List 3-dynamic coloring of graphs with small maximum average degree

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Abstract

An r-dynamic k-coloring of a graph G is a proper k-coloring such that for any vertex v, there are at least $\min\{r, \deg_G(v)\}\$ distinct colors in $N_G(v)$. The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G. The *list* r-dynamic chromatic number of a graph G is denoted by $ch_r^d(G)$.

Recently, Loeb et al. [\[11\]](#page-19-0) showed that the list 3-dynamic chromatic number of a planar graph is at most 10. And Cheng et al. [\[3\]](#page-18-0) studied the maximum average condition to have $\chi_3^d(G) \leq 4$, 5, or 6. On the other hand, Song et al. [\[14\]](#page-19-1) showed that if G is planar with girth at least 6, then $\chi_r^d(G) \leq r+5$ for any $r \geq 3$.

In this paper, we study list 3-dynamic coloring in terms of maximum average degree. We show that $ch_3^d(G) \leq 6$ if $mad(G) < \frac{18}{7}$, $ch_3^d(G) \leq 7$ if $mad(G) < \frac{14}{5}$, and $ch_3^d(G) \leq 8$ if $mad(G) < 3$. All of the bounds are tight.

1 Introduction

Let k be a positive integer. A proper k-coloring $\phi: V(G) \to \{1, 2, ..., k\}$ of a graph G is an assignment of colors to the vertices of G so that any two adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the least k such that there exists a proper k-coloring of G. An r-dynamic k-coloring of a graph G is a proper k-coloring ϕ such that for each vertex $v \in V(G)$, either the number of distinct colors in its neighborhood is at least r or the colors in its neighborhood are all distinct, that is, $|\phi(N_G(v))| = \min\{r, \deg_G(v)\}\)$. The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G.

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A *list assignment* on a graph G is a function L that assigns each vertex v a set $L(v)$ which is a list of available colors at v. For a list assignment L of a graph G, we say G is L*-colorable* if there exists a proper coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is said to be k-choosable if for any list assignment L such that $|L(v)| \geq k$ for every vertex v, G is L-colorable.

For a list assignment L of G, we say that G is r*-dynamically* L*-colorable* if there exists an r-dynamic coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is r-dynamically k-choosable if for any list assignment L with $|L(v)| \geq k$ for every vertex v, G is r-dynamically L-colorable. The *list* r-dynamic chromatic number $ch_r^d(G)$ of a graph G is the least k such that G is r-dynamically k-choosable.

The notion of r -dynamic coloring was firstly introduced in [\[12\]](#page-19-2), and then it was widely studied in $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$ $\begin{bmatrix} 1, 4, 6, 7, 8, 9, 10 \end{bmatrix}$. Note that it was also studied in $\begin{bmatrix} 3, 13, 14 \end{bmatrix}$ $\begin{bmatrix} 3, 13, 14 \end{bmatrix}$ $\begin{bmatrix} 3, 13, 14 \end{bmatrix}$ with the name of r*-hued coloring*. Similar to the Wegner's conjecture [\[15\]](#page-19-9), a conjecture about dynamic coloring of planar graphs was proposed in [\[13\]](#page-19-8).

Conjecture 1.1. *Let* G *be a planar graph. Then*

$$
\chi_r^d(G) \le \begin{cases} r+3 & \text{if } 1 \le r \le 2\\ r+5 & \text{if } 3 \le r \le 7\\ \lfloor \frac{3r}{2} \rfloor + 1 & \text{if } r \ge 8. \end{cases}
$$

Song, Lai, and Wu [\[14\]](#page-19-1) show that Conjecture [1.1](#page-1-0) is true for planar graphs with girth at least 6.

Theorem 1.2 ([\[14\]](#page-19-1)). *If* G *is a planar graph with girth at least 6,* $\chi_r^d(G) \leq r + 5$ *for any* $r \geq 3$ *.*

Recently, 3-dynamic coloring has been concerned. Loeb, Mahoney, Reiniger, and Wise [\[11\]](#page-19-0) showed that $ch_3^d(G) \leq 10$ if G is a planar graph. On the other hand, list 3-dynamic coloring was studied in [\[3\]](#page-18-0) in terms of maximum average degree, where the *maximum average degree* of a graph G , $mad(G)$, is the maximum among the average degrees of the subgraphs of G . It was showed in [\[3\]](#page-18-0) that $\chi_3^d(G) \leq 6$ if $mad(G) < \frac{12}{5}$ $\frac{12}{5}$, $\chi_3^d(G) \leq 5$ if $mad(G) < \frac{7}{3}$ $\frac{7}{3}$, and $\chi_3^d(G) \leq 4$ if G has no C_5 -component and $mad(G) < \frac{8}{3}$ $\frac{8}{3}$.

In this paper, we study list 3-dynamic coloring with maximum average degree condition. For each $k \in \{6, 7, 8\}$, we study the optimal value of maximum average degree to be $ch_3^d(G) \leq k$. First, we give an optimal value of $mad(G)$ to be $ch_3^d(G) \leq 6$, which improves a result in [\[3\]](#page-18-0).

Theorem 1.3. *If* $mad(G) < \frac{18}{7}$ *, then* $ch_3^d(G) \leq 6$ *.*

The bound on $mad(G)$ in Theorem [1.3](#page-1-1) is tight. The graph H in Figure [1](#page-2-0) is a subcubic graph and so $ch_3^d(H) = ch(H^2)$, where the *square* of H, denoted by H^2 , is the graph obtained 3 by adding to H the edges connecting two vertices having a common neighbor in H . Note that $mad(H) = \frac{18}{7}$ and H^2 is isomorphic to K_7 . Hence we have $ch(H^2) = ch_3^d(H) = 7$, which implies that the bound on $mad(G)$ in Theorem [1.3](#page-1-1) is tight.

From the graph H , one can find infinitely many tight examples for Theorem [1.3.](#page-1-1) Given a graph H' with $mad(H') \leq \frac{18}{7}$ $\frac{18}{7}$ and $ch_3^d(H') \leq 7$, let G be a graph obtained from the union of two graphs H and H' by connecting with internally disjoints paths of length at least five such that the end vertices in H of the paths have degree two. Figure [2](#page-2-1) shows a way to construct such graphs. Note that there are at most three such paths since H has three vertices of degree two.

Remark: For a given graph F with $mad(F) \leq \frac{18}{7}$ $\frac{18}{7}$, let F' be a graph obtained by adding a path of length ℓ ($\ell \geq 5$) to F such that the two end vertices x and y are in F and the other internal vertices are not. We will show that $mad(F') \leq \frac{18}{7}$.

For a graph G, let ρ_G be a function defined on the power set of $V(G)$ such that $\rho_G(A)$ $9|A| - 7|E(G[A])|$ for any $A \subset V(G)$, where |A| denotes the number of vertices in A and $|E(G[A])|$ denotes the number of edges in the subgraph induced by A. Note that $\rho_G(A) \geq 0$ for any $A \subset V(G)$ if and only if $mad(G) \leq \frac{18}{7}$ $\frac{18}{7}$.

