# STATE-MORPHISM ALGEBRAS - GENERAL APPROACH

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ABSTRACT. We present a complete description of subdirectly irreducible state BL-algebras as well as of subdirectly irreducible state-morphism BL-algebras. In addition, we present a general theory of state-morphism algebras, that is, algebras of general type with state-morphism which is an idempotent endomorphism. We define a diagonal state-morphism algebra and we show that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one. We describe generators of varieties of state-morphism algebras, in particular ones of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MValgebras.

### 1. INTRODUCTION

A state, as an analogue of a probability measure, is a basic notion of the theory of quantum structures, see e.g. [14]. However, for MV-algebras, the state as averaging the truth value in the Łukasiewicz logic was introduced firstly by Mundici in [22], 40 years after introducing MV-algebras, [6]. We recall that a state on an MV-algebra **M** is a mapping  $s: M \to [0, 1]$  such that (i)  $s(a \oplus b) = s(a) + s(b)$ , if  $a \odot b = 0$ , and (ii) s(1) = 1. The property (i) says that s is additive on mutually excluding events a and b. It is important note that every non-degenerate MV-algebra admits at least one state. The set of states is a convex set, which in the weak topology of states is a compact Hausdorff set, and every extremal state is in fact an MV-algebra homomorphism from **M** into the MV-algebra of the real interval [0, 1], and vice-versa, [22]. In addition, extremal states generate the set of all states because by the Krein-Mil'man Theorem, [18, Thm 5.17], every state is a weak limit of a net of convex combinations of these special homomorphisms.

In the last decade, the states entered into theory of MV-algebras in a very ambitious manner. In [23, 21], authors have showed a relation between states

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and de Finetti's approach to probability in terms of bets. In addition, Panti and independently Kroupa in [24, 20] have showed that every state on  $\mathbf{M}$  is an integral through a unique regular Borel probability measure concentrated on the set of extremal states on  $\mathbf{M}$ .

Nevertheless as we have seen states are not a proper notion of universal algebra, and therefore, they do not provide an algebraizable logic for probabilistic reasoning of the many-valued approach.

Recently, Flaminio and Montagna in [16] presented an algebraizable logic containing probabilistic reasoning, and its equivalent algebraic semantic is the variety of state MV-algebras. We recall that a *state MV-algebra* is an MV-algebra whose language is extended adding an operator,  $\tau$  (called also an *internal state*), whose properties are inspired by the ones of states. The analogues of extremal states are *state-morphism operators*, introduced in [7]. By definition, it is an idempotent endomorphism on an MV-algebra.

State MV-algebras generalize, for example, Hájek's approach, [19], to fuzzy logic with modality Pr (interpreted as *probably*) which has the following semantic interpretation: The probability of an event a is presented as the truth value of Pr(a). On the other hand, if s is a state, then s(a) is interpreted as averaging of appearing the many valued event a.

We note that if  $(\mathbf{M}, \tau)$  is a state MV-algebra, assuming that that the range  $\tau(\mathbf{M})$  is simple, we see that it is a subalgebra of the real interval [0, 1] and therefore,  $\tau$  can be regarded as a standard state on  $\mathbf{M}$ . On the other hand, every MV-algebra  $\mathbf{M}$  can be embedded into the tensor product  $[0, 1] \otimes \mathbf{M}$ , therefore, given a state s on  $\mathbf{M}$ , we define an operator  $\tau_s$  on  $[0, 1] \otimes \mathbf{M}$  via  $\tau_s(t \otimes a) := t \cdot s(a)$ , [16, Thm 5.3]. Then due to [7, Thm 3.2],  $\tau_s$  is a state-operator that is a state-morphism operator iff s is an extremal state. Thus, there is a natural correspondence between the notion of a state and an extremal state on one side, and a state-operator and a state-morphism operator on the other side.

