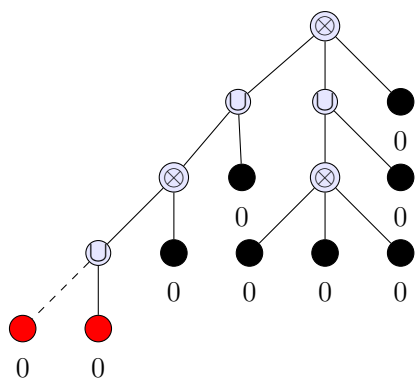$$d_k \leftarrow -2 \qquad d_l \leftarrow \frac{1}{2}$$
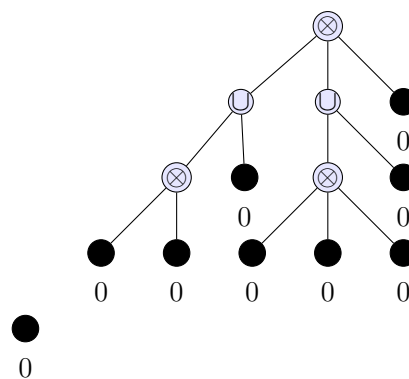
FIGURE 3.

FIGURE 4.

Figure 5 shows the cotree after these assignments, and Figure 6 shows the cotree with v_k removed and v_l relocated.
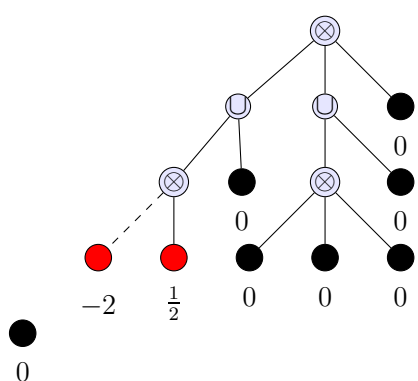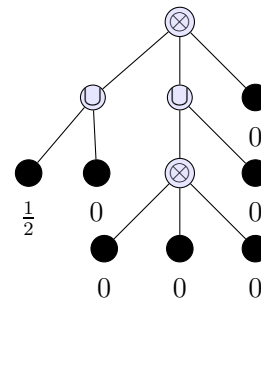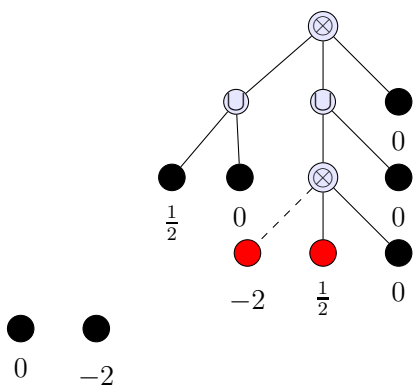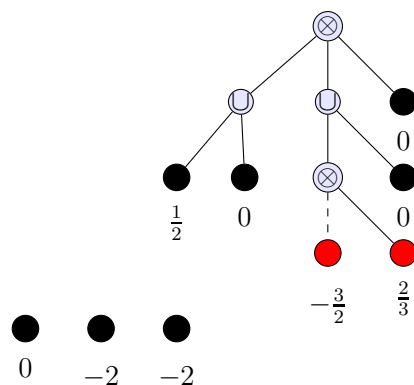

FIGURE 5.

FIGURE 6.

Next, another depth three sibling pair $\{v_k, v_l\}$ is chosen. Since $d_k = \alpha = 0$ and $d_l = \beta = 0$, **subcase 1a** is taken and assignments

$$d_k \leftarrow -2 \qquad d_l \leftarrow \frac{1}{2}$$

made. Vertex v_k is removed from the cotree as shown in Figure 7, making the value $d_k = -2$ permanent. Note v_l does not move per the rules in Lemma 4.

Depth three siblings $\{v_k, v_l\}$ are selected again. Since $d_k = \alpha = \frac{1}{2}$ and $d_l = \beta = 0$, **subcase 1a** is again executed and the assignments

$$d_k \leftarrow -\frac{3}{2} \qquad d_l \leftarrow \frac{2}{3}$$

are made as shown in Figure 8. Vertex v_k will be removed from the cotree, and v_l relocated to its parent's parent, as Figure 9 shows.

