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Efficient weight vectors from pairwise comparison matrices

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Abstract

Pairwise comparison matrices are frequently applied in multi-criteria decision making. A weight vector is called efficient if no other weight vector is at least as good in approximating the elements of the pairwise comparison matrix, and strictly better in at least one position. A weight vector is weakly efficient if the pairwise ratios cannot be improved in all non-diagonal positions. We show that the principal eigenvector is always weakly efficient, but numerical examples show that it can be inefficient. The linear programs proposed test whether a given weight vector is (weakly) efficient, and in case of (strong) inefficiency, an efficient (strongly) dominating weight vector is calculated. The proposed algorithms are implemented in Pairwise Comparison Matrix Calculator, available at pcmc.online.

Keywords: multiple criteria analysis, decision support, pairwise comparison matrix, Pareto optimality, efficiency, linear programming

1 Introduction

1.1 Pairwise comparison matrices

Pairwise comparison matrix [32] has been a popular tool in multiple criteria decision making, for weighting the criteria and evaluating the alternatives with respect to every criterion. Decision makers compare two criteria or two alternatives at a time and judge which one is more important or better, and how many times. Formally, a pairwise comparison matrix is a positive matrix $\mathbf{A}$ of size $n \times n$, where $n \geq 3$ denotes the number of items to compare. Reciprocity is assumed: $a_{ij} = 1/a_{ji}$ for all $1 \leq i, j \leq n$. A pairwise comparison matrix is called consistent, if $a_{ij}a_{jk} = a_{ik}$ for all i, j, k. Let PCM_n denote the set of pairwise comparison matrices of size $n \times n$. Once the decision maker provides all the $n(n-1)/2$ comparisons, the objective is to find a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top \in \mathbb{R}^n$ such that the pairwise ratios of the weights, w_i/w_j, are *as close as possible* to the matrix elements a_{ij}. Several methods have been suggested for this weighting problem, e.g., the eigenvector method [32], the least squares method [5, 9, 21, 23], the logarithmic least squares method [11, 12, 13], the spanning tree approach [7, 26, 30, 33, 34, 36, 37] besides many other proposals discussed and compared by Golany and Kress [22], Choo and Wedley [8], Lin [25], Fedrizzi and Brunelli [18]. Bajwa, Choo and Wedley [3] not only compare seven weighting methods with respect to four criteria, but provide a detailed list of nine earlier comparative studies, too.

1.2 Weighting as a multiple objective optimization problem

The weighting problem itself can be considered as a multi-objective optimization problem which includes $n^2 - n$ objective functions, namely $|x_i/x_j - a_{ij}|$, $1 \leq i \neq j \leq n$. Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$

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be a pairwise comparison matrix and write the multi-objective optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n_{++}} \left| \frac{x_i}{x_j} - a_{ij} \right|_{1 \le i \ne j \le n}. \tag{1}$$

Efficiency or Pareto optimality [27, Chapter 2] is a key concept in multiple objective optimization and multiple criteria decision making. See Ehrgott's historical overview [16], beginning with Edgeworth [15] and Pareto [31].

Consider the functions

$$f_{ij} : \mathbb{R}^n_{++} \to \mathbb{R}, \; i, j = 1, \ldots, n,$$

defined by

$$f_{ij}(\mathbf{x}) = \left| \frac{x_i}{x_j} - a_{ij} \right|, \; i, j = 1, \ldots, n, \tag{2}$$

as in Blanquero, Carrizosa, Conde [4, p.273]. Since $f_{ii}(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^n_{++}$ and $i = 1, \ldots, n$, these constant functions are irrelevant from the aspect of multi-objective optimization, so they will be simply left out from the investigations.

Let the vector-valued function $\mathbf{f} : \mathbb{R}^n_{++} \to \mathbb{R}^{n(n-1)}_{++}$ defined by its components $f_{ij}, i, j = 1, \ldots, n, \; i \ne j$. Consider the problem of minimizing $\mathbf{f}$ over a nonempty set $X \subseteq \mathbb{R}^n$ that can be written in the general form of the *vector optimization problem*

$$\min_{\mathbf{x} \in X} \mathbf{f}(\mathbf{x}). \tag{3}$$

With $X = \mathbb{R}^n_{++}$, where the latter denotes the positive orthant in $\mathbb{R}^n$, we get problem (1) in a bit more general form.

Recall the following basic concepts used for multiple objective or vector optimization. A point $\bar{\mathbf{x}} \in X$ is said to be an efficient solution of (3) if there is no $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) \le \mathbf{f}(\bar{\mathbf{x}})$, $\mathbf{f}(\mathbf{x}) \ne \mathbf{f}(\bar{\mathbf{x}})$, meaning that $f_{ij}(\mathbf{x}) \le f_{ij}(\bar{\mathbf{x}})$ for all $i \ne j$ with strict inequality for at least one index pair $i \ne j$. In the literature, the names Pareto-optimal, nondominated and noninferior solution are also used instead of efficient solution.

A point $\bar{\mathbf{x}} \in X$ is said to be a weakly efficient solution of (3) if there is no $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\bar{\mathbf{x}})$, i.e. $f_{ij}(\mathbf{x}) < f_{ij}(\bar{\mathbf{x}})$ for all $i \ne j$. Efficient solutions are sometimes called strongly efficient.

A point $\bar{\mathbf{x}} \in X$ is said to be a locally efficient solution of (3) if there exists $\delta > 0$ such that $\bar{\mathbf{x}}$ is an efficient solution in $X \cap B(\bar{\mathbf{x}}, \delta)$, where $B(\bar{\mathbf{x}}, \delta)$ is a δ-neighborhood around $\bar{\mathbf{x}}$. The local weak efficiency is defined similarly for a point $\bar{\mathbf{x}} \in X$, the only difference is that weakly efficient solutions are considered instead of efficient solutions.

Several multi-objective optimization models have been proposed in the research of pairwise comparison matrices. Departing from [28], Mikhailov and Knowles [29] include two objective functions, the sum of least squares, written for the upper diagonal positions, and the number of minimum violations, then apply an evolutionary algorithm to generate the Pareto frontier. A third objective function, the total deviation from second-order indirect judgments, is added in [35].

The $n(n-1)/2$ objective functions $|x_i/x_j - a_{ij}|$, $1 \le i \ne j \le n$, of the multi-objective optimization problem (1) can be aggregated into a single objective function in several ways. Their sum gives the weighting method *least absolute error* [8, Section 4, LAE]. If their maximum is taken into consideration, weighting method *least worst absolute error* [8, Section 4, LWAE] is resulted in. The sum of their squares is the classical *least squares method* [5, 9, 21, 23]. A (parametric) linear combination of the sum and the maximum is proposed by Jones and Mardle

[24] to find a compromise weight vector. A similar idea is applied in the proposal of Dopazo and Ruiz-Tagle [14], developed for group decision problems with incomplete pairwise comparison matrices.

In the rest of the paper efficiency for problem (1), including $n(n-1)/2$ objective functions, is considered. The explicit presentation will be unavoidable for the problem specific concept of internal efficiency introduced recently in [6].

1.3 Efficiency of weight vectors

Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition 1.1. *Weight vector* $\mathbf{w}$ *is called* efficient *for (1) if no positive weight vector* $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ *exists such that*

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \le \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \textit{for all } 1 \le i,j \le n, \tag{4}$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \textit{for some } 1 \le k,\ell \le n. \tag{5}$$

Weight vector $\mathbf{w}$ is called *inefficient* for (1) if it is not efficient for (1).