Subdirectly irreducible state-morphism MV-algebras were described in [7, 9] and this was extended also for state-morphism BL-algebras in [11]. A complete description of both subdirectly irreducible state MV-algebras as well as subdirectly irreducible state-morphism MV-algebras can be found in [13]. In [8], it was shown that if  $(\mathbf{M}, \tau)$  is a state MV-algebra whose image  $\tau(\mathbf{M})$  belongs to the variety generated by the  $L_1, \ldots, L_n$ , where  $L_i := \{0, 1/i, \ldots, i/i\}$ , then  $\tau$  has to be a state-morphism operator. The same is true if  $\mathbf{M}$  is linearly ordered, [7]. Recently, in [13], we have shown that the unit square  $[0, 1]^2$  with the diagonal operator generates the whole variety of state-morphism MV-algebras; it answered in positive an open problem posed in [7]. In addition, there was shown that in contrast to MV-algebras, the lattice of subvarieties is uncountable. Moreover, it was shown that every subdirectly irreducible state-morphism MV-algebra can be embedded into some diagonal one.

In this paper, we continue in the study of state BL-algebras and state-morphism BL-algebras. Because the methods developed in [13] are so general that, it is possible to study more general structures than MV-algebras or BL-algebras under a common umbrella. Hence, we introduce state-morphism algebras  $(\mathbf{A}, \tau)$ , where the algebra  $\mathbf{A}$  is an arbitrary algebra of type F and  $\tau$  is an idempotent endomorphism of  $\mathbf{A}$ . Then general results applied to special types of algebras give interesting new results.

The main goals of the paper are:

(1) Complete characterizations of subdirectly irreducible state BL-algebras and state-morphism BL-algebras.

(2) Showing that every subdirectly state-morphism algebra can be embedded into some diagonal one  $D(\mathbf{B}) := (\mathbf{B} \times \mathbf{B}, \tau_B)$ , where  $\tau(a, b) = (a, a), a, b \in B$ , which is also subdirectly irreducible.

(3) We show that if  $\mathcal{K}$  is a generator of some variety  $\mathcal{V}$  of algebras of type F, then the system of diagonal state-morphism algebras  $\{D(\mathbf{B}) : \mathbf{B} \in \mathcal{K}\}$  is a generator of the variety of state-morphism algebras whose F-reduct belongs to  $\mathcal{V}$ .

(4) We exhibit cases when the Congruence Extension Property holds for a variety of state-morphism algebras.

(5) In particular, a generator of the variety of state-morphism BL-algebras is the class of all BL-algebras of the real interval [0, 1] equipped with a continuous t-norm. Similarly, a generator of the variety of state-morphism MTL-algebras is the class of all MTL-algebras of the real interval equipped with a left-continuous tnorm, similarly for non-associative BL-algebras one is the set of all non-associative BL-algebras of the real interval [0, 1] equipped with a non-associative t-norm, and a generator of the variety of state-morphism pseudo MV-algebras is any pseudo MV-algebra  $\Gamma(G, u)$ , where (G, u) is a doubly transitive unital  $\ell$ -group.

### 2. Subdirectly Irreducible State BL-Algebras

In this section, we define state BL-algebras and state-morphism BL-algebras and we present a complete description of their subdirectly irreducible algebras. These results generalize those from [7, 9, 11, 13].

We recall that according to [19], a *BL-algebra* is an algebra  $\mathbf{M} = (M; \land, \lor, \odot, \rightarrow , 0, 1)$  of the type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  such that  $(M; \land, \lor, 0, 1)$  is a bounded lattice,  $(M; \odot, 1)$  is a commutative monoid, and for all  $a, b, c \in M$ ,

- (1)  $c \leq a \rightarrow b$  iff  $a \odot c \leq b$ ;
- (2)  $a \wedge b = a \odot (a \rightarrow b);$
- (3)  $(a \rightarrow b) \lor (b \rightarrow a) = 1.$

For any  $a \in M$ , we define a complement  $a^- := a \to 0$ . Then  $a \leq a^{--}$  for any  $a \in M$  and a BL-algebra is an MV-algebra iff  $a^{--} = a$  for any  $a \in M$ .

A non-empty set  $F \subseteq M$  is called a *filter* of **M** (or a *BL-filter* of **M**) if for every  $x, y \in M$ : (1)  $x, y \in F$  implies  $x \odot y \in F$ , and (2)  $x \in F$ ,  $x \leq y$  implies  $y \in F$ . A filter  $F \neq M$  is called a *maximal filter* if it is not strictly contained in any other filter  $F' \neq M$ . A BL-algebra is called *local* if it has a unique maximal filter.

