4-choosability of planar graphs with 4-cycles far apart via the Combinatorial Nullstellensatz*

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Abstract

By a well-known theorem of Thomassen and a planar graph depicted by Voigt, we know that every planar graph is 5-choosable, and the bound is tight. In 1999, Lam, Xu and Liu reduced 5 to 4 on C_4-free planar graphs. In the paper, by applying the famous Combinatorial Nullstellensatz, we design an effective algorithm to deal with list coloring problems. At the same time, we prove that a planar graph G is 4-choosable if any two 4-cycles having distance at least 5 in G, which extends the result of Lam et al.

Key words: planar graphs, choosable, nice path, Combinatorial Nullstellensatz.

1 Introduction

All graphs considered in the paper are simple and finite. The concepts of list coloring and choosability were introduced by Vizing [19] and independently by Erdős, Rubin and Taylor [10]. Given a graph G, a *list assignment* L for G is a function that to each vertex $v \in V(G)$ assigns a set $L(v)$ of colors, and an L-*coloring* is a proper coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. We say that G is L-*colorable* if G has an L-coloring. Moreover, G is k-*choosable* if G is L-colorable for every list assignment L with $|L(v)| \geq k$ for each $v \in V(G)$. List coloring is a fundamental object in graph theory with a wealth of results studying various aspects and variants. A variety of mathematicians have suggested imposing slightly stronger conditions in order to further explore the choosability of graphs, see [6, 9, 13]. The distance of two vertices is the shortest length (number of edges) of paths between them, and

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the distance $d(H_1, H_2)$ of two subgraphs H_1 and H_2 is the minimum of the distances between vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

The classic Four Color Theorem claims that every planar graph is 4-colorable, which was proved by Appel and Haken in 1976 [3, 4]. However, the result can not be extended to that of list colorings as Voigt [20] found a planar graph which is not 4-choosable. Fortunately, Thomassen [17] proved that every planar graph is 5-choosable by induction on the number of vertices. In order to further explore list coloring problems, forbidding certain structures within a planar graph is a common restriction used in graph coloring. Notice that all 2-choosable graphs have been characterised by Erdős, Rubin and Taylor [10]. So it remains to determine whether a given planar graph is 3- or 4-choosable. In recent years, a number of interesting results about the choosability of special planar graphs have been obtained. Alon and Tarsi [2] proved that every planar bipartite graph is 3-choosable. Thomassen [18] showed every planar graph of girth at least 5 is 3-choosable, and there exist triangle-free planar graphs which are not 3-choosable [21], so the bound 5 is tight. Very recently, Dvořák [7] showed that every planar graph in which any two (≤ 4)-cycles have distance at least 26 is 3-choosable.

Steinberg's Conjecture from 1976 states that every $\{C_4, C_5\}$-free planar graph is 3-colorable, which was disproved by Cohen-Addad et al. [5]. Previously, Voigt [22] disproved a list version of Steinberg's Conjecture by giving a $\{C_4, C_5\}$-free planar graph which is not 3-choosable. A graph G is said to be *k-degenerate* if every nonempty subgraph H of G has a vertex of degree at most k in H. Note that the list chromatic number of a k-degenerate graph is at most $k+1$. It is simple to check that every triangle-free planar graph is 3-degenerate, and so it is 4-choosable. In addition, it was proved that every C_k-free planar graph is 4-choosable for $k = 4$ in [15], for $k = 5$ in [14, 24], for $k = 6$ in [12, 14, 23], and for $k = 7$ in [11]. On the other hand, it is shown in [14] that every planar graph in which any two triangles have distance at least 2 is 4-choosable, and a conjecture was proposed in this paper, which claims that every planar graph without adjacent triangles is 4-choosable (this conjecture is still open so far). After that, Wang and Li [25] improved one of the results in [14] by showing that each planar graph without intersecting triangles is 4-choosable.

Inspired by the improvements of the results about triangle-free planar graphs, we further explore the picture when any two 4-cycles in a planar graph is far apart. A natural question can be proposed as follows.

Problem A. Does there exist a constant d such that a planar graph G is 4-choosable if any two 4-cycles have distance at least d in G?

We give a positive answer to this question with $d = 5$.

Theorem 1. *If G is a planar graph such that any two 4-cycles have distance at least 5, then G is 4-choosable.*

2 A Structural Lemma

For any positive integer r, we write $[r]$ for the set $\{1, \ldots, r\}$. Given a plane graph G, we denote its vertex set, edge set, face set by $V(G)$, $E(G)$, and $F(G)$, respectively. For any vertex $v \in V(G)$ (or any face $f \in F(G)$), the degree of v (or f), denoted by $d(v)$ (or $d(f)$), is the number of edges incident with v (or the length of the boundary walk of f, where each cut edge is counted twice). A vertex v is called a k-vertex (k^+-vertex, or k^--vertex) if $d(v) = k$ ($d(v) \geq k$, or $d(v) \leq k$, respectively). Analogously, a k-face (k^+-face, or k^--face) is a face of degree k (at least k, or at most k, respectively). Moreover, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and the minimum degree of G, respectively.

We write $f = (u_1, \ldots, u_t)$ if $u_1, \ldots, u_t$ are the boundary vertices of f in the clockwise order. Sometimes we replace u_i with $d(u_i)$ for some $i \in [t]$ in $f = (u_1, \ldots, u_t)$ to describe the face f. For example, $f = (4, 4, 5, u_4)$ denotes a 4-face with $d(u_1) = d(u_2) = 4$, $d(u_3) = 5$. For a vertex v and a face f, let $f_k(v)$, $n_k(v)$ and $n_k(f)$ denote the number of k-faces incident with v, the number of k-vertices adjacent to v, and the number of k-vertices incident with f, respectively. Let $f = (v_1, v_2, v_3, v_4, v_5)$ be a 5-face, f is called *bad* if $d(v_i) = 4$ for all $i \in [5]$. For convenience, we use $f_{5b}(v)$ to denote the number of bad 5-faces incident with a vertex v. In addition, let $\zeta_v(f_{3b})$ denote the number of 3-faces $f = (x, y, v)$ incident with v such that $d(x) = d(y) = 4$ and xy locates on a bad 5-face. Below Figure 1 shows a 6-vertex v with $\zeta_v(f_{3b}) = 3$.

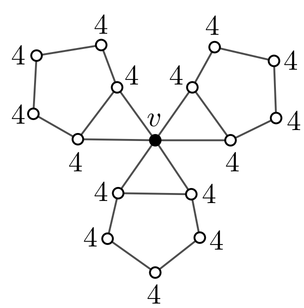

Figure 1: $d(v) = 6$ and $\zeta_v(f_{3b}) = 3$.

A 4-vertex v with $f_3(v) + f_{5b}(v) \leq 1$ of G is called *good*, whereas v is called *bad* if $f_3(v) = 1$ and $f_{5b}(v) = 1$.

Lemma 2.1. *Let G be a connected planar graph such that any two 4-cycles have distance at least 5. Then*

(a) *G has a 3^--vertex, or*

(b) *G contains one of the configurations S_1-S_{47}, see Appendix B.*

Proof. Let G be a counterexample to the lemma with $|V(G)| + |E(G)|$ as small as possible. Then $\delta(G) \geq 4$ and G contains none of the configurations S_1-S_{47} in Appendix B. Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ can be expressed in the form

$$\sum_{v \in V(G)} (d_G(v) - 2) + \sum_{f \in F(G)} (-2) = -4. \tag{1}$$

An initial charge ch_0 on $V(G) \cup F(G)$ is defined by letting $ch_0(v) = d(v) - 2$ for each $v \in V(G)$ and $ch_0(f) = -2$ for each $f \in F(G)$. Thus we have $\sum_{z \in V(G) \cup F(G)} ch_0(z) < 0$.

In the following, $c(x \to y)$ is used to denote the amount of charges transferred from an element x to an element y. For brevity, let $\gamma = \frac{2 - \frac{1}{3} n_4(f)}{n_{5^+}(f)}$.

We define the following two rounds of discharging rules. The first round contains R1-R5. Let v be a k-vertex, and let f be an ℓ-face incident with v.

R1. $c(v \to f) = \frac{2}{3}$ if $\ell = 3$, and $c(v \to f) = \frac{1}{3}$ if $\ell \geq 6$.

R2. For $k = 4$ and $\ell \in \{4, 5\}$.

R2.1. Let $T_f = \{v_i : d(v_i) = 4 \text{ and } f_3(v_i) \leq 1\}$. If $f = (v_1, v_2, v_3, v_4, v_5)$ is a bad 5-face with $f_3(v) \leq 1$, then $c(v \to f) = \frac{2}{3}$ when $|T_f| = 1$, and $c(v \to f) = \frac{1}{2}$ when $|T_f| \geq 2$.

R2.2. $c(v \to f) = \frac{1}{3}$ otherwise.

R3. For $k = 5$, $c(v \to f) = \frac{5}{9}$ if $\ell = 4$ and $n_{6^+}(f) = 1$, $c(v \to f) = \frac{4}{9}$ if $\ell = 5$ and $n_{6^+}(f) = 1$, and $c(v \to f) = \gamma$ otherwise.

R4. For $k \geq 6$, $c(v \to f) = \frac{7}{9}$ if $\ell = 4$ and $n_5(f) = 1$, $c(v \to f) = \frac{5}{9}$ if $\ell = 5$ and $n_5(f) = 1$, and $c(v \to f) = \gamma$ otherwise.

R5. Let $f = (v_1, v_2, v_3, v_4, v_5)$ be a bad 5-face with $f_3(v_i) = 2$ for each $i \in [5]$, and let $f_i = (v_i, v_{i+1}, u_i)$. Then $c(u_i \to f) = \frac{1}{9}$ if u_i is not incident with any 4-cycle.

Let $ch_1(x)$ be the new charge of x after applying R1-R5. A vertex v is called *rich* if $ch_1(v) > 0$ while it is called *poor* if $ch_1(v) < 0$ and v is incident with a 4-cycle. Given a poor vertex, we aim to get additional charge from rich vertices to keep it non-negative.

Definition 2.2. *Let u be a poor vertex with $5 \leq d(u) \leq 6$, and v be a rich vertex. A nice uv-path is a path connecting u and v of length at most two and the internal vertex (if any) has degree at most 5 in G, see Figure 2.*

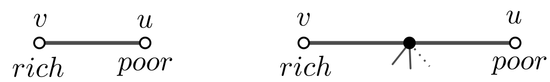

Figure 2: Nice paths.

The second round R6 can be expressed as follows.

R6. Let u be a poor vertex, and $v_1, \ldots, v_\ell$ be the rich vertices at distance at most 2 from u. Then $c(v_i \to u) = ch_1(v_i)$ if G has a nice uv_i-path.

Remark 2.3. *Since the poor vertex is incident with a 4-cycle and any two 4-cycles have distance at least 5, each rich vertex sends additional charge to at most one poor vertex. Note that the new charge of every rich vertex still keeps non-negative after applying R6.*

Let $ch_2(x)$ be the final charge of x after applying R1-R6. For convenience, we say that $S_i \not\subseteq G$ if G contains no subgraphs isomorphic to the configurations S_i ($1 \leq i \leq 47$) in Appendix B. Our goal is to show that $ch_2(z) \geq 0$ for each $z \in V(G) \cup F(G)$ and so we find a contradiction to (1), which implies that the minimum counterexample does not exist. Note that $ch_2(x) = ch_1(x)$ if R6 is not applied to x. Thus, we have that $ch_2(f) = ch_1(f)$ for any $f \in F(G)$ by R6 and $ch_2(v) = ch_1(v)$ for any v with $ch_1(v) = 0$. By Remark 2.3, we get that $ch_2(v) \geq 0$ for each rich vertex. So if $ch_1(z) \geq 0$, then we have that $ch_2(z) \geq 0$ for each $z \in V(G) \cup F(G)$.

Since G has no intersecting 4-cycles, we immediately have the following simple fact.

Fact 2.4. *For each vertex $v \in V(G)$, $f_3(v) \leq \lceil \frac{d(v)}{2} \rceil$.*

Claim 2.5. *For each face $f \in F(G)$, $ch_2(f) = ch_1(f) \geq 0$.*

Proof. If $d(f) = 3$, then $ch_1(f) \geq -2 + 3 \times \frac{2}{3} = 0$ by R1. If $d(f) \geq 6$, then $ch_1(f) \geq -2 + 6 \times \frac{1}{3} = 0$ by R1.

Suppose that $4 \leq d(f) \leq 5$ and f is not a bad 5-face. By R2.2, f gets $\frac{1}{3}$ from each of its incident 4-vertices.

(i) If $d(f) = 4$, $n_5(f) = 1$ and $n_{6^+}(f) = 1$, then f gets $\frac{5}{9}$ from its incident 5-vertex and $\frac{7}{9}$ from its incident 6^+-vertex by R3 and R4.

(ii) If $d(f) = 5$, $n_5(f) = 1$ and $n_{6^+}(f) = 1$, then f gets $\frac{4}{9}$ from its incident 5-vertex and $\frac{5}{9}$

from its incident 6^+-vertex by R3 and R4.

(iii) Otherwise, f gets γ from each of its incident 5^+-vertices by R3 and R4.

Thus, we have that $ch_1(f) \geq -2+\min\{\frac{1}{3}\times 2+\frac{5}{9}+\frac{7}{9}, \frac{1}{3}\times 3+\frac{4}{9}+\frac{5}{9}, \frac{1}{3}\cdot n_4(f)+\frac{2-\frac{1}{3}n_4(f)}{n_{5^+}(f)}\cdot n_{5^+}(f)\} = 0$.