If weight vector $\mathbf{w}$ is inefficient for (1) and weight vector $\mathbf{w}'$ fulfils (4)-(5), we say that $\mathbf{w}'$ *dominates* $\mathbf{w}$. Note that dominance is transitive.

It follows from the definition that an arbitrary rescaling does not influence (in)efficiency.

Remark 1. *A weight vector* $\mathbf{w}$ *is efficient for (1) if and only if* $c\mathbf{w}$ *is efficient for (1), where* $c > 0$ *is an arbitrary scalar.*

Example 1.1. *Consider four criteria* C_1, C_2, C_3, C_4*, pairwise comparison matrix* $\mathbf{A} \in PCM_4$ *and its principal right eigenvector* $\mathbf{w}$ *as follows:*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \qquad \mathbf{w}' = \begin{pmatrix} \mathbf{0.441126} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}.$$

In order to prove the inefficiency of the principal right eigenvector $\mathbf{w}$*, let us increase its first coordinate:* $w'_1 := 9w_4 = 0.441126$*,* $w'_i := w_i, i = 2,3,4$*. The consistent approximations generated by weight vectors* $\mathbf{w}, \mathbf{w}'$*,*

$$\left[\frac{w_i}{w_j}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}, \tag{6}$$

$$\left[\frac{w'_i}{w'_j}\right] = \begin{pmatrix} 1 & \mathbf{1.0114} & \mathbf{3.9995} & \mathbf{9} \\ \mathbf{0.9888} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2500} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1111} & 0.1124 & 0.4444 & 1 \end{pmatrix},$$

show that inequality (4) holds for all $1 \le i,j \le 4$*, and the strict inequality (5) holds for* $(k,\ell) \in \{(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)\}$*. For example, with* $k=1, \ell=2$*,* $|\frac{w'_1}{w'_2} - a_{12}| = |1.0114 - 1| =$

$0.0114 < |\frac{w_1}{w_2} - a_{12}| = |0.9274 - 1| = 0.0726$. *Weight vector* $\mathbf{w}'$ *dominates* $\mathbf{w}$. *Note that the principal right eigenvector* $\mathbf{w}$ *ranks the criteria as* $C_2 \succ C_1 \succ C_3 \succ C_4$, *while the dominating weight vector* $\mathbf{w}'$ *ranks them as* $C_1 \succ C_2 \succ C_3 \succ C_4$.

Blanquero et al. (2006) considered the local variant of efficiency:

Definition 1.2. *Weight vector* $\mathbf{w}$ *is called* locally efficient *for (1) if there exists a neighborhood of* $\mathbf{w}$, *denoted by* $V(\mathbf{w})$, *such that no positive weight vector* $\mathbf{w}' \in V(\mathbf{w})$ *fulfilling (4)-(5) exists.*

Weight vector $\mathbf{w}$ is called *locally inefficient* if it is not locally efficient.

Another variant of (in)efficiency has been introduced by Bozóki (2014):

Definition 1.3. *Weight vector* $\mathbf{w}$ *is called* internally efficient *for (1) if no positive weight vector* $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^\top$ *exists such that*

$$\left.\begin{array}{lcl} a_{ij} \le \frac{w_i}{w_j} & \Longrightarrow & a_{ij} \le \frac{w'_i}{w'_j} \le \frac{w_i}{w_j} \\ a_{ij} \ge \frac{w_i}{w_j} & \Longrightarrow & a_{ij} \ge \frac{w'_i}{w'_j} \ge \frac{w_i}{w_j} \end{array}\right\} \quad \text{for all } 1 \le i,j \le n, \tag{7}$$

$$\left.\begin{array}{lcl} a_{k\ell} \le \frac{w_k}{w_\ell} & \Longrightarrow & \frac{w'_k}{w'_\ell} < \frac{w_k}{w_\ell} \\ a_{k\ell} \ge \frac{w_k}{w_\ell} & \Longrightarrow & \frac{w'_k}{w'_\ell} > \frac{w_k}{w_\ell} \end{array}\right\} \quad \text{for some } 1 \le k,\ell \le n. \tag{8}$$

Weight vector $\mathbf{w}$ is called *internally inefficient* if it is not internally efficient.

If weight vector $\mathbf{w}$ is inefficient for (1) and weight vector $\mathbf{w}'$ fulfils (7)-(8), we say that $\mathbf{w}'$ *dominates* $\mathbf{w}$ *internally.* Note that internal dominance is transitive.

Example 1.2. *Consider the pairwise comparison matrix* $\mathbf{A} \in PCM_4$ *of Example 1.1 and its principal right eigenvector* $\mathbf{w}$. *Now let us increase the first coordinate of* $\mathbf{w}$ *until it reaches the second one,*

$$\mathbf{w}'' = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}.$$

The consistent approximation generated by weight vector $\mathbf{w}''$ *is as follows:*

$$\left[\frac{w''_i}{w''_j}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}. \tag{9}$$

Inequality (7) holds for all $1 \le i,j \le 4$, *and the strict inequality (8) holds for* $(k,\ell) \in \{(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)\}$. *Weight vector* $\mathbf{w}''$ *dominates* $\mathbf{w}$ *internally. Observe that weight vector* $\mathbf{w}'$ *in Example 1.1 does not dominate* $\mathbf{w}$ *internally. Note that the internally dominating weight vector* $\mathbf{w}''$ *ranks the criteria as* $C_1 \sim C_2 \succ C_3 \succ C_4$.

The local inefficiency of weight vector $\mathbf{w}$ *can be checked by the fact that weight vector* $(w_1 + \varepsilon, w_2, w_3, w_4)^\top$ *dominates* $\mathbf{w}$ *for all* $\varepsilon < 2(w_2 - w_1) = 0.0633$, *furthermore it dominates* $\mathbf{w}$ *internally for all* $\varepsilon < w_2 - w_1 = 0.0316$, *providing the same ranking* $C_2 \succ C_1 \succ C_3 \succ C_4$ *as of the principal right eigenvector* $\mathbf{w}$.

A natural question might arise. How can dominating weight vectors in Examples 1.1–1.2 be found? We premise that algorithmic ways of finding a dominating efficient weight vector shall be given in details in Section 4.

It follows from the definitions that if weight vector $\mathbf{w}$ is internally inefficient, then it is inefficient. Blanquero, Carrizosa and Conde proved that the two definitions are in fact equivalent:

Theorem 1.1. *[4, Theorem 3] Weight vector* $\mathbf{w}$ *is efficient for (1) if and only if it is locally efficient for (1), i.e., Definitions 1.1 and 1.2 are equivalent.*

Proposition 1.1. *Weight vector* $\mathbf{w}$ *is efficient for (1) if and only if it is internally efficient for (1), i.e., Definitions 1.1 and 1.3 are equivalent.*

