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The aim of this paper is to present a representation theorem of the Sugeno integral with minimal requirements on the domain of the functional. Using Stone condition it parallels earlier derivations of Choquet integral performed by Greco (Proc. Rend. Sem. Mat. Univ. 66 (1982) 21) and Denneberg (Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994).

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1. Introduction

The main purpose of decision making theory consists in ranking in an evaluation scale, E, mappings X from some non-empty set Ω to the evaluation scale E. In the framework of multicriteria theory X is usually called an object, Ω is the set of criteria and $X(w)$ denotes the partial score of X with respect to criterium w, whereas in the framework of decision under uncertainty, X is usually called an act, Ω is the set of states of nature, and $X(w)$ denotes the partial score of X if state of nature w occurs.

While Choquet integral is perfectly suitable for ranking real-valued mappings X through a global real-valued score $\Gamma(X)$, Sugeno integral appears to be a natural substitute to the Choquet one in a purely ordinal setting, where E is merely an ordered set, even if coded in [0, 1] for convenience.

Given a family $\mathcal{F}$ of $\mathcal{a}$-measurable mappings X from Ω to $E = [0, 1]$ (where $\mathcal{a}$ is a σ-algebra of subsets of Ω), and a (global score) function $\Gamma : \mathcal{F} \to [0, 1]$, the aim of this paper is to derive minimal richness

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requirements on $\mathcal{F}$, and minimal conditions on Γ, in order that the functional Γ can be represented as a Sugeno integral with respect to some capacity defined on $\mathcal{A}$. Our results parallel a similar derivation obtained earlier by Greco [5] and Denneberg [3] for the Choquet integral when $\mathcal{F}$ satisfies Stone condition.

The central result (Theorem 1) shows that the domain $\mathcal{F}$ of Γ can be chosen as a proper subset of V the set of all $\mathcal{A}$-measurable mappings from Ω to $E = [0, 1]$, hence differ from previous derivations of the Sugeno integral buiding upon the full domain V as those obtained in the finite case and the infinite case by Benvenuti and Mesiar [1], de Campos et al. [2], Dubois et al. [4] or else Ralescu and Sugeno [7].

Our work is an attempt to take into account the fact that in decision situations, one often has only restricted information, i.e the domain $\mathcal{F}$ of the functional may be small; as a counterpart unlike the full information case i.e. full domain V, our Sugeno integral representation allows for a set of agreeing capacities not necessarily reduced to a singleton.

The paper is organized as follows. Section 2 presents the basic notions and some preparatory results on Sugeno integral. In Section 3, we state and prove our main result of Sugeno integral representation. Section 4 is devoted to some concluding remarks.

2. Framework, notations and preparatory results

To begin with, we introduce the notations used in the following:

- Ω a non-empty set, $\mathcal{A}$ a σ-algebra of subsets of Ω.
- $V = \{X : \Omega \to [0, 1],\ X\ \mathcal{A}\text{-measurable}\}$.
- For every subset A of $\mathcal{A}$ we denote A^* the characteristic function of A.

Definition 1. A (*normalized*) *capacity* is a set-function $v : \mathcal{A} \to [0, 1]$ such that $v(\emptyset) = 0$, $v(\Omega) = 1$, $A \subseteq B \Rightarrow v(A) \leqslant v(B)$.

Now we introduce Sugeno integral [8]:

Definition 2. For all X in V and any capacity v on $\mathcal{A}$, *the Sugeno integral* of X with respect to v is defined by

$$⨍ X \,\mathrm{d}v = \bigvee_{0 \leqslant \alpha \leqslant 1} (\alpha \wedge v(X_\alpha)), \quad \text{where } X_\alpha = \{w \in \Omega \mid X(w) \geqslant \alpha\}.$$

Note that, for the sake of convenience, we write $v(X_\alpha)$ instead of $v(X \geqslant \alpha)$.

If we consider a finite step function X in V with values $x_i, i \in I = \{1, \ldots, n\}, n \in \mathbb{N}^*$, then the expression of the Sugeno integral can be reduced as follows:

$$⨍ X \,\mathrm{d}v = \bigvee_{i \in I} (x_i \wedge v(X_{x_i})).$$

Two functions $f, g : \Omega \to [0, 1]$ are *comonotonic* if

$$\forall s,\ t \in \Omega, (f(s) - f(t))(g(s) - g(t)) \geqslant 0.$$

For the main result of this article, we need two preliminary results.

