MINIMUM WIENER INDEX OF TRIANGULATIONS AND QUADRANGULATIONS

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Abstract. The Wiener index of a connected graph is the sum of the distances between all unordered pairs of vertices. We provide formulae for the minimum Wiener index of simple triangulations and quadrangulations with connectivity at least c, and provide the extremal structures, which attain those values. Our main tool is setting upper bounds for the maximum degree in highly connected triangulations and quadrangulations.

1. Definitions

Let G be a connected graph. The *Wiener index* of G, denoted by $W(G)$, is the sum of the distances between all unordered pairs of vertices. In formula,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v),$$

where $d_G(u,v)$ denotes the number of edges on the shortest path between the two vertices u and v. This index was introduced in 1947 [6] to predict the boiling point of alkanes. The Wiener index is perhaps the most frequently used graph parameter in the sciences.

Throughout this paper, every graph will be simple, finite and connected unless otherwise stated. For a graph G, the sets V and E represent the vertices and edges of G, respectively. The order of a graph is its number of vertices. The set of *neighbors* of the vertex v is denoted by $N(v)$, and the *degree* of a vertex v is denoted by $d(v) = |N(v)|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. A *cutset* is a set of vertices, whose removal makes the graph disconnected. A non-complete graph G of order $n \geq 3$ is *c-connected* for a positive integer c, if every cutset has size at least c; the *connectivity* $\kappa(G)$ of G is the largest c for which G is c-connected. Clearly, if G is not a complete graph, then G has at least $\kappa(G)+2$ vertices, the smallest cutset of G has size $\kappa(G)$, and all degrees in G are at least $\kappa(G)$. The notation $\simeq$ indicates the isomorphism of two graphs.

In this paper we will only be concerned with *planar* graphs. Those are the graphs that can be drawn in the plane (or equivalently, in the sphere), such that no edges cross. We will often rely on *Euler's formula*, which states that for any finite, connected planar graph G drawn in the plane,

$$n - e + f = 2,$$

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where n is the order, e is the number of edges, and f is the number of faces in G. In this paper, *triangulations* and *quadrangulations* are simple graphs drawn in the sphere, in which every face is a triangle or every face is a quadrangle, respectively. Euler's formula immediately implies that triangulations of order n have $3n-6$ edges and $2n-4$ faces, and quadrangulations with n vertices have $2n-4$ edges and $n-2$ faces. It is well-known that triangulations are 3-connected but Euler's formula does not allow them to be 6-connected, and quadrangulations are 2-connected but Euler's formula does not allow them to be 4-connected. Whitney's theorem [7] implies that all drawings of any 3-connected planar graph on the sphere are the same combinatorially, a conclusion that holds for all classes that we consider in this paper except general quadrangulations. We cite an elegant result of [5], although we do not use it explicitly: every 5-connected triangulation contains a spanning 3-connected quadrangulation. Comparison of Figures 5 and 9 incidentally gives an illustration for this result.

2. Results on Triangulations and Quadrangulations

Recently, there have been numerous results regarding the Wiener index on *triangulations* and *quadrangulations* of the sphere, which are edge maximal simple planar graphs, and edge maximal bipartite simple planar graphs, respectively. These recent results have mainly focused on upper bounds, see [1], [2], [3], [4]. Lower bounds for the Wiener index of such graphs were stated in [1], [2], without making extra assumptions on the connectivity. In this paper we complete the study of the minimum Wiener index of triangulations and quadrangulations, by determining the minimum Wiener index among c-connected simple triangulations and quadrangulations.

Theorem 1 ([2])**.** *Assume $n \geq 6$. The triangulation T_n^4 defined in* Figure 1 *minimizes the Wiener index among all triangulations of order n. The triangulation T_n^4 is 4-connected. Consequently, the triangulation T_n^4 minimizes the Wiener index among all 4-connected n-vertex triangulations as well.*

Remark: T_5^4 is the only triangulation of order 5, but it is not 4-connected. Gray vertices and dashed edges in the figures indicate the pattern to be repeated as n increases.

Proof. A triangulation contains $3n-6$ edges, thus there are exactly $3n-6$ pairs of vertices at distance 1 apart. If we can make sure that every remaining pair of vertices are at distance 2 apart, then we have a triangulation whose Wiener index is $2(\binom{n}{2}-(3n-6))+(3n-6)=n^2-4n+6$, and this is clearly the minimum possible Wiener index. This is the case with T_n^4. Furthermore, it is easy to see that T_n^4 is 4-connected for all $n \geq 6$. □

Triangulations and 4-connected triangulations fail to produce unique structures to minimize the Wiener index. With the aid of a computer, we evaluated the number of non-isomorphic triangulations of minimum Wiener index up to order 18, and the number of non-isomorphic 4-connected triangulations of minimum Wiener index up to order 22, see Table 1. Considering the large numbers in the Table 1, the classification of extremal structures seems hopeless. As will be shown, there are two 5-connected triangulations on 19

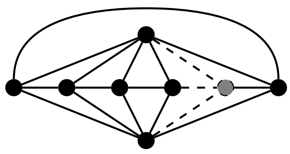

FIGURE 1. The triangulation T_n^4, which is the join of the cycle C_{n-2} with the edgeless graph on two vertices, minimizes the Wiener index among all triangulations of order $n \geq 5$ and are 4-connected for $n \geq 6$.

Order	General Triangulation Count	4-Connected Triangulation Count
4	1	0
5	1	0
6	2	1
7	5	1
8	12	2
9	36	4
10	99	6
11	255	10
12	614	10
13	1532	14
14	3908	15
15	10727	19
16	31242	21
17	96725	25
18	311735	27
19		32
20		34
21		39
22		42

TABLE 1. A summary of how many isomorphism classes, on n vertices, attain the minimum Wiener index for general and 4-connected triangulations.

vertices that minimize the Wiener index. All other graph classes studied in this paper will produce unique extremal graphs minimizing the Wiener index.

Theorem 2 ([1],[2]). *Assume $n \geq 4$. The complete bipartite graph $K_{2,n-2}$ minimizes the Wiener index among all quadrangulations.*

Proof. A quadrangulation contains $2n-4$ edges, thus exactly $2n-4$ pairs of vertices are at distance 1 apart. If we can make sure that every remaining pair of vertices are at distance 2 apart, we have a quadrangulation of Wiener index $2(\binom{n}{2}-(2n-4))+(2n-4) = n^2-3n+4$.

This is the case with the quadrangulation $K_{2,n-2}$. Clearly this is the least possible Wiener index of a quadrangulation. □

Theorem 3. *Assume $n \geq 4$. Up to isomorphism, the graph $K_{2,n-2}$ is the unique minimizer of the Wiener index among all quadrangulations of order n.*

Proof. Let Q be a quadrangulation of order n that has the same Wiener index as $K_{2,n-2}$, i.e. every non-adjacent pair of vertices are at distance 2. As quadrangulations are 2-connected, the minimum degree $\delta := \delta(Q) \geq 2$. Let v be a vertex of Q with $d(v) = \delta$, and let $u_1, \dots, u_\delta$ be the neighbors of v. The remaining $n - \delta - 1$ vertices are at distance 2 from v. As quadrangulations are bipartite, these $n - \delta - 1$ vertices can only be adjacent to $u_1, \dots, u_\delta$, and have degree at least δ. Thus we get that $Q \simeq K_{\delta,n-\delta}$. Since δ is the minimum degree, $\delta \leq n - \delta$, therefore Q contains $K_{\delta,\delta}$ as a subgraph. Since Q is planar, we get $\delta = 2$ and $Q \simeq K_{2,n-2}$. □

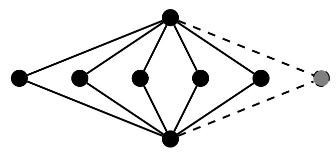

FIGURE 2. The graph $K_{2,n-2}$ which minimizes the Wiener index among all quadrangulations of order n.

Theorem 4. *Assume that $n \geq 8$, $n \neq 9$. The minimum Wiener index of 3-connected quadrangulations of order n is*

$$4\left\lceil \frac{n}{2} \right\rceil^2 + \left(\left\lceil \frac{n}{2} \right\rceil + 21\right)\left(\left\lfloor \frac{n}{2} \right\rfloor - 21\right) - 5n + 449 = \begin{cases} \frac{5n^2}{4} - 5n + 8, & \text{if } n \text{ is even,} \\ \frac{5n^2}{4} - 3n - \frac{49}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The unique minimizer of the Wiener index among 3-connected quadrangulations of order n is Q_n^3, defined in Figure 5.

The proof of this theorem is in Section 3, combining Lemma 10 (e) and Theorems 12 and 14. No 3-connected quadrangulation exists of order $n \leq 7$ by Euler's formula, and of order $n = 9$ by Lemma 7 (c).

Theorem 5. *Assume that $n \geq 12$, $n \neq 13$. The minimum Wiener index of 5-connected triangulations of order n is*

$$2n\left\lceil \frac{n}{2} \right\rceil + \left(\left\lceil \frac{n}{2} \right\rceil + 14\right)\left(\left\lfloor \frac{n}{2} \right\rfloor - 14\right) - 7n + 208 = \begin{cases} \frac{5n^2}{4} - 7n + 12, & \text{if } n \text{ is even,} \\ \frac{5n^2}{4} - 6n - \frac{9}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The unique minimizer of the Wiener index among 5-connected triangulations of order $n \neq 19$ is T_n^5, defined on Figure 9, *while for $n = 19$, exactly two minimizers exist, namely T_{19}^5, and the 5-connected triangulation X of order 19, defined on* Figure 10.

The proof of this theorem is in Section 4, combining Lemma 18 (e), Lemma 10 (e) and Theorems 20 and 24. No 5-connected triangulation exists of order $n \leq 11$ by Euler's formula, and of order $n = 13$ by Lemma 17.

3. Minimum Wiener Index of 3-Connected Quadrangulations

Note that Euler's formula implies that there are no 3-connected quadrangulations on fewer than 8 vertices.

First, we define an auxiliary drawn graph, which we will use extensively in this section. Let v be a vertex of a 3-connected quadrangulation G. We define the *sunflower graph* S_v around v (in the planar drawing of G), as v connected to its neighbors $u_1, \ldots, u_d$ (listed in the cyclic order of the drawing, $d = d(v)$), and different vertices $w_1, \ldots, w_d$ where w_i is connected to u_i and u_{i+1} (indices taken modulo d, see Figure 3). We understand S_v as a part of the drawing of G.

First we need to show that such a graph, with distinct vertices, exists in the drawing. We will also need some special properties of the sunflower graph, which will be shown in Lemma 6 below.

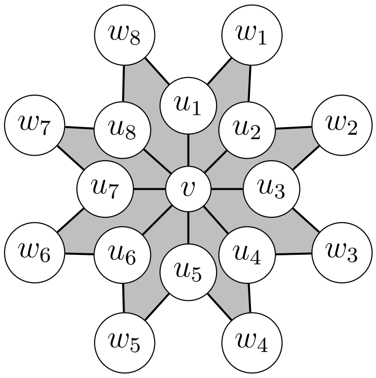

Figure 3. The sunflower graph S_v around v with $d(v) = 8$. The region $\mathcal{R}_v$ is shaded.