Proof. Sufficiency follows by definition. For necessity, it is more convenient to show that inefficiency implies internal inefficiency. Let weight vector $\mathbf{w}$ be inefficient. Theorem 1.1 implies that $\mathbf{w}$ is locally inefficient as well, i.e., there exists $\mathbf{w}'$ in any neighborhood $U(\mathbf{w})$ such that $\mathbf{w}'$ dominates $\mathbf{w}$. If $U(\mathbf{w})$ is sufficiently small, then

$$\begin{array}{lcl} a_{ij} < \frac{w_i}{w_j} & \Longrightarrow & a_{ij} < \frac{w'_i}{w'_j} \leq \frac{w_i}{w_j} \\ a_{ij} > \frac{w_i}{w_j} & \Longrightarrow & a_{ij} > \frac{w'_i}{w'_j} \geq \frac{w_i}{w_j} \\ a_{ij} = \frac{w_i}{w_j} & \Longrightarrow & a_{ij} = \frac{w'_i}{w'_j} = \frac{w_i}{w_j}, \end{array} \tag{10}$$

implying that $\mathbf{w}$ is internally inefficient. □

Corollary 1. *Efficiency (Definition 1.1), local efficiency (Definition 1.2) and internal efficiency (Definition 1.3) are equivalent.*

Definition 1.4. *Let* $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in PCM_n$ *and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ *be a positive weight vector. A directed graph* $G = (V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ *is defined as follows:* $V = \{1, 2, \ldots, n\}$ *and*

$$\overrightarrow{E} = \left\{ arc(i \to j) \left| \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right. \right\}.$$

It follows from Definition 1.4 that if $w_i/w_j = a_{ij}$, then there is a bidirected arc between nodes i, j. The fundamental theorem of Blanquero, Carrizosa and Conde using the directed graph representation above is as follows:

Theorem 1.2 ([4, Corollary 10])**.** *Let* $\mathbf{A} \in PCM_n$*. A weight vector* $\mathbf{w}$ *is efficient for (1) if and only if* $G = (V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ *is a strongly connected digraph, that is, there exist directed paths from* i *to* j *and from* j *to* i *for all pairs of nodes* i, j*.*

Blanquero, Carrizosa and Conde [4, Remark 12] and Conde and Pérez [10, Theorem 2.2] consider weak efficiency as follows:

Definition 1.5. *Weight vector* $\mathbf{w}$ *is called* weakly efficient *for (1) if no positive weight vector* $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ *exists such that*

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| < \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \textit{for all } 1 \leq i \neq j \leq n, \tag{11}$$

and weight vector $\mathbf{w}$ *is called* strongly inefficient *if it is not weakly efficient.*

If weight vector $\mathbf{w}$ is strongly inefficient for (1) and weight vector $\mathbf{w}'$ fulfils (11), we say that $\mathbf{w}'$ *strongly dominates* $\mathbf{w}$. Note that strong dominance is transitive.

Example 1.3. *Let $n \geq 3$ integer and $c, d > 0, c \neq d$ arbitrary. Let $\mathbf{A} \in PCM_n$ be a consistent pairwise comparison matrix defined as $a_{ij} = c^{j-i}$, $i, j = 1, \dots, n$. Let weight vector $\mathbf{w}$ be defined by $w_i = d^{n+1-i}, i = 1, \dots, n$. Then weight vector $\mathbf{w}'$, defined by $w'_i = c^{n+1-i}, i = 1, \dots, n$ provides strictly better approximation to all non-diagonal elements of $\mathbf{A}$ than $\mathbf{w}$ does, therefore $\mathbf{w}$ is strongly inefficient. The example is specified for $n = 4, c = 2, d = 3$ below.*

$$\mathbf{A} = \left[\frac{w'_i}{w'_j}\right]_{i,j=1,\dots,4} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 1/2 & 1 & 2 & 4 \\ 1/4 & 1/2 & 1 & 2 \\ 1/8 & 1/4 & 1/2 & 1 \end{pmatrix}, \qquad \mathbf{w}' = \begin{pmatrix} 8 \\ 4 \\ 2 \\ 1 \end{pmatrix},$$

$$\left[\frac{w_i}{w_j}\right]_{i,j=1,\dots,4} = \begin{pmatrix} 1 & 3 & 9 & 27 \\ 1/3 & 1 & 3 & 9 \\ 1/9 & 1/3 & 1 & 3 \\ 1/27 & 1/9 & 1/3 & 1 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 27 \\ 9 \\ 3 \\ 1 \end{pmatrix}.$$

1.4 Efficiency and distance minimization

Distance minimization does not necessarily induce efficiency. Blanquero, Carrizosa and Conde [4] and Fedrizzi [17] showed that if the metric is componentwise strictly increasing, then efficiency is implied.

Definition 1.6. *([17]) A metric $D : PCM_n \times PCM_n \to \mathbb{R}$ is called* strictly monotonic*, if $\left|a_{ij} - \frac{x_i}{x_j}\right| \leq \left|a_{ij} - \frac{y_i}{y_j}\right|$ for all (i, j) and the inequality is strict for at least one pair of indices (i, j), imply that $D(\mathbf{A}, \left[\frac{x_i}{x_j}\right]) < D(\mathbf{A}, \left[\frac{y_i}{y_j}\right])$.*

Theorem 1.3. *([4, Section 2],[17]) A weight vector induced by a strictly monotonic metric is efficient for (1).*

Theorem 1.3 implies that the least squares method [5, 9, 21, 23] with the objective function $\sum\limits_{i,j} \left|a_{ij} - \frac{w_i}{w_j}\right|^2$ induces efficient weight vector(s). Furthermore, power 2 can be replaced by an arbitrary $p \geq 1$, efficiency is kept.

Blanquero, Carrizosa and Conde [4, Corollary 7] proved that the logarithmic least squares method [11, 12, 13, 26] with the objective function $\sum\limits_{i,j} \left(\log a_{ij} - \log \frac{w_i}{w_j}\right)^2$ yields an efficient solution (the row geometric mean).

The eigenvector method [32] is special, because we have seen in Example 1.1 that the principal right eigenvector can be inefficient. On the other hand, Fichtner [19, 20] showed that the eigenvector method can be written as a distance minimizing problem. Note that Fichtner's metric is neither continuous, nor strictly monotonic.

1.5 Results of the paper

The rest of the paper is organized as follows. Section 2 investigates that the formally different definitions of (weak) efficiency are in fact equivalent. It is shown that the set of (strongly) dominating weight vectors is convex. Weak efficiency of the principal eigenvector is proved in Section 3. A linear program is developed in Section 4 in order to test whether a given weight vector, with respect to a fixed pairwise comparison matrix, is efficient. If it is inefficient, an efficient dominating weight vector is found. Another linear program is constructed in Section 5 for testing weak efficiency. Again, if the weight vector is found to be strongly inefficient, a strongly dominating weight vector is calculated. Linear programs are implemented in Pairwise Comparison Matrix Calculator, available at pcmc.online. Section 6 concludes and raises some open questions.

2 Equivalent definitions of efficiency and weak efficiency

In line with the efficient case, locally and internally weakly efficient points can also be defined in an explicit, problem-specific form.

Let E, E_L and E_I denote the set of the efficient, locally efficient and internally efficient solutions, respectively. Similarly, let WE, WE_L and WE_I denote the set of the weakly efficient, locally weakly efficient and internally weakly efficient solutions, respectively.