Lemma 1. *Let $X \in V$ and $a > 0$ such that $X + a \in V$, then we have*

$$\fint (X + a)\, \mathrm{d}v \leqslant \fint X\, \mathrm{d}v + a.$$

Proof. Let us first consider a finite step function X in V.

For $i \in \{1, \ldots, n\}$, we name x_i the different values of X, then we have

$$\begin{aligned}
&(x_i + a) \wedge v(\{X + a \geqslant x_i + a\}) \leqslant x_i \wedge v(\{X + a \geqslant x_i + a\}) + a \\
&\quad \Rightarrow (x_i + a) \wedge v(\{X + a \geqslant x_i + a\}) \leqslant x_i \wedge v(\{X \geqslant x_i\}) + a \\
&\quad \Rightarrow (x_i + a) \wedge v(\{X + a \geqslant x_i + a\}) \leqslant \fint X\, \mathrm{d}v + a \\
&\quad \Rightarrow \fint (X + a)\, \mathrm{d}v \leqslant \fint X\, \mathrm{d}v + a.
\end{aligned}$$

Take now a general X in V. X is the uniform limit of the standard non-decreasing sequence of finite step functions X_n in V, where

$$X_n = \sum_{i=0}^{2^n - 1} \frac{i}{2^n} \left\{ w \in \Omega,\ \frac{i}{2^n} \leqslant X(w) < \frac{i+1}{2^n} \right\}^* .$$

So, $\exists N \in \mathbb{N}$ such that $\forall n \geqslant N$, $X_n \leqslant X \leqslant X_n + \frac{1}{2^n}$, hence $\forall n \geqslant N$, $X_n + a \leqslant X + a \leqslant X_n + a + \frac{1}{2^n}$. $X + a \in V \Rightarrow X + a \leqslant 1$, therefore $X + a \leqslant (X_n + \frac{1}{2^n} + a) \wedge 1$, or equally: $X + a \leqslant (X_n + \frac{1}{2^n}) \wedge (1 - a) + a$.

Monotonicity of the Sugeno integral and the property previously proved for finite step functions give

$$\fint (X + a)\, \mathrm{d}v \leqslant \fint \left(\left(X_n + \frac{1}{2^n} \right) \wedge (1 - a) \right) \mathrm{d}v + a.$$

But for the Sugeno integral $\fint (b \wedge Y)\, \mathrm{d}v = b \wedge \fint Y\, \mathrm{d}v,\ \forall b \in [0, 1],\ \forall Y \in V$, hence

$$\fint (X + a)\, \mathrm{d}v \leqslant (1 - a) \wedge \fint \left(X_n + \frac{1}{2^n} \right) \mathrm{d}v + a,$$

$$\fint (X + a)\, \mathrm{d}v \leqslant \fint \left(X_n + \frac{1}{2^n} \right) \mathrm{d}v + a.$$

Since $X_n + 1/2^n$ is a finite step function in V and $X_n \leqslant X$ one obtains

$$\fint (X + a)\, \mathrm{d}v \leqslant \fint X\, \mathrm{d}v + \frac{1}{2^n} + a, \quad \forall n \geqslant N$$

and therefore $\fint (X + a)\, \mathrm{d}v \leqslant \fint X\, \mathrm{d}v + a$. $\square$

Lemma 2. *Let $X, Y \in V$ and $A \in \mathcal{A}$ be such that $X \leqslant A^* \leqslant Y$, then X and Y are comonotonic.*

Proof. Let us prove that $t, t' \in \Omega,\ X(t) < X(t') \Rightarrow Y(t) \leqslant Y(t')$:
- $t, t' \in A$ clearly implies $Y(t) = Y(t') = 1$.
- $t \in A$ and $t' \notin A$, implies $X(t') = 0$ which is obviously impossible.
- $t \notin A$, and $t' \in A$ implies $Y(t') = 1$ and the inequality $Y(t) \leqslant Y(t')$ is true.
- $t, t' \notin A$, implies $X(t) = X(t') = 0$ which is impossible. $\square$

3. The main result

Let $\mathcal{F}$ be a subset of V such that:

Properties of $\mathcal{F}$. A1. $a \in [0, 1],\ X \in \mathcal{F} \Rightarrow X \wedge a,\ X - X \wedge a \in \mathcal{F}$,
A2. $X, Y \in \mathcal{F}$ comonotonic $\Rightarrow X \vee Y \in \mathcal{F}$,
A3. $[a \in \mathbb{R}^+, X \in \mathcal{F}$ such that $aX \in V] \Rightarrow aX \in \mathcal{F}$,
A4. $\Omega^* \in \mathcal{F}$

and let $\Gamma : \mathcal{F} \to [0, 1]$ be a functional such that:

Properties of Γ. B1. $\Gamma(\Omega^*) = 1$,
B2. $\Gamma(a \wedge X) = a \wedge \Gamma(X),\ \forall a \in [0, 1],\ \forall X \in \mathcal{F}$,
B3. If $X, Y \in \mathcal{F}$ are comonotonic, then $\Gamma(X \vee Y) = \Gamma(X) \vee \Gamma(Y)$,
B4. $\lim_{a \searrow 0} \Gamma(X - X \wedge a) = \Gamma(X)$ for $X \in \mathcal{F}$.

Note that condition $X \wedge a \in \mathcal{F}$ for $X \in \mathcal{F}$ and $a \in [0, 1]$, of property A.1, is usually called Stone condition.

Under these hypotheses we can show a result similar to the one obtained by Greco [5] and Denneberg [3] for the Choquet integral.

Theorem 1. *Let $\mathcal{F}$ be a subset of V and $\Gamma : \mathcal{F} \to [0, 1]$ be a functional which satisfy the above properties*: A1–A4 *and* B1–B4.

Then the set-functions α and β on $(\Omega, \mathcal{A})$ defined by
- $\forall A \in \mathcal{A},\ \alpha(A) = \sup\{\Gamma(X) | X \in \mathcal{F},\ X \leqslant A^*\}$,
- $\forall A \in \mathcal{A},\ \beta(A) = \inf\{\Gamma(X) | X \in \mathcal{F},\ X \geqslant A^*\}$

are capacities and for all $X \in \mathcal{F}$, for all capacities v on $(\Omega, \mathcal{A})$ such that $\alpha \leqslant v \leqslant \beta$, one obtains $\Gamma(X) = -X\, \mathrm{d}v$.

Remark 1. It is worth noticing that when comparing our results with those obtained by Greco [5] and Denneberg [3] for the Choquet integral, the main difference consists in the fact that on one hand for the Sugeno integral the explicit monotonicity requirement of the functional Γ can be dispensed with, but on the other hand we have enforced the structure of $\mathcal{F}$ by requiring lattice property A2.

Proof. *First we prove that α and β are two capacities such that $\alpha \leqslant \beta$*:

Clearly, α and β are well-defined.

- *α is a capacity*: $\alpha(\emptyset) = 0$. Indeed, the null function is in $\mathcal{F}$ because for all a in the interval [0, 1] and for all X in $\mathcal{F}$ we have $a \wedge X$in $\mathcal{F}$. We just have to write this property for $a = 0$.

  According to A4 and B1, $\alpha(\Omega) = 1$.

  Taking $A \subseteq B$, we have $A^* \leqslant B^*$ which implies $\{X \in \mathcal{F},\ X \leqslant A^*\} \subseteq \{X \in \mathcal{F},\ X \leqslant B^*\}$, then clearly $\alpha(A) \leqslant \alpha(B)$.
- *β is a capacity*: The proof is the same as for α.
- $\alpha \leqslant \beta$: Let A be given belonging to $\mathcal{A}$. For all $X, Y \in \mathcal{F}$ such that $X \leqslant A^* \leqslant Y$ the above Lemma 2 entails that X and Y are comonotonic. Hence property B3 implies $\Gamma(X) \leqslant \Gamma(Y)$, and therefore $\alpha \leqslant \beta$.

This allows us to consider capacities v on $(\Omega, \mathcal{A})$ such that $\alpha \leqslant v \leqslant \beta$.