Next, a depth two pair $\{v_k, v_l\}$ is chosen. Since $d_k = \alpha = \frac{1}{2}$, and $d_l = \beta = 0$, **subcase 2a** is applied, and the following assignments are made:

$$d_k \leftarrow \frac{1}{2} \qquad d_l \leftarrow 0$$

FIGURE 7.

FIGURE 8.

Their values don't change, as indicated in Figure 10. Vertex v_k is removed from the cotree, and vertex v_l is relocated as shown in Figure 11. Next, a depth two pair $\{v_k, v_l\}$ is selected. Since $d_k = \alpha = \frac{2}{3}$ and $d_l = \beta = 0$. the **subcase 2a** is applied again.
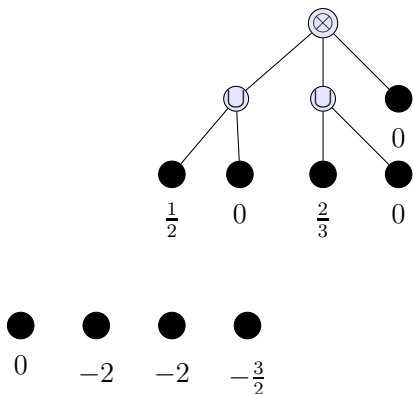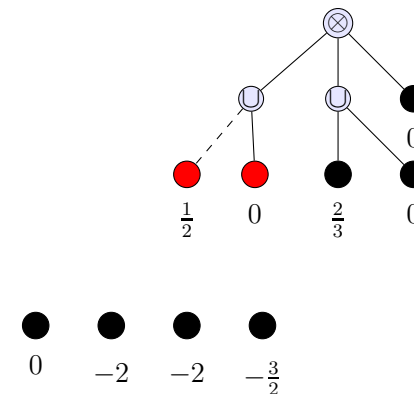
FIGURE 9.

FIGURE 10.

The assignments are made

$$d_k \leftarrow \frac{2}{3} \qquad d_l \leftarrow 0.$$

leaving the d_k and d_l unchanged, as shown in Figure 12. Vertex v_k is removed from the cotree and v_l relocated to the cotree's root, as shown in Figure 13.
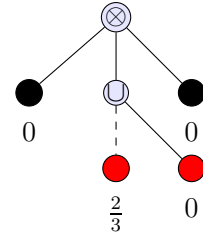
FIGURE 11.

FIGURE 12.

In the penultimate step, a depth one pair $\{v_k, v_l\}$ is chosen. Since $d_k = \alpha = 0$ and $d_l = \beta = 0$, **subcase 1a** is applied, and assignments

$$d_k \leftarrow -2 \qquad d_l \leftarrow \frac{1}{2}$$
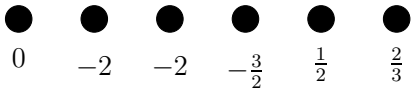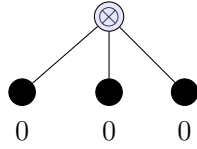
are made, as shown in Figure 14.
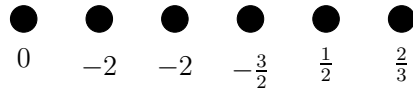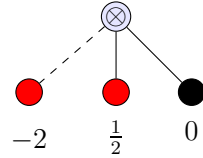

FIGURE 13.

FIGURE 14.