Lemma 6. *Assume that Q is a drawing of a 3-connected quadrangulation. Then, for any vertex v, Q contains a sunflower graph S_v with $2d(v) + 1$ distinct vertices. Furthermore, the region $\mathcal{R}_v$ that contains v and is bounded by the cycle $C_v = u_1 w_1 \ldots u_{d(v)} w_{d(v)}$ contains no vertices or edges that are not in S_v.*

Proof. We know $\delta(Q) \geq 3$ by the 3-connectedness. Label the neighbors of v by $u_1, \ldots, u_d$, in their planar cyclic order around v. For each pair of successive neighbors u_i and u_{i+1} (indices taken modulo d), let $w_i \neq v$ be their common neighbor that completes the face f_i that has u_i, v, u_{i+1} on its boundary. This means, in particular, that the interior of f_i has no vertices or edges. If y is a neighbor of u_i and $y \notin \{v, w_{i-1}, w_i\}$ then y must lie between w_{i-1} and w_i in the planar cyclic order around u_i, in particular, $w_{i-1} \neq w_i$ as $d(u_i) \geq 3$. As Q is bipartite, $u_i \neq w_j$ for all $1 \leq i, j \leq d$. We will show that each of the w_i's must be distinct. As $\mathcal{R}_v$ is the union of the faces f_i, this finishes the proof. Assume that $w_i = w_j$ for some $j \neq i$. We already know that $j \notin \{i-1, i+1\}$ and the vertices $u_i, u_{i+1}, u_j, u_{j+1}$ are all different. We consider two regions of the planar drawing of Q: $\mathcal{R}_1$ is bounded by the 4-cycle $u_{i+1}vu_jw_i$ and does not contain the vertex u_i, and $\mathcal{R}_2$ is bounded by the 4-cycle $u_ivu_{j+1}w_i$ and does not contain the vertex u_{i+1}. Thus the faces bounded by $u_ivu_{i+1}w_i$ and $u_jvu_{j+1}w_j$ are disjoint from $\mathcal{R}_1$ and $\mathcal{R}_2$. The neighbors of u_i that differ from v and w_i must lie in $\mathcal{R}_2$ and the neighbors of u_j that differ from v and w_i must lie in $\mathcal{R}_1$. Hence $\{v, w_i\}$ separates u_i from u_j (See Figure 4), contradicting the fact that Q is 3-connected. □

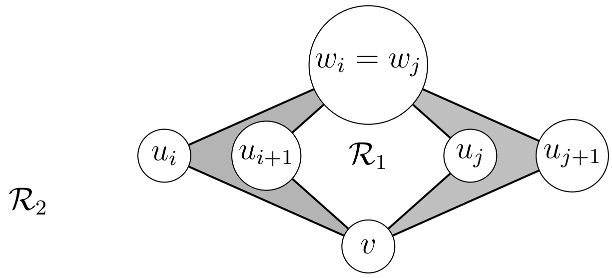

FIGURE 4. 2-element cutset appears in S_v, when $w_i = w_j$. The two faces $vu_iw_iu_{i+1}$ and $vu_jw_ju_{j+1}$ are shaded.

Lemma 7. *Assume that G is a 3-connected quadrangulation with partite sets A, B. Then*
(a) $\Delta(G) \leq \min\{|A|-1, |B|-1\}$;
(b) *If $|B| < |A|$, then for all $x \in A$ we have $d(x) \leq |B| - 2$; and*
(c) $|V(G)| \neq 9$, *i.e., no 3-connected quadrangulations exist on 9 vertices.*

Proof. (a) Let v be a vertex with degree $\Delta = \Delta(G)$, we may assume $v \in B$. As the sunflower S_v is a subgraph of the planar drawing of G, $\Delta \leq \min(|B|-1, |A|)$. We are done unless $|B| > |A| = \Delta$, so assume that is the case. As $A = N(v)$, all neighbors of the vertices of B lie in $N(v)$, in particular, every w_i has at least 3 neighbors in A. For each i let k_i be the largest positive integer such that w_i has no neighbors in the set $\{u_{i-t} : 1 \leq t \leq k_i - 1\} \cup \{u_{i+1+t} : 1 \leq t \leq k_i - 1\}$. Since for $k = 1$ the sets $\{u_{i-t} : 1 \leq t \leq k-1\}$ and $\{u_{i+1+t} : 1 \leq t \leq k-1\}$ are empty, such positive integers exist, they have an upper bound from the fact that w_i has at least 3 neighbors in A, and for the largest such integer k_i we have that at least one of u_{i-k_i}, u_{i+1+k_i} is a neighbor of w_i that is different from u_i, u_{i+1}. Choose i_0 such that $k = k_{i_0}$ is minimal amongst the k_i. By renumbering the u_i if necessary and changing the direction of the cyclic order we can assume that $i_0 = 1$ and w_1 is connected to u_1, u_2, u_{2+k} but none of u_{1-t}, u_{2+t} for all $1 \leq t \leq k-1$. Let $\mathcal{R}$ be the region of the sphere bounded by the 4-cycle $u_2vu_{2+k}w_1$ that does not contain u_1. Consider w_2. By the definition of w_2 and the minimality of k, w_2 lies in $\mathcal{R}$ and it has at least one neighbor u_j that does not lie in $\mathcal{R}$. The edge w_2u_j must cross the boundary of $\mathcal{R}$, which contradicts the planarity of G. Thus, we have $\Delta(G) \leq \min\{|A|-1, |B|-1\}$, as claimed.

To prove the case (b), assume $|B| < |A|$, i.e. $n > 2|B|$. We already know that $\Delta(G) \leq |B| - 1$. Assume that A contains a vertex v of degree $|B| - 1$. Since $|A| = n - |B|$ and all other vertices of A have degree at least 3, we have that $2n - 4 \geq (|B| - 1) + 3(n - |B| - 1) = 3n - 2|B| - 4$, so $n \leq 2|B|$, a contradiction.

To prove the case (c), assume to the contrary that G has 9 vertices and partite sets A, B. We may assume $|B| < |A|$, and therefore $|B| \leq 4$. Then every vertex in A has degree at most 2, a contradiction. □

Lemma 8. *In a 3-connected quadrangulation G of order n, the number of unordered pairs of vertices at distance 2 is at most*

$$\frac{1}{2}\sum_v d^2(v) - 4(n-2).$$

This estimate is exact precisely when G has no non-facial 4-cycles.

Proof. Euler's Formula gives us that any quadrangulation on n vertices has $2(n-2)$ edges and $n-2$ faces. The number of 2-paths in G is equal to $\sum_v \binom{d(v)}{2} = \frac{1}{2}\sum_v d^2(v) - 2(n-2)$. This sum, however, overcounts the number of pairs of vertices distance 2 apart. In a 3-connected quadrangulation, two faces cannot share two consecutive edges from their boundaries. Thus, for each face, we are double counting the two pairs of vertices distance 2 apart, and so we may safely subtract $2(n-2)$. There are pairs of vertices which we have double counted even after the substraction precisely when there are non-facial 4-cycles. □

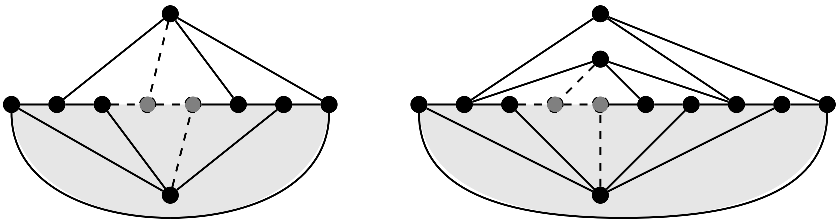

FIGURE 5. The quadrangulation Q_n^3 of order $n = 2k \geq 8$ (left) and $n = 2k+1 \geq 11$ (right), which minimizes the Wiener index among all 3-connected quadrangulations of order n. Gray vertices and dashed edges indicate the pattern to be repeated. The light gray regions are the sunflower graphs around a maximum degree vertex.

Lemma 9. *Let Q be a 3-connected quadrangulation of order n, with partite sets A, B. Then*

$$W(Q) \geq 2n^2 + 2n - 8 - |A||B| - \sum_v d^2(v). \quad (1)$$

Equality holds in (1) *precisely when the diameter of Q is at most 4 and Q has no non-facial 4-cycles.*

Proof. Let Q be an arbitrary 3-connected quadrangulation on n vertices, with partite sets A, B. Let D_i denote the number of unordered pairs of vertices at distance i in Q. Clearly $W(Q) = \sum_i i \cdot D_i$. Observe that $D_1 = 2n-4$, the number of edges; $D_2 \leq \frac{1}{2}\sum_v d^2(v) - 4(n-2)$ by Lemma 8; $D_2 + D_4 + D_6 + D_8 \cdots = \binom{|A|}{2} + \binom{|B|}{2}$, as pairs of vertices are at even distance precisely when they are from the same partite set; and finally, $D_1 + D_3 + D_5 + D_7 + \cdots = |A| \cdot |B|$, as pairs of vertices are at odd distance precisely when they are from different partite sets.

Combining all this information with the identity $|A| + |B| = n$, we obtain that

$$\begin{aligned} W(Q) &\geq (2n-4) + 2D_2 + 3\Big[|A| \cdot |B| - (2n-4)\Big] + 4\left[\binom{|A|}{2} + \binom{|B|}{2} - D_2\right] \\ &= 2n^2 - 2n - |A||B| - 2(D_2 + 2n - 4) \\ &\geq 2n^2 + 2n - 8 - |A||B| - \sum_v d^2(v). \end{aligned}$$

The first inequality in the displayed formula is an equality precisely when the diameter of Q is at most 4, and the second inequality is an equality precisely when Q has no non-facial 4-cycles. □

The 3-connected quadrangulation Q_n^3 of order $n \geq 8$, $n \neq 9$ is defined in Figure 5. The following lemma is easy to verify, and we leave the details to the reader.

Lemma 10. *Assume that $n \geq 8$, $n \neq 9$.*

(a) *Q_n^3 is a 3-connected quadrangulation.*
(b) *Q_n^3 has no non-facial 4-cycle.*
(c) *If n is even, Q_n^3 has diameter 3 and degree sequence $\frac{n}{2}-1, \frac{n}{2}-1, 3, \dots, 3$.*
(d) *If n is odd, Q_n^3 has diameter 4 and degree sequence $\lfloor\frac{n}{2}\rfloor - 1, \lfloor\frac{n}{2}\rfloor - 2, 4, 4, 3, \dots, 3$. (For $n = 11$, the terms in this sequence are not in decreasing order.)*
(e)
$$W(Q_n^3) = \begin{cases} \frac{5n^2}{4} - 5n + 8, & \text{if } n \text{ is even,} \\ \frac{5n^2}{4} - 3n - \frac{49}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The following is obvious, and we make use of it frequently

Lemma 11. *Assume that $\sum_{i=1}^n x_i = a > 0$ is given, where the x_i's are required to be integers from the interval $[b, c]$ with $0 \leq b$, and we have to maximize $\sum_{i=1}^n x_i^2$. As long as for some $(i \neq j)$ we have $b+1 \leq x_j \leq x_i \leq c-1$, we can increase the sum of squares while keeping the conditions by changing x_i to $x_i + 1$ and x_j to $x_j - 1$.*

Theorem 12. *Assume that the number $n \geq 8$ is even. The quadrangulation Q_n^3 defined in* Figure 5 *minimizes the Wiener index among all 3-connected quadrangulations of order n. Moreover, up to isomorphism, this minimizer is unique.*

Proof. Let Q be an arbitrary 3-connected quadrangulation on $n = 2k$ vertices, with partite sets A, B. Since Q is 3-connected, for all v, we have $d(v) \geq 3$, and by Lemma 7, $d(v) \leq \Delta(Q) \leq \min(|A| - 1, |B| - 1) \leq \frac{n}{2} - 1$. By Lemma 11 and Lemma 10 (c), $\sum_{v \in V(Q)} d^2(v) \leq$

$\sum_{v\in V(Q_n^3)} d^2(v)$ with equality precisely when Q has the same degree sequence as Q_n^3. Also, $|A|\cdot|B| \le \frac{n^2}{4}$ with equality precicely when $|A| = |B| = \frac{n}{2}$. Lemma 9 gives that $W(Q) \ge W(Q_n^3)$ with equality precisely when Q has the same degree sequence as Q_n^3, $|A| = |B| = \frac{n}{2}$, Q has diameter at most 4 and no nonfacial 4-cycles. In particular, Q_n^3 minimizes the Wiener index among n-vertex 3-connected quadrangulations.