Suppose that f is a bad 5-face. If there exists exactly one i ($i \in [5]$) such that $f_3(v_i) \leq 1$, then f gets at least $\frac{4}{3}$ from other incident vertices by R2.1, and so we have that $ch_1(f) \geq -2 + \frac{2}{3} + 4 \times \frac{1}{3} = 0$. If there exist at least two vertices, say v_i and v_j, such that $f_3(v_i) \leq 1$ and $f_3(v_j) \leq 1$, then f gets $\frac{1}{2}$ from each of v_i and v_j by R2.1 and gets at least 1 from other incident vertices by R2, and so we have that $ch_1(f) \geq -2 + \frac{1}{2} \times 2 + 3 \times \frac{1}{3} = 0$. Hence, we assume that each v_i satisfies $f_3(v_i) = 2$. For brevity, denote by $f_i = (v_i, v_{i+1}, u_i)$ the 3-face sharing the edge v_iv_{i+1} with f, and let $U = \{u_1, u_2, u_3, u_4, u_5\}$. Since $S_2 \nsubseteq G$, we get that $d(u_i) \geq 5$ for each $i \in [5]$. By the assumption of G, either at most one vertex in U lies on a 4-cycle, or two vertices in U lie on the same 4-cycle. Let $U^* \subseteq U$ such that each vertex in U^* does not lie on any 4-cycle. Note that $|U^*| \geq 3$ and it follows that f receives at least $3 \times \frac{1}{9}$ from U^* by R5. So we get that $ch_2(f) = ch_1(f) \geq -2 + 5 \times \frac{1}{3} + 3 \times \frac{1}{9} = 0$ by R2.2. □

Claim 2.6. *For each 4-vertex v, $ch_1(v) \geq 0$. In particular, for each good 4-vertex v, $ch_1(v) \geq \frac{1}{3}$.*

Proof. Let v be a 4-vertex. If $f_3(v) + f_{5b}(v) \leq 2$, then $ch_1(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by R1 and R2. So suppose that $f_3(v) + f_{5b}(v) \geq 3$. By Fact 2.4, $f_3(v) \leq 2$. As $S_2, S_3 \nsubseteq G$, we have that $f_{5b}(v) \leq 2$ and if $f_{5b}(v) = 2$, then $f_3(v) = 0$. It remains to consider the case that $f_3(v) = 2$ and $f_{5b}(v)$=1. By R2, v sends $\frac{1}{3}$ to each of other 4^+-faces and $\frac{2}{3}$ to each 3-face. Thus, $ch_1(v) \geq 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$.

Since $f_3(v)+f_{5b}(v) \leq 1$ holds for each good vertex v, we have that $ch_1(v) \geq 2-\frac{1}{3}\times 3-\frac{2}{3} = \frac{1}{3}$ by R1-R2. □

Claim 2.7. $ch_1(v) \geq 0$ *if v is a 7^+-vertex, or a 6-vertex with $f_{6^+}(v) \geq 1$, or a 5-vertex with $f_{6^+}(v) \geq 2$.*

Proof. Let v be a vertex. Suppose $d(v)$ is odd. Note that $f_3(v) \leq \frac{d(v)+1}{2}$ by Fact 2.4. If $f_3(v) = \frac{d(v)+1}{2}$, then by R1 and R4, we have that $ch_1(v) \geq d(v) - 2 - \frac{2}{3}d(v) = \frac{d(v)-6}{3}$. If $f_3(v) \leq \frac{d(v)-1}{2}$ and $f_4(v) = 1$, then by R1 and R4, we have that $ch_1(v) \geq d(v) - 2 - 1 - \frac{2}{3}(d(v) - 1) = \frac{d(v)-7}{3}$. If $f_3(v) \leq \frac{d(v)-1}{2}$ and $f_4(v) = 0$, then by R1 and R4-R5, we have that $ch_1(v) \geq d(v) - 2 - \frac{2}{3}d(v) - \frac{1}{9}\left(\frac{d(v)-1}{2}\right) = \frac{5d(v)-35}{18}$. Particularly, if $f_{6^+}(v) \geq 2$, then $ch_1(v) \geq \min\left\{\frac{d(v)-7}{3}, \frac{5d(v)-35}{18}\right\} + 2 \times \frac{1}{3} = \min\left\{\frac{d(v)-5}{3}, \frac{5d(v)-23}{18}\right\}$.

Suppose $d(v)$ is even. Note that $f_3(v) \le \frac{d(v)}{2}$ by Fact 2.4. If $f_4(v) = 1$, then by R1 and R4, we have that $ch_1(v) \ge d(v) - 2 - 1 - \frac{2}{3}(d(v) - 1) = \frac{d(v)-7}{3}$. If $f_4(v) = 0$, then by R1 and R4-R5, we have that $ch_1(v) \ge d(v) - 2 - \frac{2}{3}d(v) - \frac{1}{9}\left(\frac{d(v)}{2}\right) = \frac{5d(v)-36}{18}$. In particular, if $f_{6^+}(v) \ge 1$, then $ch_1(v) \ge \min\left\{\frac{d(v)-7}{3}, \frac{5d(v)-36}{18}\right\} + \frac{1}{3} = \min\left\{\frac{d(v)-6}{3}, \frac{5d(v)-30}{18}\right\}$.

Therefore, Claim 2.7 is true. □

Now it remains to consider the vertices of $W_1 = \{v : d(v) = 6 \text{ and } f_{6^+}(v) = 0\}$ and $W_2 = \{v : d(v) = 5 \text{ and } f_{6^+}(v) \le 1\}$ by Claim 2.6 and 2.7.

For $v \in W_1$, let $N(v) = \{v_1, \ldots, v_6\}$ and let $f_1, \ldots, f_6$ be the faces incident with v in clockwise such that v_i and v_{i+1} are incident with f_i. In the following Claims 2.8-2.11, we show that $ch_2(v) \ge 0$ for each vertex $v \in W_1$.

Claim 2.8. *For each vertex $v \in W_1$ with $f_3(v) \le 2$ and $f_4(v) = 1$, $ch_2(v) \ge 0$.*

Proof. W.l.o.g., let f_1 be the 4-face, denoted by v_1vv_2x. Note that v sends no charge to a bad 5-face (if it exists) which is incident with a $(4, 4, v)$-face by R5. According to R1 and R4, v sends at most 1 to each 4-face and $\frac{2}{3}$ to each 3-face and 5-face. Thus, $ch_1(v) \ge 4 - 1 - \frac{2}{3} \times 5 = -\frac{1}{3}$. If $ch_1(v) \ge 0$, then we are done. So suppose that $ch_1(v) < 0$, that is, v is poor. Clearly, if there is a good 4-vertex in $N(v)$, then $ch_2(v) \ge 4 - 1 - \frac{2}{3} \times 5 + \frac{1}{3} = 0$ by Claim 2.6 and R6. Next we only consider the case that there is no good 4-vertex in $N(v)$.

Now we first claim that f_i is not a $(4, 4, 4, 4, 6)$-face for each $i \in \{2, 6\}$ (that is, $n_{5^+}(f_i) \ge 2$). Suppose to the contrary that for some $i \in \{2, 6\}$, f_i is a $(4, 4, 4, 4, 6)$-face, say f_2. As $S_2, S_3 \not\subseteq G$, we get that $f_3(v_2) + f_{5b}(v_2) \le 1$ and v_2 is a good 4-vertex, a contradiction. Similarly, if f_6 is a $(4, 4, 4, 4, 6)$-face, then v_1 is a good 4-vertex, a contradiction.

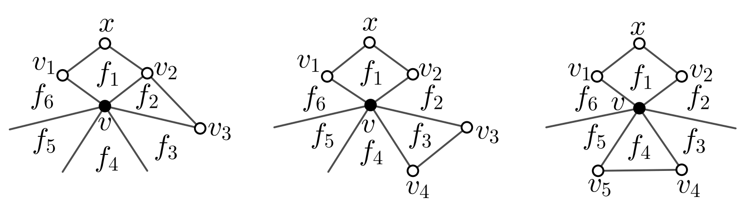

Figure 3: Configurations for 6-vertex v with $f_4(v) = 1$.

Case 1. $n_{5^+}(f_1) = 1$.

Subcase 1.1. Assume that $f_3(v) \le 1$. We will show that there are at least three 5-faces f_i such that $n_{5^+}(f_i) \ge 2$, which implies that $c(v \to f_i) \le \frac{5}{9}$ by R4, and so $ch_1(v) \ge 4 - 1 - 2 \times \frac{2}{3} - 3 \times \frac{5}{9} = 0$ by R1.

(a) Suppose that $f_3(v) = 0$. Since G has no intersecting 4-cycles, the remaining faces incident with v are all 5-faces. By $S_{24} \not\subseteq G$, there exists at least one i ($i \in \{3,4,5\}$) such that $n_{5^+}(f_i) \geq 2$. Note that $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_6) \geq 2$, so we are done.

(b) Suppose that $f_3(v) = 1$. By symmetry, three cases need to be considered (see Figure 3). In A_1, since $S_{32} \not\subseteq G$, we have that $n_{5^+}(f_3) \geq 2$. In A_2, since $S_{24} \not\subseteq G$, we have that $n_{5^+}(f_4) \geq 2$ or $n_{5^+}(f_5) \geq 2$. In A_3, since $S_{27} \not\subseteq G$, we have that $n_{5^+}(f_3) \geq 2$ or $n_{5^+}(f_5) \geq 2$. Note that if f_i is a 5-face, then $n_{5^+}(f_i) \geq 2$ for $i \in \{2,6\}$, so we are done.

Subcase 1.2. Assume that $f_3(v) = 2$. There are four subcases to be considered.

Firstly, we suppose that $d(f_2) = d(f_4) = 3$. Note that $f_3(v_1) \leq 1$ and v_1 is not good. It implies that v_1 is bad. Since $S_2, S_3 \not\subseteq G$, v_1x locates on the same bad 5-face. In this situation, $f_3(x) \leq 1$, and by R2.1, each of $\{v_1, x\}$ sends $\frac{1}{2}$ to the bad 5-face. Thus, by R2 $ch_1(u) \geq 2 - \frac{2}{3} - \frac{1}{2} - \frac{1}{3} \times 2 = \frac{1}{6}$ for each $u \in \{v_1, x\}$. Therefore, each of $\{v_1, x\}$ sends $\frac{1}{6}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Thus $ch_2(v) \geq 0$. The case that $d(f_2) = d(f_5) = 3$ is similar as above.

Next, we suppose that $d(f_3) = d(f_5) = 3$. Since v_1 and v_2 are not good and $S_2, S_3 \not\subseteq G$, v_1x locates on the same bad 5-face g_1 and v_2x locates on the same bad 5-face g_2. By $S_2 \not\subseteq G$, we have that $f_3(x) = 0$. Note that $f_3(v_i) \leq 1$ for each $i \in [2]$. It follows that $|Tg_1| \geq 2$ and $|Tg_2| \geq 2$. Thus, $ch_1(x) \geq 2 - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{1}{3}$ by R1 and R2. Hence, v (if $ch_1(v) < 0$) could receive at least $\frac{1}{3}$ from x via a nice path by R6, and $ch_2(v) \geq 0$.

It remains to consider the case where $d(f_2) = d(f_6) = 3$. Since $S_{32} \not\subseteq G$, we get that for each $i \in \{3,5\}$, $n_{5^+}(f_i) \geq 2$ and $c(v \to f_i) \leq \frac{5}{9}$ by R4. If $n_{5^+}(f_4) \geq 2$, then $ch_1(v) \geq 4 - 1 - 2 \times \frac{2}{3} - 3 \times \frac{5}{9} = 0$ by R1 and R4. Now let $n_{5^+}(f_4) = 1$, and denote by $f_4 = (v, v_4, y_1, y_2, v_5)$, that is $d(v_4) = d(v_5) = d(y_1) = d(y_2) = 4$. Note that $f_3(v_4) = f_3(v_5) \leq 1$. So we may assume that both v_4 and v_5 are not good (otherwise v receives at least $\frac{1}{3}$ from $\{v_4, v_5\}$ and $ch_2(v) \geq 0$). Since $S_2 \not\subseteq G$, v_4y_1 and v_5y_2 locate on two bad 5-faces, respectively. On the other hand, notice that $S_2, S_{47} \not\subseteq G$, and then at least one $j \in \{3,5\}$ satisfying $n_{5^+}(f_j) \geq 3$, and so $c(v \to f_j) \leq \frac{4}{9}$ for some $j \in \{3,5\}$ by R4. Thus $ch_1(v) \geq 4 - 1 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R4.