Take any subset $A' \subset V(F')$. If $\{x, y\} \subset A'$, then

$$
\rho_{F'}(A') \ge \rho_{F'}(A' \cap V(F)) + 9(\ell - 1) - 7(\ell) \ge \rho_F(A' \cap V(F)) + 2\ell - 9 > \rho_F(A' \cap V(F)) \ge 0,
$$

since $\ell \geq 5$. If $\{x, y\} \not\subset A'$, then

$$
\rho_{F'}(A') = \rho_{F'}(A' \cap V(F)) + 9|A' - V(F)| - 7|A' - V(F)| \ge \rho_F(A' \cap V(F)) \ge 0.
$$

Therefore, $mad(F') \leq \frac{18}{7}$ $\frac{18}{7}$.

Thus, from the above Remark, it follows that $mad(G) = mad(H) = \frac{18}{7}$. Since $deg_G(v) = 3$ for any $v \in V(H)$ and the distance (in G) between two vertices in $V(H)$ is at most two, all seven vertices in $V(H)$ should get distinct colors in a 3-dynamic coloring of G and so $ch_3^d(G)$ = $\chi_3^d(G) = 7.$

Figure 1: A tight example for Theorem [1.3,](#page-1-1) $mad(H) = \frac{18}{7}$ and $ch_3^d(H) = 7$

Figure 2: Construction of a large tight example G for Theorem [1.3,](#page-1-1) $mad(G) = \frac{18}{7}$ and $ch_3^d(G) = 7$

We also study the value of $mad(G)$ to be $ch_3^d(G) \leq 7$.

Theorem 1.4. *If* $mad(G) < \frac{14}{5}$ *, then* $ch_3^d(G) \leq 7$ *.*

Let H be the graph that is obtained from the Petersen graph by deleting one edge. Then $mad(H) = \frac{14}{5}$ and $ch_3^d(H) = 8$. Thus the bound in Theorem [1.4](#page-3-0) is tight.

In addition, infinitely many tight examples for Theorem [1.4](#page-3-0) are obtained by a similar way of construction shown in Figure [2.](#page-2-1) For a given graph H' with $mad(H') \leq \frac{14}{5}$ $\frac{14}{5}$ and $ch_3^d(H') \leq 8$, let G be a graph obtained from the union of H and H' connecting by internally disjoint paths of length at least four such that $\deg_G(v) \leq 3$ for any $v \in V(H)$. Note that there are at most two such paths since H has two vertices of degree two. Then $mad(G) = \frac{14}{5}$ and $ch_3^d(G) = 8$.

We also show that any graph G is 3-dynamically 8-choosable if $mad(G) < 3$.

Theorem 1.5. *If* $mad(G) < 3$ *, then* $ch_3^d(G) \leq 8$ *.*

The above result is also tight, since there are infinitely many tight examples. Note that the Petersen graph H satisfies $mad(H) = 3$ and $\chi_3^d(H) = \chi(H^2) = 10$. Now we will construct a graph W with $mad(W) = 3$ and $ch_3^d(W) = 9$. Let H_1 and H_2 be the two copies of the Petersen graph. Let W be the graph obtained by connecting H_1 and H_2 by a path of length 3. Then we can check that $mad(W) = 3$ and $ch_3^d(W) = 9$. Since H_1 and H_2 have a vertex of degree four, respectively, $V(H_1)$ and $V(H_2)$ do not have to have all distinct colors. Thus we have $\chi_3^d(W) = 9$ and also $ch_3^d(W) = 9$.

Similarly, we can find infinitely many tight examples for Theorem [1.5.](#page-3-1) For a given graph H' with $mad(H') \leq 3$ and $ch_3^d(H') \leq 9$, let G be a graph obtained from the union of H and H' connecting by exactly one path of length at least three. Then $mad(G) = 3 = mad(H)$ and $ch_3^d(G) = 9 = ch_3^d(H).$

Note that every planar graph G with grith at least g satisfies $mad(G) < \frac{2g}{g-1}$ $\frac{2g}{g-2}$. Thus from Theorem [1.3,](#page-1-1) Theorem [1.4,](#page-3-0) and Theorem [1.5,](#page-3-1) we have the following corollary. Note that Theo-rem [1.5](#page-3-1) implies Theorem [1.2](#page-1-2) when $r = 3$.

Corollary 1.6. *Let* G *be a planar graph. Then we have the following:* (1) $ch_3^d(G) \leq 6$ *if the girth of G is at least* 9*,* (2) $ch_3^d(G) \leq 7$ *if the girth of G is at least* 7*,* (3) $ch_3^d(G) \leq 8$ *if the girth of G is at least* 6*.*

It was showed in [\[5\]](#page-18-3) that $ch(G) \leq 6$ if $mad(G) < \frac{18}{7}$ and $\Delta(G) \leq 3$. And it was also showed in [\[2\]](#page-18-4) independently that $ch(G) \leq 6$ if $mad(G) < \frac{18}{7}$, $\Delta(G) \leq 3$, and the girth of G is at least 7. Thus Theorem [1.3](#page-1-1) is an extension of the results in [\[2,](#page-18-4) [5\]](#page-18-3). On the other hand, it was showed in [\[2\]](#page-18-4) that $ch(G) \leq 7$ if $mad(G) < \frac{14}{5}$ $\frac{14}{5}$ and $\Delta(G) \leq 3$. Thus Theorem [1.4](#page-3-0) is an extension of the result in [\[2\]](#page-18-4). Consequently, Corolloary [1.6](#page-3-2) is an extension of the results in [\[2\]](#page-18-4).

This paper is organized as follows. In Section 2, we give preliminaries about simple reducible configurations. In Sections 3, 4, and 5, we prove Theorems [1.3,](#page-1-1) [1.4,](#page-3-0) and [1.5,](#page-3-1) respectively.

2 Preliminaries

A vertex of degree d is called a d-*vertex*, and a vertex of degree at least d (at most d) is called a d^+ -vertex (d^- -vertex). If x is adjacent to a d -vertex y (d^+ -vertex, or d^- -vertex), then we say that y is a d-neighbor of x (d⁺-neighbor of x, or d⁻-neighbor of x). Two vertices x and y are *weakly adjacent* in G if they have a common 2-neighbor. In this case, we say that x is a *weak neighbor* of y.

For each $i \in \{0, 1, 2, 3\}$, we let $W_i(G)$ be the set of 3-vertices which have exactly i 2-neighbors. That is,

 $W_i(G) = \{v \in V(G) \mid \text{deg}(v) = 3, \text{ and exactly } i \text{ neighbors of } v \text{ are } 2\text{-vertices}\}.$

If there is no confusion, we denote $W_i(G)$ by W_i . And let $[n] = \{1, 2, \ldots, n\}$.

Lemma 2.1. Let $k \geq 6$. Let G be a graph with smallest number of vertices and edges such that $ch_3^d(G) > k$. Then the followings hold.

