STATE-MORPHISM ALGEBRAS - GENERAL APPROACH

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ABSTRACT. We present a complete description of subdirectly irreducible state BL-algebras as well as of subdirectly irreducible state-morphism BL-algebras. In addition, we present a general theory of state-morphism algebras, that is, algebras of general type with state-morphism which is an idempotent endomorphism. We define a diagonal state-morphism algebra and we show that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one. We describe generators of varieties of state-morphism algebras, in particular ones of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MValgebras.

1. INTRODUCTION

A state, as an analogue of a probability measure, is a basic notion of the theory of quantum structures, see e.g. $[14]$. However, for MV-algebras, the state as averaging the truth value in the Lukasiewicz logic was introduced firstly by Mundici in [\[22\]](#page-15-1), 40 years after introducing MV-algebras, [\[6\]](#page-15-2). We recall that a state on an MV-algebra M is a mapping $s : M \to [0, 1]$ such that (i) $s(a \oplus b) = s(a) + s(b)$, if $a \odot b = 0$, and (ii) $s(1) = 1$. The property (i) says that s is additive on mutually excluding events a and b . It is important note that every non-degenerate MV-algebra admits at least one state. The set of states is a convex set, which in the weak topology of states is a compact Hausdorff set, and every extremal state is in fact an MValgebra homomorphism from \bf{M} into the MV-algebra of the real interval [0, 1], and vice-versa, [\[22\]](#page-15-1). In addition, extremal states generate the set of all states because by the Krein-Mil'man Theorem, [\[18,](#page-15-3) Thm 5.17], every state is a weak limit of a net of convex combinations of these special homomorphisms.

In the last decade, the states entered into theory of MV-algebras in a very ambitious manner. In [\[23,](#page-15-4) [21\]](#page-15-5), authors have showed a relation between states

¹Keywords: State MV-algebra, state BL-algebra, state-morphism algebra, Congruence Extension Property, generator, diagonal subalgebra, t-norm, non-associative t-norm, MTL-algebra, non-associative BL-algebra, pseudo MV-algebra.

AMS classification: 06D35, 03G12, 03B50,

MB thanks for the support by SAIA, Slovakia, by MSM 6198959214 of the RDC of the Czech Government, and by GAČR P201/11/P346, Czech Republic, AD thanks for the support by Center of Excellence SAS - Quantum Technologies -, ERDF OP R&D Projects CE QUTE ITMS 26240120009 and meta-QUTE ITMS 26240120022, the grant VEGA No. 2/0032/09 SAV.

and de Finetti's approach to probability in terms of bets. In addition, Panti and independently Kroupa in [\[24,](#page-15-6) [20\]](#page-15-7) have showed that every state on M is an integral through a unique regular Borel probability measure concentrated on the set of extremal states on M.

Nevertheless as we have seen states are not a proper notion of universal algebra, and therefore, they do not provide an algebraizable logic for probabilistic reasoning of the many-valued approach.

Recently, Flaminio and Montagna in [\[16\]](#page-15-8) presented an algebraizable logic containing probabilistic reasoning, and its equivalent algebraic semantic is the variety of state MV-algebras. We recall that a *state MV-algebra* is an MV-algebra whose language is extended adding an operator, τ (called also an *internal state*), whose properties are inspired by the ones of states. The analogues of extremal states are *state-morphism operators*, introduced in [\[7\]](#page-15-9). By definition, it is an idempotent endomorphism on an MV-algebra.

State MV-algebras generalize, for example, Hájek's approach, [\[19\]](#page-15-10), to fuzzy logic with modality Pr (interpreted as *probably*) which has the following semantic interpretation: The probability of an event a is presented as the truth value of $Pr(a)$. On the other hand, if s is a state, then $s(a)$ is interpreted as averaging of appearing the many valued event a.

We note that if (M, τ) is a state MV-algebra, assuming that that the range $\tau(M)$ is simple, we see that it is a subalgebra of the real interval [0, 1] and therefore, τ can be regarded as a standard state on M. On the other hand, every MV-algebra **M** can be embedded into the tensor product $[0, 1] \otimes M$, therefore, given a state s on **M**, we define an operator τ_s on $[0,1] \otimes \mathbf{M}$ via $\tau_s(t \otimes a) := t \cdot s(a)$, [\[16,](#page-15-8) Thm 5.3]. Then due to [\[7,](#page-15-9) Thm 3.2], τ_s is a state-operator that is a state-morphism operator iff s is an extremal state. Thus, there is a natural correspondence between the notion of a state and an extremal state on one side, and a state-operator and a state-morphism operator on the other side.

Subdirectly irreducible state-morphism MV-algebras were described in [\[7,](#page-15-9) [9\]](#page-15-11) and this was extended also for state-morphism BL-algebras in [\[11\]](#page-15-12). A complete description of both subdirectly irreducible state MV-algebras as well as subdirectly irreducible state-morphism MV-algebras can be found in [\[13\]](#page-15-13). In [\[8\]](#page-15-14), it was shown that if (M, τ) is a state MV-algebra whose image $\tau(M)$ belongs to the variety generated by the L_1, \ldots, L_n , where $L_i := \{0, 1/i, \ldots, i/i\}$, then τ has to be a state-morphism operator. The same is true if M is linearly ordered, [\[7\]](#page-15-9). Recently, in [\[13\]](#page-15-13), we have shown that the unit square $[0, 1]^2$ with the diagonal operator generates the whole variety of state-morphism MV-algebras; it answered in positive an open problem posed in [\[7\]](#page-15-9). In addition, there was shown that in contrast to MV-algebras, the lattice of subvarieties is uncountable. Moreover, it was shown that every subdirectly irreducible state-morphism MV-algebra can be embedded into some diagonal one.

In this paper, we continue in the study of state BL-algebras and state-morphism BL-algebras. Because the methods developed in [\[13\]](#page-15-13) are so general that, it is possible to study more general structures than MV-algebras or BL-algebras under a common umbrella. Hence, we introduce state-morphism algebras (A, τ) , where the algebra **A** is an arbitrary algebra of type F and τ is an idempotent endomorphism of A. Then general results applied to special types of algebras give interesting new results.

The main goals of the paper are:

(1) Complete characterizations of subdirectly irreducible state BL-algebras and state-morphism BL-algebras.

(2) Showing that every subdirectly state-morphism algebra can be embedded into some diagonal one $D(\mathbf{B}) := (\mathbf{B} \times \mathbf{B}, \tau_B)$, where $\tau(a, b) = (a, a), a, b \in B$, which is also subdirectly irreducible.

(3) We show that if K is a generator of some variety V of algebras of type F, then the system of diagonal state-morphism algebras $\{D(\mathbf{B}) : \mathbf{B} \in \mathcal{K}\}\$ is a generator of the variety of state-morphism algebras whose F -reduct belongs to V .

(4) We exhibit cases when the Congruence Extension Property holds for a variety of state-morphism algebras.

(5) In particular, a generator of the variety of state-morphism BL-algebras is the class of all BL-algebras of the real interval $[0, 1]$ equipped with a continuous t-norm. Similarly, a generator of the variety of state-morphism MTL-algebras is the class of all MTL-algebras of the real interval equipped with a left-continuous tnorm, similarly for non-associative BL-algebras one is the set of all non-associative BL-algebras of the real interval [0, 1] equipped with a non-associative t-norm, and a generator of the variety of state-morphism pseudo MV-algebras is any pseudo MV-algebra $\Gamma(G, u)$, where (G, u) is a doubly transitive unital ℓ -group.

2. Subdirectly Irreducible State BL-algebras

In this section, we define state BL-algebras and state-morphism BL-algebras and we present a complete description of their subdirectly irreducible algebras. These results generalize those from [\[7,](#page-15-9) [9,](#page-15-11) [11,](#page-15-12) [13\]](#page-15-13).

We recall that according to [\[19\]](#page-15-10), a *BL-algebra* is an algebra $\mathbf{M} = (M; \wedge, \vee, \odot, \rightarrow)$, 0, 1) of the type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that $(M; \wedge, \vee, 0, 1)$ is a bounded lattice, $(M; \odot, 1)$ is a commutative monoid, and for all $a, b, c \in M$,

- (1) $c \leq a \rightarrow b$ iff $a \odot c \leq b$;
- (2) $a \wedge b = a \odot (a \rightarrow b);$
- (3) $(a \rightarrow b) \vee (b \rightarrow a) = 1.$

For any $a \in M$, we define a complement $a^- := a \to 0$. Then $a \leq a^{-}$ for any $a \in M$ and a BL-algebra is an MV-algebra iff $a^{--} = a$ for any $a \in M$.