Now we are going to show $\Gamma(X) \geqslant ⨍ X \,\mathrm{d}\beta$ *for all* $X \in \mathcal{F}$. Let X be in $\mathcal{F}$. For $i \in \{0, \ldots, 2^n - 1\}$ we define $T_i = X \wedge \frac{i+1}{2^n} - X \wedge \frac{i}{2^n}$. It is easy to check that $2^n T_i \geqslant A^*_{i+1}$ where $A_{i+1} = \{X \geqslant \frac{i+1}{2^n}\}$. Remark that $T_i = X \wedge \frac{i+1}{2^n} - X \wedge \frac{i+1}{2^n} \wedge \frac{i}{2^n}$ which implies, with property A1, that T_i belongs to $\mathcal{F}$. It is easy to show that if $X \leqslant \frac{i}{2^n}$, $T_i = 0$ and if $X \geqslant \frac{i}{2^n}$, $T_i \leqslant \frac{i+1}{2^n} - \frac{i}{2^n} = \frac{1}{2^n}$. Thus we have $2^n T_i \in V$ and according to property A3 we know that $2^n T_i \in \mathcal{F}$.

The definition of β implies that $\beta(A_{i+1}) \leqslant \Gamma(2^n T_i)$. Hence, $\frac{i}{2^n} \wedge \beta(X \geqslant \frac{i+1}{2^n}) \leqslant \frac{i}{2^n} \wedge \Gamma(2^n T_i)$. Then from property B2 we have

$$\Gamma\left(\frac{i}{2^n} \wedge 2^n T_i\right) \geqslant \frac{i}{2^n} \wedge \beta\left(X \geqslant \frac{i+1}{2^n}\right). \tag{1}$$

In other respects, $\frac{i}{2^n} \wedge 2^n T_i \leqslant \frac{i}{2^n}.A^*_i \leqslant X$ with $\frac{i}{2^n} \wedge 2^n T_i$ and X comonotonic.

So property B3 implies $\Gamma(X) \geqslant \Gamma(\frac{i}{2^n} \wedge 2^n T_i)$. This inequality associated with (1) gives $\Gamma(X) \geqslant \frac{i}{2^n} \wedge \beta(X \geqslant \frac{i+1}{2^n})$ for all $i \in \{0, \ldots, 2^n - 1\}$. Then

$$\Gamma(X) \geqslant \bigvee_{i=0}^{2^n-1} \left(\frac{i}{2^n} \wedge \beta\left(X \geqslant \frac{i+1}{2^n}\right)\right) = ⨍ X'_n \,\mathrm{d}\beta, \quad \text{where } X'_n = \bigvee_{i=0}^{2^n-1} \frac{i}{2^n}.\left\{X \geqslant \frac{i+1}{2^n}\right\}^*.$$

Let X_n be

$$\bigvee_{i=0}^{2^n-1} \frac{i+1}{2^n}.\left\{X \geqslant \frac{i+1}{2^n}\right\}^*,$$

we have $X_n \leqslant X'_n + \frac{1}{2^n}$ hence according to the monotonicity of the Sugeno integral and to Lemma 1 we get

$$⨍ X_n \,\mathrm{d}\beta \leqslant ⨍ \left(X'_n + \frac{1}{2^n}\right) \mathrm{d}\beta \leqslant ⨍ X'_n \,\mathrm{d}\beta + \frac{1}{2^n}.$$

In summary $\Gamma(X) \geqslant ⨍ X_n \,\mathrm{d}\beta - \frac{1}{2^n}$. $\{X_n\}_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite step functions in V such that $X_n + \frac{1}{2^n}$ belong to V and which uniformly approximates X from below, hence from Lemma 1 one easily obtains $\lim_{n\to\infty} ⨍ X_n \,\mathrm{d}\beta = ⨍ X \,\mathrm{d}\beta$. Therefore we get $\Gamma(X) \geqslant ⨍ X \,\mathrm{d}\beta$.

Let us prove now $\Gamma(X) \leqslant \fint X \, \mathrm{d}\alpha$ *for all* X *in* $\mathcal{F}$. Let X be in $\mathcal{F}$ and i be in $\{0, \ldots, 2^n - 1\}$. It is easy to check that $2^n T_i \leqslant \{X \geqslant \frac{i}{2^n}\}^*$. The property $2^n T_i \in \mathcal{F}$, and the definition of α imply that $\Gamma(2^n T_i) \leqslant \alpha(X \geqslant \frac{i}{2^n})$. According to Property B2 and the definition of the Sugeno integral we have $\Gamma(\frac{i}{2^n} \wedge 2^n T_i) \leqslant \frac{i}{2^n} \wedge \alpha(X \geqslant \frac{i}{2^n})$ $\leqslant -X \, \mathrm{d}\alpha$. These inequalities are true for all i in $\{0, \ldots, 2^n - 1\}$ so we can write

$$\bigvee_{i=0}^{2^n-1} \Gamma\left(\frac{i}{2^n} \wedge 2^n T_i\right) \leqslant \fint X \, \mathrm{d}\alpha.$$

For all j in $\{0, \ldots, 2^n - 2\}$, $\frac{j}{2^n} \wedge 2^n T_j$ and $\bigvee_{i=j+1}^{2^n-1} \frac{i}{2^n} \wedge 2^n T_i$ are comonotonic.