- (1) *There is no* 1 ⁻*-vertex.*
- (2) *No two* 2 [−]*-vertices are adjacent.*
- (3) *No two adjacent vertices share a common 2-neighbor.*
- (4) For each $i \in [3]$, every vertex in $W_i(G)$ has i distinct weak neighbors.

Figure 3: Reducible configurations for Lemma [2.1](#page-4-0)

Proof. We prove $(1) \sim (4)$ one by one. Since $ch_3^d(G) > k$, there is a list assignment L of G such that $|L(v)| \geq k$ for each vertex v of G, and G is not 3-dynamically L-colorable.

(1) Let v be a 1⁻-vertex. Since $H = G - \{v\}$ is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Note that the number of available colors at v is at least $k-3$. Since $k-3 \geq 1$, it is easy to see that ϕ can be extended to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, which is a contradiction to the choice of G.

(2) Suppose that two 2-vertices x and y are adjacent (See Figure [3-](#page-4-1)(a)). Let $H = G - \{x, y\}$. Then H is 3-dynamically L-colorable since H is smaller than G . Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Note that the number of available colors at x and y are at least $k-4$. And since $k-4 \geq 2$, it is easy to see that ϕ can be extended

to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, a contradiction to the choice of G.

(3) Suppose that two adjacent vertices x and y share a common 2-neighbor w (See Figure [3-](#page-4-1)(b)). Let $H = G - \{w\}$. Then H is 3-dynamically L-colorable since H is smaller than G. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Note that the number of available colors at w is at least $k-4$. Since $k-4 \geq 1$, it is easy to see that ϕ can be extended to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, a contradiction to the choice of G.

(4) From (3), we know that for any vertex, the set of neighbors and the set of weak neighbors are disjoint. Note that (4) trivially holds for the vertices in W_1 . Suppose that there is a vertex $x \in W_2 \cup W_3$ such that x has two 2-neighbors w_1 and w_2 and the other neighbors of w_1 and w_2 are the same as a vertex y (See Figure [3c\)](#page-4-1). Let $H = G - \{w_1\}$. H is 3-dynamically L-colorable, since H is smaller than G. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Note that the number of available colors at w_1 is at least $k-5$. Since $k-5 \geq 1$, it is easy to see that ϕ can be extended to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, a contradiction to the choice of G. \Box

The following are simple properties in list coloring, which will be often used in the paper. For a function f assigning a positive integer to each $v \in V(G)$, a graph G is said to be f-choosable if for any list assignment L such that $|L(v)| \ge f(v)$ for every vertex v, G is L-colorable.

Remark 2.2. For each $i \in [3]$, the graph H_i in Figure [4](#page-6-0) is f_i -choosable.

(a) Let $H_1 = K_4 - v_1v_4$ with $V(H_1) = \{v_1, v_2, v_3, v_4\}$, which is the graph in Figure [4-](#page-6-0)(a). Let $f_1(v_1) = 2$, $f_1(v_2) = 3$, $f_1(v_2) = 2$, and $f_1(v_4) = 2$.

Proof. If $L(v_1) \cap L(v_4) \neq \emptyset$, then color v_1 and v_4 with a color $c \in L(v_1) \cap L(v_4)$. And then color v_3 and v_2 . If $L(v_1) \cap L(v_4) = \emptyset$, then color v_2 with a color $c \notin L(v_3)$. And then, the number of available colors at the remaining three vertices in the path are 1, 2, 2. In each case, we can see that H_1 is f_1 -choosable. \Box

(b) Let H_2 be a graph with $V(H_2) = \{v_1, v_2, v_3, v_4, x_1, x_2, w\}$, which is the graph in Figure [4-](#page-6-0)(b). Let $f_2(v_1) = f_2(v_2) = f_2(v_3) = 3$, $f_2(v_4) = 2$, $f_2(x_i) = 4$, $f(x) = 5$, $f(w) = 3$.

Proof. First color the vertex x with a color $c \notin L(v_1)$. And then color the remained vertices in the order of v_4 , v_3 , x_2 , w , x_1 , v_2 , v_1 . \Box

(c) Let H_3 be a graph with $V(H_3) = \{v_1, v_2, v_3, v_4, x_1, x_2, w\}$, which is the graph in Figure [4-](#page-6-0)(c). Let $f_3(v_1) = f_3(v_2) = f_3(v_3) = 3$, $f_3(v_4) = 2$, $f_3(x_j) = 4$, $f(x) = 5$, $f_3(w) = 3$.

Proof. First color the vertex x with a color $c \notin L(v_1)$. And then color the remained vertices \Box in the order of $v_4, v_3, x_2, w, x_1, v_2, v_1$.

Figure 4: Graphs in Remark [2.2.](#page-5-0) The bold number in the parenthesis of each vertex in graph H_i denotes the value of f_i in Remark [2.2.](#page-5-0)

Notations in Figures: A hollow vertex in Figure [5](#page-6-1) stands for a 3^+ -vertex. Throughout following all figures, a hollow vertex always means a 3^+ -vertex, whereas the degree of a solid vertex is the number of incident edges drawn in the figure.

Lemma 2.3. Let $k \geq 6$. Let G be a graph with smallest number of vertices and edges such that $ch_3^d(G) > k$. The graphs in Figure [5](#page-6-1) do not appear as an induced subgraph in G.

Figure 5: Graphs in Lemma [2.3](#page-6-2) (All labelled vertices are distinct.)

Proof. Let L be a list assignment of G such that $|L(v)| \geq k$ for each vertex v of G and G is not 3-dynamically L-colorable. Note that all labelled vertices in the figure are distinct. Suppose that the graph in Figure [5-](#page-6-1)(a) appears in G as an induced subgraph. It has 7 vertices where v_1, v_2 are 3-vertices, v_3, v_5, w are 2-vertices, and v_4, u are 3⁺-vertex. Let $S = \{v_1, v_3, v_5, w\}$ and $H = G - S$. Since H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$, and v_2 and v_4 get distinct colors in ϕ (we can recolor v_2).

For $a \in S$, we denote by $L'(a)$ a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G . More precisely, $L'(a)$ is decided by the following rules.

(Rules of deciding $L'(a)$ for $a \in S$)

Let $Z = N_G(a) \cap V(H)$. That is, Z is the subset of $N_G(a)$ which are colored by the 3-dynamic coloring ϕ .

(1) Remove $\phi(u)$ from $L(a)$ for each $u \in Z$.

- (2) For $u \in Z$, if $d_H(u) = 1$ and u' is the neighbor of u in H, then remove the color $\phi(u')$ from $L(a)$.
- (3) For $u \in Z$, if $d_H(u) \geq 2$, then select two colors from the neighbors of u, say c_1 , c_2 , in $L(a) \cap {\phi(x) : x \in N_H(u)}$ and remove c_1 and c_2 from $L(a)$.
- (4) Let u be a vertex in H with $d_G(a, u) = 2$. If u and a have a common 2-vertex neighbor in G, then remove $\phi(u)$ from $L(a)$.