The final sibling pair $\{v_k, v_l\}$ is chosen. Since $d_k = \alpha = \frac{1}{2}$ and $d_l = \beta = 0$, **subcase 1a** is applied and

$$d_k \leftarrow -\frac{3}{2} \qquad d_l \leftarrow \frac{2}{3}$$

as shown in Figure 15. When we remove v_k using Lemma 4, the remaining cotree is v_l. The algorithm stops, and the final diagonal is shown in Figure 16.
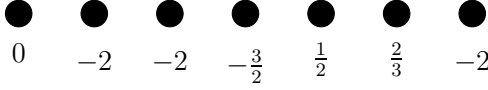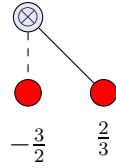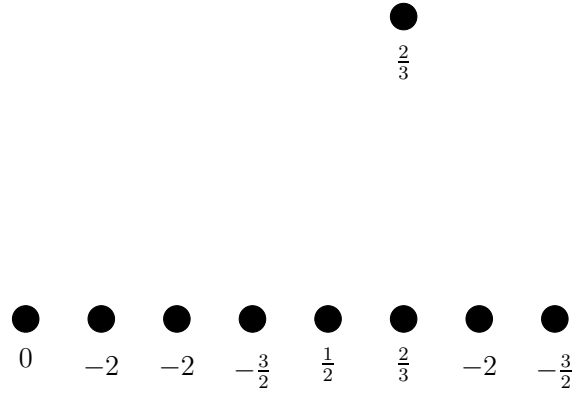
FIGURE 15.

FIGURE 16.

5. LOCATING EIGENVALUES

As an application, we can compute in $O(n)$ time the number of eigenvalues of a cograph in a given interval, as was done in [11] for trees and in [12] for threshold graphs. The following theorem is called Sylvester's Law of Inertia [3, p. 336].

Theorem 2. *Two $n \times n$ real symmetric matrices are congruent if and only if they have the same number of positive eigenvalues and the same number of negative eigenvalues.*

The proof of the following theorem is similar to Theorem 3 in [12], and is based on Theorem 2.

Theorem 3. *Let $D = [d_1, d_2, \ldots, d_n]$ be the diagonal returned by* Diagonalize$(T_G, -a)$, *and assume D has k_+ positive values, k_0 zeros, and k_- negative values.*

    i: *The number of eigenvalues of G that are greater than a is exactly k_+.*
    ii: *The number of eigenvalues of G that are less than a is exactly k_-.*
    iii: *The multiplicity of a is k_0.*

Note that when we use Diagonalize(T_G, x) and obtain $D = [d_1, d_2, \ldots, d_n]$, the order of the d_i's depends on the order in which maximum depth sibling pairs are selected. It is not clear whether the values in D are invariant, but their signs certainly are.

Example 2. *As Figure 16 indicates,* Diagonalize$(T_G, 0)$, *produces three positive entries, five negative entries and one zero. Therefore G has 3 positive eigenvalues, 5 negative eigenvalues and 0 is an eigenvalue with multiplicity one.*

Example 3. *One can check that* Diagonalize$(T_G, 1)$, *produces a diagonal with multiset $\{2, -\frac{1}{2}, 0, 0, 2, 2, -1, -\frac{1}{4}, 1\}$. Since there are four positive entries, three negative entries and two zeros, it follows that there are four eigenvalues greater than -1, three less than -1, and -1 has multiplicity two.*

Assume $a < b$, and let k_+ be the number of positive values in the diagonal of Diagonalize$(T_G, -a)$, and let j_+ be number of positive values in the diagonal of Diagonalize$(T_G, -b)$. Then $(a, b]$ must contain exactly $k_+ - j_+$ eigenvalues. The number of eigenvalues in (a, b) is $k_+ - j_+ - j_0$, where j_0 is the multiplicity of b. Thus we can find the number of eigenvalues in an interval by making two calls to the algorithm.

Example 4. *From Example 2 and Example 3, it follows that G has exactly $4 - 3 = 1$ eigenvalue in $(-1, 0]$. However, from Example 2 we know that zero is an eigenvalue, so*

there must be no eigenvalues in $(-1, 0)$. This is not an accident. Recently in [14], *the striking result was obtained that no cograph has eigenvalues in $(-1, 0)$.*

Any eigenvalue λ can be approximated by first finding an interval for which

$$a < \lambda \le b.$$

By using a divide-and-conquer approach in which the interval length is successively cut in half, one can find an interval $(a', b']$ of arbitrarily small size for which

$$a' < \lambda \le b'.$$

See [12, Sec. 4] for more detail.