A non-empty set $F \subseteq M$ is called a *filter* of **M** (or a *BL-filter* of **M**) if for every $x, y \in M$: (1) $x, y \in F$ implies $x \odot y \in F$, and (2) $x \in F$, $x \leq y$ implies $y \in F$. A filter $F \neq M$ is called a *maximal filter* if it is not strictly contained in any other filter $F' \neq M$. A BL-algebra is called *local* if it has a unique maximal filter.

We denote by $\text{Rad}_1(\mathbf{M})$ the intersection of all maximal filters of M.

Let M be a BL-algebra. A mapping $\tau : M \to M$ such that, for all $x, y \in M$, we have

 $(1)_{BL} \tau(0) = 0;$ $(2)_{BL} \tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \wedge y);$ $(3)_{BL}$ $\tau(x \odot y) = \tau(x) \odot \tau(x \rightarrow (x \odot y));$ $(4)_{BL}$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y);$ $(5)_{BL}$ $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$

is said to be a *state-operator* on M, and the pair (M, τ) is said to be a *state BLalgebra*, or more precisely, a *BL-algebra with internal state*.

If $\tau : M \to M$ is a BL-endomorphism such that $\tau \circ \tau = \tau$, then τ is a stateoperator on **M** and it is said to be a *state-morphism operator* and the couple (M, τ) is said to be a *state-morphism BL-algebra*.

A filter F of a BL-algebra M is said to be a τ -filter if $\tau(F) \subseteq F$. If τ is a state-operator on M, we denote by

$$
Ker(\tau) = \{a \in M : \tau(a) = 1\}.
$$

then Ker(τ) is a τ -filter. A state-operator τ is said to be *faithful* if Ker(τ) = {1}.

We recall that there is a one-to-one relation between congruences and τ -filters on a state BL-algebra (M, τ) as follows. If F is a τ -filter, then the relation ∼F given by $x \sim_F y$ iff $x \to y, y \to x \in F$ is a congruence of the BL-algebra M and \sim_F is also a congruence of the state BL-algebra (M, τ) .

Conversely, let ∼ be a congruence of state BL-algebra (M, τ) and set $F_{\sim} := \{x \in$ $M: x \sim 1$ }. Then F_{\sim} is a τ -filter of (M, τ) and $\sim_{F_{\sim}} = \sim$ and $F = F_{\sim_F}$.

By [\[5,](#page-15-15) Lem 3.5(k)], $(\tau(M), \tau)$ is a subalgebra of (M, τ) , τ on $\tau(M)$ is the identity, and hence, $(Ker(\tau);\to,0,1)$ is a subhoop of M. We say that two subhoops, A and B, of a BL-algebra M have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

Nevertheless a subdirectly irreducible state BL-algebra (M, τ) is not necessarily linearly ordered, according to [\[5,](#page-15-15) Thm 5.5], $\tau(M)$ is always linearly ordered.

We note that according to $[5, Prop 3.13]$, if M is an MV-algebra, then the notion of a state MV-algebra coincides with the notion of a state BL-algebra.

The following three characterizations were originally proved in [\[13\]](#page-15-13) for state MValgebras. Here we show that the original proofs from [\[13\]](#page-15-13) slightly improved work also for state BL-algebras.

Lemma 2.1. *Suppose that* (M, τ) *is a state BL-algebra. Then:*

(1) *If* τ *is faithful, then* (M, τ) *is a subdirectly irreducible state BL-algebra if and only if* τ(M) *is a subdirectly irreducible BL-algebra.*

Now let (M, τ) *be subdirectly irreducible. Then:*

- (2) $\text{Ker}(\tau)$ *is (either trivial or) a subdirectly irreducible hoop.*
- (3) $\text{Ker}(\tau)$ *and* $\tau(\mathbf{M})$ *have the disjunction property.*

Proof. (1) Suppose τ is faithful. Let F denote the least nontrivial τ -filter of (M, τ) . There are two cases: (i) If $\tau(M) \cap F \neq \{1\}$, then $\tau(M) \cap F$ is the least nontrivial filter of $\tau(\mathbf{M})$ and $\tau(\mathbf{M})$ is subdirectly irreducible. (ii) If $\tau(\mathbf{M}) \cap F = \{1\}$, then for all $x \in F$, $\tau(x) = 1$ because $\tau(x) \in \tau(M) \cap F$ and $F \subseteq \text{Ker}(\tau) = \{1\}$ is the trivial filter, a contradiction. Therefore, only the first case is possible and $\tau(M)$ is subdirectly irreducible.

Conversely, let $\tau(\mathbf{M})$ be subdirectly irreducible and let G be the least nontrivial filter of $\tau(\mathbf{M})$. Then the τ -filter F of (\mathbf{M}, τ) generated by G is the least nontrivial τ filter of (M, τ) . Indeed, if K is another nontrivial τ -filter of (M, τ) , then $K \cap \tau(M) \supset$ $F \cap \tau(M) = G$. Then K contains the τ -filter generated by G, that is $F \subseteq K$ which proves F is the least and (M, τ) is subdirectly irreducible.

Now let (M, τ) be subdirectly irreducible and let F denote the least nontrivial filter of (M, τ) .

(2) Suppose that τ is not faithful. Then $\text{Ker}(\tau)$ is a nontrivial τ -filter. If (M, τ) is subdirectly irreducible, it has a least nontrivial τ -filter, F say. So, $F \subseteq \text{Ker}(\tau)$, and hence F is the least nontrivial filter of the hoop $\text{Ker}(\tau)$. Hence, $\text{Ker}(\tau)$ is a subdirectly irreducible hoop.

(3) Suppose, by way of contradiction, that for some $x \in \text{Ker}(\tau)$ and $y = \tau(y) \in$ $\tau(M)$ one has $x < 1$, $y < 1$ and $x \vee y = 1$. It is easy to see that the BL-filters generated by x and by y, respectively, are τ -filters. Therefore they both contain F. Hence, the intersection of these filters contains F. Now let $c < 1$ be in F. Then there is a natural number n such that $x^n \leq c$ and $y^n \leq c$. It follows that $1 = (x \vee y)^n = x^n \vee y^n \leq c$, a contradiction.

Lemma 2.2. *If* (M, τ) *is a subdirectly irreducible state BL-algebra, then* $\tau(M)$ *and* $Ker(\tau)$ *are linearly ordered.*

Proof. According to [\[5,](#page-15-15) Thm 5.5], $\tau(M)$ is always linearly ordered. On the other hand, by Lemma [2.1,](#page-3-0) $\text{Ker}(\tau)$ is either a trivial hoop or a subdirectly irreducible hoop, and hence it is linearly ordered.

Theorem 2.3. Let (M, τ) be a state BL-algebra satisfying conditions (1), (2) and (3) *in Lemma* [2.1](#page-3-0)*. Then* (M, τ) *is subdirectly irreducible.*

Proof. Suppose first that τ is faithful and that $\tau(M)$ is subdirectly irreducible. Let F_0 be the least nontrivial filter of $\tau(M)$ and let F be the BL-filter of M generated by F_0 . Then F is a τ -filter. Indeed, if $x \in F$, then there is $\tau(a) \in F_0$ and a natural number *n* such that $\tau(a)^n \leq x$. It follows that $\tau(x) \geq \tau(\tau(a)^n) = \tau(a)^n$, and $\tau(x) \in F$.

We assert that F is the least nontrivial τ -filter of (M, τ) . First of all, $\tau(M)$, being a subdirectly irreducible BL-algebra, is linearly ordered. Now in order to prove that F is the least nontrivial τ -filter of (M, τ) , it suffices to prove that every τ -filter G not containing F is trivial. Now let $c < 1$ in $F \backslash G$. Then since $\text{Ker}(\tau) = \{1\}, \tau(c) < 1.$ Next, let $d \in G$. Then $\tau(d) \in G$, and for every n it cannot be $\tau(d)^n \leq \tau(c)$, otherwise $\tau(c) \in G$. Hence, for every $n, \tau(c) < \tau(d)^n$, and hence $\tau(c)$ does not belong to the τ -filter of $\tau(M)$ generated by $\tau(d)$. By the minimality of F in $\tau(\mathbf{M})$, $\tau(d) = 1$ and since τ is faithful, we conclude that $d = 1$ and G is trivial, as desired.