Then $L'(a)$ is the subset of $L(a)$ which are remained after (1), (2), (3), and (4). Now we count the number of colors in $L'(a)$, and then show that $G^2[S]$ is L'-colorable. (Throughout all proofs of the paper, we use a similar technique for obtaining such L' , we omit explanation at the other places.)

Let c_1 be a color which is colored at a neighbor of v_4 in H, that is, $c_1 \in {\phi(x) : x \in N_H(v_4)}$. And let c_2 be the color which is assigned at the neighbor of v_2 in H, that is $c_2 = \phi(v_2)$ where $N_G(v_2) = \{v_1, v_3, v_2'\}.$ Take two colors c_3 and c_4 from neighbors of u in H. We may assume the following.

$$
L'(v_1) = L(v_1) - \{\phi(v_2), \phi(v_4), \phi(u), c_2\};
$$

\n
$$
L'(v_3) = L(v_3) - \{\phi(v_2), \phi(v_4), c_1, c_2\};
$$

\n
$$
L'(v_5) = L(v_5) - \{\phi(v_2), \phi(v_4), c_1\};
$$

\n
$$
L'(w) = L(w) - \{\phi(v_2), \phi(u), c_3, c_4\}.
$$

Therefore,

$$
|L'(v_1)| \ge k - 4, \quad |L'(v_3)| \ge k - 4, \quad |L'(v_5)| \ge k - 3, \quad |L'(w)| \ge k - 4.
$$

Note that the subgraph of G^2 induced by S, $G^2[S]$, is isomorphic to K_4 minus an edge wv_3 , a graph in Figure [4-](#page-6-0)(a). Since $k-4 \geq 2$ and $k-3 \geq 3$, $G^2[S]$ is L'-colorable by (a) of Remark [2.2.](#page-5-0) Then it is easy to see that ϕ can be extended to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, a contradiction.

Next, suppose that G has the graph in Figure [5-](#page-6-1)(b) as an induced subgraph. It has 12 vertices where v_1 , v_2 , v_3 are 3-vertices, v_4 , v_6 , w_1 , w_2 , w_3 are 2-vertices, and v_5 , u_1 , u_2 , u_3 are 3⁺-vertex. Let $S = \{v_1, v_2, v_3, v_4, v_6, w_1, w_2, w_3\}$. Let $H = G - S$. Since H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. For $a \in S$, let $L'(a)$ be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Note that $G^2[S]$ is isomorphic to the graph in Figure [4-](#page-6-0)(c) and

$$
|L'(v_2)| \ge k - 1, \quad |L'(v_i)| \ge k - 2 \text{ for } i \in \{1, 3, 4, 6\}, \quad |L'(w_i)| \ge k - 3 \text{ for } i \in \{1, 2, 3\}.
$$

Note that we forbid just two colors at v_4 and v_6 since we will color v_4 and v_6 differently. By (c) of Remark [2.2,](#page-5-0) it is 3-dynamically L' -colorable. Thus G is 3-dynamically L -colorable, a contradiction. \Box

3 Proof of Theorem [1.3](#page-1-1)

In this section, we prove Theorem [1.3.](#page-1-1) We use the induction on the number of vertices and the number of edges. In the following, we let G be a minimal counterexample to Theorem [1.3.](#page-1-1) That is, G is a graph with the smallest number of vertices and edges, $mad(G) < 18/7$, and $ch_3^d(G) \geq 7$. Then there exists a list assignment L such that $|L(v)| \geq 6$ for each $v \in V(G)$ and G is not 3-dynamically L-colorable.

From now on, we show that several subgraphs can not appear in G , which are called reducible configurations. More precisely, we will show the following [C1]∼[C6]:

[C1] There is no 1^- vertex. (Lemma [2.1-](#page-4-0)(1))

[C2] No two 2-vertices are adjacent. (Lemma [2.1-](#page-4-0)(2))

 $[\mathbf{C3}]$ No two vertices in W_2 are adjacent. (Lemma [3.1,](#page-8-0) Figure [6\)](#page-8-1)

[C4] For any vertex $x \in W_3$, x has three distinct weak neighbors in W_1 . (Lemma [3.2,](#page-9-0) Figure [7\)](#page-9-1)

[C5] There is no vertex in W_1 , which has two neighbors in W_2 . (Lemma [3.3,](#page-10-0) Figure [8\)](#page-10-1)

[C6] There is no vertex 3-vertex, which has one neighbor in W_1 , one neighbor in W_2 , one weak neighbor in W_3 . (Lemma [3.4,](#page-10-2) Figure [9\)](#page-11-0)

Note that $\mathbf{[C1]}$ and $\mathbf{[C2]}$ hold by Lemma [2.1.](#page-4-0) We will see that $\mathbf{[C3]} \sim \mathbf{[C6]}$ hold.

Lemma 3.1. [C3] No two vertices in W_2 are adjacent.

Figure 6: An illustration of [C3] (Lemma [3.1\)](#page-8-0), $x, y \in W_2$

Proof. Suppose that there are two vertices x and y in W_2 that are adjacent. That is, x and y are 3-vertices and both x and y have exactly two 2-neighbors. Let v_1, v_2 be the 2-neighbors of x and let v_3, v_4 be the 2-neighbors of y. Let u_i be the 3⁺-neighbor of v_i for $i \in \{1, 2, 3, 4\}$ (see Figure [6\)](#page-8-1).

By Lemma [2.1-](#page-4-0)(4), $u_1 \neq u_2$ and $u_3 \neq u_4$. If $u_1 = u_3$, then x, y, v_3, u_1, v_1, v_2 form the induced subgraph in Figure [5-](#page-6-1)(a), this is impossible by Lemma [2.3.](#page-6-2) Thus u_1, u_2, u_3, u_4 are all distinct. Let $S = \{x, y, v_1, v_2, v_3, v_4\}$ and let $H = G - S$. Since G is a minimal counterexample and H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H

such that $\phi(a) \in L(a)$ for any $a \in V(H)$. For $a \in S$, let $L'(a)$ be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then

$$
|L'(x)| \ge 4
$$
, $|L'(y)| \ge 4$, and $|L'(v_i)| \ge 3$ for $i \in [4]$.

Since $|L'(v_1)|+|L'(v_3)|>|L'(x)|$, we can give colors to v_1 and v_3 so that the number of available colors remained at x is at least 3. Then $G^2[{x, y, v_2, v_4}]$ form K_4 minus an edge v_2v_4 as in Figure [4-](#page-6-0)(a), and the numbers of available colors are 3, 2, 2, 2, respectively. By (a) of Remark [2.2,](#page-5-0) it is colorable. This implies that G is 3-dynamically L-colorable, a contradiction. \Box

From Lemma [2.1-](#page-4-0)(4), a vertex in W_3 has three weak neighbors.

Lemma 3.2. [C4] If $x \in W_3$ and y is a weak neighbor of x, then $y \in W_1$. That is, x has three weak neighbors in W_1 .