6. Inertia of cographs

The *inertia* of a graph G is the triple (n_+, n_0, n_-), where n_+, n_0, n_- denote respectively the number of positive, zero, and negative eigenvalues of G. We wish to compute the inertia of *any* cograph G. If G is a single vertex, its inertia is $(0, 1, 0)$, so we assume that G has order $n \ge 2$ with cotree T_G. Note also that since $n = n_+ + n_0 + n_-$, it suffices to compute only two of these components. In what follows, we will obtain formulas for n_- and n_0 by using our diagonalization algorithm. Our formula for n_- given in Theorem 4 below appears to be new. The formulas for n_0 (Theorem 5) and the multiplicity of -1 (Theorem 6) can be found in [2]. Formulas relating inertia to the representation of threshold graphs can be found in the papers [1, 13].

By Theorem 3, we can obtain the inertia of a cograph by applying `Diagonalize` to T_G with $x = 0$, and then counting the number of entries in the diagonal that are positive, zero and negative. The following technical result appears in Lemma 3 of [5].

Lemma 6. *Suppose that $0 \le \alpha, \beta < 1$. Then*
(a) $0 < \frac{\alpha\beta - 1}{\alpha + \beta - 2} < 1$
(b) $0 \le \frac{\alpha\beta}{\alpha + \beta} < 1$, *provided $\alpha + \beta \ne 0$.*

Lemma 7. *If $\{v_k, v_l\}$ is a sibling pair processed by `Diagonalize`, with parent $w = \otimes$ for which $0 \le d_k, d_l < 1$, then d_k becomes permanently negative, and d_l is assigned a value in $(0, 1)$.*

Proof. By our assumption, **subcase 1a** is executed. Hence

$$
\begin{aligned}
d_k &\leftarrow \alpha + \beta - 2 \\
d_l &\leftarrow \frac{\alpha\beta - 1}{\alpha + \beta - 2}
\end{aligned}
$$

where α, β are the old values of d_k, d_l. Clearly $d_k < 0$. By Lemma 6-(a), $d_l \in (0, 1)$. $\square$

Lemma 8. *If $\{v_k, v_l\}$ is a sibling pair processed by `Diagonalize`, with parent $w = \cup$ for which $0 \le d_k, d_l < 1$, then d_k becomes permanently nonnegative and d_l is assigned a value in $[0, 1)$.*

Proof. If $d_k > 0$ or $d_l > 0$, then **subcase 2a** is executed which means

$$
\begin{aligned}
d_k &\leftarrow \alpha + \beta \\
d_l &\leftarrow \frac{\alpha\beta}{\alpha + \beta}
\end{aligned}
$$

Clearly $d_k > 0$ and by Lemma 6-(b), $d_l \in [0, 1)$. If $d_k = d_l = 0$, then **subcase 2b** is executed meaning that both d_k and d_l are assigned 0, d_k permanently so. $\square$

Lemma 9. *During the execution of* `Diagonalize`$(T_G, 0)$, *all diagonal values of vertices remaining on the cotree are in* $[0, 1)$.

Proof. Initially all values on T_G are zero. Suppose after m iterations of `Diagonalize` all diagonal values of the cotree are in $[0, 1)$, and consider iteration $m + 1$ with sibling pair $\{v_k, v_l\}$ and parent w. By assumption, $0 \leq d_k, d_l < 1$. If $w = \otimes$ then Lemma 7 guarantees the vertex d_l remaining on the tree is assigned a value in $(0, 1)$. If $w = \cup$, Lemma 8 guarantees $d_l \in [0, 1)$. This means after $m + 1$ iterations the cotree $T_G - v_k$ satisfies the desired property, completing the proof. $\square$

Remark 1. *Observe that if w is an interior node in T_G having t children, as the algorithm progresses bottom up through the rules of Lemma 4, each interior child of w eventually is replaced by a leaf. Thus when w is ready to be processed it will have t leaves as children. To simplify our analysis, without loss of generality we can assume that all $t - 1$ sibling pairs are processed consecutively.*

The following theorem shows that the quantity $n_-(G)$ can be computed in linear time from the cotree.