We will show that the extremal quadrangulation is in fact unique. Assume that $W(Q) = W(Q_n^3)$, so Q has the same degree sequence as Q_n^3 and $|A| = |B| = k$ vertices. Then in both A and B we have $k-1$ vertices of degree 3, and the remaining one vertex must have degree $k-1$ $(k-1 \ge 3)$.

As before, let v be a vertex of maximum degree $k-1$, and construct the sunflower graph S_v around v. Since S_v has exactly $n-1$ vertices, Q has one additional vertex v'. This vertex v' is in the same partite class as the u_i vertices, and differs from v. Each of the w_i has one edge not in S_v incident upon it, connecting them to either v' or one of the u_j. If all vertices u_i have degree 3, then v' has degree $k-1 \ge 3$, and it is adjacent to all w_i (in which case we have Q_n^3). Otherwise the degree of v' is 3 and exactly one of the u_i (say u_2) has degree $k-1 > 3$ in Q. Assume that the latter is the case. As w_1 and w_2 have an edge not in $E(S_v)$ incident upon them, and $w_1u_2, w_2u_2 \in E(S_v)$, both w_1 and w_2 are adjacent to v'. Thus, $v'w_1u_2w_2$ bounds a facial region $\mathcal{R}$. As $w_1u_2w_2$, of which u_2 is an internal vertex, is the common boundary of $\mathcal{R}_v$ and $\mathcal{R}$, u_2 cannot have any edge outside of S_v incident upon it, a contradiction. □

Lemma 13. *Assume $n = 2k+1$, and let Q be a 3-connected quadrangulation of order n, with partite sets A, B. If*

$$\sum_{v\in V(Q)} d^2(v) < 2k^2 + 12k + 10, \tag{2}$$

then $W(Q) > W(Q_n^3)$. If $\Delta(Q) \le k-2$, then $W(Q) > W(Q_n^3)$.

Proof. First note that by Lemma 10 (d)

$$\sum_{x\in V(Q_n^3)} d^2(x) = (k-1)^2 + (k-2)^2 + 2\cdot 4^2 + 3^2(2k-3) = 2k^2 + 12k + 10,$$

and also $|A||B| \le k(k+1)$ (note that $k, k+1$ are the sizes of the partite classes in Q_n^3), so if (2) holds, then by Lemma 9 and Lemma 10 (d) we have $W(Q) > W(Q_n)$.

Since Q has odd number of vertices and minimum degree 3, the Handshaking Lemma implies $\Delta(Q) \ge 4$. Assume now that $\Delta(Q) \le k-2$, so $k \ge 6$. Let $x_1, \ldots, x_{2k+1}$ be a sequence of integers that maximizes $\sum x_i^2$ subject to the conditions that that $\sum x_i = 4(n-2)$ and $3 \le x_i \le k-2$. If $k = 6$, the maximizing sequence is $4,4,4,4,4,3,\ldots,3$ of length 13, and if $k = 7$ the maximizing sequence by Lemma 11 is $5,5,5,4,3,\ldots 3$ of length 15. In both of these cases, we have $\sum x_i^2 < 2k^2 + 12k + 10$. For $k \ge 8$, Lemma 11 gives that the maximizing sequence is $k-2, k-2, 6, 3, 3, \ldots, 3$, so

$$\sum_{x\in V(Q)} d^2(x) \le \sum_{i=1}^{2k+1} x_i^2 = 2(k-2)^2 + 6^2 + 3^2(2k-2) = 2k^2 + 10k + 26 \le 2k^2 + 12k + 10.$$

Therefore $W(Q) > W(Q_n^3)$ unless $k = 8$ and the degree sequence of Q is $6, 6, 6, 3, 3, \ldots, 3$.

So for the rest of this proof $k = 8$, the degree sequence of Q is $6, 6, 6, 3, 3, \ldots, 3$ and is of length 17. By Lemma 9 if $W(Q) \leq W(Q_n^3)$, then $W(Q) = W(Q_{17}^3)$, the diameter of Q is 4, Q has no nonfacial 4-cycles, $|A| = 9$ and $|B| = 8$. We will show that such a Q does not exist, which finishes the proof.

Because the sum of the degrees of the vertices in each partite class must be the same (in this case, 30), B contains exactly two of the degree 6 vertices. Let $v \in B$ with degree 6, consider the sunflower S_v, and label the 4 vertices outside S_v by x, y_1, y_2, y_3 such that $B = \{v, w_1, \ldots, w_6, x\}$ and $A = \{u_1, \ldots, u_6, y_1, y_2, y_3\}$. Since $d(x) \in \{3, 6\}$, $N(x) \subseteq A$ and at most one of the u_i has an edge not from S_v incident upon it, we have $d(x) = 3$ and without loss of generality $y_1, y_2 \in N(x)$.

Assume first that $N(x) = \{y_1, y_2, y_3\}$ and consider the sunflower S_x. Let j_i be chosen such that w_{j_i} is the common neighbor of y_i and y_{i+1} (indices taken modulo 3) in S_x. Then each of the w_{j_i} are different and have degree at least 4 in Q, a contradiction.

So we can assume without loss of generality that $N(x) = \{u_1, y_1, y_2\}$. Then the unique degree 6 vertex in A is u_1, so there are two different indices t_1 and t_2 such that $w_{t_i} \in N(u_1) \setminus \{w_1, w_6\}$. For $i \in \{1, 2\}$ let z_i be the common neighbor of u_1 and y_i in the sunflower S_x, and let z_3 be the common neighbor of y_1 and y_2 in S_x. Then $\{z_1, z_2\} \subseteq \{w_1, w_2, w_{t_1}, w_{t_2}\}$ and $z_3 \in \{w_1, \ldots, w_6\} \setminus \{z_1, z_2\}$. In particular, the degree of z_3 in Q is at least 4, therefore z_1 and z_2 must have degree 3 in Q. If $w_{t_i} \in \{z_1, z_2\}$, then w_{t_i} has degree at least 4 and consequently degree 6. This gives $\{z_1, z_2\} \cap \{w_{t_1}, w_{t_2}\} = \emptyset$. Therefore without loss of generality $w_1 = z_1$, $w_6 = z_2$ and the $u_1 w_{t_i}$ edges cannot run inside $\mathcal{R}_v$ or any of the faces bounded the 4-cycles $u_1 w_1 y_1 x u_1$ and $u_1 w_6 y_2 x u_1$, which leaves them no place to be, a contradiction. ☐

Theorem 14. *Assume that the number $n \geq 11$ is odd. The quadrangulation Q_n^3 in* Figure 5 *minimizes the Wiener index among all* 3*-connected quadrangulations of order n. Moreover, up to isomorphism, this minimizer is unique.*

Proof. Let $n = 2k + 1$ and assume that Q is a 3-connected quadrangulation on n vertices of minimum Wiener index, and with partite sets A, B, such that $|A| > |B|$.

First we want to show that $|A| = k+1$, $|B| = k$, and the degree sequence of Q restricted to the partite sets is the same as the degree sequence of Q_n^3 restricted to its partite sets.

We have $|A| \geq k + 1$ and $|B| \leq k$. Lemma 7 (a) gives $\Delta(Q) \leq |B| - 1 \leq k - 1$. Lemma 13 gives $\Delta(Q) = k - 1$, which in turn shows $|B| = k$ and $|A| = k + 1$. In addition, if $d(v) = k - 1$, Lemma 7 (b) gives $v \in B$.

As the degree sequence of quadrangulations is unique for $n = 11$ under the condition that every degree is 3 or 4, we may assume now that $n \geq 13$, i.e., $k \geq 6$. As the sum of the k degrees in B is the number of edges $2n - 4 = 4k - 2$, and every degree is at least 3, only two degree sequences are possible for B: $(k - 1, 5, 3, 3, \ldots, 3)$ or $(k - 1, 4, 4, 3, \ldots, 3)$. We claim that $\Delta(A)$, the maximum degree of a vertex in A is $k - 2$. Lemma 7 (b) showed $\Delta(A) \leq |B| - 2 = k - 2$.

Assume for contradiction that $\Delta(A) \leq k-3$. Since the minimum degree is at least 3 and $\sum_{x\in A} d(x) = 4k-2$, for $k=6$ we get that $3\cdot 7 = 4\cdot 6 - 2$, a contradiction. Therefore we have that $k \geq 7$,

$$\sum_{x\in A} d^2(x) \leq (k-3)^2 + 4^2 + 3^2(k-1) = k^2 + 3k + 16,$$

and

$$\sum_{x\in V(Q)} d^2(x) \leq k^2 + 3k + 16 + (k-1)^2 + 5^2 + 3^2(k-2) = 2k^2 + 10k + 24.$$

By Lemma 13 $W(Q) > W(Q_n^3)$ when $k \geq 8$ so we may assume that $k=7$. In particular, for $k \geq 8$ the degree sequence of A is $(k-2, 3, \ldots, 3)$.

If $k=7$, the degree sequence of A is $(4,4,3,3,3,3,3,3)$ and the degree sequence of B is $(6,4,4,3,3,3,3)$ then $\sum_{x\in V(Q)} d^2(x) = 190 < 192 = 2\cdot 7^2 + 12\cdot 7 + 10$, and Lemma 13 contradicts the minimality of the Wiener index of Q.

Hence the only case that remains to be checked is when $k=7$, the degree sequence of A is $(4,4,3,3,3,3,3,3)$ and the degree sequence of B is $(6,5,3,3,3,3,3)$. In this case $\sum_{x\in V(Q)} d^2(x) = 192 = \sum_{x\in V(Q_{15}^3)} d^2(x)$, so by Lemma 9 and Lemma 10 (a), (d) the minimality of $W(Q)$ implies that Q has no nonfacial 4-cycles. Let $v \in B$, $d(v)=6$ and consider the sunflower S_v. Let x, y be the vertices outside S_v. Then $B = \{v, w_1, \ldots, w_6\}$ and $A = \{u_1, \ldots, u_6, x, y\}$, without loss of generality $d(w_1) = 5$, and the rest of the w_i have degree 3. Therefore there is an $i \in \{3,4,5,6\}$ such that w_1 is adjacent to u_i. Since w_1 and u_i cuts C_v into two paths, one contains w_2 and the other w_6, the vertices w_2 and w_6 lie inside two different regions bounded by the 4-cycle $w_1 u_j v u_1 w_1$. As this cycle is nonfacial, we have a contradiction.

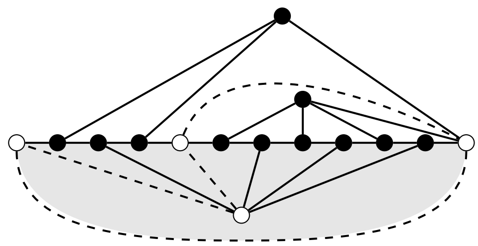

FIGURE 6. The quadrangulation Y of order 13 with Wiener index 164. The gray region shows the sunflower around a maximal degree vertex. The white vertices and the dotted edges form one of the non-facial 4-cycles.