Case 2. $n_{5^+}(f_1) \geq 2$. Since $S_2, S_{36} \not\subseteq G$, there exists at least one $i \in \{2,3,4,5,6\}$ such that $n_{5^+}(f_i) \geq 2$, and we have $c(v \to f_i) \leq \frac{5}{9}$ by R4. So $ch_1(v) \geq 4 - 4 \times \frac{2}{3} - \frac{7}{9} - \frac{5}{9} = 0$ by R1 and R4. □

Claim 2.9. *For each vertex $v \in W_1$ with $f_3(v) \leq 2$ and $f_4(v) = 0$, $ch_2(v) \geq 0$.*

Proof. Suppose that $f_4(v) = 0$. If $f_3(v) = 0$, then $ch_2(v) \geq 4 - 6 \times \frac{2}{3} = 0$ by R1 and R4. If $f_3(v) = 1$, then by $S_{35} \not\subseteq G$, either $\zeta_v(f_{3b}) = 0$ or there exists some i such that $n_{5^+}(f_i) \geq 2$, and thus $ch_1(v) \geq 4 - 5 \times \frac{2}{3} - \max\{\frac{5}{9} + \frac{1}{9}, \frac{2}{3}\} = 0$ by R1, R4-R5. Finally, we discuss the case where $f_3(v) = 2$. If the two 3-faces are consecutive, then $ch_1(v) \geq 4 - 6 \times \frac{2}{3} = 0$ by R1 and R4. Otherwise if they are not consecutive, by the fact that $S_{30}, S_{35} \nsubseteq G$, we get that $ch_1(v) \geq 4 - 2 \times \frac{5}{9} - 4 \times \frac{2}{3} - 2 \times \frac{1}{9} = 0$ by R1, R4-R5. □

Next we focus on the case $f_3(v) = 3$. Since $S_{46} \nsubseteq G$, we get $\zeta_v(f_{3b}) \leq 2$.

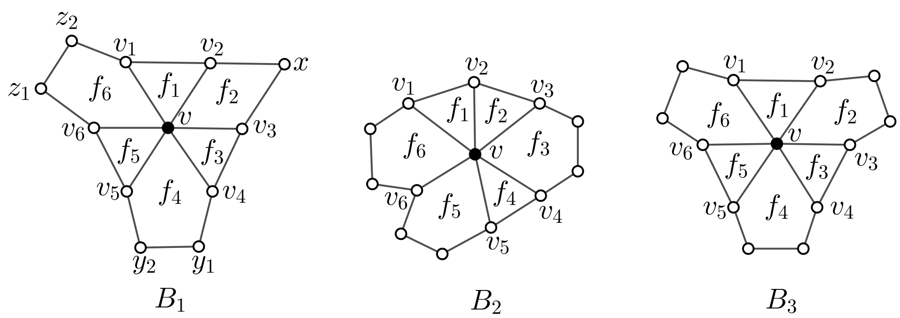

Figure 4: Configurations for 6-vertex v with $f_3(v) = 3$.

Recall that v sends no charge to a bad 5-face which is incident with a $(4, 4, v)$-face by R5.

Claim 2.10. *For each vertex $v \in W_1$ with $f_3(v) = 3$ and $f_4(v) = 1$ (see Figure 4(B_1)), $ch_2(v) \geq 0$.*

Proof. We divide the proof into four possibilities depending on $n_4(v) \in \{4, 5, 6\}$ or $n_4(v) \leq 3$.

(i) Suppose $n_4(v) = 6$. (a). $d(x) = 4$. As $S_{32} \nsubseteq G$, for each $i \in \{4, 6\}$, we have $n_{5^+}(f_i) \geq 2$, and so $c(v \to f_i) \leq \frac{5}{9}$ by R4. If $n_{6^+}(f_i) \geq 2$ or $n_{5^+}(f_i) \geq 3$ for each $i \in \{4, 6\}$, then $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - 2 \times \frac{1}{2} = 0$ by R1 and R4. Assume $n_6(f_4) = 1$ and $n_{5^+}(f_4) = 2$. Denote by $f_4 = (v, v_4, y_1, y_2, v_5)$. First, let $d(y_1) = 5$ and $d(y_2) = 4$. If $f_3(v_4) = 1$, then according to $S_{25} \nsubseteq G$, v_4 can not locate on a bad 5-face. Thus v_4 is good, and $ch_1(v_4) \geq \frac{1}{3}$. If $f_3(v_4) = 2$, then by $S_{25}, S_{41}, S_{44} \nsubseteq G$, we have $\zeta_{y_1}(f_{3b}) = 0$. Thus $ch_1(y_1) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{1}{2} = \frac{1}{18}$ by R1 and R3. In both cases, $\{v_4, y_1\}$ could send at least $\frac{1}{18}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Second, let $d(y_1) = 4$ and $d(y_2) = 5$. If $f_3(v_5) = 1$, then v_5 can not locate on a bad 5-face by $S_{38} \nsubseteq G$. Thus v_5 is good, and $ch_1(v_5) \geq \frac{1}{3}$. If $f_3(v_5) = 2$, then by $S_{38}, S_{39}, S_{42} \nsubseteq G$, we have $\zeta_{y_2}(f_{3b}) = 0$. Thus $ch_1(y_2) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{1}{2} = \frac{1}{18}$ by R1 and R3. In both cases, $\{v_5, y_2\}$ could send at least $\frac{1}{18}$ to v (if $ch_1(v) < 0$) via a nice path by R6. In conclusion, v could receive at least $\frac{1}{18}$ from $\{v_4, v_5, y_1, y_2\}$. By symmetry, the same arguments also hold for the vertices on f_6 (i.e. $\{v_1, v_6, z_1, z_2\}$). If $n_6(f_6) = 1$ and

$n_{5^+}(f_6) = 2$, then $ch_2(v) \geq 4 - 3 \times \frac{2}{3} - 1 - 2 \times \frac{5}{9} + 2 \times \frac{1}{18} = 0$ by R1, R4 and R6. Otherwise $ch_2(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{1}{2} + \frac{1}{18} = 0$.

(b). $d(x) \geq 5$. Since $S_{30} \not\subseteq G$, $n_{5^+}(f_i) \geq 2$ for some $i \in \{4, 6\}$, and $c(v \to f_i) \leq \frac{5}{9}$ by R4. Thus $ch_1(v) \geq 4 - 4 \times \frac{2}{3} - \frac{7}{9} - \frac{5}{9} = 0$ by R1 and R4.

(ii) Suppose $n_4(v) = 5$. By symmetry, we only need to consider three subcases: $d(v_1) \geq 5$, $d(v_2) \geq 5$ and $d(v_5) \geq 5$.

(a). $d(x) = 4$. Assume that $d(v_1) \geq 5$. If $d(v_1) = 5$, then $n_{5^+}(f_6) \geq 3$ by $S_{33} \not\subseteq G$, and we have $c(v \to f_i) \leq \frac{4}{9}$ by R4. Thus $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R4. If $d(v_1) = 6$, then $ch_1(v_1) \geq 4 - 4 \times \frac{2}{3} - \frac{1}{2} - \frac{5}{9} - 2 \times \frac{1}{9} = \frac{1}{18}$ by R1, R4-R5 because of $S_{40} \not\subseteq G$. If $d(v_1) \geq 7$, then by Claim 2.7, $ch_1(v_1) \geq \frac{5d(v)-36}{18} + \frac{1}{9} + \frac{1}{18} \geq \frac{1}{18}$. Hence, when $d(v_1) \geq 6$, v (if $ch_1(v) < 0$) could receive at least $\frac{1}{18}$ from v_1 via a nice path by R6. Thus $ch_2(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{1}{2} + \frac{1}{18} = 0$ by R1, R4 and R6.

Assume that $d(v_2) \geq 5$. Then $n_{5^+}(f_i) \geq 2$ for some $i \in \{4, 6\}$ by $S_{30} \not\subseteq G$, and we have $c(v \to f_i) \leq \frac{5}{9}$ by R4. Thus $ch_1(v) \geq 4 - 4 \times \frac{2}{3} - \frac{7}{9} - \frac{5}{9} = 0$ by R1 and R4.

Assume that $d(v_5) \geq 5$. Since $S_{32} \not\subseteq G$, $n_{5^+}(f_6) \geq 2$ and $c(v \to f_6) \leq \frac{5}{9}$ by R4. According to $S_{26} \not\subseteq G$, either $d(v_5) \geq 6$ or $n_{5^+}(f_4) \geq 3$. If $n_{5^+}(f_4) \geq 3$, then $c(v \to f_4) \leq \frac{4}{9}$ by R4, and thus $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R4; if $d(v_5) \geq 6$, then by the similar arguments as above, we have that v (if $ch_1(v) < 0$) could receive at least $\frac{1}{18}$ from v_5 via a nice path by R6, and thus $ch_2(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{1}{2} + \frac{1}{18} = 0$ by R1, R4 and R6.

(b). $d(x) \geq 5$. In all three cases, it is easy to check that $ch_1(v) \geq 4 - 4 \times \frac{2}{3} - \max\{\frac{7}{9} + \frac{5}{9}, \frac{5}{9} + \frac{2}{3}\} = 0$ by R1 and R4.

(iii) Suppose $n_4(v) = 4$. That is, $n_{5^+}(v) = 2$. If the pair of two 5^+-vertices fall in $\{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_5), (v_2, v_6)\}$, then we have $ch_1(v) \geq 4 - 4 \times \frac{2}{3} - \max\{\frac{7}{9} + \frac{5}{9}, \frac{2}{3} + \frac{5}{9}\} = 0$ by R1 and R4. By symmetry, it remains to discuss the following cases.

Assume that $d(v_1) \geq 5$ and $d(v_4) \geq 5$. (a). $d(x) = 4$. Note that v_1 and v_4 are symmetric to some extent. If $d(v_1) \geq 6$ and $d(v_4) \geq 6$, then $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - 2 \times \frac{1}{2} = 0$ by R1 and R4. If $d(v_i) = 5$ for some $i \in \{1, 4\}$, then $n_{5^+}(f_{8-2i}) \geq 3$ by $S_{31} \not\subseteq G$, and so $c(v \to f_{8-2i}) \leq \frac{4}{9}$ by R4. Thus $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \max\{\frac{1}{2} + \frac{4}{9}, 2 \times \frac{4}{9}\} = \frac{1}{18} > 0$ by R1 and R4. (b). $d(x) \geq 5$. Then $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - \frac{7}{9} - 2 \times \frac{5}{9} = \frac{1}{9} > 0$ by R1 and R4.

Assume that $d(v_1) \geq 5$ and $d(v_5) \geq 5$. (a). $d(x) = 4$. If $d(v_1) \geq 6$ and $d(v_5) \geq 6$, then $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - 2 \times \frac{1}{2} = 0$ by R1 and R4. If $d(v_i) = 5$ for some $i \in \{1, 5\}$, then $n_{5^+}(f_{\frac{13-i}{2}}) \geq 3$ by $S_{26}, S_{33} \not\subseteq G$, and so $c(v \to f_{\frac{13-i}{2}}) \leq \frac{4}{9}$ by R4. Thus $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 1 - \max\{\frac{1}{2} + \frac{4}{9}, 2 \times \frac{4}{9}\} = \frac{1}{18} > 0$ by R1 and R4. (b). $d(x) \geq 5$. Then $ch_1(v) \geq 4 - 3 \times \frac{2}{3} - 2 \times \frac{5}{9} - \frac{7}{9} = \frac{1}{9} > 0$ by R1 and R4.

Assume that $d(v_1) \geq 5$ and $d(v_6) \geq 5$. (a). $d(x) = 4$. Since $S_{32} \not\subseteq G$, we get $n_{5^+}(f_4) \geq 2$, and $c(v \to f_4) \le \frac{5}{9}$ by R4. Then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R4. (b). $d(x) \ge 5$. Then we have $ch_1(v) \ge 4 - 4 \times \frac{2}{3} - \frac{7}{9} - \frac{4}{9} = \frac{1}{9} > 0$ by R1 and R4.

Assume that $d(v_5) \ge 5$ and $d(v_6) \ge 5$. (a). $d(x) = 4$. If $d(v_5) \ge 6$ and $d(v_6) \ge 6$, then $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - 1 - 2 \times \frac{1}{2} = 0$ by R1 and R4. If $d(v_i) = 5$ and $d(v_{11-i}) \ge 6$ for some $i \in \{5, 6\}$, then by $S_2 \nsubseteq G$, we get that $ch_1(v_i) \ge 3 - \frac{2}{3} - 2 \times \frac{4}{9} - \max\{\frac{2}{3} + \frac{1}{2} + \frac{1}{9}, 2 \times \frac{2}{3}\} = \frac{1}{9}$ by R1, R3 and R5. Hence, v_i could send at least $\frac{1}{9}$ to v via a nice path by R6, and $ch_2(v) \ge 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{1}{2} + \frac{1}{9} > 0$ by R1, R4 and R6. If $d(v_5) = 5$ and $d(v_6) = 5$, then there is at least one $i \in \{4, 6\}$ such that $n_{5^+}(f_i) \ge 3$ by $S_{34} \nsubseteq G$, and so $c(v \to f_i) \le \frac{4}{9}$ by R4. Hence, $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - 1 - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R4. (b). $d(x) \ge 5$. Then $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - \frac{7}{9} - 2 \times \frac{5}{9} = \frac{1}{9} > 0$ by R1 and R4.