Now suppose that $Ker(\tau)$ is nontrivial. By condition (2), $Ker(\tau)$ is subdirectly irreducible. Thus, let F be the least nontrivial filter of $\text{Ker}(\tau)$. Then F is a non trivial $τ$ -filter, and we have to prove that F is the least nontrivial $τ$ -filter of $(M, τ)$. Let G be any non trivial τ -filter of (M, τ) . If $G \subseteq \text{Ker}(\tau)$, then it contains the least filter, F, of Ker(τ), and F \subseteq G. Otherwise, G contains some $x \notin \text{Ker}(\tau)$, and hence it contains $\tau(x)$ < 1. Now by the disjunction property, for all $y < 1$ in Ker(τ), $\tau(x) \vee y < 1$ and $\tau(x) \vee y \in \text{Ker}(\tau) \cap G$. Thus, G contains the filter generated by $\tau(x) \vee y$, which is a non trivial filter of the hoop Ker(τ), and hence it contains F, the least nontrivial filter of $\text{Ker}(\tau)$. This proves the claim.

By $[13, Thm 3.5]$, conditions $(1), (2),$ and (3) from Lemma [2.1](#page-3-0) are independent ones even for state BL-algebras. In addition, Theorem [2.3](#page-4-0) gives a characterization of subdirectly irreducible state BL-algebras. If (M, τ) is a state-morphism BLalgebra, combining [\[11,](#page-15-12) Thm 4.5] we can say more about subdirectly irreducible state-morphism BL-algebras. The following examples are from [\[11\]](#page-15-12).

Example 2.4. Let M be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2 , as follows

$$
\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M. \tag{2.0}
$$

Then τ_1 and τ_2 are two state-morphism operators on $\mathbf{M} \times \mathbf{M}$. Moreover, $(\mathbf{M} \times \mathbf{M}, \tau_1)$ and $(\mathbf{M} \times \mathbf{M}, \tau_2)$ are isomorphic state BL-algebras under the isomorphism $(a, b) \mapsto$ $(b, a).$

We say that an element $a \in M$ is *Boolean* if $a^{--} = a$ and $a \odot a = a$. Let $B(M)$ be the set of Boolean elements. Then $0, 1 \in B(M), B(M)$ is a subset of the MVskeleton MV(M) := { $x \in M : x^{--} = x$ }, and $a \in B(M)$ implies $a^- \in B(M)$. We recall that according to [\[26,](#page-15-16) Thm 2], $MV(M)$ is an MV-algebra, therefore, $B(M)$ is a Boolean subalgebra of $MV(M)$.

Example 2.5. Let **B** be a local MV-algebra such that $Rad_1(B) \neq \{1\}$ is a unique nontrivial filter of B. Let M be a subalgebra of $\mathbf{B} \times \mathbf{B}$ that is generated by $\text{Rad}_1(\mathbf{B}) \times$ $\text{Rad}_1(\mathbf{B})$, that is $M = (\text{Rad}_1(\mathbf{B}) \times \text{Rad}_1(\mathbf{B}) \cup (\text{Rad}_1(\mathbf{B}) \times \text{Rad}_1(\mathbf{B}))^-$. Let $\tau(x, y) :=$ (x, x) for all $x, y \in M$. Then τ is a state-morphism operator on M, Ker (τ) = $\{1\} \times \text{Rad}_{1}(\mathbf{B}) \subset \text{Rad}_{1}(\mathbf{M}) = \text{Rad}_{1}(\mathbf{B}) \times \text{Rad}_{1}(\mathbf{B}), \mathbf{M}$ has no Boolean nontrivial elements, and (M, τ) is a subdirectly irreducible state-morphism MV-algebra that is not linear.

Example 2.6. Let A be a linear nontrivial BL-algebra and B a nontrivial subdirectly irreducible BL-algebra with the smallest nontrivial BL-filter F_B and let $h: \mathbf{A} \to \mathbf{B}$ be a BL-homomorphism. On $M = \mathbf{A} \times \mathbf{B}$ we define a mapping $\tau_h: M \to M$ by

$$
\tau_h(a, b) = (a, h(a)), \quad (a, b) \in M. \tag{2.2}
$$

If we set $y = (0, 1)$ and $y^- = (1, 0)$, then y and y^- are unique nontrivial Boolean elements.

Then τ_h is a state-morphism operator on M and (M, τ_h) is a subdirectly irreducible state-morphism BL-algebra iff $\text{Ker}(h) = \{a \in A : h(a) = 1\} = \{1\}.$ In such a case, $\text{Ker}(\tau_h) = \{1\} \times B$ and $F := \{1\} \times F_B$ is the least nontrivial state-morphism filter on M.

Now we present the main result on the complete characterization of subdirectly irreducible state-morphism BL-algebras which is a combination of [\[11,](#page-15-12) Thm 4.5] and Theorem [2.3.](#page-4-0)

Theorem 2.7. *A state-morphism BL-algebra* (M, τ) *is subdirectly irreducible if and only if one of the following three possibilities holds.*

- (i) M *is linear,* $\tau = \text{Id}_M$ *is the identity on* M, and the BL-reduct M *is a subdirectly irreducible BL-algebra.*
- (ii) *The state-morphism operator* τ *is not faithful,* M *has no nontrivial Boolean elements, and the BL-reduct* **M** *of* (**M**, τ) *is a local BL-algebra,* Ker(τ) *is a subdirectly irreducible irreducible hoop, and* $\text{Ker}(\tau)$ *and* $\tau(\mathbf{M})$ *have the disjunction property.*

Moreover, **M** *is linearly ordered if and only if* $Rad_1(M)$ *is linearly ordered, and in such a case,* M *is a subdirectly irreducible BL-algebra such that if* F *is the smallest nontrivial state-filter for* (M, τ) , *then* F *is the smallest nontrivial BL-filter for* M.

If $\text{Rad}_1(\mathbf{M}) = \text{Ker}(\tau)$, *then* **M** *is linearly ordered.*

(iii) *The state-morphism operator* τ *is not faithful,* M *has a nontrivial Boolean element. There are a linearly ordered BL-algebra* A, *a subdirectly irreducible BL-algebra* **B**, and an injective BL-homomorphism $h : A \rightarrow B$ *such that* (M, τ) *is isomorphic as a state-morphism BL-algebra with the* *state-morphism BL-algebra* $(\mathbf{A} \times \mathbf{B}, \tau_h)$, *where* $\tau_h(x, y) = (x, h(x))$ *for any* $(x, y) \in A \times B$.

Proof. It follows from [\[11,](#page-15-12) Thm 4.5] and Theorem [2.3.](#page-4-0)

We recall that a *t-norm* is a function $t : [0,1] \times [0,1] \rightarrow [0,1]$ such that (i) t is commutative, associative, (ii) $t(x, 1) = x, x \in [0, 1]$, and (iii) t is nondecreasing in both components. If t is continuous, we define $x \odot_t y = t(x, y)$ and $x \rightarrow_t y =$ $\sup\{z \in [0,1]: t(z,x) \leq y\}$ for $x, y \in [0,1]$, then $\mathbb{I}_t := ([0,1]; \min, \max, \odot_t, \rightarrow_t, 0, 1)$ is a BL-algebra. Moreover, according to [\[3,](#page-15-17) Thm 5.2], the variety of all BL-algebras is generated by all \mathbb{I}_t with a continuous t-norm t. Let $\mathcal T$ denote the system of all BL-algebras \mathbb{I}_t , where t is any continuous t-norm.

The proof of the following result will follow from Theorem [5.2.](#page-13-0)

Theorem 2.8. *The variety of all state-morphism BL-algebras is generated by the system* $\{D(\mathbb{I}_t): t \in \mathcal{T}\}.$

3. General State-Morphism Algebras

In this section, we generalize the notion of state-morphism BL-algebras to an arbitrary variety of algebras of some type. It is interesting that many results known only for state-morphism MV-algebras or state-morphism BL-algebras have a very general presentation as state-morphism algebras. The main result of this section, Theorem [3.7,](#page-9-0) says that every subdirectly irreducible state-morphism algebra can be embedded into some diagonal one.

Let **A** be any algebra of type F and let Con **A** be the system of congruences on **A** with the least congruence $\Delta_{\mathbf{A}}$. An endomorphisms $\tau : \mathbf{A} \longrightarrow \mathbf{A}$ satisfying $\tau \circ \tau = \tau$ is said to be a *state-morphism* on **A** and a couple (A, τ) is said to be a *state-morphism algebra* or an algebra with internal state-morphism. Clearly, if K is a variety of algebras of type F, then the class \mathcal{K}_{τ} of all state-morphism algebras (\mathbf{A}, τ) , where $\mathbf{A} \in \mathcal{K}$ and τ is any state-morphism on \mathbf{A} , forms a variety, too.