We denote by  $\operatorname{Rad}_1(\mathbf{M})$  the intersection of all maximal filters of  $\mathbf{M}$ .

Let  ${\bf M}$  be a BL-algebra. A mapping  $\tau: M \to M$  such that, for all  $x,y \in M,$  we have

 $(1)_{BL} \tau(0) = 0;$   $(2)_{BL} \tau(x \to y) = \tau(x) \to \tau(x \land y);$   $(3)_{BL} \tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y));$   $(4)_{BL} \tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y);$  $(5)_{BL} \tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$ 

is said to be a *state-operator* on  $\mathbf{M}$ , and the pair  $(\mathbf{M}, \tau)$  is said to be a *state BL-algebra*, or more precisely, a *BL-algebra with internal state*.

If  $\tau : M \to M$  is a BL-endomorphism such that  $\tau \circ \tau = \tau$ , then  $\tau$  is a stateoperator on **M** and it is said to be a *state-morphism operator* and the couple  $(\mathbf{M}, \tau)$  is said to be a *state-morphism BL-algebra*.

A filter F of a BL-algebra **M** is said to be a  $\tau$ -filter if  $\tau(F) \subseteq F$ . If  $\tau$  is a state-operator on **M**, we denote by

$$\operatorname{Ker}(\tau) = \{a \in M : \tau(a) = 1\}.$$

then  $\operatorname{Ker}(\tau)$  is a  $\tau$ -filter. A state-operator  $\tau$  is said to be *faithful* if  $\operatorname{Ker}(\tau) = \{1\}$ .

We recall that there is a one-to-one relation between congruences and  $\tau$ -filters on a state BL-algebra  $(\mathbf{M}, \tau)$  as follows. If F is a  $\tau$ -filter, then the relation  $\sim_F$ given by  $x \sim_F y$  iff  $x \to y, y \to x \in F$  is a congruence of the BL-algebra  $\mathbf{M}$  and  $\sim_F$  is also a congruence of the state BL-algebra  $(\mathbf{M}, \tau)$ .

Conversely, let ~ be a congruence of state BL-algebra  $(\mathbf{M}, \tau)$  and set  $F_{\sim} := \{x \in M : x \sim 1\}$ . Then  $F_{\sim}$  is a  $\tau$ -filter of  $(\mathbf{M}, \tau)$  and  $\sim_{F_{\sim}} = \sim$  and  $F = F_{\sim_F}$ .

By [5, Lem 3.5(k)],  $(\tau(\mathbf{M}), \tau)$  is a subalgebra of  $(\mathbf{M}, \tau)$ ,  $\tau$  on  $\tau(M)$  is the identity, and hence,  $(\text{Ker}(\tau); \rightarrow, 0, 1)$  is a subhoop of  $\mathbf{M}$ . We say that two subhoops, A and B, of a BL-algebra  $\mathbf{M}$  have the *disjunction property* if for all  $x \in A$  and  $y \in B$ , if  $x \lor y = 1$ , then either x = 1 or y = 1.

Nevertheless a subdirectly irreducible state BL-algebra  $(\mathbf{M}, \tau)$  is not necessarily linearly ordered, according to [5, Thm 5.5],  $\tau(\mathbf{M})$  is always linearly ordered.

We note that according to [5, Prop 3.13], if **M** is an MV-algebra, then the notion of a state MV-algebra coincides with the notion of a state BL-algebra.

The following three characterizations were originally proved in [13] for state MValgebras. Here we show that the original proofs from [13] slightly improved work also for state BL-algebras.

**Lemma 2.1.** Suppose that  $(\mathbf{M}, \tau)$  is a state BL-algebra. Then:

(1) If  $\tau$  is faithful, then  $(\mathbf{M}, \tau)$  is a subdirectly irreducible state BL-algebra if and only if  $\tau(\mathbf{M})$  is a subdirectly irreducible BL-algebra.

Now let  $(\mathbf{M}, \tau)$  be subdirectly irreducible. Then:

- (2)  $\operatorname{Ker}(\tau)$  is (either trivial or) a subdirectly irreducible hoop.
- (3)  $\operatorname{Ker}(\tau)$  and  $\tau(\mathbf{M})$  have the disjunction property.