According to Definition (1.5),

$$WE = \{\mathbf{w} > \mathbf{0} \mid \text{there exists no } \mathbf{w}' > \mathbf{0} \text{ for which (11) holds}\}.$$

In the same way,

$$\begin{aligned} WE_L = \{\mathbf{w} > \mathbf{0} \mid & \text{ there exists a neighbourhood } U(\mathbf{w}) \text{ such that} \\ & \text{there exists no } \mathbf{w}' \in U(\mathbf{w}) \text{ for which (11) holds}\} \end{aligned}$$

and

$$\begin{aligned} WE_I = \{\mathbf{w} > \mathbf{0} \mid & \text{ there exists no } \mathbf{w}' > \mathbf{0} \text{ such that} \\ & a_{ij} \leq \tfrac{w_i}{w_j} \implies a_{ij} \leq \tfrac{w'_i}{w'_j} < \tfrac{w_i}{w_j} \quad \text{for all } 1 \leq i \neq j \leq n, \\ & a_{ij} \geq \tfrac{w_i}{w_j} \implies a_{ij} \geq \tfrac{w'_i}{w'_j} > \tfrac{w_i}{w_j} \quad \text{for all } 1 \leq i \neq j \leq n\}. \end{aligned}$$

The above relations imply that if for a given $\mathbf{w} > \mathbf{0}$, there exists an index pair $(k, \ell), k \neq \ell$, such that $a_{k\ell} = \frac{w_k}{w_\ell}$, then $\mathbf{w} \in WE$, $\mathbf{w} \in WE_L$ and $\mathbf{w} \in WE_I$.

It is evident that $E \subseteqq WE$, $E_L \subseteqq WE_L$ and $E_I \subseteqq WE_I$. We show below that the relations $E = E_L = E_I$ and $WE = WE_L = WE_I$ also hold. This means that the three definitions given, regarding both the stronger and the weaker cases of efficiency, are equivalent. Example 1.1 demonstrates that $E \subsetneqq WE$.

For a given $\mathbf{w} > \mathbf{0}$, let $D(\mathbf{w})$ denote the set of the points *dominating* the point $\mathbf{w}$, i.e.

$$\begin{aligned} D(\mathbf{w}) = \{\mathbf{x} > \mathbf{0} \mid & f_{ij}(\mathbf{x}) \leq f_{ij}(\mathbf{w}) \text{ for all } i \neq j \text{ and} \\ & f_{k\ell}(\mathbf{x}) < f_{k\ell}(\mathbf{w}) \text{ for some } k \neq \ell\}. \end{aligned}$$

Similarly, let $SD(\mathbf{w})$ denote the set of the points *strongly dominating* the point $\mathbf{w}$, i.e.

$$SD(\mathbf{w}) = \{\mathbf{x} > \mathbf{0} \mid f_{ij}(\mathbf{x}) < f_{ij}(\mathbf{w}) \text{ for all } i \neq j\}.$$

It is easy to see that if $SD(\mathbf{w}) \neq \emptyset$, then $SD(\mathbf{w}) = \text{int}(D(\mathbf{w}))$ and $\text{cl}(SD(\mathbf{w})) = \text{cl}(D(\mathbf{w}))$, where int and cl denote, the interior and closure, respectively, of the relating set.

Proposition 2.1. *$D(\mathbf{w})$ and $SD(\mathbf{w})$ are convex sets, and if any of them is nonempty, then $\mathbf{w}$ lies in its boundary.*

Proof. We start with the proof of the simpler case of $SD(\mathbf{w})$. Clearly,

$$\begin{aligned}\mathbf{x} \in SD(\mathbf{w}) \Longleftrightarrow & \left|\frac{x_i}{x_j} - a_{ij}\right| < f_{ij}(\mathbf{w}) \quad \text{for all } i \neq j \Longleftrightarrow \\ & \frac{x_i}{x_j} - a_{ij} < f_{ij}(\mathbf{w}), \ -\frac{x_i}{x_j} + a_{ij} < f_{ij}(\mathbf{w}) \quad \text{for all } i \neq j \Longleftrightarrow \\ & x_i + (-a_{ij} - f_{ij}(\mathbf{w}))x_j < 0, \ -x_i + (a_{ij} - f_{ij}(\mathbf{w}))x_j < 0, \quad \text{for all } i \neq j.\end{aligned}$$

The set of points fulfilling the last system of strict inequalities is an intersection of finitely many open halfspaces, it is thus an open convex set. At the same time, with $\mathbf{x} = \mathbf{w}$, the linear inequalities above hold as equalities, consequently, $\mathbf{w}$ lies in the boundary of $SD(\mathbf{w})$, of course, if it is nonempty.

By applying similar rearranging steps, we also get that

$$\mathbf{x} \in D(\mathbf{w}) \Longleftrightarrow x_i + (-a_{ij} - f_{ij}(\mathbf{w}))x_j \leq 0, \ -x_i + (a_{ij} - f_{ij}(\mathbf{w}))x_j \leq 0, \quad \text{for all } i \neq j, \text{ and} \tag{12}$$

$$x_k + (-a_{k\ell} - f_{k\ell}(\mathbf{w}))x_\ell < 0, \ -x_k + (a_{k\ell} - f_{k\ell}(\mathbf{w}))x_\ell < 0, \text{ for some } k \neq \ell. \tag{13}$$

We show that $D(\mathbf{w})$ is a convex set. Let $\mathbf{y} \neq \mathbf{z} \in D(\mathbf{w})$, $0 < \lambda < 1$ and $\hat{\mathbf{x}} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$. The linear inequalities of (12) hold at the points $\hat{\mathbf{x}}, \mathbf{y}$ and $\mathbf{z}$.

Let $(\hat{k}, \hat{\ell}), \hat{k} \neq \hat{\ell}$ denote the index pair for which (13) also holds at the point $\mathbf{x} = \mathbf{y}$. Then, with $\mathbf{x} = \hat{\mathbf{x}}$, (13) also holds for the index pair $(\hat{k}, \hat{\ell})$. This implies $\hat{\mathbf{x}} \in D(\mathbf{w})$ and the convexity of $D(\mathbf{w})$.

The point $\mathbf{x} = \mathbf{w}$ fulfils (12) as equalities. Thus, $\mathbf{w} \notin D(\mathbf{w})$ but $\mathbf{w}$ is boundary point of $D(\mathbf{w})$ if it is nonempty. □

Proposition 2.2. *$E = E_L = E_I$ and $WE = WE_L = WE_I$.*

Proof. Obviously,

$E = \{\mathbf{w} > \mathbf{0} \mid D(\mathbf{w}) = \emptyset\}$,
$E_L = \{\mathbf{w} > \mathbf{0} \mid D(\mathbf{w}) \cap U(\mathbf{w}) = \emptyset\}$, where $U(\mathbf{w})$ is a suitably small neighborhood around $\mathbf{w}$, and
$E_I = \{\mathbf{w} > \mathbf{0} \mid D(\mathbf{w}) \cap V_I(\mathbf{w}) = \emptyset\}$, where
$V_I(\mathbf{w}) = \{\mathbf{x} > \mathbf{0} \mid a_{ij} \leq \frac{x_i}{x_j} \leq \frac{w_i}{w_j} \text{ for } a_{ij} \leq \frac{w_i}{w_j}, \forall i \neq j, a_{ij} \geq \frac{x_i}{x_j} \geq \frac{w_i}{w_j} \text{ for } a_{ij} \geq \frac{w_i}{w_j}, \forall i \neq j\}$ is a convex set containing $\mathbf{w}$.

If $\mathbf{w} \in E$, then $D(\mathbf{w}) = \emptyset$, thus $\mathbf{w} \in E_L$ and $\mathbf{w} \in E_I$, therefore, $E \subseteq E_L$ and $E \subseteq E_I$.