(iv) Suppose $n_4(v) \le 3$. If $n_4(v) = 3$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - \max\{2 \times \frac{5}{9} + \frac{7}{9}, \frac{2}{3} + \frac{4}{9} + \frac{7}{9}, \frac{2}{3} + 2 \times \frac{5}{9}, 1 + \frac{4}{9} + \frac{5}{9}, \frac{4}{9} + \frac{5}{9} + \frac{7}{9}\} = 0$ by R1 and R4. If $n_4(v) = 2$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - \max\{\frac{7}{9} + \frac{4}{9} + \frac{5}{9}, 3 \times \frac{5}{9}, \frac{2}{3} + \frac{4}{9} + \frac{5}{9}, 1 + 2 \times \frac{4}{9}\} = \frac{1}{9} > 0$ by R1 and R4. If $n_4(v) = 1$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - \max\{2 \times \frac{5}{9} + \frac{4}{9}, 2 \times \frac{4}{9} + \frac{7}{9}\} = \frac{4}{9} > 0$ by R1 and R4. If $n_4(v) = 0$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - \frac{5}{9} - 2 \times \frac{4}{9} = \frac{5}{9} > 0$ by R1 and R4. □

Claim 2.11. *For each vertex $v \in W_1$ with $f_3(v) = 3$ and $f_4(v) = 0$, $ch_2(v) \ge 0$.*

Proof. If v is incident with a 4-cycle (see Figure 4(B_2)), then v also sends no charge to a bad 5-face (if it exists) which is incident with a $(4, 4, v)$-face by R5. Thus $ch_1(v) \ge 4 - 6 \times \frac{2}{3} = 0$ by R1 and R4. Next we turn to the case that v is not incident with any 4-cycle, see Figure 4(B_3). Recall that $\zeta_v(f_{3b}) \le 2$.

(i) Suppose $n_4(v) = 6$. Then there are at least two faces f_i, f_j in $\{f_2, f_4, f_6\}$ satisfying $f_i \ne (4, 4, 4, 4, 6)$ and $f_j \ne (4, 4, 4, 4, 6)$ by $S_{30} \nsubseteq G$. Thus $ch_1(v) \ge 4 - 4 \times \frac{2}{3} - 2 \times \frac{5}{9} - 2 \times \frac{1}{9} = 0$ by R1 and R4-R5.

(ii) Suppose $n_4(v) = 5$. By symmetry, say $d(v_1) \ge 5$. Since $S_{30} \nsubseteq G$, we get $n_{5^+}(f_i) \ge 2$ when $d(f_i) = 5$ for some $i \in \{2, 4\}$, and so $c(v \to f_i) \le \frac{5}{9}$ by R4. Thus $ch_1(v) \ge 4 - 4 \times \frac{2}{3} - 2 \times \frac{5}{9} - 2 \times \frac{1}{9} = 0$ by R1 and R4-R5.

(iii) Suppose $n_4(v) \le 4$. If $n_4(v) = 4$, then $ch_1(v) \ge 4 - \max\{4 \times \frac{2}{3} + 2 \times \frac{5}{9} + 2 \times \frac{1}{9}, 5 \times \frac{2}{3} + \frac{4}{9} + \frac{1}{9}\} = 0$ by R1 and R4-R5. If $n_4(v) = 3$, then $ch_1(v) \ge 4 - \max\{4 \times \frac{2}{3} + \frac{4}{9} + \frac{5}{9} + \frac{1}{9}, 3 \times \frac{2}{3} + 3 \times \frac{5}{9} + \frac{1}{9}\} = \frac{2}{9} > 0$ by R1, R4-R5. If $n_4(v) = 2$, then $ch_1(v) \ge 4 - \max\{3 \times \frac{2}{3} + \frac{4}{9} + 2 \times \frac{5}{9} + \frac{1}{9}, 4 \times \frac{2}{3} + 2 \times \frac{5}{9}\} = \frac{2}{9} > 0$ by R1, R4-R5. If $n_4(v) = 1$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - 2 \times \frac{4}{9} - \frac{5}{9} = \frac{5}{9} > 0$ by R1 and R4. If $n_4(v) = 0$, then we have $ch_1(v) \ge 4 - 3 \times \frac{2}{3} - 3 \times \frac{4}{9} = \frac{2}{3} > 0$ by R1 and R4. □

For each vertex $v \in W_2$, denote by f_i ($i \in [5]$) the faces incident with v. If $d(f_i) = 3$ for some i, then denote by $f_i = (v, v_i, v_{i+1})$. The following Claims 2.12-2.16 imply that

$ch_2(v) \geq 0$, for each vertex $v \in W_2$.

Claim 2.12. *For each vertex $v \in W_2$ with $f_3(v) = 3$, $ch_2(v) \geq 0$.*

Proof. In this case, $f_4(v) = 0$ since G does not contain intersecting 4-cycles. Let f_1, f_2 and f_4 be the 3-faces incident with v. If $d(f_i) \geq 6$ for some $i \in \{3, 5\}$, then $ch_1(v) \geq 3-4\times\frac{2}{3}-\frac{1}{3} = 0$ by R1 and R3. Next, we consider the situation where $d(f_i) = 5$ for each $i \in \{3, 5\}$, see Figure 5(C_1).

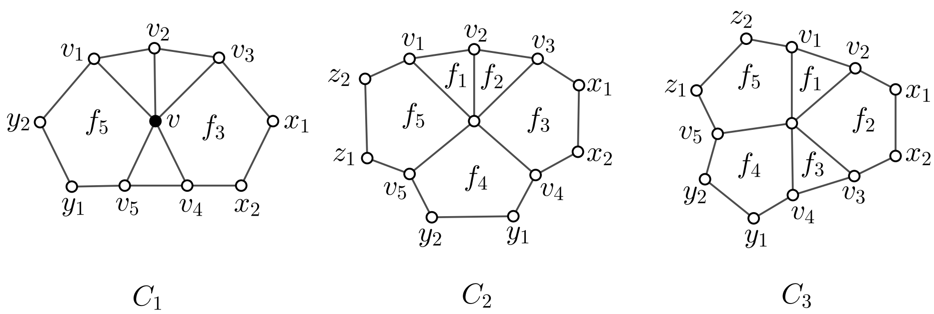

Figure 5: Configurations for 5-vertex v.

(i) Suppose $n_4(v) = 5$. Then $n_{5^+}(f_3) \geq 2$ and $n_{5^+}(f_5) \geq 2$ hold by $S_2 \nsubseteq G$, and so $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{3, 5\}$ by R3. Thus $ch_1(v) \geq 3-3\times\frac{2}{3}-2\times\frac{1}{2} = 0$ by R1 and R3.

(ii) Suppose $n_4(v) = 4$, that is $n_{5^+}(v) = 1$. By symmetry, there are only three cases need to be considered: $d(v_1) \geq 5$; $d(v_2) \geq 5$; $d(v_4) \geq 5$. In all three cases, since $S_2 \nsubseteq G$, we have $ch_1(v) \geq 3-3\times\frac{2}{3}-2\times\frac{1}{2} = 0$ by R1 and R3.

(iii) Suppose $n_4(v) = 3$, that is $n_{5^+}(v) = 2$. If the pair of two 5^+-vertices fall in $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_4, v_5)\}$, then $ch_1(v) \geq 3-3\times\frac{2}{3}-\max\{2\times\frac{1}{2}, \frac{4}{9}+\frac{1}{2}\} = 0$ by R1 and R3. By symmetry, it remains to consider the pair (v_2, v_4) with $d(v_2) \geq 5$ and $d(v_4) \geq 5$. We may assume that $n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 0$). Since $S_{22} \nsubseteq G$, $d(v_2) \geq 6$. If $d(v_2) = 6$, then by $S_{29}, S_{35}, S_{37} \nsubseteq G$, there are at least two faces $\tilde{f}$ incident with v_2 such that $n_{5^+}(\tilde{f}) \geq 2$, and so $c(v \to \tilde{f}) \leq \frac{5}{9}$. Thus $ch_1(v_2) \geq 4-4\times\frac{2}{3}-2\times\frac{5}{9} = \frac{2}{9}$ by R1 and R4. If $d(v_2) \geq 7$, then $ch_1(v_2) \geq \frac{d(v)-6}{3} > \frac{2}{9}$. Hence, v_2 could send at least $\frac{2}{9}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3-4\times\frac{2}{3}-\frac{1}{2}+\frac{2}{9} > 0$ by R1, R3 and R6.

(iv) Suppose $n_4(v) = 2$. If the pair of two 4-vertices fall in $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_4), (v_4, v_5)\}$, then $ch_1(v) \geq 3-3\times\frac{2}{3}-\max\{2\times\frac{1}{2}, \frac{4}{9}+\frac{1}{2}\} = 0$ by R1 and R3. By symmetry, it remains to consider the pair (v_1, v_5) with $d(v_1) = d(v_5) = 4$. We may assume that $n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 0$). If $d(v_2) = 6$, then $ch_1(v) \geq 4-5\times\frac{2}{3}-\frac{5}{9} = \frac{1}{9}$ by R1 and R4; if $d(v_2) \geq 7$, then $ch_1(v_2) \geq \frac{d(v)-6}{3} > \frac{1}{9}$. Hence, v_2 could send at least $\frac{1}{9}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{4}{9} + \frac{1}{9} = 0$ by R1, R3 and R6. The same results hold for v_3. We now turn to the case $d(v_2) = d(v_3) = 5$. For simplicity, denote by f_6, f_7 and f_8 the remaining faces incident with v_3 in clockwise. If $n_{5^+}(f_6) \geq 3$, then $c(v_3 \to f_6) \leq \frac{4}{9}$ by R4, and thus $ch_1(v_3) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{4}{9} = \frac{1}{9}$ by R1 and R4. Otherwise, $n_{5^+}(f_6) = 2$. If $d(f_7) = 3$, then $n_{5^+}(f_8) \geq 2$ as $S_2 \nsubseteq G$; if $d(f_8) = 3$, then by S_{20}, $n_{5^+}(f_7) \geq 2$; if none of f_7 and f_8 are 3-faces, then by S_3, $n_{5^+}(f_i) \geq 2$ for some $i \in \{7, 8\}$. In all cases, we have $ch_1(v_3) \geq 3 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} - \frac{4}{9} = \frac{2}{9}$ by R1 and R3. Thus v_3 could send at least $\frac{2}{9}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{4}{9} + \frac{2}{9} > 0$ by R1, R3 and R6.

(v) Suppose $n_4(v) \leq 1$. If $n_4(v) = 1$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \max\{\frac{4}{9} + \frac{1}{2}, 2 \times \frac{4}{9}\} = \frac{1}{18} > 0$ by R1 and R3. If $n_4(v) = 0$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{4}{9} = \frac{1}{9} > 0$ by R1 and R3. □

Claim 2.13. *For each vertex $v \in W_2$ with $f_3(v) = 2$ and $f_4(v) = 0$, $ch_2(v) \geq 0$.*

Proof. Firstly, suppose that the two 3-faces are consecutive and denote them by f_1 and f_2. Assume that there exists one 6^+-face in $\{f_4, f_5, f_6\}$, then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Next we consider the situation where $d(f_i) = 5$ for each $i \in \{4, 5, 6\}$, see Figure 5(C_2).

If $d(v_i) \geq 5$ for some $i \in \{4, 5\}$, then $\max\{c(v \to f_{i-1}), c(v \to f_i)\} \leq \frac{1}{2}$ by R3, and thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. Now let $d(v_4) = d(v_5) = 4$. Since $S_3 \nsubseteq G$, $n_{5^+}(f_i) \geq 2$ for some $i \in \{3, 4\}$ and $n_{5^+}(f_j) \geq 2$ for some $j \in \{4, 5\}$. If $i \neq j$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. If $i = j = 4$, then we may assume that $n_{5^+}(f_3) = n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 0$). Note that $f_3(v_k) \leq 1$ for each $k \in \{4, 5\}$. If v_k is good for some $k \in \{4, 5\}$, then $ch_1(v_k) \geq \frac{1}{3}$ by R1-R2. Hence, v_k sends at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Since $S_2, S_{12} \nsubseteq G$, we get that at least one vertex in $\{v_4, v_5\}$ is good, and we are done.

Secondly, suppose that the two 3-faces are not consecutive, say f_1 and f_3 are the 3-faces. By $S_{16} \nsubseteq G$, $\zeta_v(f_{3b}) \leq 1$. If $d(f_2) \geq 6$, then according to S_3, we have that $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{9} = \frac{1}{18} > 0$ by R1, R3 and R5. If $d(f_4) \geq 6$ and $\zeta_v(f_{3b}) = 1$, then $n_{5^+}(f_2) \geq 2$ by $S_2 \nsubseteq G$, and so $c(v \to f_2) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{9} = \frac{1}{18} > 0$ by R1, R3 and R5. In the following, we may assume $d(f_i) = 5$ for each $i \in \{2, 4, 5\}$, see Figure 5(C_3).

Assume $\zeta_v(f_{3b}) = 1$, and let v_1v_2 be the edge incident with a bad 5-face. By $S_2 \nsubseteq G$, we get $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_5) \geq 2$, and so $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{2, 5\}$ by R3. If $f_3(v_i) \leq 1$ for some $i \in [2]$, then v need not send any charge to the bad 5-face by R5 (since v_i sends $\frac{2}{3}$ to the bad 5-face), and thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. It remains to consider $f_3(v_i) = 2$ for each $i \in [2]$. If $n_{5^+}(f_4) \geq 2$, then $c(v \to f_4) \leq \frac{1}{2}$ by R3, and thus $ch_1(v) \geq 3-2\times\frac{2}{3}-3\times\frac{1}{2}-\frac{1}{9}=\frac{1}{18}>0$ by R1, R3 and R5. Otherwise, $n_{5^+}(f_4)=1$. We have that $n_{5^+}(f_2)\geq 3$ since $S_2 \nsubseteq G$, and $n_{6^+}(f_5)\geq 1$ or $n_{5^+}(f_5)\geq 3$ since $S_{28}\nsubseteq G$, and so $c(v\to f_i)\leq \frac{4}{9}$ for each $i\in\{2,5\}$. Hence, $ch_1(v)\geq 3-3\times\frac{2}{3}-2\times\frac{4}{9}-\frac{1}{9}=0$ by R1, R3 and R5.