Figure 7: An illustration of [C4] (Lemma [3.2\)](#page-9-0), $x \in W_3$, $u_1 \notin W_1$

Proof. Let x be a 3-vertex in W_3 and let v_1, v_2, v_3 be the 2-neighbors of x. Let u_i be the other neighbor of v_i for each $i \in [3]$ (so they are weak neighbors of x). By [C2], each u_i is a 3⁺-vertex. By Lemma [2.1-](#page-4-0)(4), three vertices u_1, u_2, u_3 are distinct.

First, we will show that u_i is a 3-vertex for each $i \in [3]$. Suppose that some u_i is not a 3-vertex for some $i \in [3]$. Without loss of generality, assume that u_1 is not a 3-vertex. Then u_1 is a 4⁺-vertex by [C2]. Let $S = \{x, v_1, v_2, v_3\}$ and $H = G - S$. Then H is 3-dynamically L-colorable, since G is a minimal counterexample and H is smaller than G . Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. For $a \in S$, let $L'(a)$ be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then

$$
|L'(v_1)| \ge 5
$$
, $|L'(v_2)| \ge 3$, $|L'(v_3)| \ge 3$ and $|L'(x)| \ge 3$.

Since $G^2[S]$ is K_4 , $G^2[S]$ is f_S-choosable. This implies that ϕ can be extended to a 3-dynamic coloring of G so that G is 3-dynamically L-colorable, a contradiction. Hence u_i is a 3-vertex for each $i \in [3]$.

Next, we will show that every u_i is in W_1 , which means that x is the only weak neighbor of u_1 . Without loss of generality, we may assume that u_1 is has another 2-neighbor w other than v_1 . Then all 8 vertices x, v_i 's, u_i 's, w are distinct (See Figure [7\)](#page-9-1). Note that x and w cannot have a common neighbor as all neighbors of x are v_i 's. Let $S = \{x, v_1, u_1, w\}$. Since G is a minimal counterexample and H is smaller than G , H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. For $a \in S$, let $L'(a)$

be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then $G^2[S]$ is the graph in Figure [4-](#page-6-0)(a) and

$$
|L'(x)| \ge 2, |L'(v_1)| \ge 3, |L'(u_1)| \ge 2, |L'(w)| \ge 2.
$$

By (a) of Remark [2.2,](#page-5-0) $G^2[S]$ is L'-colorable. This implies that G is 3-dynamically L-colorable, a contradiction. \Box

Lemma 3.3. [C5] There is no vertex in W_1 which has two neighbors in W_2 .

Figure 8: An illustration of [C5] (Lemma [3.3\)](#page-10-0), $x \in W_1, x_1, x_2 \in W_2$

Proof. Suppose that there is a vertex $x \in W_1$ such that x has two neighbors $x_1, x_2 \in W_2$. Then all 5 vertices in $N_G(x) \cup N_G(x_1) \cup N_G(x_2) - \{x_1, x_2\}$ are 2-vertices. We label those vertices as in Figure [8.](#page-10-1) Let $S = \{x, x_1, x_2, v_1, v_2, v_3, v_4, w\}$. Let u_i be the neighbor of v_i other than x_1 and x_2 for each $i \in [4]$, and w' be the neighbor of w other than x. Note that $u_1 \neq u_2$ and $u_3 \neq u_4$ by Lemma $2.1-(4)$.

Then $w' \neq u_1$, otherwise the five vertices w', w, v_1, x_1, x form a cycle C_5 and together with the three vertices v_2, u_2, x_2 , they form the induced subgraph in Figure [5-](#page-6-1)(a). Hence, $w' \neq u_i$ for all $i \in [4]$. If $u_1 = u_3$, then we have the graph in Figure [5-](#page-6-1)(b), which is a contradiction. Thus u_1, u_2, u_3, u_4 , and w' are distinct.

Let $H = G - S$. Since G is a minimal counterexample and H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. For $a \in S$, let $L'(a)$ be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then

$$
|L'(x)| \ge 5, |L'(x_1)| \ge 4, |L'(x_2)| \ge 4, |L'(w)| \ge 3, |L'(v_i)| \ge 3 \text{ (for } i \in [4]),
$$

and $G^2[S]$ is graph H_2 in Figure [4-](#page-6-0)(b). By (b) of Remark [2.2,](#page-5-0) it is colorable. This implies that G is 3-dynamically L-colorable, a contradiction. □

Lemma 3.4. [C6] There is no 3-vertex which has one neighbor in W_1 , one neighbor in W_2 , and one weak neighbor in W_3 .

Proof. Suppose that there exists a 3-vertex x which has one neighbor y in W_2 , one neighbor z in W_1 , one weak neighbor u in W_3 . Let v_1 be the 2-neighbor of x, let v_2 and v_3 be the other 2-neighbors of u. Let y_1 and y_2 be two 2-neighbors of y other than v_1 , and let z_1 and z_2 be two neighbors of z other than x (See Figure [9\)](#page-11-0).

Figure 9: An illustration of [C6] (Lemma [3.4\)](#page-10-2), $x, z \in W_1, y \in W_2, u \in W_3$

Figure 10: The local structure of $H = G - \{v_1, v_2, u\}$ near the vertex x

Since $H = G - \{v_1, v_2, u\}$ is smaller than G, there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. In the graph H, the vertex x can be recolored without changing the colors of the other vertices, except 5 vertices y, z, y_1, y_2, z_1 (see Figure [10\)](#page-11-1) by Claim [3.5.](#page-11-2) (The following claim appeared in Lemma 17 in [\[2\]](#page-18-4). But, we include here for the sake of completeness.)

Claim 3.5. *There is a 3-dynamic coloring* ϕ' *of* H *such that* $\phi'(a) \in L(a)$ *for all* $a \in V(H)$ *, and* $\phi(x) \neq \phi'(x)$, $\phi(a) = \phi'(a)$ *for any vertex* $a \in V(G) \setminus \{y, z, y_1, y_2, z_1\}.$

Proof of Claim [3.5.](#page-11-2) We uncolor the colors of 6 vertices x, y, z, y₁, y₂, z₁ from ϕ . Then we will show that we can recolor the vertices so that the new color of x is not different from $\phi(x)$. For $a \in S$, we denote by $L'(a)$ a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then

$$
|L'(x)| \ge 5, |L'(y)| \ge 4, |L'(z)| \ge 2, |L'(y_1)| \ge 3, |L'(y_2)| \ge 3, |L'(z_1)| \ge 2.
$$

Color y by a color $c \notin L'(y_1)$. Redefine $L'(v)$ by the set of available colors at v after coloring y. Then

$$
|L'(x)| \ge 4, \ |L'(z)| \ge 1, \ |L'(y_1)| \ge 3, \ |L'(y_2)| \ge 2, \ |L'(z_1)| \ge 2.
$$