Theorem 4. *Let G be a cograph with cotree T_G having $\otimes$-nodes $\{w_1, \ldots, w_j\}$, and assume each w_i has t_i children in T_G. Then*

$$n_-(G) = \sum_{i=1}^{j}(t_i - 1).$$

Proof. In executing `Diagonalize`$(T_G, 0)$, consider an interior node w_i in T_G of type $\otimes$ with t_i children. By Remark 1, when it becomes eligible to be processed, it will have t_i leaves, and the algorithm will process $t_i - 1$ sibling pairs. By Lemma 9 all diagonal values on the cotree remain in $[0, 1)$. By Lemma 7 each of the $t_i - 1$ sibling pairs will produce a permanent negative value before w_i is removed. This shows

$$n_-(G) \geq \sum_{i=1}^{j}(t_i - 1).$$

However Lemma 8 shows that processing a sibling pair with parent $\cup$ can only produce nonnegative permanent values. Hence the inequality is tight, completing the proof. $\square$

We now consider $n_0(G)$. A formula by Bıyıkoğlu, Simić and Stanić is known [2, Cor. 3.2] but we give an alternate proof using our algorithm.

Remark 2. *Consider an interior node w of type $\cup$ with $k = s + t$ children, where s children are interior (of type $\otimes$) and t children are terminal. In the execution of* `Diagonalize`$(T_G, 0)$*, from Lemmas 7 and 9 each $\otimes$ node will become positive. Thus when w is processed, it will have s leaves with positive values and t leaves with zero.*

Theorem 5. *Let G be a cograph with cotree T_G having $\cup$-nodes $\{w_1, \ldots, w_m\}$, where w_i has t_i terminal children. If G has $j \geq 0$ isolated vertices, then*

$$n_0(G) = j + \sum_{i=1}^{m}(t_i - 1).$$

Proof. Consider the execution of `Diagonalize`$(T_G, 0)$, and first consider the case when $j = 0$. Let w_i be an interior node of type $\cup$ having k_i children where s_i children are interior nodes and t_i are terminal. By Remark 2, when w_i is ready to be processed

it will have s_i positive children and t_i zeros. From Lemma 8 we see that **subcase 2a** will be executed s_i times and **subcase 2b** $t_i - 1$ times. Each execution of **subcase 2b** produces a permanent zero on the diagonal, and so w_i contributes to $t_i - 1$ zeros. This shows $n_0(G) \geq \sum_{i=1}^{m}(t_i - 1)$. To obtain equality, we note that no zero can be created when processing a sibling pair whose parent is $\otimes$. If G has $j > 0$ isolates, then the root of T_G has type $\cup$ with j children as leaves. We claim j additional zeros are created. Indeed, $j - 1$ are created through **subcase 2b**. The last iteration, either **subcase 2a** or **subcase 2b**, creates an additional zero. $\square$

It is possible to frame Theorem 5 in terms of duplicate vertices. Let $\{V_i\}$ be a partition over the set of all vertices that are in duplicate pairs, such that each V_i contains mutually pairwise duplicate vertices, and for $i \neq j$, $v \in V_i$ and $w \in V_j$ imply $N(v) \neq N(w)$. Then $|V_i| = t_i$.

We mention two other known theorems that can be obtained through our algorithm. While we omit the details, their proofs involve algorithm analysis when $x = 1$. One such result [14] is that no cograph has an eigenvalue in $(-1, 0)$. Another [2] involves the multiplicity of -1:

Theorem 6. *Let G be a cograph with cotree T_G having $\otimes$-nodes $\{w_1, \ldots, w_m\}$, where w_i has t_i terminal children. Then the multiplicity of -1 is $\sum_{i=1}^{m}(t_i - 1)$.*

We apply the theorems in this section to the graph of Figure 1. By Theorem 5, we have $n_0(G) = 2 - 1 = 1$. Applying Theorem 4, $n_-(G) = (3-1) + (3-1) + (2-1) = 5$. Therefore $n_+(G) = 9 - 5 - 1 = 4$. By Theorem 6 the multiplicity of -1 in G is two. These numbers agree with those given in Section 5.