We show that $E_L \subseteq E$ holds, as well. Let $\mathbf{w} \in E_L$ and assume that $\mathbf{w} \notin E$, i.e. $D(\mathbf{w}) \neq \emptyset$. Let $\hat{\mathbf{x}} \in D(\mathbf{w})$. Since $\mathbf{w}$ is a boundary point of the convex set $D(\mathbf{w})$, every point of the half-open line segment $[\hat{\mathbf{x}}, \mathbf{w})$ is in $D(\mathbf{w})$. However, the points of $[\hat{\mathbf{x}}, \mathbf{w})$ that are close enough to $\mathbf{w}$ are also in $U(\mathbf{w})$. This contradicts $\mathbf{w} \in E_L$ since $D(\mathbf{w}) \cap U(\mathbf{w}) \neq \emptyset$. Consequently, $\mathbf{w} \in E$, thus, $E_L \subseteq E$, and then $E_L = E$.

The proof of $E_I \subseteq E$ is similar. Let $\mathbf{w} \in E_I$ and assume that $\mathbf{w} \notin E$. Let $\hat{\mathbf{x}} \in D(\mathbf{w})$. Now, if $a_{ij} = \frac{w_i}{w_j}$, then also $a_{ij} = \frac{x_i}{x_j}$ for every $\mathbf{x} \in [\hat{\mathbf{x}}, \mathbf{w}]$. If $a_{ij} < \frac{w_i}{w_j}$, then $a_{ij} < \frac{x_i}{x_j} < \frac{w_i}{w_j}$ for the points $\mathbf{x} \in [\hat{\mathbf{x}}, \mathbf{w}]$ being close enough to $\mathbf{w}$. The same holds in case of $a_{ij} > \frac{w_i}{w_j}$ with opposite sign.

These imply that $[\hat{\mathbf{x}}, \mathbf{w}) \cap D(\mathbf{w}) \cap V_I(\mathbf{w}) \neq \emptyset$, leading again to a contradiction. Consequently, $E_I \subseteq E$, so $E_I = E$.

The proof of the relations $WE = WE_L = WE_I$ can be carried out in the same way, we have simply use the set $SD(\mathbf{w})$ instead of $D(\mathbf{w})$. The remainder part of the proof is left to the reader. □

3 The principal right eigenvector is weakly efficient

Blanquero, Carrizosa and Conde [4, p. 279] stated (without proof) that weak efficiency is equivalent to that the directed graph, according to Definition 1.4, includes at least one cycle. Here we rephrase the proposition and give a proof.

Lemma 3.1. *Let* $\mathbf{A}$ *be an arbitrary pairwise comparison matrix of size* $n \times n$ *and* $\mathbf{w}$ *be an arbitrary positive weight vector. Weight vector* $\mathbf{w}$ *is strongly inefficient for (1) if and only if its digraph is isomorphic to the acyclic tournament on* n *vertices (including arc*(i,j) *if and only if* $i < j$*).*

Proof. Sufficiency. Assume without loss of generality that the rows and columns of pairwise comparison matrix $\mathbf{A}$ are permuted such that digraph G includes $arc(i,j)$ if and only if $i < j$. Then

$$\frac{w_i}{w_j} > a_{ij} \qquad \text{for all } 1 \leq i < j \leq n, \tag{14}$$

and, equivalently, $\frac{w_i}{w_j} < a_{ij}$ for all $1 \leq j < i \leq n$. We shall find a weight vector $\mathbf{w}'$ such that (4), moreover, $\frac{w_i}{w_j} > \frac{w'_i}{w'_j} \geq a_{ij}$ hold for all $1 \leq i < j \leq n$.

Let

$$p_j := \max_{i=1,2,\dots,j-1} \left\{ \frac{a_{ij}}{\frac{w_i}{w_j}} \right\}, \; j = 2, 3, \dots, n. \tag{15}$$

It follows from (14) that $p_j < 1$ for all $2 \leq j \leq n$. Let $w'_k := w_k \cdot \prod_{j=k+1}^{n} p_j$ for all $1 \leq k \leq n-1$, and $w'_n := w_n$. It follows from the construction that

$$\frac{w'_k}{w'_\ell} = \frac{w_k}{w_\ell} \frac{\prod_{j=k+1}^{n} p_j}{\prod_{j=\ell+1}^{n} p_j} = \frac{w_k}{w_\ell} \prod_{j=k+1}^{\ell} p_j < \frac{w_k}{w_\ell}.$$

On the other hand, (15) ensures that $\frac{w'_k}{w'_\ell} \geq a_{k\ell}$. Furthermore, for every $1 \leq k \leq n-1$ there exists a (not necessarily unique) $\ell > k$ such that $\frac{w'_k}{w'_\ell} = a_{k\ell}$. Especially $\frac{w'_1}{w'_2} = a_{12}$.

For necessity let us suppose that digraph G includes a directed 3-cycle (i,j,k): $\frac{w_i}{w_j} > a_{ij}$, $\frac{w_j}{w_k} > a_{jk}$, $\frac{w_k}{w_i} > a_{ki}$. Assume for contradiction that weight vector $\mathbf{w}$ is strongly inefficient, that is, there exists another weight vector $\mathbf{w}'$ such that (4) holds. Then

$$\frac{w_i}{w_j} > \frac{w'_i}{w'_j}, \tag{16}$$

$$\frac{w_j}{w_k} > \frac{w'_j}{w'_k}, \tag{17}$$

$$\frac{w_k}{w_i} > \frac{w'_k}{w'_i}, \tag{18}$$

otherwise none of

$$\left|\frac{w'_i}{w'_j} - a_{ij}\right| < \left|\frac{w_i}{w_j} - a_{ij}\right|, \quad \left|\frac{w'_j}{w'_k} - a_{jk}\right| < \left|\frac{w_j}{w_k} - a_{jk}\right|, \quad \left|\frac{w'_k}{w'_i} - a_{ki}\right| < \left|\frac{w_k}{w_i} - a_{ki}\right|$$

could hold. Multiply inequalities (16)-(18) to get the contradiction $1 > 1$. □

Corollary 2. *Weight vector is strongly inefficient for (1) if and only if the set of outdegrees in the associated directed graph is* $\{0, 1, 2, \ldots, n-1\}$.

Theorem 3.1. *The principal eigenvector of a pairwise comparison matrix is weakly efficient for (1).*

Proof. The principal right eigenvector $\mathbf{w}$ satisfies the equation

$$\mathbf{A}\mathbf{w} = \lambda_{\max}\mathbf{w}. \tag{19}$$

Assume for contradiction that weight vector $\mathbf{w}$ is strongly inefficient. Apply Lemma 3.1 and consider the acyclic tournament associated to $\mathbf{A}$ and $\mathbf{v}$. We can assume without loss of generality that the Hamiltonian path is already $1 \to 2 \to \ldots \to n$. Then

$$\frac{w_i}{w_j} > a_{ij} \text{ for all } 1 \leq i < j \leq n. \tag{20}$$

The i-th equation of (19) is

$$\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i, \tag{21}$$

the left hand side of which is bounded above due to (20):

$$\sum_{j=1}^{n} a_{ij} w_j < \sum_{j=1}^{n} \frac{w_i}{w_j} w_j = n w_i$$

which contradicts $\lambda_{\max} \geq n$. □

4 Efficiency test and search for an efficient dominating weight vector by linear programming

Let a pairwise comparison matrix $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ and a positive weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be given as before. First we shall verify whether $\mathbf{w}$ is efficient for (1) by solving an appropriate linear program. Furthermore, if $\mathbf{w}$ is inefficient, the optimal solution of the linear program provides an efficient weight vector that dominates $\mathbf{w}$ internally.