So we have that the degree sequence of Q restricted to A is the same as the degree sequence of Q_n^3 restricted to its A, i.e. $(k-2, 3, \ldots, 3)$. We need to figure out what the degree sequence of Q restricted to B is. Assume $k \geq 6$, and B has degree sequence $(k-1, 5, 3, 3, \ldots, 3)$. Referring to the sunflower graph S_v at vertex v, where $d(v) = k-1$, we have $B = \{v, w_1, \ldots, w_{k-1}\}$ and $A = \{u_1, \ldots, u_{k-1}, x, y\}$. We can assume without loss of generality that $w_1 \in B$ has degree 5, and for $i : 2 \leq i \leq k-1$ set z_i be the unique vertex in $N(w_i) \setminus \{u_i, u_{i+1}\}$. Since w_1 is adjacent to 3 vertices of $A \setminus \{u_1, u_2\}$, it is adjacent to at least one (and at most three) vertices in $\{u_i : 3 \leq i \leq k-1\}$, and consequently these vertices have degree at least 4. As A has a single vertex with degree more than 3, we conclude that there is a unique $j : 3 \leq j \leq k-1$ that w_1 is adjacent to u_j, $d(u_j) = k-2$ and $d(x) = d(y) = 3$, and w_1 is adjacent to x and y. In addition, for $i : 2 \leq i \leq k-1$ we have that $z_i \in \{x, y, u_j\}$; in particular $z_{j-1}, z_j \in \{x, y\}$. We may assume without loss of generality that $z_{j-1} = x$.

Let $\mathcal{P}$ be the region bounded by C_v that is different from $\mathcal{R}_v$, and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two subregions that the edge w_1u_j cuts $\mathcal{P}$ into; without loss of generality the boundary of $\mathcal{R}_1$ is the cycle $w_1u_2w_2u_3 \dots u_j$. Now $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_v$ share only vertices on the boundary, and the common boundary of $\mathcal{R}_1$ and $\mathcal{R}_2$ is the edge u_jw_1. $\mathcal{R}_1$ has $j-2 \geq 1$ vertices $w_2, \dots w_{j-1}$ from $B \setminus \{w_1\}$ on its common boundary with $\mathcal{R}_v$, and for $i : 2 \leq i \leq j-1$ the vertex z_i lies in $\mathcal{R}_1$ (inside or on the boundary). Since $z_{j-1} = x$, x is inside $\mathcal{R}_1$. Let $\mathcal{Q}$ be the subregion of $\mathcal{R}_1$ bounded by the cycle $w_1u_2w_2 \cdots u_{j-1}w_{j-1}xw_1$. Then for $i : 2 \leq i \leq j-2$ the vertex z_i lies in $\mathcal{Q}$, so $z_i \in \{x, y\}$. $\mathcal{R}_2$ has $k-j \geq 1$ vertices $w_j, w_{j+1}, \dots, w_{k-1}$ from $B \setminus \{w_1\}$ on its common boundary with $\mathcal{R}_v$ and for $i : j \leq i \leq k-1$, z_i lies in $\mathcal{R}_2$ (inside or on the boundary). Since $z_j \in \{x, y\}$ and x is inside $\mathcal{R}_1$, this implies that $z_j = y$, y is inside $\mathcal{R}_2$, for all $i : j \leq i \leq k-1$ we have $z_i \in \{u_j, y\}$ and for all $i : 2 \leq i \leq j-1$ we have $z_i = x$. Similar logic as before gives that for all $i : j \leq i \leq k-1$ we have $z_i = y$ Since $d(x) = d(y) = 3$, this means $3 = j-1 = k-j+1$, so $j = 4$ and $k = 6$. We have $Q \simeq Y$ (see Figure 6) and $W(Q) = 164 > 160 = W(Q^3_{13})$, a contradiction.

For the rest of the proof we assume that $n \geq 11$, so $k \geq 5$. The integer sequence that maximizes the sum of squares, and satisfies the conditions we have for the degree sequence of G in A (respectively B) is $k-2, 3, \dots, 3$ (respectively $k-1, 4, 4, 3, \dots, 3$), the degree sequence of Q^3_n. Since Q^3_n is a 3-connected quadrangulation with diameter at most 4, this shows that $W(Q^3_n)$ is minimal, and the degree sequence of Q is the same as the degree sequence of Q^3_n, and furthermore, the degree sequences of their respective partite sets are the same. Last, we need to show that $Q \simeq Q^3_n$.

Let $v \in Q$ with $d(v) = k-1$, and consider the sunflower S_v around v. Let x, y be the vertices of Q not in S_v. Then $B = \{v, w_1, w_2, \dots, w_{k-1}\}$, $A = \{u_1, \dots, u_{k-1}, x, y\}$, and without loss of generality the two vertices of degree 4 in B are w_1 and w_j.

If every u_i has degree 3 (this must happen in particular when $k = 5$ and vertices of A all have degree 3), then none of the u_i has a neighbor outside of S_v. In this case w_1 and w_j must both be adjacent to x and y. Without loss of generality the region $\mathcal{R}$ bounded by the cycle $xw_1u_2w_2 \dots u_jw_jx$ that does not contain v contains y. (Otherwise we exchange the name of x and y). If $j = k-1$, then x is on the interior of the 4-cycle $yw_{k-1}u_1w_1y$ that does not contain v, and the degree of x can only be 2, which is a contradiction. If $j = 2$, then $\mathcal{R}$ is bounded by a 4-cycle and y can have only degree 2, a contradiction. So $3 \leq j \leq k-2$, the $k-1-j \geq 1$ vertices $w_{j+1}, w_{j+2}, \dots, w_{k-1}$ must have x as their third neighbor, and the the $j-2 \geq 1$ vertices $w_2, w_3, \dots, w_{j-1}$ must have y as their third neighbor. So $\{d(x), d(y)\} = \{k+1-j, j\} = \{3, k-2\}$, which gives $j \in \{3, k-2\}$. This is precisely the graph Q^3_n.

Now let i be chosen so u_i have degree greater than 3. As all but one of the vertices of A have degree 3, for $j \neq i$ we have $d(u_j) = 3$, u_j has the same neighbors in Q and S_v, $d(u_i) = k-2$, $d(x) = d(y) = 3$, and $k \geq 6$. This means that u_i is adjacent to precisely $k-5 \geq 1$ of the vertices in $\{w_s : s \notin \{i-1, i\}, 1 \leq s \leq k-1\}$, so it is adjacent to at least one w_ℓ such that $\ell \notin \{i-1, i\}$. If $i < \ell \leq k-1$ then $u_i v u_\ell w_\ell u_i$ is a non-facial 4-cycle (as $u_i w_i u_{i+1} \ldots w_{\ell-1} u_\ell$ lies in one of the regions bounded by this cycle while $w_\ell u_{\ell+1} w_{\ell+1} \ldots w_{i-1} u_i$ lies in the other region). If $1 \leq \ell < i-1$ then $u_i v u_{\ell+1} w_\ell u_i$ is a non-facial 4-cycle. Since Q can not have non-facial 4 cycles by Lemma 9, this is a contradiction □

4. Minimum Wiener Index of 5-Connected Triangulations

Euler's formula shows that there are no 5-connected triangulations of order less than 12.

First we state some facts about triangulations of a simple n-gon not using additional vertices. Triangulations of an n-gon can be viewed as planar graphs, where the outer face is bounded by an n cycle and all other faces are bounded by a 3-cycle (we will refer to such faces as *triangles*).

Lemma 15. *Let $n \geq 4$. Any triangulation of a simple n-gon uses $n-3$ additional edges (i.e. edges which are not edges of the n-gon), and has at least 2 triangles with exactly two of their boundary edges on the n-gon.*

Proof. The fact that the triangulation has $n-3$ edges (and consequently $n-2$ triangles) is easy to prove by induction on n. When $n \geq 4$, all these triangles have at most 2 boundary edges on the n-gon. As there are $n-2$ triangles inside and n edges on the n-gon itself, by the pigeonhole principle some two triangles must have two edges from edges of the n-gon. □

We need the following basic facts about 5-connected triangulations:

Lemma 16. *Let T be a 5-connected triangulation of order n. The following are true:*

- (a) *Every 3-cycle is the boundary of a face and every 4-cycle is the boundary of a region whose interior does not contain vertices of the graph, and contains exactly one edge.*
- (b) *Every edge lies on exactly two triangles. If abc and bcd are triangles of T, then ad is not an edge of T.*
- (c) *For every edge xy of T, there is precisely one 4-cycle in T that goes through its vertices, but does not use the xy edge; hence the number of 4-cycles in T is $3(n-2)$.*
- (d) *If x, y are non-adjacent vertices in T, then there is at most one 4-cycle that contains them.*
- (e) *Let D_i denote the number of unordered pairs of vertices at distance i in T. We have $D_1 = 3(n-2)$ and*

$$D_2 = \frac{1}{2} \sum_{x \in V(T)} d^2(x) - 12(n-2).$$

(f)

$$W(T) \geq 3\binom{n}{2} + 6(n-2) - \frac{1}{2}\sum_{x\in V(T)} d^2(x),$$

with equality if and only if T *has diameter at most* 3.

Proof. (a): if C is a cycle that separates two regions that both contain vertices in their interior, then the vertices of C form a cutset, therefore C has at least 5 vertices.

(b): An edge bounds two faces that are triangles, and if there is a third 3-cycle using the edge, the other two edges of two of these 3-cycles give a 4-cycle that has vertices in both of its regions. If abc and bcd are triangles such that ad is an edge, then one of abd, acd would be a non-facial triangle unless $n = 4$. Both of these contradict (a), and (b) follows.

(c): Since every 4-cycle $abcd$ bounds a region that has no vertices but has an edge (say ac), and if ac is an edge then bd cannot be an edge by (b), for every 4-cycle there is a unique edge that is not part of the cycle and connects two of its vertices. So we can map 4-cycles to edges by assigning this edge to each cycle. This map is injective. If two different 4-cycles would map to the same edge, this edge is part of three triangles, contradicting (b).

As each edge lies on two triangles which together form a 4 cycle, every edge is assigned to precisely one of these 4-cycles, so the map is surjective as well. Thus, the number of 4-cycles is the same as the number of edges, which is $3(n-2)$ in any planar triangulation. (c) follows.

(d): Assume x, y are non-adjacent vertices that appear on two 4-cycles. As each 4-cycle containing x, y has two x-y paths of length 2, we have at least three x-y paths of length 2, say xa_1y, xa_2y, xa_3y. By (a), for each $i, j \in \{1, 2, 3\}$, $i \neq j$ the cycle $C_{ij} = xa_iya_jx$ bounds a region $\mathcal{R}_{ij}$ that contains no vertices in its interior. But the three regions $\mathcal{R}_{12}, \mathcal{R}_{13}\mathcal{R}_{23}$ together with their boundaries cover the entire plane, so T has no other vertices besides x, y, a_1, a_2, a_3. This is a contradiction, as 5-connected triangulations must have at least 12 vertices.

(e): Observe that D_1 is exactly the number of edges of T, $3(n-2)$. The formula $\sum \binom{d(x)}{2} = \frac{1}{2}\sum d^2(x) - 3(n-2)$ counts the number of paths of length 2 between unordered pairs of vertices. If an unordered pair of vertices has more than 1 such path, it appears on a 4-cycle, and by (c) and (d) this 4-cycle is unique. As each 4-cycle contains exactly two such unordered pairs of vertices, the number of unordered pairs of vertices that have a path of length 2 between them is $\frac{1}{2}\sum d^2(x) - 9(n-2)$ by (c). As every edge is contained in exactly one 4-cycle, this equals $D_1 + D_2$, proving (e).