In the rest of the paper, we will assume that A is an arbitrary algebra with a fixed type F ; if \bf{A} is of a specific type, it will be said that and specified.

Definition 3.1. Let $B \in \mathcal{K}$. Then an algebra $D(D) := (B \times B, \tau_B)$, where τ_B is defined by $\tau_B(x, y) = (x, x), x, y \in B$, is a state-morphism algebra (more precisely $(\mathbf{B} \times \mathbf{B}, \tau_B) \in \mathcal{K}_{\tau}$; we call τ_B also a *diagonal state-operator*. If a state-morphism algebra (C, τ) can be embedded into some diagonal state-morphism algebra, $(B \times$ \mathbf{B}, τ_B), (\mathbf{C}, τ) is said to be a *subdiagonal* state-morphism algebra, or, more precisely, B*-subdiagonal.*

Let (A, τ) be a state-morphism algebra. We introduce the following sets:

$$
\theta_{\tau} = \{(x, y) \in A \times A : \tau(x) = \tau(y)\},\tag{3.1}
$$

$$
\tau(A) = \{\tau(x) : x \in A\}.
$$

The subalgebra which is an image of **A** by τ is denoted by $\tau(A)$ and thus $\tau(A) \in$ K and $(\tau(A), \mathrm{Id}_{\tau(A)}) \in \mathcal{K}_{\tau}$, where $\mathrm{Id}_{\tau(A)}$ is the identity on $\tau(A)$; we have also $\tau | \tau(A) = \mathrm{Id}_{\tau(A)}.$

If $\phi \in \text{Con }\tau(\mathbf{A})$, we define

$$
\theta_{\phi} := \{(x, y) \in A \times A : (\tau(x), \tau(y)) \in \phi\}.
$$
\n
$$
(3.2)
$$

Finally, if $\phi \subseteq A^2$ then the congruence on **A** generated by ϕ is denoted by Θ(φ) and the congruence on (A, τ) generated by φ is denoted by $\Theta_{\tau}(\phi)$. Clearly Con $(\mathbf{A}, \tau) \subseteq \text{Con } \mathbf{A}$ and also $\Theta(\phi) \subseteq \Theta_{\tau}(\phi)$.

Lemma 3.2. *Let* (A, τ) *be a state-morphism algebra. For any* $\phi \in \text{Con } \tau(A)$, *we have* $\theta_{\phi} \in \text{Con}(\mathbf{A}, \tau)$, *and* $\theta_{\phi} \cap \tau(A)^2 = \phi$. *In addition*, $\theta_{\tau} \in \text{Con}(\mathbf{A}, \tau)$, $\phi \subseteq \theta_{\phi}$, *and* $\Theta_{\tau}(\phi) \subseteq \theta_{\phi}$.

Proof. Clearly, θ_{ϕ} is reflexive and symmetric. Moreover, if $(x, y), (y, z) \in \theta_{\phi}$, then $(\tau(x), \tau(y)), (\tau(y), \tau(z)) \in \phi$ and thus $(\tau(x), \tau(z)) \in \phi$ which gives $(x, z) \in \theta_{\phi}$.

Let $f^{\mathbf{A}}$ be an *n*-ary operation on **A** and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ be such that $(x_i, y_i) \in \theta_\phi$ for any $i = 1, \ldots, n$. Then $(\tau(x_i), \tau(y_i)) \in \phi$ holds for any $i = 1, \ldots, n$. Due to $\phi \in \text{Con}\,\tau(\mathbf{A}),\,$ we obtain $(f^{\tau(\mathbf{A})}(\tau(x_1),\ldots,\tau(x_n)),f^{\tau(\mathbf{A})}(\tau(y_1),\ldots,\tau(y_n)))\in$ φ.

Because τ is an endomorphism, $\tau(f^{\mathbf{A}}(x_1,\ldots,x_n)) = f^{\tau(\mathbf{A})}(\tau(x_1),\ldots,\tau(x_n))$ and $\tau(f^{\mathbf{A}}(y_1,\ldots,y_n)) = f^{\tau(\mathbf{A})}(\tau(y_1),\ldots,\tau(y_n))$ which gives $(\tau(f^{\mathbf{A}}(x_1,\ldots,x_n)),$ $\tau(f^{\mathbf{A}}(y_1,\ldots,y_n))) \in \phi$ and finally also $(f^{\mathbf{A}}(x_1,\ldots,x_n), f^{\mathbf{A}}(y_1,\ldots,y_n)) \in \theta_{\phi}$.

Moreover, take an arbitrary $(x, y) \in \theta_{\phi}$. Then $(\tau(\tau(x)), \tau(\tau(y))) = (\tau(x), \tau(y)) \in$ ϕ which gives $(\tau(x), \tau(y)) \in \theta_{\phi}$.

Hence, $\theta_{\phi} \in \text{Con}(\mathbf{A}, \tau)$ and if $\phi = \Delta_{\tau(\mathbf{A})}$, then $\theta_{\phi} = \theta_{\tau}$.

It is clear that $\theta_{\phi} \cap \tau(A)^2 \supseteq \phi$. Now let $(x, y) \in \theta_{\phi} \cap \tau(A)^2$. Then $x, y \in \tau(A)$, $(\tau(x), \tau(y)) \in \phi \subseteq \tau(A)^2$, so that $x = \tau(x) \in \tau(A)$, $y = \tau(y) \in \tau(A)$, and consequently, $(x, y) \in \phi$.

It is evident that θ_{τ} is a congruence on (\mathbf{A}, τ) .

Finally, if $(x, y) \in \phi$ then $\tau(x) = x$ and $\tau(y) = y$ which gives $(\tau(x), \tau(y)) =$ $(x, y) \in \phi$. Thus $(x, y) \in \theta_{\phi}$ which finishes the proof that $\phi \subseteq \theta_{\phi}$ and $\Theta_{\tau}(\phi) \subseteq$ θ_{ϕ} .

Lemma 3.3. *Let* $\theta \in \text{Con } A$ *be such that* $\theta \subseteq \theta_{\tau}$ *. Then* $\theta \in \text{Con } (A, \tau)$ *holds. Moreover, if* $x, y \in A$ *are such that* $(x, y) \in \theta_{\tau}$, *then* $\Theta(x, y) = \Theta_{\tau}(x, y)$ *.*

Proof. If $(x, y) \in \theta \subseteq \theta_{\tau}$, then $\tau(x) = \tau(y)$ and thus $(\tau(x), \tau(y)) = (\tau(x), \tau(x)) \in \theta$ proves that $\theta \in \text{Con}(\mathbf{A}, \tau)$.

Moreover, if $(x, y) \in \theta_{\tau}$, then $\Theta(x, y) \subseteq \theta_{\tau}$. Due to the first part of Lemma, we obtain $\Theta(x, y) \in \text{Con}(\mathbf{A}, \tau)$ and thus $\Theta_{\tau}(x, y) \subseteq \Theta(x, y)$ holds. The second inclusion is trivial.

Lemma 3.4. *If* $x, y \in \tau(\mathbf{A})$, *then* $\Theta(x, y) = \Theta_{\tau}(x, y)$ *. Consequently,* $\Theta(\phi) = \Theta_{\tau}(\phi)$ *whenever* $\phi \subseteq \tau(A)^2$.

Proof. Let us denote by ϕ the congruence on $\tau(A)$ generated by (x, y) . Clearly we obtain the chain of inclusions $\phi \subseteq \Theta(x, y) \subseteq \Theta(\phi) \subseteq \theta_{\phi}$ (because $(x, y) \in \phi$ and $\phi \subseteq \theta_{\phi}$, see Lemma [3.2\)](#page-7-0).

Assume $(a, b) \in \Theta(x, y)$, then $(a, b) \in \theta_{\phi}$ and thus $(\tau(a), \tau(b)) \in \phi \subseteq \Theta(x, y)$. Thus $\Theta(x, y) \in \text{Con}(\mathbf{A}, \tau)$ and $\Theta_{\tau}(x, y) \subseteq \Theta(x, y)$ holds. The second inclusion is trivial.