Recall the double inequality (7) in Definition (1.3). For every positive weight vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$

$$a_{ij} \leq \frac{x_i}{x_j} \underset{(<)}{\leq} \frac{w_i}{w_j} \iff \left(\frac{a_{ij}x_j}{x_i} \leq 1, \; \frac{x_i}{x_j}\frac{w_j}{w_i} \underset{(<)}{\leq} 1 \right) \iff \\ \iff \left(\frac{a_{ij}x_j}{x_i} \leq 1, \; \frac{x_i}{x_j}\frac{w_j}{w_i}\frac{1}{t_{ij}} \leq 1 \text{ for some } 0 < t_{ij} \underset{(<)}{\leq} 1 \right), \tag{22}$$

and

$$a_{ij} \geq \frac{x_i}{x_j} \underset{(>)}{\geq} \frac{w_i}{w_j} \iff \left(\frac{x_i}{a_{ij}x_j} \leq 1, \; \frac{x_j}{x_i}\frac{w_i}{w_j} \underset{(<)}{\leq} 1 \right) \iff \\ \iff \left(\frac{x_i}{a_{ij}x_j} \leq 1, \; \frac{x_j}{x_i}\frac{w_i}{w_j}\frac{1}{t_{ij}} \leq 1 \text{ for some } 0 < t_{ij} \underset{(<)}{\leq} 1 \right), \tag{23}$$

and

$$a_{ij} = \frac{x_i}{x_j} \iff \frac{x_i}{a_{ij}x_j} = 1. \tag{24}$$

This leads us to develop the following optimization problem.

Define index sets

$$I = \left\{ (i,j) \,\middle|\, a_{ij} < \frac{w_i}{w_j} \right\}$$
$$J = \left\{ (i,j) \,\middle|\, a_{ij} = \frac{w_i}{w_j}, i < j \right\}$$

The index set I is empty if and only if pairwise comparison matrix $\mathbf{A}$ is consistent. In this case weight vector $\mathbf{w}$ is efficient and $|J| = n(n-1)/2$. It is assumed in the sequel that I is nonempty. No assumptions are needed for the (non)emptiness of J.

$$\begin{aligned} \min \prod_{(i,j)\in I} & t_{ij} & \\ \frac{x_j}{x_i} a_{ij} &\leq 1 & \text{for all } (i,j) \in I, \\ \frac{x_i}{x_j}\frac{w_j}{w_i}\frac{1}{t_{ij}} &\leq 1 & \text{for all } (i,j) \in I, \\ 0 < t_{ij} &\leq 1 & \text{for all } (i,j) \in I, \\ a_{ji}\frac{x_i}{x_j} &= 1 & \text{for all } (i,j) \in J, \\ x_1 &= 1 & \end{aligned} \tag{25}$$

Variables are $x_i > 0$, $1 \leq i \leq n$ and t_{ij}, $(i,j) \in I$.

Proposition 4.1. *The optimum value of the optimization problem (25) is at most 1 and it is equal to 1 if and only if weight vector* $\mathbf{w}$ *is efficient for (1). Denote the optimal solution to (25) by* $(\mathbf{x}^*, \mathbf{t}^*) \in \mathbb{R}_+^{n+|I|}$. *If weight vector* $\mathbf{w}$ *is inefficient, then weight vector* $\mathbf{x}^*$ *is efficient and dominates* $\mathbf{w}$ *internally.*

Proof. The constraints in (22)-(24) are obtained by simple rearrangements. It is obvious that in (22) $\frac{x_i}{x_j}\frac{w_j}{w_i} \leq 1$ if and only there exists a scalar $0 < t_{ij} \leq 1$ such that $\frac{x_i}{x_j}\frac{w_j}{w_i}\frac{1}{t_{ij}} \leq 1$. In addition, the inequalities hold as strict inequalities simultaneously on both sides. The reasoning is similar for (23), and (24) is evident.

In (25), only the constraints belonging to the index pairs from I and J appear. Due to the reciprocity property, the remainder constraints are now redundant. First, we show that the feasible set of problem (25) is a nonempty compact set. Therefore, since the objective function is continuous, (25) has a finite optimum value and an optimal solution.

Problem (25) has a feasible solution, e.g. $\mathbf{x} = \frac{1}{w_1}\mathbf{w}$ and $t_{ij} = 1$, for all $(i,j) \in I$ fulfill the constraints. Due to the normalization constraint $x_1 = 1$, the other variables $x_i, i \neq 1$, have finite positive lower and upper bounds over the feasible set. This comes from the property that for all $i \neq 1$, either $(i,1)$ or $(1,i)$ is in $I \cup J$. The fourth constraint gives a fixed value for x_i, and positive lower and upper bounds can be computed from the first and second constraints. Since the components of $\mathbf{x}$ have positive upper and lower bound, from the second constraint, positive lower bounds can be computed for the variables $t_{ij}, (i,j) \in I$, too.

The objective function serves for testing the internal efficiency of $\mathbf{w}$. Its value cannot exceed 1. If its value is less than 1, then there exists an index pair (i_0, j_0) for which $\frac{x_{i_0}}{x_{j_0}}\frac{w_{j_0}}{w_{i_0}} \leq t_{i_0 j_0} < 1$, hence $\frac{x_{i_0}}{x_{j_0}} < \frac{w_{i_0}}{w_{j_0}}$. From this and the equivalent forms in (22) and (24), we get that $\mathbf{x}$ internally dominates $\mathbf{w}$. Conversely, assume that $\mathbf{x}$ internally dominates $\mathbf{w}$. It is easy to see that the normalized vector $\mathbf{x}$ with $t_{ij} = \frac{x_i}{x_j}\frac{w_j}{w_i}$, $(i,j) \in I$, is feasible to (25). In addition, for every index pair (i_0, j_0) for which, due to the internal dominance, $\frac{x_{i_0}}{x_{j_0}} < \frac{w_{i_0}}{w_{j_0}}$ holds, we have $t_{i_0 j_0} < 1$, thus, the considered feasible solution has an objective function value less than 1, implying that the optimal value is also less than 1.

It remains to deal with the case when $\mathbf{w}$ turns out to be inefficient. It is obvious that the $\mathbf{x}$-part of the optimal solution $(\mathbf{x}^*, \mathbf{t}^*)$ of (25) internally dominates $\mathbf{w}$, and $t^*_{ij} = \frac{x^*_i}{x^*_j}\frac{w_j}{w_i}$, for all $(i,j) \in I$. Assume that $\mathbf{x}^*$ is inefficient. Then it is internally dominated by a vector $\bar{\mathbf{x}}$, For $\bar{\mathbf{x}}$, we have $a_{ij} = \frac{\bar{x}_i}{\bar{x}_j}$, for all $(i,j) \in J$. Also, $a_{ij} \leq \frac{\bar{x}_i}{\bar{x}_j} \leq \frac{x^*_i}{x_j{}^*} \leq \frac{w_i}{w_j}$, for all $(i,j) \in I$ and there exists at least one index pair $(i_0, j_0) \in I$ for which the second inequality is strict. Let $\bar{t}_{ij} = \frac{\bar{x}_i}{\bar{x}_j}\frac{w_j}{w_i}$, for all $(i,j) \in I$. It is easy to see that after a normalization, $(\bar{\mathbf{x}}, \bar{\mathbf{t}})$ is feasible to (25). However, we also have $\bar{t}_{ij} \leq t^*_{ij}$, for all $(i,j) \in I$ and $\bar{t}_{i_0 j_0} < t^*_{i_0 j_0}$. This implies that the objective function value at $(\bar{\mathbf{x}}, \bar{\mathbf{t}})$ is less than that at $(\mathbf{x}^*, \mathbf{t}^*)$. This contradicts the fact that $(\mathbf{x}^*, \mathbf{t}^*)$ is an optimal solution to (25). Consequently, $\mathbf{x}^*$ is an efficient solution.