Color z and z_1 , then redefine $L'(v)$ by the set of available colors at v after coloring z and z_1

$$
|L'(x)| \ge 2, \ |L'(y_1)| \ge 3, \ |L'(y_2)| \ge 2,
$$

the number of available colors remained at x is at least 2. Thus we can recolor x with a color distinct from $\psi(x)$. This completes the proof of Claim [3.5.](#page-11-2) 口

For $a \in S$, let $L'(a)$ be a subset of $L(a)$, which makes ϕ extended to a 3-dynamic coloring of G. Then

$$
|L'(v_1)| \ge 2, \quad |L'(v_2)| \ge 2, \quad |L'(u)| \ge 2.
$$

Let u_2 and u_3 be the neighbors of v_2 and v_3 other than u, respectively. Select and fix two colors c_1 and c_2 in $\{\phi(q): q \in N_G(u_2) \setminus \{v_2\}\}\)$, and then we may let

$$
L'(v_1) = L(v_1) - \{\phi(v_3), \phi(x), \phi(y), \phi(z)\};
$$

\n
$$
L'(v_2) = L(v_2) - \{\phi(v_3), \phi(u_2), c_1, c_2\};
$$

\n
$$
L'(u) = L(u) - \{\phi(v_3), \phi(x), \phi(u_2), \phi(u_3)\}.
$$

By Claim [3.5,](#page-11-2) we can assume that a set of available colors at v_2 is not equal to that of u by recoloring x. As each has two available colors and all of them are not same, we can color v_1, v_2, u from the lists. Thus G is 3-dynamically L -colorable, a contradiction. \Box

We use discharging technique. We define the charge of each vertex v of G by its degree deg(v). Note that the average charge is less than $\frac{18}{7}$. Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least $\frac{18}{7}$, which leads a contradiction.

Recall that W_2 is the set of 3-vertices which have two 2-neighbors, and W_3 is the set of 3-vertices which have three 2-neighbors. See also Figure [11](#page-13-0) for discharging rules.

Discharging Rules

- **R1.** A 3⁺-vertex gives $\frac{2}{7}$ to each of its 2-neighbors.
- **R2.** A 3⁺-vertex gives $\frac{1}{7}$ to each of its weak neighbors in W_3 .
- **R3.** A 3⁺-vertex gives $\frac{1}{7}$ to each of its neighbors in W_2 .
- **R4.** A 3-vertex in W_0 gives $\frac{1}{7}$ to each of its neighbors x in W_1 if x has a neighbor in W_2 and a weak neighbor in W_3 .

Let $d^*(u)$ be the new charge after discharging. We will show that $d^*(u) \geq \frac{18}{7}$ $\frac{18}{7}$ for all $u \in V(G)$. Note that by [C1] each vertex of G is a 2^+ -vertex.

(1) Suppose that $deg(u) = 2$. By [C2] the neighbors of u are 3⁺-vertices and so it receives $\frac{2}{7}$ from each of its neighbors by $R1$, and so the new charge of u is

$$
d^*(u) = 2 + \frac{2}{7} + \frac{2}{7} = \frac{18}{7}.
$$

(2) Suppose that $deg(u) = 3$. If $u \in W_0$, then u does not have a 2-neighbor and a weak neighbor, and so u might give charge $\frac{1}{7}$ to each of neighbors by **R3** and **R4**. Thus the new charge of u satisfies

$$
d^*(u) \ge 3 - 3 \times \frac{1}{7} = \frac{18}{7}.
$$

Next, suppose that $u \in W_1 \cup W_2 \cup W_3$. Then u gives $\frac{2}{7}$ to each of its 2-neighbors by **R1**.

(a) R1 A 3⁺-vertex u gives $\frac{2}{7}$ to each of its 2-neighbors.

(c) **R3** A 3⁺-vertex u gives $\frac{1}{7}$ to each of its neighbors in W_2 .

(b) **R2** A 3⁺-vertex u gives $\frac{1}{7}$ to each of its weak neighbors in W_3 .

(d) **R4** A 3-vertex u in W_0 gives $\frac{1}{7}$ to each of its neighbors x in W_1 if x has a neighbor in W_2 and a weak neighbor in W_3 .

Figure 11: An illustration of Discharging Rules

Figure 12: An illustration of case (2-2), $u \in W_2$

 $(2-1)$. Suppose that $u \in W_3$.

By Lemma [3.2](#page-9-0) ([C4]), u has three distinct weak neighbors x_1, x_2, x_3 in W_1 . Then u receives 1 $\frac{1}{7}$ from each x_i by **R2**. Since each x_i is not in $W_2 \cup W_3$ and so u does not give any charge to them by $R2$ or $R3$. Thus the new charge of u satisfies

$$
d^*(u) \ge 3 - 3 \times \frac{2}{7} + 3 \times \frac{1}{7} = 3 - \frac{3}{7} = \frac{18}{7}.
$$

 $(2-2)$. Suppose that $u \in W_2$.

Let x be the 3⁺-neighbor of u, and x_1 and x_2 be the two weak neighbors of u. See Figure [12](#page-13-1) for an illustration. Then u receives $\frac{1}{7}$ from x by **R3**. By [C4], each x_i is not in W_3 . Thus u does not give any charge to them by R2. By [C3], $x \notin W_2$ and so u does not send any charge to x by R3. Since $u \notin W_0$, u does not send any charge to x by R4. Thus, in total,

$$
d^*(u) \ge 3 - 2 \times \frac{2}{7} + \frac{1}{7} = 3 - \frac{3}{7} = \frac{18}{7}.
$$

(2-3). Suppose that $u \in W_1$.

Figure 13: An illustration for case (2-3-2), $u \in W_1$, $x \in W_2$, and $w \in W_3$.

Let x and z be the two 3^+ -neighbors and w be the weak neighbor of u. Then by [C5], we may assume that $z \notin W_2$. Thus u gives at most $\frac{1}{7}$ to x and z in total by **R3**. By **R2**, u gives at most $\frac{1}{7}$ to w.

(2-3-1). Suppose that $x \notin W_2$ or $w \notin W_3$. If $x \notin W_2$, then u does not give any charge to x by **R3**. If $w \notin W_3$, then u does not give any charge to w by **R2**. Thus u gives at most $\frac{1}{7}$ to x, z, and w in total,

$$
d^*(u) \ge 3 - \frac{2}{7} - \frac{1}{7} = \frac{18}{7}.
$$

(2-3-2). Suppose that $x \in W_2$ and $w \in W_3$. See Figure [13](#page-14-0) for an illustration. Then u gives $\frac{1}{7}$ to w by **R2**. By [C6], $z \notin W_1$, which implies that $z \in W_0$. Then u receives $\frac{1}{7}$ from z and u does not send any charge to z by R4. (Note that u is a vertex in W_1 , which has a neighbor in W_2 and one weak neighbor in W_3 .) Thus

$$
d^*(u) \ge 3 - \frac{2}{7} - 2 \cdot \frac{1}{7} + \frac{1}{7} = \frac{18}{7}.
$$

(3) Suppose that deg(u) ≥ 4 .