Finally, let $\phi \subseteq \tau(A)^2$. By [\[2,](#page-15-18) Thm 5.3], the both congruence lattices of **A** and (\mathbf{A}, τ) are complete sublattices of the lattice of equivalencies on \mathbf{A} , and therefore, they have the same infinite suprema. Hence, by the first part of the lemma,

$$
\Theta(\phi) = \bigvee_{(x,y)\in\phi} \Theta(x,y) = \bigvee_{(x,y)\in\phi} \Theta_{\tau}(x,y) = \Theta_{\tau}(\phi).
$$

Remark 3.5. By Lemma [3.2,](#page-7-0) if ϕ is a congruence on $\tau(A)$, then θ_{ϕ} is an extension of ϕ on (\mathbf{A}, τ) and $\Theta(\phi) = \Theta_{\tau}(\phi) \subseteq \theta_{\phi}$. There is a natural question whether $\Theta(\phi) = \theta_{\phi}$? The answer is positive if and only if τ is the identity on A. Indeed, if τ is the identity on A, the statement is evident, in the opposite case, we have $\theta_{\Delta_{\tau(A)}} = \theta_{\tau} \neq \Delta_{\mathbf{A}} = \Theta(\Delta_{\tau(\mathbf{A})}).$

Theorem 3.6. Let (A, τ) be a subdirectly irreducible state-morphism algebra such *that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* **B**-subdiagonal.

Proof. First, if $\theta_{\tau} = \Delta_{\mathbf{A}}$, then for any $x \in A$, the equality $\tau(x) = x$ holds and thus $Con \mathbf{A} = Con(\mathbf{A}, \tau)$ which is absurd because **A** is subdirectly irreducible and (A, τ) is not subdirectly irreducible.

The subdirect irreducibility of (A, τ) implies that there is a least proper congruence $\theta_{\min} \in \text{Con}(\mathbf{A}, \tau)$. Moreover, due to Lemma [3.3,](#page-7-1) the congruence θ_{\min} is also a least proper congruence θ on **A** with $\theta \subseteq \theta_{\tau}$ and thus θ_{\min} is an atom in Con **A**. Let us denote

$$
\theta_\tau^\perp = \{\theta \in \mathop{\mathrm{Con}}\nolimits {\mathbf{A}} : \theta \cap \theta_\tau = \Delta_{\mathbf{A}}\}.
$$

First, we prove that there exists proper $\theta \in \theta_{\tau}^{\perp}$. The subdirect reducibility of **A** shows that there exists proper $\theta \in \text{Con } A$ with $\theta_{\min} \nsubseteq \theta$. Hence, $\theta_{\tau} \cap \theta = \Delta_{A}$ holds (because if $\theta_{\tau} \cap \theta \neq \Delta_{\mathbf{A}}$, then $\theta_{\tau} \cap \theta$ is a proper congruence contained in θ_{τ} and minimality of θ_{\min} yields $\theta_{\min} \subseteq \theta \cap \theta_{\tau} \subseteq \theta$, which is absurd).

Moreover, let us have $\theta_n \in \theta_{\tau}^{\perp}$ for any $n \in \mathbb{N}$ with $\theta_n \subseteq \theta_{n+1}$, then clearly $\bigvee_{n\in\mathbb{N}}\theta_n=\bigcup_{n\in\mathbb{N}}\theta_n\in\theta^{\perp}_\tau.$ Due to Zorn's Lemma, there is maximal $\theta^*\in\theta^{\perp}_\tau.$

We have proved that both θ_{τ} and θ^* are proper congruences on **A** with $\theta_{\tau} \cap \theta^* =$ $\Delta_{\mathbf{A}}$. By the Birkhoff Theorem about subdirect reducibility, \mathbf{A} is a subdirect product of two algebras \mathbf{A}/θ_{τ} and \mathbf{A}/θ^* with an embedding $h : \mathbf{A} \longrightarrow \mathbf{A}/\theta_{\tau} \times \mathbf{A}/\theta^*$ defined by $h(x) = (x/\theta_\tau, x/\theta^*)$.

Now we define the mapping $\psi : A/\theta_{\tau} \longrightarrow A/\theta^*$ by $\psi(x/\theta_{\tau}) = \tau(x)/\theta^*$. Clearly ψ is well-defined because $x/\theta_{\tau} = y/\theta_{\tau}$ yields $\tau(x) = \tau(y)$ and thus $\psi(x/\theta_{\tau}) =$ $\tau(x)/\theta^* = \tau(y)/\theta^* = \psi(y/\theta_\tau)$. Let us suppose that $\psi(x/\theta_\tau) = \psi(y/\theta_\tau)$. Then $\tau(x)/\theta^* = \tau(y)/\theta^*$ and $(\tau(x), \tau(y)) \in \theta^*$. Hence, $\Theta(\tau(x), \tau(y)) \subseteq \theta^*$ holds. Finally, if $\tau(x) \neq \tau(y)$ (thus $\Theta(\tau(x), \tau(y))$ is a proper congruence), then $\tau(x), \tau(y) \in \tau(\mathbf{A})$ and Lemma [3.4](#page-7-2) yields $\Theta(\tau(x), \tau(y)) \in \text{Con}(\mathbf{A}, \tau)$ and thus $\theta_{\min} \subseteq \Theta(\tau(x), \tau(y)) \subseteq$ θ^* which is absurd $(\theta_{\min} \subseteq \theta_{\tau} \cap \theta^* = \Delta_{\mathbf{A}})$. Therefore, the mapping ψ is injective.

We shall prove that ψ is a homomorphism (and thus an embedding). If $f^{\mathbf{A}}$ is an *n*-ary operation and $x_1/\theta_\tau, \ldots, x_n/\theta_\tau \in \mathbf{A}/\theta_\tau$, then

$$
\psi(f^{\mathbf{A}/\theta_{\tau}}(x_1/\theta_{\tau},\ldots,x_n/\theta_{\tau})) = \psi(f^{\mathbf{A}}(x_1,\ldots,x_n)/\theta_{\tau})
$$

\n
$$
= \tau(f^{\mathbf{A}}(x_1,\ldots,x_n))/\theta^*
$$

\n
$$
= f^{\mathbf{A}}(\tau(x_1),\ldots,\tau(x_n))/\theta^*
$$

\n
$$
= f^{\mathbf{A}/\theta^*}(\tau(x_1)/\theta^*,\ldots,\tau(x_n)/\theta^*)
$$

\n
$$
= f^{\mathbf{A}/\theta^*}(\psi(x_1/\theta_{\tau}),\ldots,\psi(x_n/\theta_{\tau}))
$$

Now we prove that **A** is \mathbf{A}/θ^* -diagonal. Let $g: A \longrightarrow (A/\theta^*)^2$ be defined via $g(x) = (\psi(x/\theta_{\tau}), x/\theta^*) = (\tau(x)/\theta^*, x/\theta^*)$. Because the mapping g is the composition of two functions h and ψ which are embeddings, g is also an embedding of A into $(\mathbf{A}/\theta^*)^2$. Now we can compute:

$$
g(\tau(x)) = (\tau(\tau(x))/\theta^*, \tau(x)/\theta^*)
$$

\n
$$
= (\tau(x)/\theta^*, \tau(x)/\theta^*)
$$

\n
$$
= \tau_{\mathbf{A}/\theta^*}(\tau(x)/\theta^*, x/\theta^*)
$$

\n
$$
= \tau_{\mathbf{A}/\theta^*}(g(x)),
$$

where $\tau_{\mathbf{A}/\theta^*}$ is the diagonal state-morphism on the product $\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*$. Therefore, $g:(\mathbf{A},\tau)\rightarrow (\mathbf{A}/\theta^*\times \mathbf{A}/\theta^*, \tau_{\mathbf{A}/\theta^*})$ is an embedding and (\mathbf{A},τ) is \mathbf{A}/θ^* -diagonal.

Finally, we prove the subdirect irreducibility of \mathbf{A}/θ^* . Of course, $\theta_{\min} \cap \theta^* = \Delta_{\mathbf{A}}$ yields $\theta_{\min} \nsubseteq \theta^*$ and thus $\theta^* \subset \theta^* \vee \theta_{\min}$. Moreover, if $\theta^* \subset \theta$, from maximality of θ^* we obtain $\theta \cap \theta_\tau \neq \Delta_A$ and thus $\theta_{\min} \subseteq \theta_\tau \cap \theta$. Finally, $\theta_{\min} \vee \theta^* \subseteq (\theta_\tau \cap$ θ) $\forall \theta^* \subseteq (\theta_\tau \cap \theta) \lor \theta = \theta$ holds. Hence, for any congruence $\theta \in \text{Con } \mathbf{A}$, the inequality $\theta^* \subset \theta^* \cap \theta_{\min} \subseteq \theta$ holds. Due to the Birkhoff's Theorem and the Second Homomorphism Theorem, an algebra \mathbf{A}/θ^* is subdirectly irreducible.

Theorem [3.6](#page-8-0) can be extended as follows.