Assume $\zeta_v(f_{3b})=0$. Since $S_3\nsubseteq G$, we know that at least one of $f_i\in\{f_4,f_5\}$ satisfies $n_{5^+}(f_i)\geq 2$. If $n_{5^+}(f_i)\geq 2$ for each $i\in\{4,5\}$, then $c(v\to f_i)\leq\frac{1}{2}$ for each $i\in\{4,5\}$ by R3, and thus $ch_1(v)\geq 3-3\times\frac{2}{3}-2\times\frac{1}{2}=0$ by R1 and R3. Otherwise we assume that $n_{5^+}(f_4)=1$ (which means $n_{5^+}(f_5)\geq 2$), then $n_{5^+}(f_2)\geq 2$ by $S_2\nsubseteq G$, and thus $ch_1(v)\geq 3-3\times\frac{2}{3}-2\times\frac{1}{2}=0$ by R1 and R3. □

Claim 2.14. *For each vertex $v\in W_2$ with $f_3(v)=2$ and $f_4(v)=1$, $ch_2(v)\geq 0$.*

Proof. There are two subcases to be considered, see Figure 6. Recall that v sends no charge to any bad 5-face by R5.

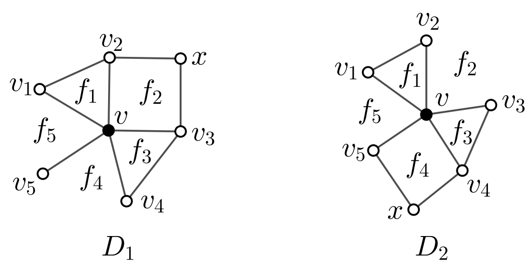

Figure 6: Configuration for 5-vertex v.

We consider the configuration D_1 first. (i) Suppose $n_4(v)=5$, that is $d(v_i)=4$ for each $i\in[5]$. Since $S_1\nsubseteq G$, we obtain that $d(x)\geq 5$. If $d(f_i)=5$ for $i\in\{4,5\}$, then by $S_2\nsubseteq G$, $n_{5^+}(f_i)\geq 2$. Thus $ch_1(v)\geq 3-3\times\frac{2}{3}-\max\{2\times\frac{1}{2},\frac{1}{2}+\frac{1}{3}\}=0$ by R1 and R3.

(ii) Suppose $n_4(v)=4$. If $d(v_1)\geq 5$, then $d(x)\geq 5$ by $S_1\nsubseteq G$, and $f_4\neq(4,4,4,4,5)$ by $S_2\nsubseteq G$. If $d(v_5)\geq 5$, then $d(x)\geq 5$ by $S_1\nsubseteq G$. In both cases, $ch_1(v)\geq 3-3\times\frac{2}{3}-\max\{2\times\frac{1}{2},\frac{1}{2}+\frac{1}{3}\}=0$ by R1 and R3. At last, we study the case where $d(v_2)\geq 5$. If $f_{6^+}(v)\geq 1$, then $ch_1(v)\geq 3-4\times\frac{2}{3}-\frac{1}{3}=0$ by R1 and R3. We now turn to the situation $f_{6^+}(v)=0$. In this situation, we may assume that $n_{5^+}(f_5)=1$ (otherwise $ch_1(v)\geq 0$). Let us see v_5. Note that $f_3(v_5)\leq 1$. Denote by f_6 and f_7 the remaining faces incident with v_5 in clockwise. If v_5 is good, then v_5 sends at least $\frac{1}{3}$ to v (if $ch_1(v)<0$) via a nice path by R6. Otherwise $d(f_6)=3$ and f_7 is a bad 5-face, then by $S_2\nsubseteq G$, we have $d(x_1)\geq 5$, see Figure 7(E_1). By the assumption, $d(v_2)\geq 5$. If $d(v_2)\geq 6$, then v sends at most $\frac{5}{9}$ to f_2 by R4; if $d(v_2)=5$, then by $S_6\nsubseteq G$, $d(x)\geq 5$, and v sends at most $\frac{5}{9}$ to f_2 by R3. On the other hand, if

$d(x_1) \geq 6$, then v sends at most $\frac{4}{9}$ to f_4 by R3; if $d(x_1) = 5$, then by $S_7 \nsubseteq G$, $d(x_2) \geq 5$, and v sends at most $\frac{4}{9}$ to f_4 by R3. In conclusion, $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{5}{9} = 0$ by R1 and R3.

(iii) Suppose $n_4(v) = 3$, that is $n_{5^+}(v) = 2$. If the pair of two 5^+-vertices fall in $\{(v_3, v_4), (v_3, v_5), (v_4, v_5)\}$, then by $S_1, S_2 \nsubseteq G$, we get $ch_1(v) \geq 3 - 3\times\frac{2}{3} - \max\{\frac{1}{2}+\frac{4}{9}, 2\times\frac{1}{2}\} = 0$ by R1 and R3. It remains to consider the pairs: $\{(v_1, v_4), (v_2, v_4), (v_2, v_3)\}$.

Assume $d(v_2) \geq 5$ and $d(v_4) \geq 5$. If $f_{6^+}(v) \geq 1$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \max\{\frac{2}{3} + \frac{1}{3}, \frac{1}{2} + \frac{1}{3}\} = 0$ by R1 and R3. It remains to discuss the case where $f_{6^+}(v) = 0$. Here, we can let $n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 0$) and v_5 be not good (otherwise v_5 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6 and $ch_2(v) \geq 0$). Denote by f_6 and f_7 the remaining faces incident with v_5 in clockwise. Note that $d(f_6) = 3$ and f_7 is a bad 5-face. By $S_2 \nsubseteq G$, we get $n_{5^+}(f_4) \geq 3$, and so $c(v \to f_4) \leq \frac{4}{9}$. On the other hand, recall that $d(v_2) \geq 5$. If $d(v_2) \geq 6$, then v sends at most $\frac{5}{9}$ to f_2 by R3; if $d(v_2) = 5$, then $d(x) \geq 5$ holds because of $S_{21} \nsubseteq G$. Hence, $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{5}{9} = 0$ by R1 and R3.

Assume $d(v_2) \geq 5$ and $d(v_3) \geq 5$. If $f_{6^+}(v) = 1$, then $ch_1(v) \geq 3 - 3\times\frac{2}{3} - \frac{5}{9} - \frac{1}{3} = \frac{1}{9} > 0$ by R1 and R3. Suppose that $f_{6^+}(v) = 0$. Since $S_3 \nsubseteq G$, $n_{5^+}(f_i) \geq 2$ holds for some $i \in \{4, 5\}$. If $n_{5^+}(f_i) \geq 2$ for each $i \in \{4, 5\}$, then $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{4, 5\}$ by R3, and thus $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} - \frac{5}{9} = \frac{1}{9} > 0$ by R1 and R3. Otherwise let $n_{5^+}(f_5) = 1$, that is $d(x_1) = d(x_2) = 4$, see Figure 7(E_2). If $d(x) \geq 5$, then $c(v \to f_2) \leq \frac{2}{3}$ by R3, and thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. Otherwise if $d(x) = 4$, then $f_3(v_5) \leq 1$ as f_2 is a 4-face and any two 4-faces in G are at distance at least 2. If v_5 is good, then v_5 sends at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Otherwise $f_3(v_5) = 1$ and v_5 is incident with a bad 5-face. Denote by f_6 and f_7 the faces incident with v_5 in clockwise. Since $S_2 \nsubseteq G$, we get $d(f_6) = 3$ and f_7 is a bad 5-face. If $d(z_2) = 5$, then $n_{5^+}(f_4) \geq 3$ by $S_{14} \nsubseteq G$, and so $c(v \to f_4) \leq \frac{4}{9}$. Otherwise $d(z_2) \geq 6$, in this situation $c(v \to f_4) \leq \frac{4}{9}$ by R3. Hence, $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3.

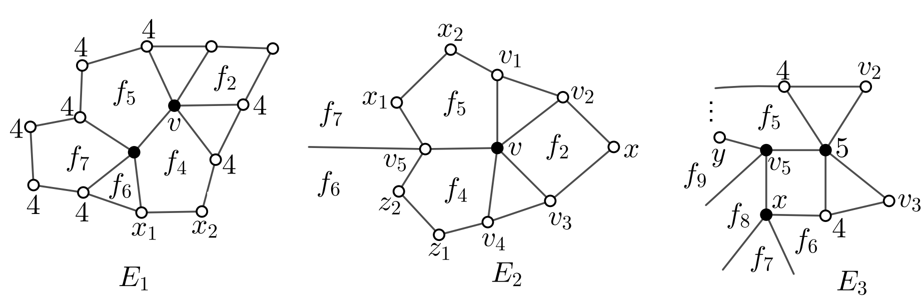

Figure 7: Specified Configuration.

Finally, we consider the case where $d(v_1) \geq 5$ and $d(v_4) \geq 5$. (a). $d(x) = 4$. Let us start to claim that v sends at most $\frac{8}{9}$ in total to $\{f_4, f_5\}$. Assume that $f_{6^+}(v) = 0$. If $d(v_1) \geq 6$, then we are done by R3. Otherwise if $d(v_1) = 5$, then $n_{5^+}(f_5) \geq 3$ holds because of $S_8 \nsubseteq G$, and so $c(v \to f_5) \leq \frac{4}{9}$. The above arguments can also be applied to v_4. So the same result holds for $f_{6^+}(v) \geq 1$, as claimed. Note that v_1 is symmetric to v_4. So we only discuss v_4 in the following, and we would like to claim that v_4 could send at least $\frac{1}{9}$ to v (if $ch_1(v) < 0$) when $d(v_4) \geq 6$ via a nice path.

Assume $d(v_4) \geq 7$. By Claim 2.7, we have $ch_1(v_4) \geq \frac{5d(v)-36}{18} + \frac{1}{9} + \frac{2}{9} = \frac{5d(v)-30}{18} > \frac{1}{9}$. Assume $d(v_4) = 6$. Since $S_{45} \nsubseteq G$, $\zeta_{v_4}(f_{3b}) \leq 1$. If $\zeta_{v_4}(f_{3b}) = 0$, then $ch_1(v_4) \geq \frac{5d(v)-36}{18} + 4 \times \frac{1}{9} \geq \frac{1}{9}$. Otherwise if $\zeta_{v_4}(f_{3b}) = 1$, then by $S_{43} \nsubseteq G$, we get $n_{5^+}(f_4) \geq 3$ and thus v_4 sends at most $\frac{4}{9}$ to f_4 by R4. Hence $ch_1(v_4) \geq \frac{5d(v)-36}{18} + 2 \times \frac{1}{9} + \frac{2}{9} \geq \frac{1}{9}$, and v_4 could send at least $\frac{1}{9}$ to v (if $ch_1(v) < 0$) via a nice path by R6, as claimed. So when $\min\{d(v_2), d(v_4)\} \geq 6$, v (if $ch_1(v) < 0$) could receive at least $\frac{2}{9}$ in total from $\{v_1, v_4\}$ via two nice paths by R6, and $ch_2(v) \geq 3 - 2 \times \frac{2}{3} - 1 - 2 \times \frac{4}{9} + 2 \times \frac{1}{9} = 0$ by R1, R3 and R6.

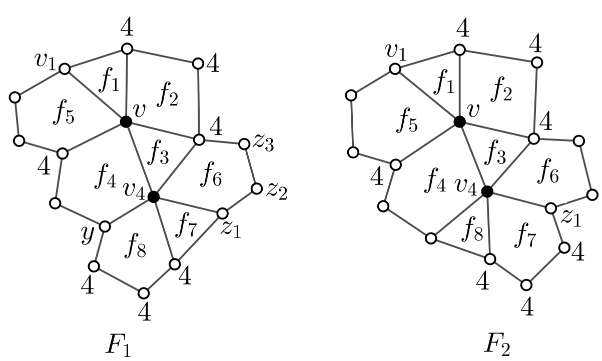

Figure 8: Specified Configuration.

Now we consider $\min\{d(v_2), d(v_4)\} = 5$. W.l.o.g., we assume $d(v_4) = 5$. Since $S_{17} \nsubseteq G$, $\zeta_{v_4}(f_{3b}) = 0$. Denote by f_6, f_7 and f_8 the remaining faces incident with v_4 in clockwise. As $S_{10} \nsubseteq G$, $n_{5^+}(f_6) \geq 2$ when $d(f_6) = 5$. We may assume that $f_{6^+}(v_4) = 0$ (otherwise $ch_1(v_4) \geq 3 - \frac{2}{3} - \frac{1}{3} - \max\{\frac{2}{3} + 2 \times \frac{1}{2}, \frac{2}{3} + \frac{1}{2} + \frac{4}{9}\} > \frac{2}{9}$ by R1 and R3, and v_4 could send at least $\frac{2}{9}$ to v (if $ch_1(v) < 0$) via a nice path by R6. So $ch_2(v) \geq 0$). First, let $d(f_7) = 3$. If $n_{5^+}(f_8) = 1$, see Figure 8(F_1) ($d(y) = 4$), then by $S_8, S_{13} \nsubseteq G$, y can not be incident with a bad 5-face. Note that $f_3(y) \leq 1$, thus y is good, and $ch_1(y) \geq \frac{1}{3}$. Hence, y could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Otherwise if $n_{5^+}(f_8) \geq 2$, then $ch_1(v_4) \geq 3 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} - \frac{4}{9} = \frac{2}{9}$ by R1 and R3. Second, let $d(f_8) = 3$. If $n_{5^+}(f_7) = 1$, see Figure 8(F_2) ($d(z_1) = 4$), then z_1 is good since $S_{13} \nsubseteq G$, and thus z_1 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6. Otherwise if $n_{5^+}(f_7) \geq 2$, then

$ch_1(v_4) \geq 3 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} - \frac{4}{9} = \frac{2}{9}$ by R1 and R3.