In this case, u gives charge at most $\frac{2}{7}$ to its neighbors by **R1**, **R2** and **R3**. Note that any weak neighbor of u is not in W_3 by $\mathbf{[C4]}$ and so u does not give any charge to its weak neighbor by **. Thus**

$$
d^*(u) \ge \deg(u) - \deg(u) \times \frac{2}{7} = \frac{5}{7} \deg(u) > \frac{18}{7}.
$$

This completes the proof of Theorem [1.3.](#page-1-1)

4 Proof of Theorem [1.4](#page-3-0)

We use the induction on the number of vertices and the number of edges. In the following, we let G be a minimal counterexample to Theorem [1.4.](#page-3-0) That is, G is a graph with the smallest number of vertices and edges, $mad(G) < 14/5$, and $ch_3^d(G) \geq 8$. Then there exists a list assignment L such that $|L(v)| \ge 7$ for each $v \in V(G)$ and G is not 3-dynamically L-colorable.

Lemma 4.1. *For* $k \in \{3, 4\}$ *, any k*-vertex has at most $(k − 2)$ 2-neighbors.

Proof. Let $k \in \{3, 4\}$ and let v be a k-vertex, and v_1, v_2, \ldots, v_k be its neighbors. Suppose that v has at least $(k-1)$ 2-neighbors v_1, \ldots, v_{k-1} . Let $H = G - vv_k$. Then $mad(H) < 14/5$. Since G is a minimal counterexample and H is smaller than G , H is 3-dynamically L-colorable. Thus

Figure 14: An illustration for Lemma [4.2.](#page-15-0)

there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Then uncolor the vertex v and its 2-neighbors v_1, \ldots, v_{k-1} .

Note that the number of forbidden colors at v is at most $(k-1)+3=k+2\leq 6$. Thus v has at least one available color. We color v first with an available color. Then we recolor each 2-neighbor of v one by one. Since the number of available colors at each 2-neighbor of v is two, and so they are colorable so that v has three distinct colored neighbors. Thus G is 3-dynamically L-colorable, a contradiction. \Box

Lemma 4.2. *No two* 3-vertices x and y in W_1 are adjacent.

Proof. Let x and y be 3-vertices such that $x, y \in W_1$ and $xy \in E(G)$. Let x' and y' be 2-neighbors of x and y, respectively. And let w_1 and w_2 be the other neighbor of x' and y' , respectively. See Figure [14.](#page-15-1) Let $H = G - \{x', y'\}$. Then $mad(H) < 14/5$. Since G is a minimal counterexample and H is smaller than G , H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Then uncolor the colors of x and y. Then the number of available colors at x is at least 3, and that of y is also at least 3. Color x with a color which is different from the color assigned at w_1 , and y with a color which is different from the color assigned at w_2 . Let $L'(x')$ and $L'(y')$ be the set of available colors at x' and y' , respectively.

Now, we consider two cases.

Case 1: $w_1 \neq w_2$ (See Figure [14-](#page-15-1)(a)).

Since $|L'(x')| \geq 1$ and $|L'(y')| \geq 1$, we can color x' and y' to have a dynamic 3-coloring.

Case 2: $w_1 = w_2$ (See Figure [14-](#page-15-1)(b)).

If the degree of w_1 in H is at least three, then x' and y' do not have to use different color and so we have a 3-dynamic coloring. Next, if the degree of w_1 is 2 in H, then $|L'(x')| \geq 2$ and $|L'(y')| \geq 2$. So they are colorable. Thus G is 3-dynamically L-colorable, a contradiction. \Box

Lemma 4.3. *No* 3*-vertex has three neighbors in* W1*.*

Proof. Suppose that there is a vertex x having three neighbors x_1, x_2, x_3 in W_1 . Let x'_i be the 2-neighbor of x_i for each $i \in [3]$. See Figure [15](#page-16-0) for an illustration. Let $H = G \{x, x_1, x_2, x_3, x'_1, x'_2, x'_3\}$. Since G is a minimal counterexample and H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Then the number of available colors at x is at least 4, that of x_i is at least

(a) Case when x'_1, x'_2 and x'_3 do not share a neighbor

(b) Case when x'_1 , x'_2 and x'_3 share a neighbor w

(c) Case when x'_1 and x'_2 share a neighbor w but x'_3 does not

Figure 15: An illustration for Lemma [4.3.](#page-15-2)

3 for each $i \in [3]$. We give a color to x_1, x_2, x_3, x with their available colors so that they get distinct colors. Then in the resulting coloring, the number of available colors at x_i' is at least 1. We color x'_1, x'_2, x'_3 by that available colors. Here, the only thing that we have to concern is the case where x'_i and x'_j share a common neighbor and they get the same color.

Suppose that x_1', x_2', x_3' share a neighbor w. See Figure [15-](#page-16-0)(b). Then w has at least three 2-neighbors and so by Lemma [4.1,](#page-14-1) w is a 5^+ -vertex. Thus in the 3-dynamic coloring ϕ of H, w has already at least two distinct colors in its neighbors other than the colors of x'_1, x'_2 , x'_3 . Thus eventually, the extended coloring of G results that G is 3-dynamically L-colorable, a contradiction.

Suppose that x'_1 and x'_2 share a neighbor w and x'_3 does not. See Figure [15-](#page-16-0)(c). Then w has at least two 2-neighbors by Lemma [4.1,](#page-14-1) w is a 4⁺-vertex. Thus in the 3-dynamic coloring ϕ of H, w has already at least two distinct colors in its neighbors other than the colors of x'_1 and x'_2 . Thus the extended coloring of G results that G is 3-dynamically L-colorable, a contradiction. \Box

We use discharging technique. We define the charge of each vertex v of G by its degree deg(v). Note that the average charge is less than $\frac{14}{5}$. Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least $\frac{14}{5}$, which leads a contradiction. The rules are as follows.

R1. A 3⁺-vertex gives $\frac{2}{5}$ to its each of 2-neighbors.

R2. A 3⁺-vertex gives $\frac{1}{10}$ to its each of 3-neighbors in W_1 .