Theorem 3.7. For every subdirectly irreducible state-morphism algebra (A, τ) , *there is a subdirectly irreducible algebra* **B** *such that* (A, τ) *is* **B**-subdiagonal.

Proof. There are two cases: (1) (A, τ) and A are subdirectly irreducible, and (2) (\mathbf{A}, τ) is a subdirectly irreducible state-morphism algebra and \mathbf{A} is a subdirectly reducible algebra

(1) Assume that (A, τ) and A are subdirectly irreducible. Define two statemorphism algebras $(\tau(\mathbf{A}) \times \mathbf{A}, \tau_1)$ and $(\mathbf{A} \times \mathbf{A}, \tau_2)$, where $\tau_1(a, b) = (a, a), (a, b) \in$ $\tau(A) \times A$, and $\tau_2(a, b) = (a, a), a, b \in A$. Then the first one is a subalgebra of the second one.

Define a mapping $\phi: A \to \tau(A) \times A$ defined by $\phi(a) = (\tau(a), a), a \in A$. Then ϕ is injective because if $\phi(a) = \phi(b)$ then $(\tau(a), a) = (\tau(b), b)$ and $a = b$. We show that ϕ is a homomorphism. Let $f^{\mathbf{A}}$ be an *n*-ary operation on **A** and let $a_1, \ldots, a_n \in A$. Then

$$
\begin{array}{rcl}\n\phi(f^{\mathbf{A}}(a_1,\ldots,a_n)) & = & \left(\tau(f^{\mathbf{A}}(a_1,\ldots,a_n)),f^{\mathbf{A}}(a_1,\ldots,a_n)\right) \\
& = & \left(f^{\mathbf{A}}(\tau(a_1),\ldots,\tau(a_n)),f^{\mathbf{A}}(a_1,\ldots,a_n)\right) \\
& = & f^{\tau(\mathbf{A})\times\mathbf{A}}\left((\tau(a_1),a_1),\ldots,(\tau(a_n),a_n)\right) \\
& = & f^{\tau(\mathbf{A})\times\mathbf{A}}(\phi(a_1),\ldots,\phi(a_n)).\n\end{array}
$$

Since $\phi : \mathbf{A} \to \tau(\mathbf{A}) \times \mathbf{A} \subseteq \mathbf{A} \times \mathbf{A}$, ϕ can be assumed also as an injective homomorphism from the state-morphism algebra (\mathbf{A}, τ) into the state-morphism algebra $D(\mathbf{B})$, where $\mathbf{B} := \mathbf{A}$ is a subdirectly irreducible algebra.

(2) This case was proved in Theorem [3.6.](#page-8-0)

For example, a state-morphism algebra (A,Id_A) , where Id_A is the identity on A, is subdirectly irreducible if and only if A is subdirectly irreducible. Therefore, $(\mathbf{A}, \mathrm{Id}_A)$ can be embedded into $(\mathbf{A} \times \mathbf{A}, \tau_A)$ under the mapping $a \mapsto (a, a), a \in A$. Consequently, every subdirectly irreducible state-morphism algebra (A,Id_A) is A subdiagonal with **A** subdirectly irreducible.

We note that in the same way as in [\[13,](#page-15-13) Lem 6.1], it is possible to show that the class of subdiagonal state-morphism algebras is closed under subalgebras and ultraproducts, and not closed under homomorphic images, see [\[13,](#page-15-13) Lem 6.6].

4. Varieties of State-Morphism Algebras and Their Generators

In this section, we study varieties of state-morphism algebras and their generators. It is interesting to note that some similar results proved for state-morphism MV-algebras in [\[13\]](#page-15-13) can be obtained in an analogous way also for a general variety of algebras.

Let τ be a state-morphism operator on an algebra **A**. We set

$$
Ker(\tau) := \{(x, y) \in A \times A : \tau(x) = \tau(y)\},\
$$

the *kernel* of τ . We say that τ is *faithful* if Ker(τ) = $\Delta_{\mathbf{A}}$. It is evident that τ is faithful iff $\tau(x) = x$ for each $x \in A$. In addition, τ is faithful iff τ is injective.

For every class K of same type algebras, we set $D(K) = \{D(A) : A \in K\}$, where $D(\mathbf{A}) = (\mathbf{A} \times \mathbf{A}, \tau_A).$

As usual, given a class K of algebras of the same type, $I(K)$, $H(K)$, $S(K)$ and $P(K)$ and $P_U(K)$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from K , respectively. Moreover, $V(K)$ will denote the variety generated by K.

Lemma 4.1. (1) Let K be a class of algebras of the same type F. Then $VD(K)$ ⊆ $V(\mathcal{K})_{\tau}$.

(2) Let V be any variety. Then $V_\tau = \text{ISD}(\mathcal{V})$.

Proof. (1) If $D(A) \in D(\mathcal{K})$ (thus $A \in \mathcal{K}$), then the F-reduct of the algebra $D(A)$ is the algebra $\mathbf{A} \times \mathbf{A}$ which belongs to the variety $\mathsf{V}(\mathcal{K})$. Due to definition of $\mathsf{V}(\mathcal{K})_{\tau}$, we obtain also $D(\mathbf{A}) \in V(\mathcal{K})_{\tau}$. We have proved that $D(\mathcal{K}) \subseteq V(\mathcal{K})_{\tau}$. Because $V(\mathcal{K})_{\tau}$ is a variety then also $VD(\mathcal{K}) \subseteq V(\mathcal{K})_{\tau}$

(2) Let $(A, \tau) \in V_{\tau}$. As we have seen in the proof of Theorem [3.7,](#page-9-0) the map $\phi: a \mapsto (\tau(a), a)$ is an injective homomorphism of (\mathbf{A}, τ) into $D(\mathbf{A})$. Hence, ϕ is compatible with τ , and $(\mathbf{A}, \tau) \in \text{ISD}(\mathcal{V})$. Conversely, the F-reduct of any algebra in $D(V)$ is in V, (being a direct product of algebras in V), and hence the F-reduct of any member of $\text{ISD}(\mathcal{V})$ is in $\text{IS}(\mathcal{V}) = \mathcal{V}$. Hence, any member of $\text{ISD}(\mathcal{V})$ is in \mathcal{V}_τ .

Lemma 4.2. Let K be a class of algebras of the same type F . Then:

 (1) DH (K) \subseteq HD (K) . (2) DS $(K) \subseteq$ ISD (K) *.* (3) DP $(K) \subseteq$ IPD (K) .

(4) $VD(\mathcal{K}) = ISD(V(\mathcal{K}))$.

Proof. (1) Let $D(C) \in DH(K)$. Then there are $A \in K$ and a homomorphism h from **A** onto **C**. Let for all $a, b \in A$, $h^*(a, b) = (h(a), h(b))$. We claim that h^* is a homomorphism from $D(A)$ onto $D(C)$. That h^* is a homomorphism is clear. We verify that h^* is compatible with τ_A . We have $h^*(\tau_A(a, b)) = h^*(a, a) =$ $(h(a), h(a)) = \tau_C(h(a), h(b)) = \tau_C(h^*(a, b))$. Finally, since h is onto, given $(c, d) \in$ $C \times C$, there are $a, b \in A$ such that $h(a) = c$ and $h(b) = d$. Hence, $h^*(a, b) = (c, d)$, h^* is onto, and $D(\mathbf{C}) \in \mathsf{HD}(\mathcal{K})$.

(2) It is trivial.

(3) Let $\mathbf{A} = \prod_{i \in I} (\mathbf{A}_i) \in \mathsf{P}(\mathcal{K})$, where each \mathbf{A}_i is in \mathcal{K} . Then the map

$$
\Phi: \big((a_i : i \in I), (b_i : i \in I)\big) \mapsto \big((a_i, b_i) : i \in I\big)
$$

is an isomorphism from $D(\mathbf{A})$ onto $\prod_{i\in I} D(\mathbf{A}_i)$. Indeed, it is clear that Φ is an *F*-isomorphism. Moreover, denoting the state-morphism of $\prod_{i\in I} D(\mathbf{A}_i)$ by τ^* , we

get:

$$
\Phi(\tau_A((a_i : i \in I), (b_i : i \in I))) = \Phi((a_i : i \in I), (a_i : i \in I)) =
$$

= ((a_i, a_i) : i \in I) = (\tau_{\mathbf{A}_i}(a_i, b_i) : i \in I) = \tau^*(\Phi((a_i : i \in I), (b_i : i \in I))),

and hence Φ is an isomorphism.