In conclusion, when $d(v_4) = 5$, v (if $ch_1(v) < 0$) could receive at least $\frac{2}{9}$ from one vertex in $\{v_4, y_1, z_1\}$ via a nice path by R6. Thus $ch_2(v) \geq 3 - 2 \times \frac{2}{3} - 1 - 2 \times \frac{4}{9} + \frac{2}{9} = 0$ by R1, R3 and R6.

(b). $d(x) \geq 5$. Then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

(iv) Suppose $n_4(v) = 2$. If the pair of two 4-vertices fall in $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_5)\}$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

Assume $d(v_1) = d(v_5) = 4$. If $f_{6^+}(v) \geq 1$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{1}{9} = \frac{1}{3} > 0$ by R1 and R3. It remains to consider $f_{6^+}(v) = 0$. If $n_{5^+}(f_5) \geq 2$, then $c(v \to f_5) \leq \frac{1}{2}$ by R3, and thus $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - \frac{5}{9} - 2 \times \frac{1}{2} = \frac{1}{9} > 0$ by R1 and R3. Otherwise $n_{5^+}(f_5) = 1$. We may let v_5 is not good (otherwise v_5 could send $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and thus $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{1}{2} + \frac{1}{3} > 0$ by R1, R3 and R6). Denote by f_6 and f_7 the faces incident with v_5 in clockwise. Since $S_2 \nsubseteq G$, $d(f_6) = 3$ and f_7 is a bad 5-face. Moreover, by $S_2 \nsubseteq G$ again, $n_{5^+}(f_4) \geq 3$, and so $c(v \to f_4) \leq \frac{4}{9}$. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3.

Assume $d(v_2) = d(v_3) = 4$. (a). $d(x) = 4$. By the same arguments as the case $d(v_1) \geq 5$ and $d(v_4) \geq 5$, we have that v (if $ch_1(v) < 0$) could receive at least $\frac{2}{9}$ from $\{v_2, x\}$ via a nice path, and thus $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - 2 \times \frac{4}{9} + \frac{2}{9} = 0$ by R1, R3 and R6. (b). $d(x) \geq 5$. Then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{4}{9} = \frac{1}{9} > 0$ by R1 and R3.

(v) Suppose $n_4(v) \leq 1$. If $n_4(v) = 1$, then $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - \max\{\frac{1}{2} + \frac{4}{9} + \frac{5}{9}, \frac{2}{3} + 2 \times \frac{4}{9}, 2 \times \frac{1}{2} + \frac{5}{9}\} = \frac{1}{9} > 0$ by R1 and R3. If $n_4(v) = 0$, then $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - 2 \times \frac{4}{9} - \frac{5}{9} = \frac{2}{9} > 0$ by R1 and R3.

Now we consider the configuration D_2. (i) Suppose $n_4(v) = 5$. Since $S_2 \nsubseteq G$, $f_i \neq (4,4,4,4,5)$ for each $i \in \{2, 5\}$. By $S_1 \nsubseteq G$, we get $f_4 \neq (4,4,4,4,5)$, and so $c(v \to f_4) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \max\{2 \times \frac{1}{2}, \frac{1}{2} + \frac{1}{3}\} = 0$ by R1 and R3.

(ii) Suppose $n_4(v) = 4$, that is $n_{5^+}(v) = 1$. Assume $d(v_1) \geq 5$, then $f_4 \neq (4,4,4,4,5)$ holds because of $S_1 \nsubseteq G$. Moreover, $f_2 \neq (4,4,4,4,5)$. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

Assume $d(v_2) \geq 5$, then $f_4 \neq (4,4,4,4,5)$ holds by $S_1 \nsubseteq G$, and so $c(v \to f_4) \leq \frac{1}{2}$ by R3. If $f_{6^+}(v) \geq 1$, then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Now we discuss $d(f_2) = d(f_5) = 5$. We may assume that $n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$). On the other hand, we may assume that v_5 is not good (otherwise v_5 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path and thus $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{2} + \frac{1}{3} > 0$). Denote by f_6 and f_7 the faces incident with v_5 in clockwise. Since $S_2 \nsubseteq G$, we have that $d(f_6) = 3$ and f_7 is a bad 5-face. By $S_{18} \nsubseteq G$, we know that $n_{6^+}(f_4) = 1$, and thus v sends at most $\frac{5}{9}$ to f_4

by R3. Next we claim that v sends at most $\frac{4}{9}$ to v. If $d(v_2) \geq 6$, then we are done by R3; if $d(v_2) = 5$, then $n_{5^+}(f_2) \geq 3$ by $S_{11} \nsubseteq G$, and so $c(v \to f_2) \leq \frac{4}{9}$ by R3, as claimed. Hence, $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{5}{9} = 0$ by R1 and R3.

Assume $d(v_3) \geq 5$. (a). $d(x) = 4$. Since $S_2 \nsubseteq G$, $f_5 \neq (4,4,4,4,5)$. For brevity, denote by f_6 and f_7 the faces incident with v_5 in clockwise. We may assume that v_5 is not good (otherwise, v_5 sends at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path, and $ch_2(v) \geq 3 - 2 \times \frac{2}{3} - 1 - 2 \times \frac{1}{2} + \frac{1}{3} = 0$). Since $S_2 \nsubseteq G$, $d(f_7) = 3$ and f_6 is a bad 5-face, which is impossible because S_5 is reducible. (b). $d(x) \geq 5$. By $S_2 \nsubseteq G$, we get $f_5 \neq (4,4,4,4,5)$, so $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

Assume $d(v_4) \geq 5$. If $f_{6^+}(v) \geq 1$, then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Now we consider $f_{6^+}(v) = 0$. Notice that $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_5) \geq 2$ holds because of $S_2 \nsubseteq G$, and so $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{2,5\}$ by R3. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. Assume $d(v_5) \geq 5$, then by $S_2 \nsubseteq G$, we get $n_{5^+}(f_2) \geq 2$, and so $c(v \to f_2) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

(iii) Suppose $n_4(v) = 3$, that is $n_{5^+}(v) = 2$. If the pair of two 5^+-vertices fall in $\{(v_1,v_2),(v_2,v_5),(v_3,v_4),(v_3,v_5),(v_4,v_5)\}$, then by $S_2 \nsubseteq G$, $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - \max\{2 \times \frac{1}{2} + \frac{5}{9}, 2 \times \frac{1}{2} + \frac{2}{3}\} = 0$ by R1 and R3.

Assume $d(v_2) \geq 5$ and $d(v_4) \geq 5$. If $f_{6^+}(v) \geq 1$, then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Next we discuss $f_{6^+}(v) = 0$. We may assume that $n_{5^+}(f_5) = 1$ (otherwise $ch_1(v) \geq 0$). (a). $d(x) = 4$. By $S_9 \nsubseteq G$, we get $d(v_2) \geq 6$, and then $c(v \to f_2) \leq \frac{4}{9}$ by R3. By $S_{15} \nsubseteq G$, we know that $d(v_4) \geq 6$, and then $c(v \to f_4) \leq \frac{5}{9}$ by R3. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3. (b). $d(x) \geq 5$. Similarly as above, we have $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3.

Assume $d(v_2) \geq 5$ and $d(v_3) \geq 5$. (a). $d(x) = 4$. Let y be the neighbor of v_5 which locates on f_5. If $d(y) = 4$, then by $S_2 \nsubseteq G$, v_5 is good and thus v_5 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6. When $d(f_5) \geq 6$, and we have $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - \frac{1}{3} - \frac{4}{9} + \frac{1}{3} = \frac{2}{9} > 0$ by R1 and R3. When $d(f_5) = 5$, $n_{5^+}(f_5) \geq 2$ holds because of $S_4 \nsubseteq G$, and we have $c(v \to f_5) \leq \frac{1}{2}$ by R3. Thus $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - \frac{1}{2} - \frac{4}{9} + \frac{1}{3} = \frac{1}{18} > 0$ by R1 and R3. It remains to consider $d(y) \geq 5$. Note that $f_5 \neq (4,4,4,4,5)$. Denote by f_8 and f_9 the faces incident with v_5 in clockwise, and f_6, f_7 the remaining faces incident with x in clockwise, see Figure 7(E_3). We may assume that v_5 is not good (otherwise v_5 could send at least $\frac{1}{3}$ to v via a nice path and $ch_2(v) \geq 0$). We immediately have $d(f_9) = 3$ and f_8 is a bad 5-face. In this situation, $f_3(x) \leq 1$, and by $S_2 \nsubseteq G$, x is good. Thus x could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path and $ch_2(v) \geq 0$.

(b). $d(x) \geq 5$. If $f_{6^+}(v) \geq 1$, then $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. It remains to discuss $f_{6^+}(v) = 0$. In this situation, we may assume that $d(y) = 4$ (otherwise $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} - \frac{4}{9} > 0$). Denote by f_6 and f_7 the remaining faces incident with v_5 in clockwise. Let v_5 be a vertex which is not good (otherwise v_5 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path and $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{4}{9} + \frac{1}{3} > 0$). Note that $d(f_6) = 3$ and f_7 is a bad 5-face by $S_2 \not\subseteq G$. Since $S_{18} \not\subseteq G$, we have that $d(x) \geq 6$, and v sends at moat $\frac{5}{9}$ to f_4 by R3. Hence, $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3.

Assume $d(v_1) \geq 5$ and $d(v_3) \geq 5$. (a). $d(x) = 4$. Denote by f_6, f_7 and f_8 the remaining faces incident with x in clockwise and f_9 another faces incident with v_5. If v_5 is good, then v (if $ch_1(v) < 0$) could receive at least $\frac{1}{3}$ from v_5 via a nice path by R6. Now we assume v_5 is not good. Since $S_2 \not\subseteq G$, $d(f_9) = 3$ and f_8 is a bad 5-face. In this situation, x must be good by $S_2 \not\subseteq G$ again. Hence, v (if $ch_1(v) < 0$) could receive at least $\frac{1}{3}$ from x via a nice path by R6. Thus $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} + \frac{1}{3} = 0$ by R1, R3 and R6. (b). $d(x) \geq 5$. Then $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

(iv) Suppose $n_4(v) = 2$. If the pair of two 4-vertices fall in $\{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_5)\}$, then $ch_2(v) \geq 3 - 2\times\frac{2}{3} - \max\{2\times\frac{1}{2}+\frac{5}{9}, \frac{2}{3}+\frac{1}{2}+\frac{4}{9}, \frac{2}{3}+\frac{5}{9}+\frac{4}{9}, \frac{2}{3}+2\times\frac{1}{2}\} = 0$ by R1 and R3.

Assume $d(v_1) = d(v_5) = 4$. If $f_{6^+}(v) \geq 1$, then $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Otherwise $f_{6^+}(v) = 0$, then we may assume that $n_{5^+}(f_5) = 1$ (otherwise $ch_2(v) \geq 0$). If $d(x) \geq 5$ or $d(v_4) \geq 6$, then $c(v \to f_4) \leq \frac{4}{9}$ by R3, and thus $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - \frac{4}{9} - \frac{5}{9} = 0$ by R1 and R3. So we consider $d(x) = 4$ and $d(v_4) = 5$. Denote by f_6 and f_7 the remaining faces incident with v_5 in clockwise. Since $S_2 \not\subseteq G$, v_5 is good. Hence, v_5 could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3 - 4 \times \frac{2}{3} - \frac{4}{9} + \frac{1}{3} > 0$ by R1, R3 and R6.

Assume $d(v_4) = d(v_5) = 4$. (a). $d(x) = 4$. Denote by f_6, f_7 the remaining faces incident with v_5 in clockwise. Since $S_2 \not\subseteq G$, $d(f_7) = 3$ and f_6 is a bad 5-face. We also have $n_{5^+}(f_5) \geq 3$ because of $S_2 \not\subseteq G$. Denote by f_8, f_9 the remaining faces incident with x in clockwise. By $S_2 \not\subseteq G$, x is good and x could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path. Thus $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - 2 \times \frac{4}{9} + \frac{1}{3} > 0$ by R1 and R3. (b). $d(x) \geq 5$. Then $ch_2(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} - \frac{4}{9} = \frac{1}{18} > 0$ by R1 and R3.

(v) Suppose $n_4(v) \leq 1$. If $n_4(v) = 1$, then $ch_2(v) \geq 3 - 2 \times \frac{2}{3} - \max\{\frac{1}{2} + \frac{5}{9} + \frac{4}{9}, \frac{2}{3} + 2 \times \frac{4}{9}, \frac{2}{3}+\frac{1}{2}+\frac{4}{9}\} = \frac{1}{18} > 0$ by R1 and R3. If $n_4(v) = 0$, then $ch_2(v) \geq 3 - 2\times\frac{2}{3} - \frac{5}{9} - 2\times\frac{4}{9} = \frac{2}{9} > 0$ by R1 and R3. □

Claim 2.15. *For each vertex $v \in W_2$ with $f_3(v) = 1$, $ch_2(v) \geq 0$.*

Proof. W.l.o.g., let $d(f_1) = 3$.