7. Equienergetic cographs

Recall that the *energy* $E(G)$ of a graph G is defined to be $\sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are its eigenvalues. If $E(G) = E(H)$, we say G and H are *equienergetic*. It is known that the complete graph K_n has energy $2(n - 1)$. We finish this paper by exhibiting an infinite class $\mathcal{G} = \{G_1, G_2, \ldots, G_r, \ldots\}$ where G_r is a cograph of order $n = 3r + 4$ and $E(G_r) = E(K_{3r+4})$. In [13] the authors gave similar examples of threshold graphs, however our present example involves non-threshold graphs. This is of interest because equienergetic examples seem to often involve non-integer energy.

The following two technical lemmas describe the diagonalization algorithm when multiple leaves of the same parent have the *same* diagonal value $d_i = y$. In particular, when an interior vertex w of the cograph has m terminal children, $v_1, \ldots, v_m$, the algorithm will generally make $m - 1$ iterations. If all d_i are equal, then under certain conditions, can say exactly what assignments are made at each iteration. We can assume sibling pairs are chosen to be the leftmost pair. In Lemma 10, we prove the case when $w = \otimes$, and the case $w = \cup$ in Lemma 11 is handled in a similar way.

Lemma 10. *If $v_1, \ldots, v_m$ have parent $w = \otimes$, each with diagonal value $y > 1$, then the algorithm performs $m - 1$ iterations of* **subcase 1a** *assigning, during iteration j:*

$$d_k \quad \leftarrow \quad \frac{j+1}{j}(y - 1) \tag{3}$$

$$d_l \quad \leftarrow \quad \frac{y+j}{j+1} \tag{4}$$

Proof. It is easy to check that when $j = 1$, since $\alpha = \beta = y > 1$, **subcase 1a** will be chosen and perform assignments $d_k \leftarrow 2(y - 1)$ and $d_l \leftarrow \frac{y+1}{2}$. Now suppose iteration j

makes assignments (3) and (4). Then during iteration $j + 1$, we will have $\alpha = \frac{y+j}{j+1} > 1$ and $\beta = y > 1$. Therefore $\alpha + \beta \neq 2$ and **subcase 1a** will execute again. From the rules of the algorithm we get

$$d_k \;\leftarrow\; \alpha + \beta - 2 = \frac{y+j}{j+1} + y - 2 = \frac{j+2}{j+1}(y-1)$$

$$d_l \;\leftarrow\; \frac{\alpha\beta - 1}{\alpha + \beta - 2} = \Big(\frac{y^2 + jy}{j+1} - 1\Big)\frac{j+1}{(j+2)(y-1)} = \frac{j+y+1}{j+2}$$

which completes the induction. $\square$

Lemma 11. *If $v_1, \ldots, v_m$ have parent $w = \cup$, each with diagonal value $y > 0$, then the algorithm performs $m - 1$ iterations of* **subcase 2a**, *assigning during iteration j:*

$$d_k \;\leftarrow\; \frac{(j+1)y}{j} \tag{5}$$

$$d_l \;\leftarrow\; \frac{y}{j+1} \tag{6}$$

Proof. Similar to Lemma 10. $\square$

For each integer $r \geq 1$, we now define the cograph G_r to be the join of K_{r+2} with the disjoint union of $r + 1$ copies of K_2, that is

$$G_r = \underbrace{(K_2 \cup K_2 \cup \ldots \cup K_2)}_{r+1} \otimes K_{r+2}.$$

G_r has order $n = 3r + 4$ and its cotree is shown in Figure 17. Let $m(\lambda; G)$ denote the multiplicity of an eigenvalue in G.