(f): As $\sum_i D_i = \binom{n}{2}$, we get

$$\begin{aligned} W(T) &= \sum_i iD_i \geq D_1 + 2D_2 + 3\left(\binom{n}{2} - D_1 - D_2\right) = 3\binom{n}{2} - 2D_1 - D_2 \\ &= 3\binom{n}{2} + 6(n-2) - \frac{1}{2}\sum_{x\in V(T)} d^2(x), \end{aligned}$$

and equality holds precisely when the diameter is at most 3. □

Analogously to Section 3, we define an auxiliary drawn graph, which we will use extensively. Let T be a 5-connected triangulation, and let $v \in V(T)$ have degree d. We define the *mosaic graph* M_v at vertex v, together with its planar drawing, in the following way. M_v contains the neighbors of v in G, $u_1, u_2, \ldots, u_d$, with the edges vu_i, such that vertices u_i are labeled according the clockwise cyclic order of the edges. We include the edges $u_iu_{i+1} \in E(T)$ for every $1 \le i \le d$ (indices are taken modulo d) in M_v, following the drawing of T. We also add a vertex $w_i \ne v$, which is a common neighbor of u_i and u_{i+1}, together with edges joining them to u_i and u_{i+1} in T, following the drawing of T, for every i. We understand M_v as a part of the drawing of T. We will show that M_v has $2d+1$ distinct vertices.

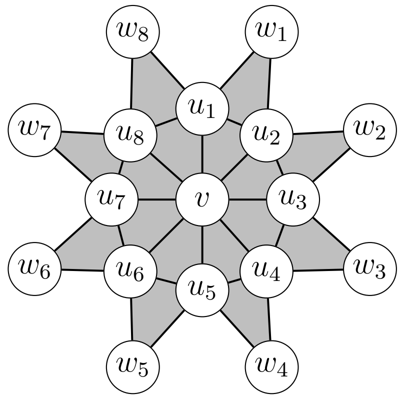

FIGURE 7. The mosaic graph M_v around v, with $d = 8$. The grey region is $\mathcal{R}_v$.

Lemma 17. *If T is a drawing of a 5-connected triangulation of order n, and v is any vertex of T with degree $d(v) = d$, the mosaic graph M_v in T has $2d+1$ distinct vertices. Furthermore, the region $\mathcal{R}_v$ that is bounded by the cycle $u_1w_1u_2w_2 \ldots u_dw_d$ and contains the vertex v, contains edges and vertices from T if and only if they are edges and vertices in the mosaic graph M_v. In addition, T contains at least one vertex not in M_v, consequently $\Delta(T) \le \lfloor \frac{n}{2} \rfloor - 1$. Moreover, $n \ne 13$.*

Proof. Since T is 5-connected, $\delta(T) \ge 5$. As before, label the neighbors of v by $u_1, \ldots, u_d$, in their planar clockwise cyclic order around v. We get for free that u_iu_{i+1} is an edge in T, since we have a triangulation. For each pair of successive neighbors u_i and u_{i+1} (indices taken modulo d), let $w_i \ne v$ be their common neighbor that completes the face that has u_iu_{i+1} on its boundary, but not v. This means, in particular, that $\mathcal{R}_v$ will satisfy the required property, so we just need to show that the vertices listed in M_v are all distinct.

If y is a neighbor of u_i and $y \notin \{v, w_{i-1}, w_i, u_{i-1}, u_{i+1}\}$ then y must lie between w_{i-1} and w_i in the planar cyclic order around u_i, In particular, as $d(u_i) \ge 5$, we have that $w_{i-1} \ne w_i$.

Also, $u_i \ne w_j$ for all $1 \le i, j \le d$. For $j \in \{i-1, i\}$ this is obvious, and for other values of j if $u_i = w_j$ then v, u_i, u_j is a 3-element cutset.

Assume now that $w_i = w_j$ for some $j \neq i$. We already have that $j \notin \{i-1, i+1\}$ and hence the vertices $u_i, u_{i+1}, u_j, u_{j+1}$ are all distinct. We consider two regions of the planar drawing of T: $\mathcal{R}_1$ is bounded by the 4-cycle $u_{i+1}vu_jw_i$ and does not contain the vertex u_i, and $\mathcal{R}_2$ is bounded by the 4-cycle $u_ivu_{j+1}w_i$ and does not contain the vertex u_{i+1}.

The neighbors of u_i that differ from v, u_{i+1} and w_i, lie in $\mathcal{R}_2$ and the neighbors of u_j that differ from v, u_{j+1} and w_i, must lie in $\mathcal{R}_1$. Hence $\{v, u_{i+1}, u_{j+1}, w_i\}$ separates u_i from u_j (see Figure 8), contradicting that T is 5-connected.

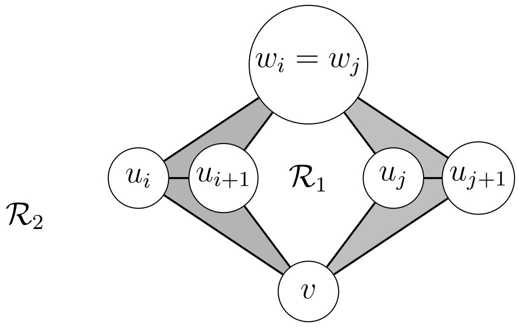

FIGURE 8. 4-element cutset $\{v, u_{i+1}, u_{j+1}, w_i\}$ in M_v when $w_i = w_j$. The shaded regions are unions of faces, so they have no additional vertices.

Now let v be a vertex of T with maximum degree, i.e., $d(v) = \Delta(T) = \Delta$. The mosaic graph M_v around v contains $2\Delta + 1$ vertices. If T contains a vertex that is not in M_v, then $2\Delta + 2 \leq n$, and the claimed inequality follows.

If every vertex of T is in M_v, then set of edges F not in M_v form a triangulation of the 2Δ-cycle $u_1w_1u_2w_2 \dots u_\Delta w_\Delta u_1$ on the region different from $\mathcal{R}_v$; consequently $|F| = 2\Delta - 3$. Note that for any $1 \leq i < j \leq \Delta$, if $u_iu_j \in F$, then the 3-cycle u_iu_jv separates the vertices w_i and w_j, so u_i, u_j, v would be a cutset of size 3, a contradiction. If for any $j \notin \{i, i-1\}$, $u_iw_j \in F$, then the 4-cycle $u_iw_ju_jv$ has the vertices w_i and w_{i-1} on its different sides, giving a cutset of size 4, which is also a contradiction. Therefore every edge in F connects two vertices of $W = \{w_1, \dots, w_\Delta\}$.

But then for every i, the edges $w_{i-1}u_i$ and u_iw_i lie on the boundary of the same face, giving $w_{i-1}w_i \in F$. Hence $w_1, \dots, w_\Delta$ determines a Δ-gon (all of the sides are in F), and this Δ-gon is triangulated by the remaining edges of F. Lemma 15 applies. Say, w_{i-1}, w_i, w_{i+1} is a triangle with two edges on the boundary of the Δ-gon. Then $d(w_i) = 4$, contradicting the fact that T is 5-connected.

Finally, assume to the contrary that T has 13 vertices. Then $\Delta(T) \leq 5$, therefore T is 5-regular. The sum of degrees of T is odd, contradicting the Handshaking Lemma. □

For every $n \geq 12$, $n \neq 13$, the n-vertex triangulation T_n^5 is defined by Figure 9 (these will be our minimizers of the Wiener index). The following lemma is easy to verify and we leave the details to the reader.

Lemma 18. *Assume that $n \geq 12$, $n \neq 13$.*

(a) *T_n^5 is a 5-connected triangulation.*

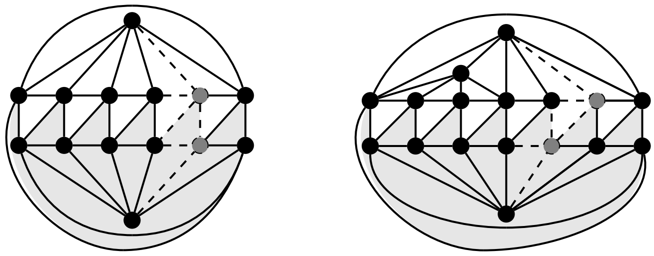

FIGURE 9. The triangulation T_n^5, which minimizes the Wiener index among all 5-connected triangulations of order $n = 2k \geq 12$ (left) and of order $n = 2k+1 \geq 15$ (right). Gray vertices and dashed edges indicate the pattern to be repeated. The shaded region shows the mosaic graph around a degree $\lfloor \frac{n}{2} \rfloor - 1$ vertex.

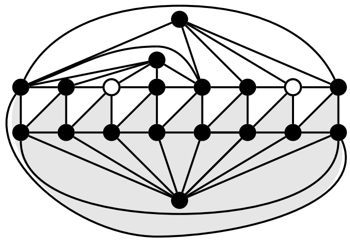

FIGURE 10. The 5-connected triangulation X. The two white vertices are at distance 4. The shaded region shows the mosaic graph around the degree 8 vertex.

(b) *T_n^5 has diameter 3.*
(c) *For n even, the degree sequence of T_n^5 is $\frac{n}{2} - 1, \frac{n}{2} - 1, 5, \ldots, 5$.*
(d) *For n odd, the degree sequence of T_n^5 is $\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor - 2, 6, 6, 5, \ldots, 5$. (For $n = 15$, the terms in this sequence are* not *in decreasing order.)*
(e)

$$W(T_n^5) = \begin{cases} \frac{5n^2}{4} - 7n + 12, & \textit{if } n \textit{ is even,} \\ \frac{5n^2}{4} - 6n - \frac{9}{4}, & \textit{if } n \textit{ is odd.} \end{cases}$$

The 5-connected triangulation X of order 19, defined by Figure 10, has

$$W(X) = 335 = W(T_{19}^5). \tag{3}$$

We will also show that X is the *only* 5-connected triangulation that is not isomorphic to any T_n^5 and achieves the minimum Wiener index for its order. Note that as X is of diameter 4, the lower bound in Lemma 16 (f) cannot be used to compute $W(X)$. The different diameter, and also the different degree sequence, implies that $X \not\simeq T_{19}^5$.

We define the *extended mosaic graph* $M_v^\star$ by adding edges to M_v. Given a 5-connected triangulation G and a vertex v with mosaic graph M_v, we introduce the graph $M_v^\star$, on the vertex set of M_v, by setting

$$E(M_v^\star) = E(M_v) \cup \{w_i w_{i+1} : 1 \le i \le d(v), w_i w_{i+1} \in E(G)\}.$$

Note that $w_i w_{i+1} \in E(G)$ if an only if $d(u_{i+1}) = 5$. Let $\mathcal{R}_v^\star$ denote the extension of $\mathcal{R}_v$ by adding to it the faces bounded by the 3-cycles $u_{i+1} w_i w_{i+1}$ for all edges $w_i w_{i+1} \in E(G)$; let C_v denote the boundary cycle of $\mathcal{R}_v^\star$ and let $\mathcal{Q}_v^\star$ denote the other domain defined by the cycle C_v. Now all vertices of G that are not vertices of M_v and all edges of $E(G) \setminus E(M_v^\star)$ lie in the region $\mathcal{Q}_v^\star$ of the drawing of G.

We will use the following notation in the rest of the section. Given a 5-connected triangulation G and a vertex v, if a, b are vertices of C_v, then $P_v(a, b)$ denotes the path on the cycle C_v from a to b that follows the clockwise cyclic order. (So if $C_v = (w_1, w_2, \dots, w_d)$ in clockwise cyclic order, then $P_v(w_1, w_2)$ is just the edge $w_1 w_2$ with its endpoints, while $P_v(w_2, w_1)$ goes through all vertices of the cycle and misses only the edge $w_1 w_2$.)