Case 1. Suppose $f_4(v) = 0$. Assume that $\zeta_v(f_{3b}) = 0$ firstly. If $f_{6^+}(v) = 1$, then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3. Otherwise $f_{6^+}(v) = 0$. Since $S_3 \nsubseteq G$, there exists at least one 5-face f_i ($i \in \{2, 3, 4, 5\}$) such that $n_{5^+}(f_i) \geq 2$, and $c(v \to f_i) \leq \frac{1}{2}$ by R3. If $n_{5^+}(f_1) = 3$, then $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{2, 5\}$ by R3, and thus $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - 3 \times \frac{1}{2} = \frac{1}{6} > 0$ by R1 and R3. If $n_{5^+}(f_1) = 2$, say $d(v_1) \geq 5$, then we claim that v sends at most $\frac{10}{9}$ in total to $\{f_1, f_5\}$. Obviously, the claim holds when $d(v_1) \geq 6$; when $d(v_1) = 5$, then by $S_{11} \nsubseteq G$, either $n_{5^+}(f_1) \geq 2$ or $n_{5^+}(f_5) \geq 3$, as claimed. Hence, $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - \frac{1}{2} - \frac{10}{9} = \frac{1}{18} > 0$ by R1 and R3. If $n_{5^+}(f_1) = 0$, that is $d(v_1) = d(v_2) = 4$, then by $S_2 \nsubseteq G$, $n_{5^+}(f_1) \geq 2$ and $n_{5^+}(f_5) \geq 2$, and so $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{2, 5\}$ by R3. Thus $ch_1(v) \geq 3 - 2 \times \frac{2}{3} - 3 \times \frac{1}{2} = \frac{1}{6} > 0$ by R1 and R3.

We now turn to the case $\zeta_v(f_{3b}) = 1$, which means that $d(v_1) = d(v_2) = 4$. Since $S_2, S_3 \nsubseteq G$, we get that $ch_1(v) \geq 3 - \frac{2}{3} - \frac{1}{2} - \max\{\frac{1}{3} + \frac{1}{2} + \frac{2}{3}, 2 \times \frac{1}{2} + \frac{2}{3}\} - \frac{1}{9} = \frac{1}{18} > 0$ by R1, R3 and R5.

Case 2. Suppose $f_4(v) = 1$ and let the other vertex on 4-face is x. Recall that v sends no charge to a bad 5-face (if it exists) which is incident with a $(4, 4, v)$-face by R5. By symmetry, we only need to consider the cases $d(f_2) = 4$ and $d(f_3) = 4$.

Subcase 2.1. $f_{6^+}(v) = 1$. (a). $n_{5^+}(f_i) = 1$ for some $i \in \{2, 3\}$. Then we have $ch_1(v) \geq 3 - 1 - 3 \times \frac{2}{3} - \frac{1}{3} = -\frac{1}{3}$. If $ch_1(v) \geq 0$, then we are done. So $ch_1(v) < 0$, that is, v is poor. Clearly, if there is a good 4-vertex in $N(v)$, then $ch_2(v) \geq -\frac{1}{3} + \frac{1}{3} = 0$ by Claim 2.6 and R6. Next we discuss the case that there is no good 4-vertex in $N(v)$.

Assume that $d(f_2) = 4$, then $d(v_1) \geq 5$ as $S_1 \nsubseteq G$. If $d(f_3) \geq 6$, then we may assume that $d(v_5) = 4$ (otherwise if $d(v_5) \geq 5$, then $ch_1(v) \geq 3 - 1 - \frac{2}{3} - \frac{1}{3} - 2 \times \frac{1}{2} = 0$). Recall that $v_5 \in N(v)$ is not good. By $S_2, S_{23} \nsubseteq G$, we get that $n_{5^+}(f_4) \geq 2$, and then $c(v \to f_4) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. If $d(f_4) \geq 6$, then $n_{5^+}(f_3) \geq 2$ because of $S_4 \nsubseteq G$, and so $c(v \to f_3) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. If $d(f_5) \geq 6$, then we may assume that $d(v_4) = 4$ (otherwise $ch_1(v) \geq 0$) and $v_4 \in N(v)$ is not good. Similarly, by $S_2, S_{23} \nsubseteq G$, we get that $n_{5^+}(f_4) \geq 2$, and then $c(v \to f_4) \leq \frac{1}{2}$ by R3. Thus $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

Assume $d(f_3) = 4$, then by $S_2, S_3 \nsubseteq G$, there is at least one $j \in \{2, 4, 5\}$ such that $f_j \neq (4, 4, 4, 4, 5)$, and so $c(v \to f_j) \leq \frac{1}{2}$. If $d(f_2) \geq 6$, then we may assume that $d(v_5) = 4$ (otherwise $ch_1(v) \geq 0$) and $v_5 \in N(v)$ is not good. Since $S_2, S_{23} \nsubseteq G$, we get that $n_{5^+}(f_5) \geq 2$, and $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. If $d(f_4) \geq 6$, then by the similar arguments, $n_{5^+}(f_5) \geq 2$, and thus $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$. If $d(f_5) \geq 6$, then $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_4) \geq 2$ hold because of $S_4 \nsubseteq G$, and so $c(v \to f_i) \leq \frac{1}{2}$ for each $i \in \{2, 4\}$ by R3. Thus $ch_1(v) \geq 3 - 1 - \frac{1}{3} - \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

(b). $n_{5^+}(f_i) \geq 2$ for some $i \in \{2,3\}$. Then $ch_1(v) \geq 3 - 4 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R3.

Subcase 2.2. $f_{6^+}(v) = 0$. (a). $n_{5^+}(f_i) = 1$ for some $i \in \{2,3\}$. If $d(f_2) = 4$, then by $S_1 \nsubseteq G$, $d(v_1) \geq 5$ holds, and by $S_3 \nsubseteq G$, there is at least one face f_j ($j \in \{3,4\}$) satisfying $f_j \neq (4,4,4,4,5)$, and so $c(v \to f_j) \leq \frac{1}{2}$ and $c(v \to f_5) \leq \frac{1}{2}$ by R3. If $d(f_3) = 4$, then $f_i \neq (4,4,4,4,5)$ holds for each $i \in \{2,4\}$ by $S_4 \nsubseteq G$, and so $c(v \to f_i) \leq \frac{1}{2}$. By the similar arguments as above, we may assume that each vertex in $N(v)$ is not good (otherwise $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} + \frac{1}{3} = 0$, and we are done).

Assume that $d(f_2) = 4$. For brevity, let $x_1 \in N(v_3)$ such that x_1 locates on f_3, and denote by f_6, f_7 the faces incident with v_3 in clockwise. Since $S_2 \nsubseteq G$, $d(f_7) = 3$ and f_6 is a bad 5-face, and $d(x_1) \geq 5$. Moreover, if $d(x_1) = 5$, then $n_{5^+}(f_3) \geq 3$ because of $S_{19} \nsubseteq G$. So v sends at most $\frac{4}{9}$ to f_3 by R3. However, x is good in this situation and $ch_1(x) \geq \frac{1}{3}$. Hence, x could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - \frac{1}{2} - \frac{4}{9} + \frac{1}{3} > 0$ by R1, R3 and R6.

Assume that $d(f_3) = 4$. Note that $v_3, v_4 \in N(v)$ are not good. By $S_2 \nsubseteq G$, x is good and $ch_1(x) \geq \frac{1}{3}$. Hence, x could send at least $\frac{1}{3}$ to v (if $ch_1(v) < 0$) via a nice path by R6, and $ch_2(v) \geq 3 - 1 - 2 \times \frac{2}{3} - 2 \times \frac{1}{2} + \frac{1}{3} = 0$ by R1, R3 and R6.

(b). $n_{5^+}(f_i) \geq 2$ for some $i \in \{2,3\}$. Similarly, we may assume that each vertex in $N(v)$ is not good (otherwise $ch_2(v) \geq 3 - \frac{5}{9} - 4 \times \frac{2}{3} + \frac{1}{3} = \frac{1}{9} > 0$, and we are done). Assume $d(f_2) = 4$. Since $S_3 \nsubseteq G$, there exists at least one face f_i and f_j in $\{f_3, f_4\}$ and $\{f_4, f_5\}$, respectively such that $n_{5^+}(f_i) \geq 2$, $n_{5^+}(f_j) \geq 2$. If $i = j = 4$, that is $n_{5^+}(f_3) = n_{5^+}(f_5) = 1$, recall that both v_4 and v_5 are not good, then by $S_2, S_{12} \nsubseteq G$, there exists at least one face f_k ($k \in \{3,5\}$) such that $n_{5^+}(f_k) \geq 2$, and thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. Otherwise if $i \neq j$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3.

Assume that $d(f_3) = 4$. If $d(v_5) = 5$, then $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R1 and R3. Next we discuss $d(v_5) = 4$, and recall that $v_5 \in N(v)$ is not good. On the other hand, since $S_3 \nsubseteq G$, there exists at least one face f_i in $\{f_4, f_5\}$ such that $n_{5^+}(f_i) \geq 2$. We may also assume that $n_{5^+}(f_2) = 1$ (otherwise $ch_1(v) \geq 0$). If $n_{5^+}(f_4) \geq 2$, then by $S_{11} \nsubseteq G$, $d(v_1) \geq 6$, and thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} - \frac{4}{9} = \frac{1}{18} > 0$ by R1 and R3. If $n_{5^+}(f_5) \geq 2$, then we are going to claim that $n_{5^+}(f_5) \geq 3$. Note that $d(v_3) = d(v_4) = d(v_5) = 4$, and we may assume none of them is rich (otherwise $ch_2(v) \geq 0$). By $S_2 \nsubseteq G$, we get $n_{5^+}(f_5) \geq 3$, as claimed. Recall that $n_{5^+}(f_3) \geq 2$, we get $d(x) \geq 5$. On the other hand, by $S_{16} \nsubseteq G$, we get $d(x) \geq 6$. Thus $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{5}{9} - \frac{4}{9} = 0$ by R1 and R3. □

Claim 2.16. *For each vertex $v \in W_2$ with $f_3(v) = 0$, $ch_2(v) \geq 0$.*

Proof. Assume that $f_4(v) = 0$, then by $S_3 \nsubseteq G$, there is at least one f_i ($i \in [5]$) satisfying

$f_i \neq (4,4,4,4,5)$, and so $c(v \to f_i) \leq \frac{1}{2}$. Hence, $ch_1(v) \geq 3 - 3 \times \frac{2}{3} - \frac{1}{2} = 0$ by R1 and R3. Assume that $f_4(v) = 1$. W.l.o.g., let $f_1 = (v, v_1, x, v_2)$ be the 4-face.

Case 1. $n_{5^+}(f_1) = 1$. Assume that $f_{6^+}(v) = 1$. If $d(f_2) \geq 6$, then $n_{5^+}(f_5) \geq 2$ by $S_4 \nsubseteq G$, and $n_{5^+}(f_i) \geq 2$ for some $i \in \{3,4\}$ by $S_3 \nsubseteq G$. Thus $ch_1(v) \geq 3-1-\frac{2}{3}-\frac{1}{3}-2\times\frac{1}{2} = 0$ by R3. If $d(f_3) \geq 6$, then $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_5) \geq 2$ by $S_4 \nsubseteq G$. Thus $ch_1(v) \geq 3-1-\frac{2}{3}-\frac{1}{3}-2\times\frac{1}{2} = 0$ by R1 and R3.

Assume that $f_{6^+}(v) = 0$. Since $S_4 \nsubseteq G$, $n_{5^+}(f_2) \geq 2$ and $n_{5^+}(f_5) \geq 2$ hold. If $d(v_4) \geq 5$, then $ch_1(v) \geq 3 - 1 - 4 \times \frac{1}{2} = 0$ by R3. If $d(v_1) = 4$, then we may assume that v_4 is not good (otherwise $ch_2(v) \geq 0$). By $S_2, S_{23} \nsubseteq G$, we get that $n_{5^+}(f_3) \geq 2$ and $n_{5^+}(f_4) \geq 2$, and thus $ch_1(v) \geq 3 - 1 - 4 \times \frac{1}{2} = 0$ by R3.

Case 2. $n_{5^+}(f_1) \geq 2$. Then $ch_1(v) \geq 3 - \frac{2}{3} - \max\{2 \times \frac{2}{3} + \frac{1}{2} + \frac{1}{3}, 2 \times \frac{2}{3} + 2 \times \frac{1}{2}\} = 0$ by R3. □

According to all above claims, we know that the minimum counterexample does not exist. □

3 Proof of Theorem 1

Let G be a counterexample to Theorem 1 with fewest vertices and edges, that is, there is a list assignment L of G satisfying $|L(v)| \geq 4$ for any $v \in V(G)$ such that G is not L-colorable but any proper subgraph of G is L-colorable. Firstly, we present the well-known Combinatorial Nullstellensatz initiated by Alon which is essential to produce reducible subgraphs.