□

Optimization problem (25) is nonlinear but it can be transformed to a linear program. Let denote $y_i = \log x_i$, $v_i = \log w_i$, $1 \leq i \leq n$; $s_{ij} = -\log t_{ij}$, $(i,j) \in I$; and $b_{ij} = \log a_{ij}$, $1 \leq i,j \leq n$. Taking the logarithm of the objective function and the constraints in (25), we arrive at an equivalent linear program

$$\min \sum_{(i,j)\in I} -s_{ij} \tag{26}$$

$$y_j - y_i \le -b_{ij} \quad \text{for all } (i,j) \in I, \tag{27}$$

$$y_i - y_j + s_{ij} \le v_i - v_j \quad \text{for all } (i,j) \in I, \tag{28}$$

$$y_i - y_j = b_{ij} \quad \text{for all } (i,j) \in J, \tag{29}$$

$$s_{ij} \ge 0 \quad \text{for all } (i,j) \in I, \tag{30}$$

$$y_1 = 0 \tag{31}$$

Variables are y_i, $1 \le i \le n$ and $s_{ij} \ge 0$, $(i,j) \in I$.

Theorem 4.1. *The optimum value of the linear program (26)-(31) is at most 0 and it is equal to 0 if and only if weight vector* $\mathbf{w}$ *is efficient for (1). Denote the optimal solution to (26)-(31) by* $(\mathbf{y}^*, \mathbf{s}^*) \in \mathbb{R}^{n+|I|}$. *If weight vector* $\mathbf{w}$ *is inefficient, then weight vector* $\exp(\mathbf{y}^*)$ *is efficient and dominates* $\mathbf{w}$ *internally.*

An example is given in the Appendix.

5 Test of weak efficiency and search for an efficient dominating weight vector by linear programming

The test of weak efficiency and searching for a dominating weakly efficient point can be carried out similarly to the case of efficiency. Consider a vector $\mathbf{w} > \mathbf{0}$. Obviously, if $J \neq \emptyset$, i.e. $f_{ij}(\mathbf{w}) = 0$ for an index pair $i \neq j$, then $\mathbf{w} \in WE$, so we are ready with the test of weak efficiency.

Now, examine the case $J = \emptyset$. Then $|I| = n(n-1)/2$. Note that if the rows and columns of pairwise comparison matrix $\mathbf{A}$ are permuted according to Lemma 3.1, then $I = \{(i,j) | 1 \le i < j \le n\}$. Here are some equivalent forms for strong inefficiency. For all $(i,j) \in I$

$$a_{ij} \le \frac{x_i}{x_j} < \frac{w_i}{w_j} \iff \left(\frac{a_{ij} x_j}{x_i} \le 1, \; \frac{x_i}{x_j} \frac{w_j}{w_i} < 1 \right)$$
$$\iff \left(\frac{a_{ij} x_j}{x_i} \le 1, \; \frac{x_i}{x_j} \frac{w_j}{w_i} \frac{1}{t} \le 1, 0 < t < 1 \right). \tag{32}$$

Based on the last form of (32), we can establish a modification of problem (25), adapting it to the case of weak efficiency.

$$\begin{aligned} &\min t \\ &\frac{x_j}{x_i} a_{ij} \le 1 \quad \text{for all } (i,j) \in I, \\ &\frac{x_i}{x_j} \frac{w_j}{w_i} \frac{1}{t} \le 1 \quad \text{for all } (i,j) \in I, \\ &0 < t \le 1 \\ &x_1 = 1. \end{aligned} \tag{33}$$

Variables are $x_i > 0$, $1 \le i \le n$ and t.

Proposition 5.1. *The optimum value of the optimization problem (33) is at most 1 and it is equal to 1 if and only if weight vector* $\mathbf{w}$ *is weakly efficient for (1). Denote the optimal solution to (33) by* $(\mathbf{x}^*, t^*) \in \mathbb{R}_+^{n+1}$*. If weight vector* $\mathbf{w}$ *is strongly inefficient, then weight vector* $\mathbf{x}^*$ *is weakly efficient and dominates* $\mathbf{w}$ *internally and strictly.*

Proof. The statements can be proved by analogy with the proof of Proposition 4.1. By using the same reasoning as there, one can easily show that the feasible set of (33) is not empty, and positive upper and lower bounds can be determined for each variable. Thus (33) has an optimal solution and a positive optimal value $t^* \leq 1$.

If $t^* < 1$, then $\frac{x_i}{x_j}\frac{w_j}{w_i} \leq t^* < 1$ for all $i \neq j$, implying that $\mathbf{x}$ internally strongly dominates $\mathbf{w}$. Conversely, assume that $\mathbf{x}$ internally strongly dominates $\mathbf{w}$. It is easy to see that the normalized vector $\mathbf{x}$ with $t = \max_{i\neq j} \frac{x_i}{x_j}\frac{w_j}{w_i}$ is feasible to (33). It is obvious that $0 < t < 1$ at this feasible solution, implying that $t^* < 1$ at the optimal solution.

Consider the case when $\mathbf{w}$ has turned out to be weakly inefficient, i.e. it is strongly dominated. It is obvious that the $\mathbf{x}$-part of the optimal solution $(\mathbf{x}^*, \mathbf{t}^*)$ of (33) internally dominates $\mathbf{w}$, and $t^* = \max_{i\neq j} \frac{x_i^*}{x_j^*}\frac{w_j}{w_i}$. Assume that $\mathbf{x}^*$ is inefficient. Then it is internally strongly dominated by a vector $\bar{\mathbf{x}}$. For $\bar{\mathbf{x}}$, $a_{ij} \leq \frac{\bar{x}_i}{\bar{x}_j} < \frac{x_i^*}{x_j{}^*} \leq \frac{w_i}{w_j}$, for all $i \neq j$. Let $\bar{t} = \max_{i\neq j} \frac{\bar{x}_i}{\bar{x}_j}\frac{w_j}{w_i}$. It is easy to see that after a normalization, $(\bar{\mathbf{x}}, \bar{\mathbf{t}})$ is feasible to (33). It is however obvious that $\bar{t} < t^*$, implying that the objective function value at $(\bar{\mathbf{x}}, \bar{t})$ is less than that at $(\mathbf{x}^*, t^*)$ contradicting the optimality of the latter one. Consequently, $\mathbf{x}^*$ is a weakly efficient solution. □

By using the same idea that was applied to get problem (26)-(31) from (25), problem (33) can also be transformed to a linear program. By using the same notations as there, and introducing the variable $s = -\log t$, we arrive at an equivalent linear program

$$\begin{aligned} \min -s & \\ y_j - y_i \leq -b_{ij} & \quad \text{for all } (i,j) \in I \\ y_i - y_j + s \leq v_i - v_j & \quad \text{for all } (i,j) \in I \\ s \geq 0 & \\ y_1 = 0 & \end{aligned} \tag{34}$$

Variables are y_i, $1 \leq i \leq n$ and s.