Lemma 19. *Let G be a 5-connected triangulation of order $n \ge 12$, and let v be a vertex of G with $d(v) = d$. Consider the extended mosaic graph $M_v^\star$. The following are true:*

(a) *Every vertex $z \in V(C_v)$ has an edge of $E(G) \setminus E(M_v^\star)$ incident upon it.*
(b) *If for some $z_1, z_2 \in V(C_v)$ we have $z_1 z_2 \in E(G) \setminus E(M_v^\star)$, then $z_1 z_2$ cuts $\mathcal{Q}_v^\star$ into two subregions, each containing a vertex of G in its interior, and $z_1, z_2 \in \{w_1, w_2 \dots, w_d\}$.*
(c) *If n is even and $d(v) = \frac{n}{2} - 1$, then $G \simeq T_n^5$.*

Proof. Set $W = \{w_1, \dots, w_d\}$ and $U = \{u_1, \dots, u_d\}$.

(a): Observe that $W \subseteq V(C_v) \subseteq W \cup U$. Vertices in W have degree at most 4 in $M_v^\star$, and vertices of U have degree 5 in $M_v^\star$. If a vertex of U has degree 5 in G, then it is not a vertex of C_v. (a) follows.

(b): Let$z_1, z_2 \in V(C_v)$ where $z_1 z_2 \in E(G) \setminus E(M_v^\star)$. Assume first that $z_1 z_2$ cuts $Q_v^\star$ into two subregions, one of which (say $\mathcal{R}$) contains no vertices in its interior. We will show that $\mathcal{R}$ contains a (triangular) face f such that the boundary of f has two edges e_1, e_2 that are on the boundary of $\mathcal{R}$ and $z_1 z_2 \notin \{e_1, e_2\}$. This is obviously true when the boundary of $\mathcal{R}$ is a 3-cycle. Otherwise the edges lying in the interior of $\mathcal{R}$ are edges of $E(G) \setminus E(M_v^\star)$ that form a triangulation of $\mathcal{R}$, and by Lemma 15 this triangulation contains two faces with two boundary edges on the boundary of $\mathcal{R}$. One of these faces, f, does not have the edge $z_1 z_2$ on its boundary. Let c be the common endvertex of the two edges e_1, e_2 on the boundary of $\mathcal{R}$. Then $c \notin \{z_1, z_2\}$ and c cannot have an edge from $E(G) \setminus E(M_v^\star)$ incident upon it, contradicting (a). So $z_1 z_2$ cuts $Q_v^\star$ into two subregions, both of which contains a vertex in its interior. Now assume to the contrary that $\{z_1, z_2\} \cap U \ne \emptyset$. We may assume that $z_1 = u_1$. Then $z_2 = u_\ell$ for some $3 \le \ell \le d - 1$ or $z_2 = w_j$ for some $3 \le j \le d - 2$ ($j \ne 2$ and $j \ne d-1$, using Lemma 16 (b) for edges $u_1 u_2$ and $u_1 u_d$). If $z_2 = u_\ell$, then $u_1 v u_\ell$ is a separating 3-cycle (as w_2 and w_d are in different regions of this cycle), and if $z_2 = w_j$ then $u_1 w_j u_j v$ is a separating 4-cycle (as w_2 and w_d are in different regions); both of which contradict the 5-connectedness of G.

(c): Assume now that n is even and $d(v) = \frac{n}{2} - 1$. Lemma 17 gives $\Delta(G) = d(v)$. G has exactly one vertex, say x, not in M_v, and hence in the region $\mathcal{Q}_v^\star$. Then the already proven parts (a) and (b) imply that $E(G) \setminus E(M_v^\star) = \{z_1x : z_1 \in V(C_v)\}$. As $W \subseteq C_v$, $d(x) \geq |W| = d(v) = \Delta(G)$, we get $d(x) = \frac{n}{2} - 1$ and $W = C_v$, and each edge of the form w_iw_{i+1} is an edge of $M_v^\star$. (c) follows. □

Theorem 20. *Assume that $n \geq 12$ and n is even. The triangulation T_n^5, which was defined in Figure 9, is the unique minimizer of the Wiener index among all 5-connected triangulations of order n.*

Proof. Let $n \geq 12$ be even and assume T is a 5-connected triangulation on $n = 2k$ vertices ($k \geq 6$). The degree sum of T is $2(3n-6) = 6n - 12$, and Lemma 17 gives $\Delta(T) \leq \frac{n}{2} - 1$. By Lemma 11 the integer sequence $y_1, \dots, y_n$ that sums to $6n - 12$, satisfies $5 \leq y_i \leq \frac{n}{2} - 1$ and has the largest sum of squares is the sequence $\frac{n}{2} - 1, \frac{n}{2} - 1, 5, 5, \dots, 5$, which is exactly the degree sequence of T_n^5 by Lemma 18 (c). As T_n^5 has diameter 3, by Lemma 16 (f) T_n^5 indeed has the minimum Wiener index among all 5-connected n-vertex triangulations. We know that the degree sequence of T is the same as the degree sequence of T_n^5, so $T \simeq T_n^5$ follows from Lemma 19 (c). □

To characterize extremal triangulations of odd order, we need a bit more information about their structure.

Lemma 21. *There are no 5-connected triangulations on 21 vertices with degree sequence $8, 8, 8, 5, ..., 5$.*

Proof. Assume that G is a 5-connected triangulation on 21 vertices with degree sequence $8, 8, 8, 5, \dots 5$. Let v be a degree 8 vertex, and let x_1, x_2, x_3, x_4 be the vertices in $V(G) \setminus V(M_v^\star)$. Set $X = \{x_1, x_2, x_3, x_4\}$, $U = \{u_i : 1 \leq i \leq 8\}$ and $W = \{w_i : 1 \leq i \leq 8\}$. Let b be the number of connected components of the subgraph of G induced by X and D_i be the component containig x_i, $c_i = |N(x_i) \cap X|$ and χ be the number of vertices of degree 8 in X.

Clearly $W \subseteq V(C_v)$, and by Lemma 19 (a) and (b) all vertices of $V(C_v) \cap U$ have degree 8 (consequently $|V(C_v) \cap U| \leq 2$). Assume that $z \in V(C_v) \cap U$; then $|N(z) \cap X| = 3$. Let $\{x_i, x_j, x_k\} = N(z) \cap X$, then without loss of generality $x_ix_jx_k$ is a path in G whose edges form faces with the edges x_iz, x_jz, x_kz. Moreover, if e, f are the two edges on C_v that are incident upon z, then the cyclic order of the edges that lie in or on the boundary of $\mathcal{Q}_v^\star$ around z is e, zx_i, zx_j, zx_k, f or f, zx_i, zx_j, zx_k, e; otherwise one of the triangles zx_ix_j or zx_jx_k is a separating triangle, which is a contradiction. Finally, $x_ix_k \notin E(G)$, as otherwise zx_ix_k is a separating triangle.

Assume first that $U \cap V(C_v) \neq \emptyset$; without loss of generality $u_1 \in U \cap V(C_v)$, u_1 is adjacent to x_2, x_3, x_4 and $x_2x_3x_4$ is a path in G, consequently $x_2x_4 \notin E(G)$. We have that either $b = 2$ and $D_1 = \{x_1\}$, or $b = 1$. Without loss of generality we may assume that the cyclic order of edges around u_1 in $\mathcal{Q}_v^\star$ is $u_1w_1, u_1x_2, u_1x_3, u_1x_4, u_1w_8$. As u_1w_1, u_1x_2 (and also u_1x_4, u_1w_8) bound a common face, we have $w_1x_2, w_8x_4 \in E(G)$. Let $\mathcal{P}^\star$ be the region we get if we leave out from $\mathcal{Q}_v^\star$ the faces with u_1 on their boundary.

Consider the case when $|U \cap V(C_v)| = 2$, i.e. for some $j \neq 1$ the vertex u_j also has degree 8. If $N(u_j) \cap X = N(u_1) \cap X = \{x_2, x_3, x_4\}$ (which must happen when $b = 2$), we have that the path $x_2x_3x_4$ is induced in D_2. But then $u_1x_2u_jx_4$ is a separating 4-cycle, which is a contradiction. Therefore we must have $b = 1$, and without loss of generality for some $t \in \{2, 3\}$, $N(u_j) \cap X = \{x_1, x_t, x_{t+1}\}$ where $x_1x_t \in E(G)$ and $x_1x_{t+1} \notin E(G)$. This gives $|E(X)| \geq 3$, $|E(X, U)| = 6$, and (since the vertices of X have degree 5) $|E(X, W)| = 20 - 6 - 2|E(X)| \leq 8$. On the other hand, since $b = 1$, Lemma 19 (b) implies that the set of edges in $E(G) \setminus E(M_v^\star)$ that are incident upon W form the set $E(X, W)$, and (as w_1 and w_8 have degree 3 in $M_v^\star$) consequently $|E(X, W)| \geq 10$, a contradiction. Therefore we must have $|U \cap V(C_v)| < 2$, i.e. $U \cap V(C_v) = \{u_1\}$.

Now we have that $U \cap V(C_v) = \{u_1\}$. Assume that $b = 2$, so $D_1 = \{x_1\}$. Let s and t be the smallest and largest indices such that $w_s, w_t \in N(x_1)$; $1 \leq s < s + 4 \leq t \leq 8$. The path $w_sx_1w_t$ cuts the region $\mathcal{Q}_v^\star$ into two regions $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where $\mathcal{Q}_1$ has u_1 on its boundary. Therefore x_2, x_3, x_4 lie inside $\mathcal{Q}_1$, $w_sw_t \in E(G) \setminus E(M_v^\star)$, and by Lemma 19 (b) for each $i : s \leq i \leq t$ $x_1w_i \in E(G)$. If $s \neq 1$ then $\{w_{s-1}, w_{s+1}, u_s, u_{s+1}, w_t, x_1\} \subseteq N(w_s)$, so w_s must have degree 8. If $s = 1$, then $\{u_1, u_2, w_2, x_2, w_t, x_1\} \subseteq N(w_s)$, so w_s has degree 8. Similar arguments imply that w_t also has degree 8. But then u_1, v, w_s, w_t all have degree 8, a contradiction. So we must have $b = 1$, i.e. x_1 is connected to at least one other vertex in X. If x_1 is connected to both x_2 and x_4, then (as the edges x_1x_2 and x_1x_4 lie in $\mathcal{P}^\star$) $u_1x_2x_1x_4$ is a separating 4-cycle, which is a contradiction. We may assume $x_1x_4 \notin E(G)$, so $c_1 \in \{1, 2\}$. Then $|E(X, W)| = 13 + 3\chi - 2c_1$. Since the number of degree 8 vertices in W is $1 - \chi$, $|E(X, W)| = 10 + 3(1 - \chi) = 13 - 3\chi$. This gives $2c_1 = 6\chi$, so 3 divides c_1, which is a contradiction. Thus we must have $U \cap V(C_v) = \emptyset$.

Therefore $W = V(C_v)$ and every vertex in W has degree 4 in $M_v^\star$, so every vertex in W has either one or 4 edges incident upon it from $E(G) \setminus E(M_v^\star)$. Set $F = E(W) \setminus E(M_v^\star)$ and $m_x = |E(X)|$. We have $2|F| + |E(W, X)| = \sum_{z \in W}(d(z) - 4) = 14 - 3\chi$ and $|E(X, W)| = (\sum_{z \in X} d(z)) - 2m_x = 20 + 3\chi - 2m_x$.

If $\chi = 2$, then all vertices of W have at most one neighbor in X. Since the two vertices in X that have degree 8 have at least 5 neighbors in W, this implies that $5 + 5 \leq |W| = 8$, a contradiction. Therefore $\chi \in \{0, 1\}$; at least one vertex in W has degree 8.