Hence Property B3 leads to $\Gamma\left(\bigvee_{i=0}^{2^n-1} \frac{i}{2^n} \wedge 2^n T_i\right) \leqslant -X \, \mathrm{d}\alpha$.

But $X - X \wedge \frac{2}{2^n} \leqslant \bigvee_{i=0}^{2^n-1} \frac{i}{2^n} \wedge 2^n T_i$, $X - X \wedge \frac{2}{2^n}$ belongs to $\mathcal{F}$, moreover $X - X \wedge \frac{2}{2^n}$ and $\bigvee_{i=0}^{2^n-1} \frac{i}{2^n} \wedge 2^n T_i$ are comonotonic. Therefore Property B3 implies $\Gamma(X - X \wedge \frac{2}{2^n}) \leqslant \fint X \, \mathrm{d}\alpha$. This inequality is true for all n so $\lim_{n\to\infty} \Gamma(X - X \wedge \frac{2}{2^n}) \leqslant \fint X \, \mathrm{d}\alpha$ and according to Property B4 we obtain $\Gamma(X) \leqslant \fint X \, \mathrm{d}\alpha$.

Finally, for each capacity v such that $\alpha \leqslant v \leqslant \beta$ we have $\Gamma(X) \leqslant \fint X \, \mathrm{d}\alpha \leqslant \fint X \, \mathrm{d}v \leqslant \fint X \, \mathrm{d}\beta \leqslant \Gamma(X)$ which implies $\Gamma(X) = \fint X \, \mathrm{d}v$, and completes the proof. □

4. Concluding remarks

Let us mention that properties B1–B4 under which we derived our Sugeno integral representation for the functional Γ can be easily proved to be necessary conditions. So in fact, this paper offers a characterization of the Sugeno integral with respect to a set of possible capacities, when the domain of the functional Γ is restricted to a subset $\mathcal{F}$ of V satisfying properties A1–A4. Obviously, if A^* belongs to $\mathcal{F}$ for any $A \in \mathcal{A}$, the set of possible capacities shrinks into a singleton through the requirement $v(A) = \Gamma(A^*)$.

To end, note that as suggested by a referee, it would be interesting to know whether B3 could be replaced by the weaker max-homogeneity version $B3^*$: $\Gamma(a \vee X) = a \vee \Gamma(X),\ \forall a \in [0, 1],\ \forall X \in \mathcal{F}$. In the finite case (see [6]) as well as, in the general case (see [1]), this proves to be possible when the domain $\mathcal{F}$ of the functional Γ is domain V. It is a challenging question to know whether this remains true in our framework when $\mathcal{F}$ is a proper subset of V.

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References

[1] P. Benvenuti, R. Mesiar, A note on Sugeno and Choquet integrals, in: Information Processing and Management of Uncertainty in Knowledge Bases Systems (IPMU), Madrid, Spain, July 3–7 2000, pp. 582–585.

[2] L. de Campos, M. Lamata, S. Moral, A unified approach to define fuzzy integrals, Fuzzy Sets ans Systems 39 (1991) 75–90.

[3] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.

[4] D. Dubois, J.-L. Marichal, H. Prade, M. Roubens, R. Sabbadin, The use of the discrete Sugeno integral in decision making, Internat. J. Uncertainty Fuzziness Knowledge-Based Systems 9 (2001) 539–561.
[5] G. Greco, Sulla rappresentazione di funzionali mediante integrali, Proc. Rend. Sem. Mat. Univ. Padova 66 (1982) 21–46.
[6] J.L. Marichal, On Sugeno integral as an aggregation function, Fuzzy Sets and Systems 114 (2000) 347–365.
[7] D.A. Ralescu, M. Sugeno, Fuzzy integral representation, Fuzzy Sets and Systems 84 (1996) 127–133.
[8] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institute of Technology, 1974.