Let $d^*(u)$ be the new charge after discharging. We will show that $d^*(u) \geq \frac{14}{5}$ $\frac{14}{5}$ for all $u \in V(G)$. Note that by Lemma [2.1-](#page-4-0)(1), each vertex of G is a 2^+ -vertex. If deg(u) = 2, by Lemma 2.1-(2), the neighbors of u are 3^+ -vertices and so it receives $\frac{2}{5}$ from each of its neighbors by **R1**, which implies that

$$
d^*(u) = 2 + \frac{2}{5} + \frac{2}{5} = \frac{14}{5}.
$$

If $deg(u) = 3$, then either $u \in W_0$ or $u \in W_1$ by Lemma [4.1.](#page-14-1) If $u \in W_0$, u has at most two neighbors W_1 by Lemma [4.3](#page-15-2) and so u gives $\frac{1}{10}$ to each of its 3-neighbors in W_1 by R2 and so

$$
d^*(u) \ge 3 - 2 \times \frac{1}{10} = \frac{14}{5}.
$$

If $u \in W_1$, then by Lemma [4.2,](#page-15-0) the u has two 3⁺-neighbors and so it receives $\frac{1}{10}$ from each of them by $R2$ and so

$$
d^*(u) \ge 3 - \frac{2}{5} + \frac{1}{10} + \frac{1}{10} = \frac{14}{5}.
$$

If deg(u) = 4, then by Lemma [4.1,](#page-14-1) u has at most two 2-neighbors, and so

$$
d^*(u) \ge 4 - 2 \times \frac{2}{5} - 2 \times \frac{1}{10} = 3 > \frac{14}{5}.
$$

If deg(u) \geq 5, then

$$
d^*(u) \ge \frac{3}{5} \deg(u) > \frac{14}{5}.
$$

5 Proof of Theorem [1.5](#page-3-1)

We use the induction on the number of vertices and the number of edges. In the following, we let G be a minimal counterexample to Theorem [1.5.](#page-3-1) That is, G is a graph with the smallest number of vertices and edges, $mad(G) < 3$, and $ch_3^d(G) \ge 9$. Then there exists a list assignment L such that $|L(v)| \geq 8$ for each $v \in V(G)$ and G is not 3-dynamically L-colorable.

Lemma 5.1. *Any* 3 [−]*-vertex has no* 2*-neighbors.*

Proof. Let x be a 3⁻-vertex and has a 2-neighbor y. Consider $H = G - xy$, deleting the edge xy from G. Then $mad(H) < 3$. Since G is a minimal counterexample and H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Then uncolor the vertices x and y.

Note that the number of forbidden colors at x is at most $3 + 3 + 1 = 7$. Thus x has at least one available color. We color x first with that color. Then we recolor y , since the number of forbidden colors at y is at most $3 + 3 = 6$. \Box

Lemma 5.2. *For* $k \in \{4, 5\}$ *, any* k -vertex has at most $(k − 2)$ 2-neighbors.

Proof. Let $k \in \{4, 5\}$ and let v be a k-vertex, and v_1, v_2, \ldots, v_k be its neighbors. Suppose that v has at least $(k-1)$ 2-neighbors, v_1, \ldots, v_{k-1} . Let $H = G - vv_1$. Then $mad(H) < 3$. Since G is a minimal counterexample and H is smaller than G, H is 3-dynamically L-colorable. Thus there is a 3-dynamic 8-coloring ϕ of H such that $\phi(a) \in L(a)$ for any $a \in V(H)$. Then uncolor the vertex v and its all 2-neighbors.

Note that the number of forbidden colors at v is at most $(k-1)+3=k+2\leq 7$. Thus v has at least one available color and we color v first with that color. Then we recolor each 2-neighbor of v one by one. Since the number of available colors at each 2-neighbor of v is 3, and so they are colorable so that v has three distinct colored neighbors. \Box

We use discharging technique. We define the charge of each vertex v of G by its degree $deg(v)$. Note that the average charge is less than 3. Next, we distribute their charges by the following rules, and then we show that the new charge of each vertex is at least 3, which leads a contradiction. The rule is as follows.

R1. A 4⁺-vertex gives $\frac{1}{2}$ to its each of 2-neighbors.

Let $d^*(u)$ be the new charge after discharging. We will show that $d^*(u) \geq 3$ for all $u \in V(G)$. By Lemma [2.1-](#page-4-0)(1), each vertex of G is a 2⁺-vertex. If deg(u) = 2, by Lemma [5.1,](#page-17-0) then the neighbors of u are 4^+ -vertices and so u receives $\frac{1}{2}$ from each of its neighbors by **R1** and so

$$
d^*(u) = 2 + \frac{1}{2} + \frac{1}{2} = 3.
$$

If $deg(u) = 3$ then the charge of u is not changed and so $d^*(u) = deg(u) = 3$. If $deg(u) = 4$ then by Lemma [5.2,](#page-17-1) it has at most two 2-neighbors and so

$$
d^*(u) \ge 4 - 2 \times \frac{1}{2} = 3.
$$

If deg(u) = 5 then by Lemma [5.2,](#page-17-1) it has at most three 2-neighbors and so

$$
d^*(u) \ge 5 - 3 \times \frac{1}{2} > 3.
$$

If deg $(u) \geq 6$, then

$$
d^*(u) \ge \deg(v) - \deg(v) \times \frac{1}{2} \ge \frac{\deg(v)}{2} \ge 3.
$$

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References

- [1] S. Akbari, M. Ghanbari, and S. Jahanbekam, On the list dynamic coloring of graphs, *Discrete Appl. Math.* 157 (2009), 3005–3007.
- [2] D. Cranston and S.-J. Kim, List-coloring the square of a subcubic grap, *J. Graph Theory* 57 (2008), 65–87.
- [3] J. Cheng, H.-J. Lai, K. Lorenzen, R. Luo, J. Thompson, and C.Q. Zhang, r-hued coloring of sparse graphs, manuscript.
- [4] L. Esperet, Dynamic list coloring of bipartite graphs, *Discrete Appl. Math.* 158 (2010), 1963–1965.
- [5] F. Havet, Choosability of the square of planar subcubic graphs with large girth, *Discrete Math.* 309 (2009), 3553–3563.
- [6] S. Jahanbekam, J. Kim, S. O, and D.B. West, On r-dynamic coloring of graphs, *Discrete Appl. Math.* 206 (2016), 65–72.
- [7] R. Kang, T. M¨uller, and D.B. West, On r-dynamic coloring of grids, *Discrete Appl. Math.* 186 (2015), 286–290.
- [8] J. Kim and S. Ok, Dynamic choosability of triangle-free graphs and sparse random graphs, *J. Graph Theory*, to appear.
- [9] S.-J. Kim, S. J. Lee, W.-J. Park, Dynamic coloring and list dynamic coloring of planar graphs, *Discrete Appl. Math.* 161 (2013), 2207–2212.
- [10] S.-J. Kim, W.-J. Park, List dynamic coloring of sparse graphs, Combinatorial optimization and applications, *Lect. Notes Comput. Sci.* 6831 (2011), 156–162.
- [11] S. Loeb, T. Mahoney, B. Reiniger, and J. Wise, Dynamic coloring parameters for graphs with given genus, manuscript.
- [12] B. Montgomery, Dynamic coloring of graphs, Ph.D. Thesis, West Virginia University, 2001.
- [13] H. Song, S. Fan, Y. Chen, L. Sun, H.-J. Lai, On r-hued coloring of K_4 -minor free graphs, *Discrete Math.* 315 (2014), 47–52.
- [14] H. Song, H.-J. Lai, and J.-L. Wu, On r-hued coloring of planar graphs with girth at least 6, *Discrete Appl. Math.* 198 (2016), 251–263.
- [15] G. Wegner, Graphs with given diameter and a coloring problem, Technical Report, University of Dortmund, 1977.