Lemma 12. *G_r has inertia $(r + 1, 0, 2r + 3)$ and $m(-1; G_r) = 2(r + 1)$.*

Proof. This follows from Theorem 4, Theorem 5 and Theorem 6. $\square$

Lemma 13. *The eigenvalue 1 in G_r has multiplicity r.*

Proof. By Theorem 3 it suffices to show that $\texttt{Diagonalize}(T_{G_r}, -1)$ creates exactly r zeros. After initializing vertices with -1, the algorithm performs **subcase 1a** on each of the $r + 1$ sibling pairs $\{v_k, v_l\}$ of depth three. Each operation

$$d_k \;\leftarrow\; -4$$
$$d_l \;\leftarrow\; 0$$

leaves a zero on the tree at depth two. After all depth three pairs have been processed, the algorithm applies **subcase 2b** to r sibling pairs $\{v_k, v_l\}$ where $d_k = d_l = 0$. This creates r permanent zeros, and so $m(1; G_r) \geq r$. By Lemma 12, G_r has exactly $r + 1$
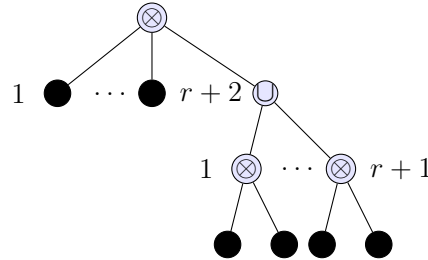

FIGURE 17. The cotree T_{G_r}

positive eigenvalues. However $m(1; G_r) = r + 1$ would contradict the well-known fact that a graph's largest eigenvalue is simple. So we have $m(1; G_r) = r$. $\square$

Since $n = 3r + 4$, Lemmas 12 and 13 account for all but two eigenvalues.

Lemma 14. *The largest and smallest eigenvalues of G_r are*

$$\lambda_1 = 2r + 3 \tag{7}$$
$$\lambda_n = -(r + 1). \tag{8}$$

Proof. We will prove (8), and (7) will follow since a graph's eigenvalues sum to zero. To show (8), it suffices to prove that $\texttt{Diagonalize}(T_{G_r}, r + 1)$ creates a non-negative diagonal with a single zero. Consider the $r + 1$ sibling pairs at depth three. Since $\alpha = \beta = r + 1$, for each pair **subcase 1a** is executed. The assignments

$$d_k \leftarrow 2r$$
$$d_l \leftarrow \frac{r + 2}{2}$$

are made, leaving $r + 1$ vertices with a permanent positive value of $2r$.

At depth two, there are $r + 1$ leaves all with positive value $y = \frac{r+2}{2}$, and r iterations are performed. By Lemma 11, each iteration generates the positive permanent diagonal value of (5). On iteration r, the assignment in (6)

$$d_l \leftarrow \frac{y}{j + 1} = \frac{r + 2}{2(r + 1)}$$

is made to a vertex u which then gets moved under the root.

At depth one, there are $r + 3$ leaves: $r + 2$ with diagonal value $r + 1$, and a single leaf u whose diagonal value is $\frac{r+2}{2(r+1)}$. Assume the algorithm processes the leaves with identical value first. Letting $y = r + 1$ in Lemma 10, we see that on each of the $r + 1$ iterations the algorithm generates the permanent positive diagonal value in (3). Letting $j = r + 1$ in (4) we see that the last iteration leaves the value $\frac{2(r+1)}{r+2}$ on the tree. For the last step of the algorithm, we process the two remaining vertices whose diagonal values are $\alpha = \frac{2(r+1)}{r+2}$ and $\beta = \frac{r+2}{2(r+1)}$. Since $\alpha > 2$ and $\beta > 0$, on the last iteration **subcase 1a** assigns

$$d_k \leftarrow \alpha + \beta - 2 > 0$$
$$d_l \leftarrow \frac{\alpha\beta - 1}{\alpha + \beta - 2} = 0$$

creating a positive value and zero for the last two diagonal entries. $\square$

From Lemma 12, Lemma 13, and Lemma 14 it follows that

$$E(G_r) = 2(r + 1) + r + (2r + 3) + (r + 1) = 2(3r + 4) - 2 = 2n - 2$$

so we have:

Theorem 7. *For each $r \geq 1$, G_r and K_{3r+4} are equienergetic.*

Clearly G_r and K_n are noncospectral. Note G_r is not a threshold graph since its cotree is not a caterpillar. Three other infinite classes of cographs, equienergetic to complete graphs, were discovered. In all cases, the cotrees had depth three. Since the cotree structure and proofs are similar to the above example, we omit them.

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