(4) By (1), (2) and (3), $DV(\mathcal{K}) = DHSP(\mathcal{K}) \subseteq HSPD(\mathcal{K}) = VD(\mathcal{K})$, and hence $\mathsf{IBDV}(\mathcal{K}) \subseteq \mathsf{ISVD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$. Conversely, by Lemma [4.1\(](#page-10-0)1), $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{\tau}$, and by Lemma [4.1\(](#page-10-0)2), $V(K)_{\tau} = \text{ISDV}(K)$. This proves the claim.

Theorem 4.3. (1) For every class K of algebras of the same type F, $V(D(K))$ = $V(\mathcal{K})_{\tau}$.

(2) Let \mathcal{K}_1 and \mathcal{K}_2 be two classes of same type algebras. Then $\mathsf{V}(D(\mathcal{K}_1))$ = $V(D(\mathcal{K}_2))$ *if and only if* $V(\mathcal{K}_1) = V(\mathcal{K}_2)$ *.*

Proof. (1) By Lemma [4.2\(](#page-10-1)4), $VD(K) = ISD(V(K))$. Moreover, by Lemma [4.1\(](#page-10-0)2), $V(\mathcal{K})_{\tau} = \text{ISDV}(\mathcal{K})$. Hence, $V(D(\mathcal{K})) = V(\mathcal{K})_{\tau}$.

(2) We have $V(D(\mathcal{K}_1)) = V(\mathcal{K}_1)_{\tau}$ and $V(D(\mathcal{K}_2)) = V(\mathcal{K}_2)_{\tau}$. Clearly, $V(\mathcal{K}_1) =$ $V(\mathcal{K}_2)$ implies $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$, and hence $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$. Conversely, $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$ implies $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$. But any algebra $\mathbf{A} \in V(\mathcal{K}_1)$ is the F-reduct of a state-morphism algebra in $V(\mathcal{K}_1)_{\tau}$, namely of (A,Id_A) .

It follows that, if $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$, then the classes of F-reducts of $V(\mathcal{K}_1)_{\tau}$ and of $V(\mathcal{K}_2)$ _τ coincide, and hence $V(\mathcal{K}_1) = V(\mathcal{K}_2)$.

As a direct corollary of Theorem [4.3,](#page-11-0) we have:

Theorem 4.4. If a system K of algebras of the same type F generates the whole *variety* $V(F)$ *of all algebras of type* F, *then the variety* $V(F)$ _r *of all state-morphism algebras* (A, τ) , *where* $A \in V(F)$, *is generated by the class* $\{D(A) : A \in \mathcal{K}\}.$

Some applications of the latter theorem for different varieties of algebras will be done in Section 5.

Theorem 4.5. *If* A *is a subdirectly irreducible algebra, then any state-morphism algebra* (A, τ) *is subdirectly irreducible.*

Proof. Let **A** be a subdirectly irreducible algebra and let τ be a state-morphism operator on **A**. If τ is the identity on A, then Con $\mathbf{A} = \text{Con}(\mathbf{A}, \tau)$ and, consequently, (\mathbf{A}, τ) is subdirectly irreducible. If τ is not the identity on A, then θ_{τ} , defined by (3.1), is a nontrivial congruence on **A**, and thus $\theta_{\min} \subseteq \theta_{\tau}$, where $\theta_{\min} \in \text{Con } A$ is the least nontrivial congruence. Hence, θ_{\min} belongs to the set Con (\mathbf{A}, τ) , see Lemma [3.3.](#page-7-1) Therefore, Con $(A, \tau) \subseteq$ Con A yields the subdirect irreducibility of the algebra (\mathbf{A}, τ) , more precisely, θ_{\min} is also the least proper congruence in $Con(\mathbf{A}, \tau).$

We remind the following Mal'cev Theorem, [\[2,](#page-15-18) Lem 3.1].

Theorem 4.6. Let **A** be an algebra and $\phi \subseteq A^2$. Then $(a, b) \in \Theta(\phi)$ if and *only if there exist two finite sequences of terms* $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$ *and pairs* $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ *with*

$$
a = t_1(\overline{x}_1, a_1), t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1})
$$
 and $t_n(\overline{x}_n, b_n) = b$

for some $\overline{x}_1, \ldots, \overline{x}_n \in A$ *.*

We say that an algebra B has the Congruence Extension Property (CEP for short) if, for any algebra \bf{A} such that \bf{B} is a subalgebra of \bf{A} and for any congruence $\theta \in \text{Con } \mathbf{B}$, there is a congruence $\phi \in \text{Con } \mathbf{A}$ such that $\theta = (B \times B) \cap \phi$. A variety $\mathcal K$ has the CEP if every algebra in $\mathcal K$ has the CEP. For example, the variety of MValgebra, or the variety of BL-algebras or the variety of state-morphism MV-algebras (see [\[13,](#page-15-13) Lem 6.1]) satisfies the CEP.

Theorem 4.7. A variety V_{τ} satisfy the CEP if and only if V satisfies the CEP.

Proof. Let us have a variety V with the CEP. If $A \in V$ is such that (A, τ) is an algebra with state-morphism, for any subalgebra $(\mathbf{B}, \tau) \subset (\mathbf{A}, \tau)$ and any $\phi \in$ Con (\mathbf{B}, τ) , the condition $\phi = B^2 \cap \Theta(\phi)$ holds.

Now we prove $\Theta(\phi) = \Theta_{\tau}(\phi)$. To show that, assume $(a, b) \in \Theta(\phi)$. Mal'cev's Theorem shows the existence of finite sequences of terms $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$ and pairs $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ with

$$
a = t_1(\overline{x}_1, a_1), t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1})
$$
 and $t_n(\overline{x}_n, b_n) = b$

for some $\overline{x}_1, \ldots, \overline{x}_n \in A$. Because τ is an endomorphism, we obtain also equalities

$$
\tau(a) = t_1(\tau(\overline{x}_1), \tau(a_1)), \ t_i(\tau(\overline{x}_i), \tau(b_i)) = t_{i+1}(\tau(\overline{x}_{i+1}), \tau(a_{i+1}))
$$

and

$$
t_n(\tau(\overline{x}_n), \tau(b_n)) = \tau(b).
$$

We have assumed that $\phi \in \text{Con}(\mathbf{B}, \tau)$, thus $(a_i, b_i) \in \phi$ yields $(\tau(a_i), \tau(b_i)) \in \phi$ for any $i = 1, \ldots, n$. Now, we have obtained $(\tau(a), \tau(b)) \in \Theta(\phi)$. In other words, $\Theta(\phi) \in \text{Con}(\mathbf{A}, \tau)$ and thus $\Theta(\phi) = \Theta_{\tau}(\phi)$.

If \mathcal{V}_{τ} has the CEP, then for any $\mathbf{A} \in \mathcal{V}$, we have Con $\mathbf{A} = \text{Con}(\mathbf{A}, \text{Id}_A)$. Clearly, the CEP on (A, Id_A) yields the CEP on A .

5. Applications to Special Types of Algebras

In this section, we apply a general result concerning generators of some varieties of state-morphism algebras, Theorem [4.3,](#page-11-0) to the variety of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BLalgebras, and state-morphism pseudo MV-algebras, when we use different systems of t-norms on the real interval $[0, 1]$ and a special type of pseudo MV-algebras, respectively.

Algebras for which the logic MTL is sound are called MTL-algebras. They can be characterized as prelinear commutative bounded integral residuated lattices. In more detail, according to [\[15\]](#page-15-19), an algebraic structure $\mathbf{A} = (A; \wedge, \vee, *, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is an *MTL-algebra* if

- (M1) $(A; \wedge, \vee, 0, 1)$ is a bounded lattice with the top element 0 and bottom element 1,
- $(M2)$ $(A;*,1)$ is a commutative monoid,
- (M3) $*$ and \rightarrow form an adjoint pair, that is, $z * x \leq y$ if and only if $z \leq x \rightarrow y$, where \leq is the lattice order of $(A; \wedge, \vee)$ for all $x, y, z \in A$, (the residuation condition),
- (M4) $(x \to y) \lor (y \to x) = 1$ holds for all $x, y \in A$ (the prelinearity condition).

If t is any left-continuous t-norm on [0, 1], we define two binary operations $*_t \rightarrow_t$ on [0, 1] via $x*_ty = t(x, y)$ and $x \to_t y = \sup\{z \in [0, 1] : t(z, x) \leq y\}$ for $x, y \in [0, 1]$, then $\mathbb{I}_t = ([0, 1]; \min, \max, *, , \rightarrow_t, 0, 1)$ is an example of an MTL-algebra. An MTLalgebra \mathbb{I}_t is a BL-algebra iff t is continuous.