Suppose $b = 1$. Then $3 \leq m_x \leq 5$, and $|E(X, W)| = 20 - 2m_x + 3\chi$. On the other hand by Lemma 19 (b) $F = \emptyset$, so $|E(X, W)| = 14 - 3\chi$. This gives $m_x = 3 + 3\chi$, consequently $\chi = 0$, $m_x = 3$, and exactly two vertices (say w_ℓ, w_q) in W have degree 8. But then at least one of the 4-cycles of the form $w_\ell x_i w_q x_j w_\ell$ is separating, which is a contradiction. Thus we must have $b > 1$.

Since $b > 1$, we must have either that the components spanned by X are a K_1 and a K_3, or the subgraph generated by X has exactly $4 - b$ edges (as all of its components are trees). In the former case $|E(X, W)| = 14 + 3\chi$, in the latter $|E(X, W)| = 12 + 2b + 3\chi \geq 16 + 3\chi > 14 - 3\chi \geq |E(X, W)|$, which is a contradiction. Therefore the components spanned by X are a K_1 and a K_3. We get that $2|F| + 14 + 3\chi = 14 - 3\chi$, which gives $\chi = 0$, and $F = \emptyset$. So exactly two vertices (say w_ℓ, w_q) in W have degree 8, and $E(W, V(G)) \setminus E(M_v^\star) = E(X, W)$. Without loss of generality the K_3 in X is formed by the vertices x_2, x_3, x_4. But then $X \subseteq N(w_\ell)$, so the subgraph generated by $\{x_2, x_3, x_4, w_\ell\}$ is a

K_4. This is a contradiction, as one of the triangles $w_\ell x_2 x_3$, $w_\ell x_3 x_4$, $w_\ell x_2 x_4$ is separating, contradicting the 5-connectedness of G. □

Lemma 22. *Let $n \geq 15$ be odd, and let G be a 5-connected triangulation of order n with degree sequence $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$. If $W(G) \leq W(T_n^5)$, then $d_1 = \lfloor \frac{n}{2} \rfloor - 1$, and one of the following holds:*

(a) *$n = 23$ and the degree sequence of G is $10, 8, 8, 5, \ldots, 5$*
(b) *$d_2 \geq \lfloor \frac{n}{2} \rfloor - 2$, $d_3 + d_4 \leq 12$, and consequently $d_3 \leq 7$, $d_4 \leq 6$ and $d_5 = 5$.*

Proof. Set $n = 2k+1$, then $k = \lfloor \frac{n}{2} \rfloor \geq 7$. As $d_1 \leq k-1$, the only possible degree sequence for $k = 7$ is $(6, 6, 6, 5, \ldots, 5)$, which satisfies the conclusion. Hence we may assume $k \geq 8$. As T_n^5 has diameter 3, by Lemma 16 (f) and Lemma 18 (d) we must have

$$\sum_{x \in V(G)} d^2(x) \geq \sum_{x \in V(T_n^5)} d^2(x) = (k-1)^2 + (k-2)^2 + 2 \cdot 6^2 + 5^2(2k-3) = 2k^2 + 44k + 2.$$

Assume first that $d_1 \leq k - 2$.

If $k = 8$, the only sequence possible is $6, 6, 6, 6, 6, 5, \ldots, 5$ whose sum of squares is $480 < 2 \cdot 8^2 + 44 \cdot 8 + 2$, which is a contradiction.

If $k = 9$, then

$$\sum_{x \in V(G)} d^2(x) \leq 3 \cdot 7^2 + 6^2 + 15 \cdot 5^2 = 558 < 560 = 2 \cdot 9^2 + 44 \cdot 9 + 2,$$

a contradiction.

If $k = 10$, then By Lemma 21

$$\sum_{x \in V(G)} d^2(x) \leq 2 \cdot 8^2 + 7^2 + 6^2 + 17 \cdot 5^2 = 638 < 642 = 2 \cdot 10^2 + 44 \cdot 10 + 2,$$

a contradiction.

If $k \geq 11$, then

$$\sum_{x \in V(G)} d^2(x) \leq 2(k-2)^2 + 8^2 + 5^2(2k-2) = 2k^2 + 42k + 22 < 2k^2 + 44k + 2,$$

a contradiction. So we proved that $d_1 = k - 1$. If $k = 8$, the only sequences possible are $7, 7, 6, 5, \ldots, 5$ and $7, 6, 6, 6, 5 \ldots, 5$, which satisfy the conclusion. Hence we may assume $k \geq 9$.

Assume next that $d_2 \leq k - 3$. If $k = 9$, the only sequence possible is $8, 6, 6, 6, 6, 5, \ldots, 5$ with degree square sum $558 < 2 \cdot 9^2 + 44 \cdot 9 + 2$. If $k = 10$,

$$\sum_{x \in V(G)} d^2(x) \leq 9^2 + 2 \cdot 7^2 + 6^2 + 17 \cdot 5^2 = 640 < 2 \cdot 10^2 + 44 \cdot 10 + 2.$$

If $k \geq 11$, then

$$\sum_{x \in V(G)} d^2(x) \leq (k-1)^2 + (k-3)^2 + 8^2 + 5^2(2k-2) = 2k^2 + 42k + 24 \leq 2k^2 + 44k + 2.$$

with equality only if $k = 11$ and the degree sequence is $10, 8, 8, 5, \ldots, 5$, i.e. G satisfies (a). Therefore we may assume that $d_2 \geq k - 2$.

As we already know that $d_1 = k - 1$ and $d_2 \geq k - 2$, using that $\sum d_i = 6(2k - 1)$, $d_3 + d_4 \leq 6(2k - 1) - (2k - 3) - 5(2k - 3) = 12$, so from $d_4 \leq d_3$ we get $d_4 \leq 6$ and from $d_4 \geq 5$ we get $d_3 \leq 7$. If $d_4 = 5$ then $d_5 = 5$, otherwise we have that $d_4 = d_3 = 6$ and $d_5 \leq 6(2k - 1) - (2k - 3) - 12 - 5(2k - 4) = 5$. □

Lemma 23. *Let $n \geq 15$ be odd, $k = \lfloor \frac{n}{2} \rfloor$, and let G be a 5-connected triangulation of order n, with $W(G) \leq W(T_n^5)$. Let v be a vertex with $d(v) = \Delta(G) =: \Delta$. Consider the extended mosaic graph $M_v^\star$, and the sets $W = \{w_1, \ldots, w_\Delta\}$ and $U = \{u_1, \ldots, u_\Delta\}$. Let x_1, x_2 denote the two vertices not in $M_v^\star$. The following statements are true:*

(a) *$\Delta = k - 1$, at most 4 degrees of G are larger than 5, and for the largest 4 degrees $\Delta \geq d_2 \geq d_3 \geq d_4$ of G we have either $d_2 \geq k - 2$, $d_3 \leq 7$, $d_4 \leq 6$ and $d_3 + d_4 \leq 12$, or $n = 23$, $d_2 = d_3 = 8 = k - 3$, $d_4 = 5$.*

(b) *For $i \in \{1, 2\}$, there are vertices $a_i, b_i \in V(C_v)$ such that $N(x_i) \setminus \{x_{3-i}\} = V(P_v(a_i, b_i))$. (We refer to these a_i, b_i vertices in the forthcoming claims.) Furthermore, if $c \in V(P_v(a_1, b_1)) \cap V(P_v(a_2, b_2))$, then $c = a_1 = b_2$ or $c = a_2 = b_1$.*

(c) *$C_v = w_1 w_2 \ldots w_\Delta$, $d(x_i) \leq \Delta - 1$, and if $x_1 x_2 \notin E(G)$ then $d(x_i) \leq \Delta - 2$ and $n \geq 19$.*

(d) *If $x_1 x_2 \in E(G)$, then $G \simeq T_n^5$*

(e) *If $x_1 x_2 \notin E(G)$ then $a_1 b_1, a_2 b_2 \in E(G) \setminus E(M_v^\star)$. Moreover, for $i \in \{1, 2\}$, if $c_i \in V(P_v(b_i, a_{3-i}))$ and $z \in V(G) \setminus \{x_1, x_2\}$, such that $c_i z \in E(G) \setminus E(M_v^\star)$, then $z \in V(P_v(b_{3-i}, a_i))$, and the neighbors of c_i in $P_v(b_{3-i}, a_i)$ form a consecutive sequence of vertices on this path.*

(f) *If $x_1 x_2 \notin E(G)$, then $a_1 = b_2$ or $a_2 = b_1$.*

(g) *If $x_1 x_2 \notin E(G)$, then $a_1 = b_2$ and $a_2 = b_1$.*

(h) *If $x_1 x_2 \notin E(G)$, then $G \simeq X$ and $W(G) = W(T_{19}^5)$.*

Proof. Note that $n \geq 15$, so $k \geq 7$. Let G, v, x_1, x_2 be as in the conditions. Lemma 22 yields (a) .

(b): Assume $i \in \{1, 2\}$. As $d(x_i) \geq 5$, x_i has at least 4 neighbors on C_v, so there are vertices $a_i, b_i \in N(x_i) \cap V(C_v)$ such that all vertices in $N(x_i) \setminus \{x_{3-i}\}$ lie on the path $P_v(a_i, b_i)$ and x_{3-i} does not lie in the interior of the subregion of $\mathcal{Q}_v^\star$ bounded by the cycle $x_i P_v(a_i, b_i)$. By Lemma 19 (b), no two vertices in $P_v(a_i, b_i)$ can be joined by an edge that is not in $M_v^\star$. As every vertex of C_v has at least one edge incident upon it from $E(G) \setminus E(M_v^\star)$, we must have $V(P_v(a_i, b_i)) \subseteq N(x_i)$, therefore $V(P_v(a_i, b_i)) = N(x_i) \setminus \{x_{3-i}\}$. As all edges incident upon x_1 or x_2 lie in the region $\mathcal{Q}_v^\star$ and do not cross, the two paths share at most their endvertices.

(c): Assume to the contrary that $c \in U \cap V(P_v(a_i, b_i))$. As $d(x_i) \geq 5$, $P_v(a_i, b_i)$ has at least 4 vertices. Therefore, there exists an internal vertex $c^\star$ of the path $P_v(a_i, b_i)$, such that the edge $cc^\star$ is an edge of this path. This means $c^\star \in W$ (see Lemma 19 (b)), $c^\star$ has at most 3 edges incident upon it in $E(M_v^\star)$, and the only edge in $E(G) \setminus E(M_v^\star)$ incident upon $c^\star$ is $c^\star x_i$, contradicting $d(c^\star) \geq 5$. Thus, $V(P_v(a_i, b_i)) \subseteq W$. By Lemma 19 (a) the internal vertices of the paths $P(b_i, a_{3-i})$ each have at least one edge of $E(G) \setminus E(M_v^\star)$ incident upon them. Thus, by the definition of a_i, b_i and $M_v^\star$, the internal vertices of the paths $P(b_i, a_{3-i})$ have to be adjacent to at least one other internal vertex of the paths $P(b_i, a_{3-i})$. Lemma 19 (b) implies that $V(P(b_i, a_{3-i})) \subseteq W$. So $V(C_v) = W$. By part (b), we have $|V(P_v(a_{3-i}, b_{3-i})) \setminus V(P_v(a_i, b_i))| \geq 2$. Hence $\Delta = |W| \geq |V(P_v(a_i, b_i))| + |V(P_v(a_{3-i}, b_{3-i})) \setminus V(P_v(a_i, b_i))|$. As $N(x_i) = V(P_v(a_i, b_i))$ or $V(P_v(a_i, b_i)) \cup \{x_{3-i}\}$, depending on whether x_1x_2 is non-edge or edge, the claimed upper bounds on $d(x_i)$ follow. Assume $x_1x_2 \notin E(G)$. As we have $5 \leq d(x_i) \leq \Delta - 2$, we have $\Delta \geq 7$, so $n \geq 17$. If $n = 17$, then we must have $d(x_1) = d(x_2) = 5$. As $8 = d(x_1) + d(x_2) - 2 \leq |W| = 7$, this is a contradiction. (c) follows.