Theorem 5.1. *The optimum value of the linear program (34) is at most 0 and it is equal to 0 if and only if weight vector* $\mathbf{w}$ *is weakly efficient for (1). Denote the optimal solution to (34) by* $(\mathbf{y}^*, s) \in \mathbb{R}^{n+1}$*. If weight vector* $\mathbf{w}$ *is strongly inefficient, then weight vector* $\exp(\mathbf{y}^*)$ *is weakly efficient and dominates* $\mathbf{w}$ *internally and strictly.*

Remark 2. *If weight vector* $\mathbf{w}$ *is strongly inefficient for (1), then weight vector* $\exp(\mathbf{y}^*)$ *in Theorem 5.1 is weakly efficient, but not necessarily efficient. However, linear program (26)-(31) in Section 4 can test its efficiency, and it if is inefficient, (26)-(31) find a dominating efficient weight vector, that obviously dominates (internally and strictly) the strongly inefficient weight vector* $\mathbf{w}$*, too.*

6 Conclusions and open questions

6.1 Conclusions

The key problem of weighting is to approximate the elements of a pairwise comparison matrix, filled in by the decision maker. The multi-objective optimization problem (1) has a unique solution only for consistent pairwise comparison matrices. Numerical examples show that certain weighting methods, such as the eigenvector, result in inefficient (for (1)) solutions. Less formally, the pairwise ratios do not approximate the matrix elements in the *best* possible way, since some of the estimations can be strictly improved without impairing any other one. Nevertheless, the weak efficiency of the principal eigenvector has been proved in Section 3.

Linear programs have been developed in Sections 4 and 5 in order to test efficiency, and to find an efficient dominating weight vector.

6.2 Open questions

Efficiency for (1) is a potential criterion in future comparative studies of the weighting methods. Our opinion is that an inefficient weight vector is less preferred to any of its dominating weight vectors, and, especially to the efficient dominating weight vector(s).

The use of models developed in Sections 4 and 5 enables the decision maker to improve a possibly inefficient weight vector, however, the problem of generating the whole set of efficient dominating weight vectors is open.

An extended analysis of numerical examples could show how often inefficiency occurs and how large differences there are between an inefficient and an efficient dominating weight vector.

The efficiency analysis of the principal eigenvector is still incomplete. Sufficient conditions are discussed in [1, 2]: if the pairwise comparison matrix can be made consistent by a modification of one or two elements (and their reciprocal), then the eigenvector is efficient for (1). It is shown in [6] that the eigenvector can be inefficient even if the level of inconsistency (as proposed by Saaty [32], a positive linear transformation of the maximal eigenvalue) is arbitrarily small. However, the necessary and sufficient condition for the efficiency of the principal eigenvector is a challenging open problem.

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Appendix

Consider pairwise comparison matrix $\mathbf{A} \in PCM_4$ and its principal right eigenvector $\mathbf{w}$ in Example 1.1, and the consistent pairwise comparison matrix $\left[\frac{w_i}{w_j}\right]$ as in (6).

The linear program (26)-(31) is specified below. $I = \{(2,1),(2,4),(3,1),(3,2),(4,1),(4,3)\}$ and $J = \emptyset$.

y_1	y_2	y_3	y_4	s_{21}	s_{24}	s_{31}	s_{32}	s_{41}	s_{43}			remark
0	0	0	0	-1	-1	-1	-1	-1	-1	$\to$	min	(26)
1	-1	0	0	0	0	0	0	0	0	$\leq$	0	(27), $i=2$, $j=1$
-1	1	0	0	1	0	0	0	0	0	$\leq$	0.0753	(28), $i=2$, $j=1$
0	-1	0	1	0	0	0	0	0	0	$\leq$	-1.6094	(27), $i=2$, $j=4$
0	1	0	-1	0	1	0	0	0	0	$\leq$	2.1859	(28), $i=2$, $j=4$
1	0	-1	0	0	0	0	0	0	0	$\leq$	1.3863	(27), $i=3$, $j=1$
-1	0	1	0	0	0	1	0	0	0	$\leq$	-1.2995	(28), $i=3$, $j=1$
0	1	-1	0	0	0	0	0	0	0	$\leq$	1.9459	(27), $i=3$, $j=2$
0	-1	1	0	0	0	0	1	0	0	$\leq$	-1.3749	(28), $i=3$, $j=2$
1	0	0	-1	0	0	0	0	0	0	$\leq$	2.1972	(27), $i=4$, $j=1$
-1	0	0	1	0	0	0	0	1	0	$\leq$	-2.1106	(28), $i=4$, $j=1$
0	0	1	-1	0	0	0	0	0	0	$\leq$	1.3863	(27), $i=4$, $j=3$
0	0	-1	1	0	0	0	0	0	1	$\leq$	-0.8111	(28), $i=4$, $j=3$
0	0	0	0	1	0	0	0	0	0	$\geq$	0	(30), $i=2$, $j=1$
0	0	0	0	0	1	0	0	0	0	$\geq$	0	(30), $i=2$, $j=4$
0	0	0	0	0	0	1	0	0	0	$\geq$	0	(30), $i=3$, $j=1$
0	0	0	0	0	0	0	1	0	0	$\geq$	0	(30), $i=3$, $j=2$
0	0	0	0	0	0	0	0	1	0	$\geq$	0	(30), $i=4$, $j=1$
0	0	0	0	1	0	0	0	0	1	$\geq$	0	(30), $i=4$, $j=3$
1	0	0	0	0	0	0	0	0	0	$=$	0	(31)

The first two inequalities belong to $i = 2, j = 1$. Right hand sides are calculated as $-b_{21} = -\log(a_{21}) = -\log 1 = 0$ in (27) and $v_2 - v_1 = \log(w_2) - \log(w_1) \approx 0.0753$ in (28).

The optimal solution is $y_1^* = y_2^* = 0, y_3^* = -1.3749, y_4^* = -2.1859, s_{21}^* = s_{31}^* = s_{41}^* = 0.0753, s_{24}^* = s_{32}^* = s_{43}^* = 0$, and the optimum value is $-s_{21}^* - s_{31}^* - s_{41}^* = -0.226$. In order to make the weight vector $\mathbf{x}^* = \exp(\mathbf{y}^*) = (1, 1, 0.2529, 0.1124)$ comparable to $\mathbf{w}$, renormalize it by a multiplication by $w_2/x_2^* = w_3/x_3^* = w_4/x_4^* = 0.436173$ to get

$$\mathbf{w}^* = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

that differs from $\mathbf{w}$ in its first coordinate only. Furthermore, $\mathbf{w}^*$ equals to weight vector $\mathbf{w}''$ in Example 1.2. One can check from the consistent approximation generated by weight vector $\mathbf{w}^*$,

$$\left[\frac{w_i^*}{w_j^*}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix} \qquad \left(= \left[\frac{w_i''}{w_j''}\right] \text{ as in (9)} \right).$$

that weight vector $\mathbf{w}^*$ internally dominates $\mathbf{w}$, moreover, Theorem 4.1 guarantees that weight vector $\mathbf{x}^*$ is efficient, so is weight vector $\mathbf{w}^*$.