Lemma 3.1 ([1], Combinatorial Nullstellensatz)**.** *Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree* $\deg(f)$ *of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $C_1, \ldots, C_n$ are subsets of F with $|C_i| > t_i$, there are $c_1 \in C_1, c_2 \in C_2, \ldots, c_n \in C_n$ so that*

$$f(c_1, \ldots, c_n) \neq 0.$$

If G has a vertex v of degree at most three, then we can extend an L-coloring φ of $G \setminus v$ to an L-coloring ϕ of G by setting $\phi(v) \in L(v) \setminus \{\varphi(u) : uv \in E(G)\}$, a contradiction. So $\delta(G) \geq 4$. By Lemma 2.1, G must contain a subgraph isomorphic to one of the configurations in $\mathcal{S}$ (see Appendix B). Next, we prove that all these subgraphs do not exist, that is, all configurations S_1-S_{47} in $\mathcal{S}$ are reducible, which leads to a contradiction.

Lemma 3.2. *S_1-S_{47} in $\mathcal{S}$ are reducible.*

Proof. By the minimality of G, there is an L-coloring of $G-S_i$ for each $i\in[47]$. Fix some i, say i_0, there is an L-coloring φ of $G-S_{i_0}$. Let $S_{i_0}=\{x_0,x_1,\ldots,x_{n-1}\}$ and $C_\varphi(v)=\{\varphi(u): uv\in E(G) \textit{ and } u\in V(G-S_{i_0})\}$. Let $C_j=L(x_j)\backslash C_\varphi(x_j)$ for $j\in\{0,1,\ldots,n-1\}$. Now we extend φ to G and let ϕ denote the coloring after all vertices in S_{i_0} are colored. Let $c_0,c_1,\ldots,c_{n-1}$ correspond to the colors of $x_0,x_1,\ldots,x_{n-1}$ respectively. If $c_i-c_j\neq 0$ for any $x_ix_j\in E(G)$, then ϕ is a proper L-coloring of G. Next let $P=P(x_0,x_1,\ldots,x_{n-1})$ be the following polynomial:

$$P(x_0,x_1,\ldots,x_{n-1})=\prod_{x_ix_j\in E(G)}(x_i-x_j).$$

That is, if there are $c_0\in C_0,c_1\in C_1,\ldots,c_{n-1}\in C_{n-1}$ such that $P(x_0,x_1,\ldots,x_{n-1})\neq 0$, then we can extend φ to an L-coloring ϕ of G by choosing $x_0=c_0,x_1=c_1,\ldots,x_{n-1}=c_{n-1}$.

Based on Lemma 3.1, we present an algorithm in Appendix A which effectively calculates reducible configurations. Let us take S_1 as an example. Let $S_1=\{x_0,x_1,\ldots,x_4\}$ such that $x_0x_1x_4$ is a triangle and $x_1x_2x_3x_4$ is a 4-face, where $d(x_i)=4$ for each $i\in\{0,1,2,3\}$ and $d(x_4)=5$. Then

$$P(x_0,x_1,\ldots,x_4)=(x_0-x_1)(x_0-x_4)(x_1-x_2)(x_1-x_4)(x_2-x_3)(x_3-x_4).$$

That is, input “vve = [(0, 1), (0, 4), (1, 2), (1, 4), (2, 3), (3, 4)]”. Note that $|C_1|>2$ and $|C_i|>1$ for each $i\in\{0,2,3,4\}$ as x_1 has one neighbor in $V(G-S_1)$ and each x_i has two neighbors in $V(G-S_1)$. Thus, we input “v_List = [1,2,1,1,1]”. Through the computation of the algorithm in Appendix A, we get the 1st valid expansion is [1,2,1,1,1], that is, the coefficient of $x_0x_1^2x_2x_3x_4$ in P is nonzero. Therefore, S_1 is reducible by Lemma 3.1. □

This completes the proof.

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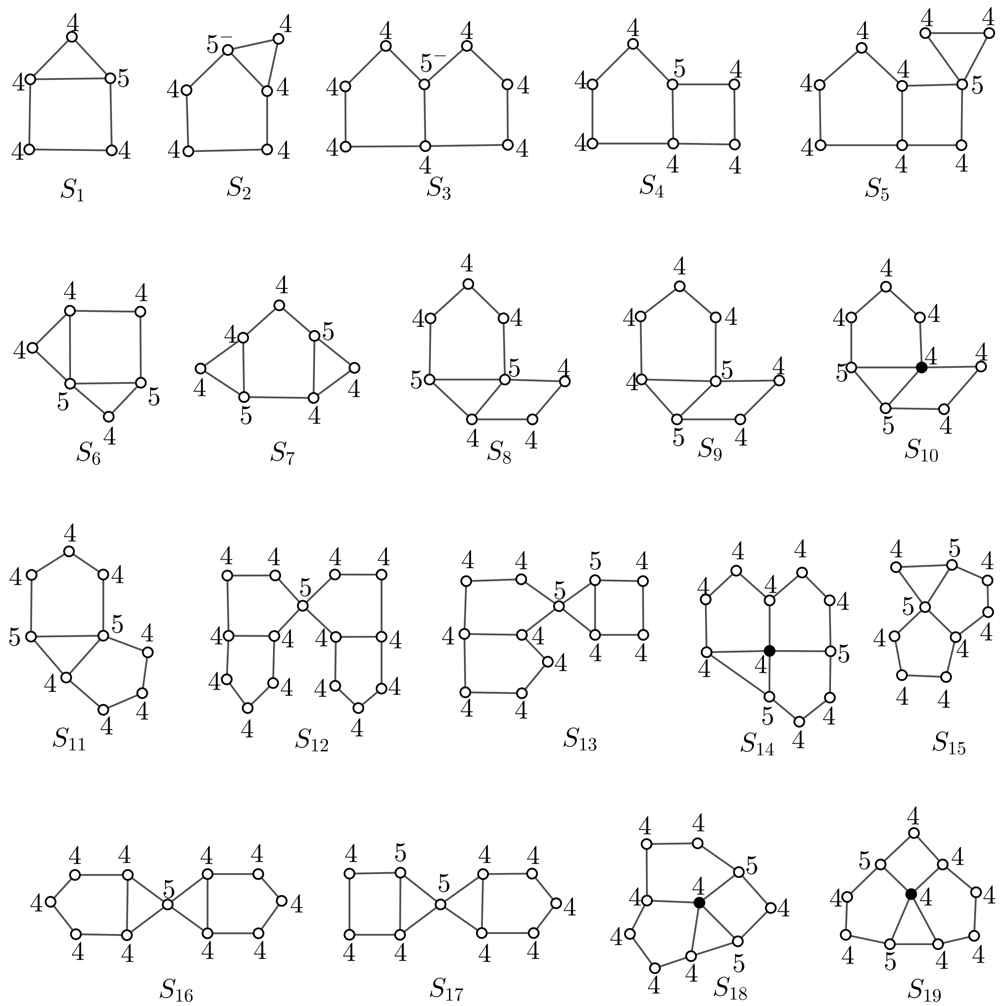
A Algorithm

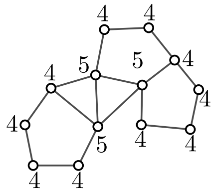
```
# -*- coding: utf-8 -*-
#!/usr/bin/env python
import copy

def choosable(n,v_List,edges): # Determine whether satisfying \
Combinatorial Nullstellensatz, back to the remainder of the expansion!
    # n: the number of vertices, v_List[0..n-1]: |L(v)|-1, edges: |L(e)|
    zks={}
    zks['0'*n]=1
    len_edges=len(edges)
    for i in range(len_edges):
        v1,v2=edges[i]
```

```
        List_zks=[]
        while zks:
            List_zks.append(zks.popitem())
        while List_zks:
            a,b=List_zks.pop()
            if ord(a[v1])-ord("0")<v_List[v1]:
                a1=a[:v1]+chr(ord(a[v1])+1)+a[v1+1:]
                if a1 in zks.keys():
                    zks[a1]=zks[a1]+b
                    if zks[a1]==0:
                        del zks[a1]
                else:
                    zks[a1]=b
            if ord(a[v2])-ord("0")<v_List[v2]:
                a2=a[:v2]+chr(ord(a[v2])+1)+a[v2+1:]
                if a2 in zks.keys():
                    zks[a2]=zks[a2]-b
                    if zks[a2]==0:
                        del zks[a2]
                else:
                    zks[a2]=-b
    return zks

# The main program
def Comb_Null(vve, v_List):
    # ==================================================================
    # List coloring.
    # vve: Labelling vertices must start at 0. \
    e.g. 3-cycle: vve=[(0, 1),(1, 2),(2, 0)]
    # v_List: |L(v)|-1, must be integers. \
    e.g. 3-cycle: v_List=[1, 1, 1]

    #  Apply Combinatorics Nullstellensatz ================================================
    v_no=len(v_List)
    zks=choosable(v_no,v_List, vve)

    #  Output part. If there are too many expansions that \
    satisfy the criteria, we print up to 10 ======================
    size_zks=len(zks)
    if size_zks>0:
        print("\n\nThe total number of valid expansions= "+str(size_zks)+",\
        among them:")
        i=percent = 0
        for a in zks.keys():
            if i/size_zks>=percent:
                if i == 0:
                    print("The 1st valid expansion is: [",end="")
                elif i == 1:
                    print("The 2nd valid expansion is: [",end="")
                else:
                    print("The "+str(i+1)+"th valid expansion is: [",end="")
```

```
                for j in range(v_no-1):
                    print(str(ord(a[j])-ord("0"))+",",end="")
                print(str(ord(a[v_no-1])-ord("0"))+"]")
                percent+=0.1
            i+=1
    else: print("\n\n No valid expansion!!")

#Example
#Input
vve = [(0, 1), (0, 4), (1, 2), (1, 4), (2, 3), (3, 4)]
v_List = [1, 2, 1, 1, 1]
Comb_Null(vve, v_List)

# Output
# The total number of valid expansions= 1, among them:
# The 1st valid expansion is: [1,2,1,1,1]
```

B All configurations in $\mathcal{S}$





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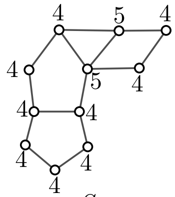

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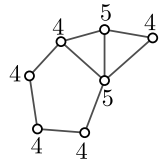

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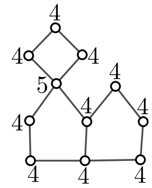

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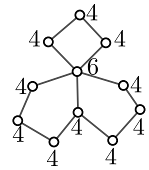

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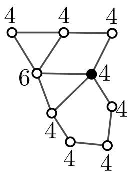

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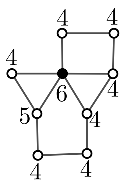

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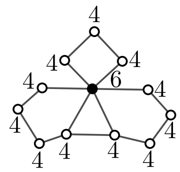

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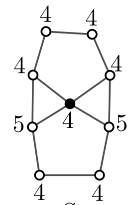

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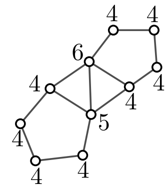

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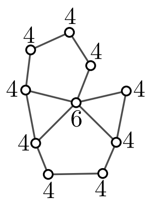

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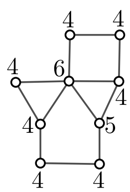

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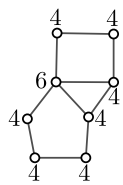

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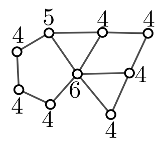

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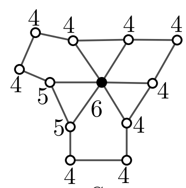

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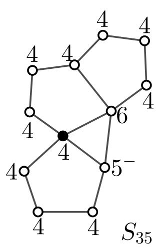

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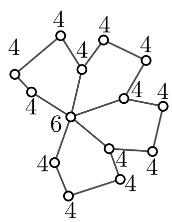

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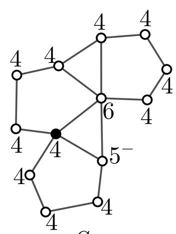

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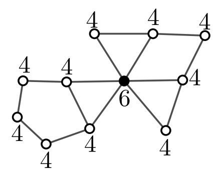
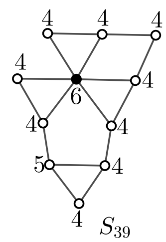
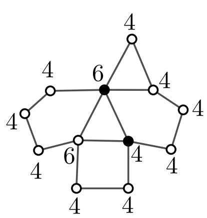


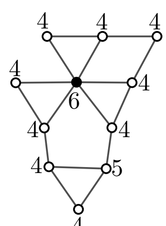
S_{38}



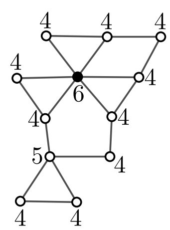
S_{39}



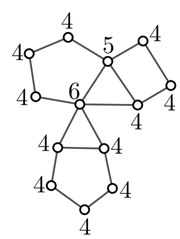
S_{40}



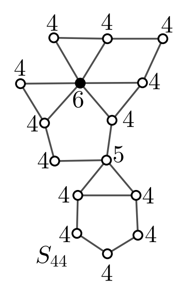
S_{41}



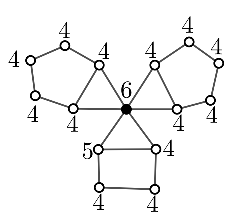
S_{42}



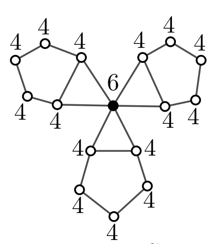
S_{43}



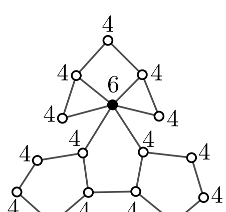
S_{44}



S_{45}



S_{46}



S_{47}