(d): Let $x_1x_2 \in E(G)$. Assume that $b_2 \neq a_1$. Then either $P_v(b_2, a_1)$ has an internal vertex c or $b_2a_1 \in E(G)$. In the first case, as $c \in C_v$, there is at least one edge in $E(G) \setminus E(M_v^\star)$ incident upon c by Lemma 19 (a). All edges of $E(G) \setminus E(M_v^\star)$ incident upon c must lie in the subregion of $\mathcal{Q}_v^\star$ bounded by the cycle $P_v(b_2, a_1)x_1x_2$. By Lemma 19 (b) these edges must be of the form cx_1 or cx_2. But none of these are edges of G, which is a contradiction. In the second case $b_2a_1 \in E(G)$, and the subregion of $\mathcal{Q}_v^\star$ bounded by the 4-cycle $b_2a_1x_1x_2$ has no vertices in its interior, so we must have either $x_1b_2 \in E(G)$ or $x_2a_1 \in E(G)$, a contradiction. So $a_1 = b_2$, and $a_2 = b_1$, and $V(C_v) = W$ by (c). All $w \in W$ are incident to 4 edges of $M_v^\star$, but $a_1, a_2 \in W$ are incident to 2 more edges, and vertices of $W \setminus \{a_1, a_2\}$ are incident to one more, so $d(a_1) = d(a_2) = 6$, and (a) gives $d_2 \geq k - 2$. Since a_1, a_2 and v are vertices with degree greater than 5, and G has at most 4 vertices with degree greater than 5, we get $\min(d(x_1), d(x_2)) = 5$, which gives that $G \simeq T_n^5$ as claimed.

For the remaining cases assume that x_1 and x_2 are not adjacent, so $n \geq 19$ and $\Delta \geq 8$. This also implies that $N(x_i) = V(P(a_i, b_i))$, so the paths $P(a_i, b_i)$ have at least 5 vertices.

(e): In this case the edges x_ia_i, x_ib_i lie on the boundary of the same face, so $a_ib_i \in E(G)$, and $a_ib_ix_i$ is a boundary of a face. Moreover, as $|V(P(a_i, b_i)| \geq 5$, $a_ib_i \notin E(M_v^\star)$. The rest of the statement is trivial if $a_1 = b_2$ and $a_2 = b_1$, so assume that is not the case. Consider the connected subregion $\mathcal{R}$ of $\mathcal{Q}_v^\star$ bounded by the cycle $P(b_1, a_2)P(b_2, a_1)$ (that has length at least 3 by the assumption); it has no vertices in its interior. Any edges between vertices of the cycle $P(b_1, a_2)P(b_2, a_1)$ are edges of this cycle or lie inside $\mathcal{R}$. This finishes the proof unless $a_1 \neq b_2$ and $a_2 \neq b_1$, so consider that to be the case. Let $c_i \in V(P_v(b_i, a_{3-i}))$ and $z \in V(G) \setminus \{x_1, x_2\}$ such that $c_iz \in E(G) \setminus E(M_v^\star)$. As c_i lies on the boundary of the connected subregion $\mathcal{R}$, Lemma 19 (b) gives that $z \in V(P(b_{3-i}, a_i))$, as claimed. Also, if $z_1, z_2 \in V(P(b_{3-i}, a_i))$ are different neighbors of c_i where z_1z_2 is not an edge of the path $P_v(b_{3-i}, a_i)$, then by Lemma 19 (b) any internal vertex z_3 of the $z_1 - z_2$ subpath of $P(b_{3-i}, a_i)$ can only have the edge z_3c_i incident upon it from $E(G) \setminus E(M_v^\star)$. Since by Lemma 19 (a) z_3 must have an edge from $E(G) \setminus E(M_v^\star)$ incident upon it, (e) follows.

(f): Assume to the contrary that $a_1 \neq b_2$ and $a_2 \neq b_1$. By (e) we have $a_1b_1, a_2b_2 \in E(G) \setminus E(M_v^\star)$. Let $\mathcal{R}$ be the connected subregion of $\mathcal{Q}_v^\star$ bounded by the cycle $P_v(b_2, a_1)P_v(b_1, a_2)$. G has at most 4 vertices with degree greater than 5. As $d(v) = \Delta > 6$, $V(G) \setminus \{v\}$ has at most 3 vertices with degree greater than 5. In particular, C_v contains at most 3 vertices with degree greater than 5. As by (c) $V(C_v) = W$, each of a_1, a_2, b_1, b_2 has at least 4 edges incident upon them in $E(M_v^\star)$, and two edges incident upon them from $E(G) \setminus E(M_v^\star)$ (the edges a_ib_i, a_ix_i, b_ix_i). This gives that a_1, b_1, a_2, b_2 have degree at least 6, a contradiction. (f) follows.

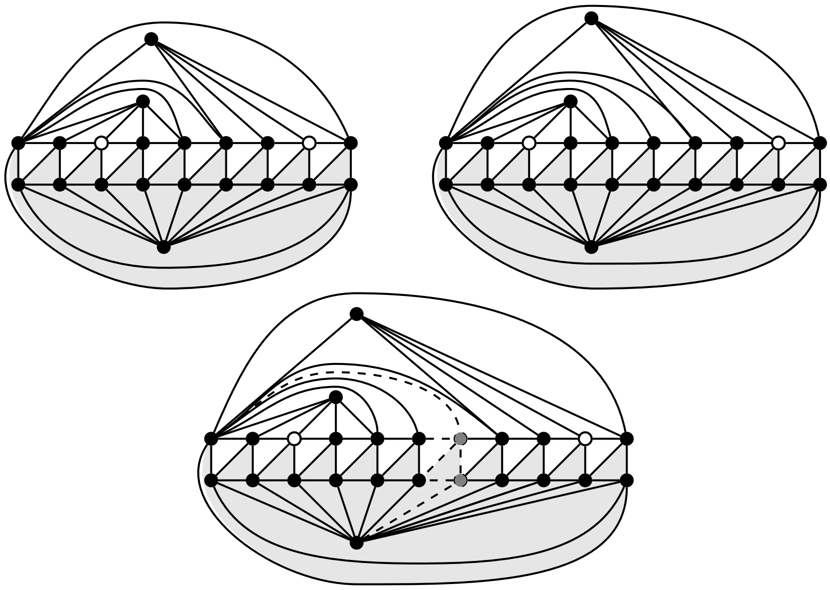

FIGURE 11. 5-connected triangulations of order $n = 21$, 23 and $n \geq 25$, which have the same degree sequence as T_n^5. The gray regions show the mosaic graphs around the vertex of degree $k-1$. The gray vertices and dashed edges on the triangulation of order 25 indicate the pattern to be repeated to get the construction for higher odd order. The two white vertices are at distance 4.

(g): Assume to the contrary that $a_1 \neq b_2$ or $a_2 \neq b_1$. By (f), without loss of generality we have $a_1 = b_2$ and $a_2 \neq b_1$. By definition $a_1x_1, a_1x_2 \in E(G)$ and by (e) all vertices of $P_v(b_1, a_2)$ are neighbors of a_1. All other neighbors of a_1 are one of the 4 neighbors of a_1 in $M_v^\star$. Consequently $d(a_1) = 6 + |V(P_v(b_1, a_2))| \geq 8$, so $d(a_1) = d_2 \in \{k-1, k-2, k-3\}$, and if $d(a_1) = k-3$, then G contains no degree 6 vertices. As $V(C_v) = W$, we must have $d(a_2) = d(b_1) = 6$, as a_2, b_1 each have 4 neighbors in $M_v^\star$, and both are joined to $a_1 = b_2$, and to a single x_i, and not joined to anything else. By (a) $d(a_1) \geq k-2$, and, as v, a_1, a_2, b_1 are the 4 vertices of degree greater than 5, all other vertices (including x_1 and x_2) have degree 5. So every $w \in W \setminus \{a_1, a_2, b_1\}$ has 4 neighbors in $M_v^\star$, and is joined by an edge in $E(G) \setminus E(M_v^\star)$ to exactly one of the vertices x_1, a_1, x_2, and the paths $P(a_i, b_i)$ have 5 vertices each. As the sum of degrees is $6n - 12 = k - 1 + d(a_1) + 12 + 5(n-4)$, we get $d(a_1) = k-2$. As $d(a_1) \geq 8$, this gives $n \geq 21$. Figure 11 has the graph G for all $n \geq 21$. Since G has the same degree sequence as T_n^5 and $W(G) \leq W(T_n^5)$, by Lemma 16 (f) we must have $W(G) = W(T_n^5)$ and the diameter of G is at most 3. However, G has diameter at least 4, as demonstratred on Figure 11, a contradiction. (g) follows.

(h): By (g), $a_1 = b_2$ and $a_2 = b_1$. By (e) $a_1a_2 \in E(G)$. By (c) $V(C_v) = W$, and any edge from $E(G) \setminus E(M_v^\star)$ incident upon a vertex $w \in W \setminus \{a_1, a_2\}$ connects w to exactly one of x_1, x_2. Each a_i has 4 incident edges in $E(M_v^\star)$, and in addition, it is joined to exactly 3 more vertices: x_1, x_2, a_{3-i}. So $d(a_1) = d(a_2) = 7 = d_2 = d_3$. By (a), $d_3 + d_4 \leq 12$, consequently all vertices of $V(G) \setminus \{v, a_1, a_2\}$ (including x_1 and x_2) have degree 5. As $6n - 12 = k - 1 + 14 + 5(n - 3)$, $n = 19$. We have that $G \simeq X$ and $W(G) = W(T^5_{19})$ (see Figure 10). □

The following theorem now follows:

Theorem 24. *Let $n \geq 15$ be odd. If $n \neq 19$, then the unique minimizer of the Wiener index among 5-connected triangulations of order n is T^5_n. If $n = 19$, then there are precisely two minimizers, T^5_{19} and X.*

References

[1] Z. Che, K.L. Collins, An upper bound on Wiener indices of maximal planar graphs. *Discrete Appl. Math.* **258** 76–86 (2019) `https://doi.org/10.1016/j.dam.2018.11.026`

[2] É. Czabarka, P. Dankelmann, T. Olsen and L. A. Székely, Wiener index and remoteness in triangulations and quadrangulations. *Discrete Mathematics & Theoretical Computer Science*, **23** (1) (2021)

[3] D. Ghosh, E. Győri, A. Paulos, N. Salia, O. Zamora, The maximum Wiener index of maximal planar graphs, *Journal of Combinatorial Optimization* **40**(4) 1121–1135 (2020) `https://doi.org/10.1007/s10878-020-00655-4`

[4] E. Győri, A. Paulos and Chuanqi Xiao, Wiener index of quadrangulation graphs, *Discrete Applied Mathematics* **289**, 262–269 (2021) `https://doi.org/10.1016/j.dam.2020.11.016`

[5] K. Noguchi, Y. Suzuki, Relationship among triangulations, quadrangulations and optimal 1-planar graphs. *Graphs and Combinatorics* **31** 1965–1972 (2015) `https://doi.org/10.1007/s00373-015-1568-8`

[6] H. Wiener, Structural determination of paraffin boiling points. *J. Amer. Chem. Soc.* **69**, 17–20 (1947).

[7] H. Whitney, 2-isomorphic graphs. *Amer. J. Math.* **55**(1–4), 245–254 (1933).

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