Due to [\[15\]](#page-15-19), the class \mathcal{T}_{lc} , which denotes the system of all BL-algebras \mathbb{I}_t , where t is a left-continuous t-norm on the interval $[0, 1]$, generates the variety of MTLalgebras. This result was strengthened in [\[27\]](#page-15-20) who introduced the class of regular left-continuous t-norms which is strictly smaller than the class of left-continuous t-norms, but they generate the variety of MTL-algebras.

According to [\[1\]](#page-15-21), we say that an algebra $\mathbf{A} = (A; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is a *non-associative BL-algebra* (naBL-algebra in short) if

- (A1) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
- $(A2)$ $(A; \cdot, 1)$ is a commutative groupoid with the neutral element 1,
- (A3) any $x, y, z \in A$ satisfy $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$,
- (A4) algebra satisfy the divisibility axiom $(x \cdot (x \rightarrow y) = x \land y)$,
- (A5) algebra satisfy the α -prelinearity and β -prelinearity $(x \to y \lor \alpha_b^a(y \to x) =$ $x \to y \lor \beta_b^a(y \to x) = 1$, where $\alpha_b^a(x) = (a \cdot b) \to (a \cdot (b \cdot x))$ and $\beta_b^a(x) = b \rightarrow (a \rightarrow ((a \cdot b) \cdot x)).$

A function $t : [0,1] \times [0,1] \rightarrow [0,1]$ on the interval $[0,1]$ of reals is said to be a *non-associative* t-norm (nat-norm briefly) if

- (nat1) $([0, 1]; t, 1)$ is a commutative groupoid with the neutral element 1,
- $(nat2)$ t is continuous in the usual sense,
- (nat3) if $x, y, z \in [0, 1]$ are such that $x \leq y$, then $t(x, z) \leq t(y, z)$.

According to [\[1,](#page-15-21) Thm 5], for any nat-norm there is a unique binary operation \rightarrow_t satisfying the adjointness condition, i.e. $t(x, y) \leq z$ if and only if $x \leq y \rightarrow_t z$. Moreover, an algebra $\mathbb{I}^{na}_{t} := ([0, 1]; \min, \max, t, \rightarrow_t, 0, 1)$ is an naBL-algebra.

The class of all naBL-algebras is denoted by $na\mathcal{BL}$ and $na\mathcal{T}$ denotes the class of all naBL-algebras \mathbb{I}^{na}_t for any non-associative t-norm. The main result on nonassociative BL-algebras says that $n\pi$ is the generating class for the variety $n\pi\beta\mathcal{L}$, $[1, Thm 8]:$ $[1, Thm 8]:$

Theorem 5.1. *There hods*

$$
na\mathcal{BL} = IP_{S}SP_{U}(na\mathcal{T}).
$$

Finally, we recall that a noncommutative generalization of MV-algebras was introduced in [\[17\]](#page-15-22) as *pseudo MV-algebras* or in [\[25\]](#page-15-23) as *generalized MV-algebras*. According to [\[10\]](#page-15-24), every pseudo MV-algebra $(M, \oplus, \neg, \sim, 0, 1)$ of type $\langle 2, 1, 1, 0, 0 \rangle$ is an interval in a unital ℓ -group (G, u) with strong unit u , i.e. $M \cong \Gamma(G, u) := [0, u]$, where $x \oplus y = (x + y) \wedge, x^- = u - x, x^* = -x + u, 0 = 0$, and $1 = u$. If (G, u) is double transitive (for definitions and details see [\[12\]](#page-15-25)), then $\Gamma(G, u)$ generates the variety of pseudo MV-algebras, [\[12,](#page-15-25) Thm 4.8]. For example, if $Aut(\mathbb{R})$ is the set of all automorphisms of the real line R preserving the natural order in R and $u(t) := t + 1, t \in \mathbb{R}$, let ${\rm Aut}_u(\mathbb{R}) = \{g \in {\rm Aut}(\mathbb{R}) : g \leq nu$ for some integer $n \geq 1$. Then $\Gamma(\text{Aut}_{u}(\mathbb{R}), u)$ is double transitive and it generates the variety of pseudo MV-algebras, see [\[12,](#page-15-25) Ex 5.3].

Now we apply the general statement, Theorem [4.4,](#page-11-1) on generators to different types of state-morphism algebras. We recall that $\mathcal T$ was defined as the class of all BL-algebras \mathbb{I}_t , where t is a continuous t-norm on [0, 1].

Theorem 5.2. (1) *The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra* $D([0,1]_{MV})$.

(2) *The variety of all state-morphism BL-algebras is generated by the class* $\{D(\mathbb{I}_t): \mathbb{I}_t \in \mathcal{T}\}.$

(3) *The variety of all state-morphism MTL-algebras is generated by the class* $\{D(\mathbb{I}_t): \mathbb{I}_t \in \mathcal{T}_{lc}\}.$

(4) *The variety of all state-morphism naBL-algebras is generated by the class* $\{D(\mathbb{I}_t^{na}): \mathbb{I}_t \in na\mathcal{T}\}.$

(5) If a unital ℓ -group (G, u) is double transitive, then $D(\Gamma(G, u))$ generates the *variety of state-morphism pseudo MV-algebras.*

Proof. (1) It follows from the fact that the MV-algebra of the real interval $[0,1]$ generates the variety of MV-algebras, see e.g. [\[4,](#page-15-26) Prop 8.1.1], and then apply Theorem [4.4.](#page-11-1)

(2) The statement follows from the fact that $V(\mathcal{T})$ is by [\[3,](#page-15-17) Thm 5.2] the variety \mathcal{BL} of all BL-algebras. Now it suffices to apply Theorem [4.4.](#page-11-1)

(3) By [\[15\]](#page-15-19), the class \mathcal{T}_{lc} of all \mathbb{I}_t , where t is any left-continuous t-norms on the interval [0, 1], generates the variety of MTL-algebras; then apply Theorem [4.4.](#page-11-1)

(4) By [\[1,](#page-15-21) Thm 8] or Theorem [5.1,](#page-13-1) the class $na\mathcal{T}$ of all \mathbb{I}_t , where t is any nonassociative t-norms on the interval $[0, 1]$, generates the variety of non-associative BL-algebras; then apply again Theorem [4.4.](#page-11-1)

(5) By the above, $\Gamma(G, u)$ generates the variety of pseudo MV-algebras, see also [\[12,](#page-15-25) Thm 4.8]; then apply Theorem [4.4.](#page-11-1)

We note that the case (1) in Theorem [4.4](#page-11-1) was an open problem posed in [\[7\]](#page-15-9) and was positively solved in [\[13,](#page-15-13) Thm 5.4(3)].

6. Conclusion

In the paper, we have presented a general approach to theory of state-morphism algebras which generalizes state-morphism MV-algebras and state-morphism BLalgebras as pairs (A, τ) , where A is an algebra of type F and τ is an endomorphism of **A** such that $\tau \circ \tau = \tau$.

This enables us to present complete characterizations of subdirectly irreducible state BL-algebras and subdirectly irreducible state-morphism BL-algebras, Theorem [2.7,](#page-5-0) which generalizes the results from [\[7,](#page-15-9) [9,](#page-15-11) [11,](#page-15-12) [13\]](#page-15-13).

A general approach is studied in the third section where the main result, Theorem [3.7,](#page-9-0) says that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one.

The fourth section describes some generators of the varieties of state-morphism algebras, and Theorem [4.4](#page-11-1) shows that if a class $\mathcal K$ generates a variety $\mathcal V$ of algebras of the same type F , then the variety of state-morphism algebras whose F -reduct belongs to the class K is generated by the class of diagonal state-morphism algebras $D(A)$, where $A \in \mathcal{K}$. In addition, Theorem [4.7](#page-12-0) deals with the CEP for the variety of state-morphism algebras.

In Theorem [5.2,](#page-13-0) Theorem [4.4](#page-11-1) was applied to the special class of algebras: MValgebras, BL-algebras, MTL-algebras, non-associative BL-algebras, and pseudo MValgebras to obtain the generators of the corresponding varieties of state-morphism algebras.

During the study on this paper, we found some interesting open problems like: (1) find a characterization of an analogue of a state-operator that is not necessarily a state-morphism operator, (2) if the lattice of varieties of some variety is countable, how big is the lattice of corresponding state-morphism algebras, e.g. in the case of MV-algebras, the lattice under question is uncountable [\[13\]](#page-15-13), (3) decidability of the variety of state-morphism algebras.

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