*Proof.* (1) Suppose  $\tau$  is faithful. Let F denote the least nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ . There are two cases: (i) If  $\tau(M) \cap F \neq \{1\}$ , then  $\tau(M) \cap F$  is the least nontrivial filter of  $\tau(\mathbf{M})$  and  $\tau(\mathbf{M})$  is subdirectly irreducible. (ii) If  $\tau(\mathbf{M}) \cap F = \{1\}$ , then for all  $x \in F$ ,  $\tau(x) = 1$  because  $\tau(x) \in \tau(M) \cap F$  and  $F \subseteq \text{Ker}(\tau) = \{1\}$  is the trivial filter, a contradiction. Therefore, only the first case is possible and  $\tau(\mathbf{M})$  is subdirectly irreducible.

Conversely, let  $\tau(\mathbf{M})$  be subdirectly irreducible and let G be the least nontrivial filter of  $\tau(\mathbf{M})$ . Then the  $\tau$ -filter F of  $(\mathbf{M}, \tau)$  generated by G is the least nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ . Indeed, if K is another nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ , then  $K \cap \tau(M) \supseteq F \cap \tau(M) = G$ . Then K contains the  $\tau$ -filter generated by G, that is  $F \subseteq K$  which proves F is the least and  $(\mathbf{M}, \tau)$  is subdirectly irreducible.

Now let  $(\mathbf{M}, \tau)$  be subdirectly irreducible and let F denote the least nontrivial filter of  $(\mathbf{M}, \tau)$ .

(2) Suppose that  $\tau$  is not faithful. Then  $\operatorname{Ker}(\tau)$  is a nontrivial  $\tau$ -filter. If  $(\mathbf{M}, \tau)$  is subdirectly irreducible, it has a least nontrivial  $\tau$ -filter, F say. So,  $F \subseteq \operatorname{Ker}(\tau)$ ,

and hence F is the least nontrivial filter of the hoop  $\text{Ker}(\tau)$ . Hence,  $\text{Ker}(\tau)$  is a subdirectly irreducible hoop.

(3) Suppose, by way of contradiction, that for some  $x \in \text{Ker}(\tau)$  and  $y = \tau(y) \in \tau(M)$  one has x < 1, y < 1 and  $x \lor y = 1$ . It is easy to see that the BL-filters generated by x and by y, respectively, are  $\tau$ -filters. Therefore they both contain F. Hence, the intersection of these filters contains F. Now let c < 1 be in F. Then there is a natural number n such that  $x^n \leq c$  and  $y^n \leq c$ . It follows that  $1 = (x \lor y)^n = x^n \lor y^n \leq c$ , a contradiction.

**Lemma 2.2.** If  $(\mathbf{M}, \tau)$  is a subdirectly irreducible state BL-algebra, then  $\tau(M)$  and  $\operatorname{Ker}(\tau)$  are linearly ordered.

*Proof.* According to [5, Thm 5.5],  $\tau(M)$  is always linearly ordered. On the other hand, by Lemma 2.1, Ker( $\tau$ ) is either a trivial hoop or a subdirectly irreducible hoop, and hence it is linearly ordered.

**Theorem 2.3.** Let  $(\mathbf{M}, \tau)$  be a state *BL*-algebra satisfying conditions (1), (2) and (3) in Lemma 2.1. Then  $(\mathbf{M}, \tau)$  is subdirectly irreducible.

*Proof.* Suppose first that  $\tau$  is faithful and that  $\tau(\mathbf{M})$  is subdirectly irreducible. Let  $F_0$  be the least nontrivial filter of  $\tau(\mathbf{M})$  and let F be the BL-filter of  $\mathbf{M}$  generated by  $F_0$ . Then F is a  $\tau$ -filter. Indeed, if  $x \in F$ , then there is  $\tau(a) \in F_0$  and a natural number n such that  $\tau(a)^n \leq x$ . It follows that  $\tau(x) \geq \tau(\tau(a)^n) = \tau(a)^n$ , and  $\tau(x) \in F$ .

We assert that F is the least nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ . First of all,  $\tau(\mathbf{M})$ , being a subdirectly irreducible BL-algebra, is linearly ordered. Now in order to prove that F is the least nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ , it suffices to prove that every  $\tau$ -filter G not containing F is trivial. Now let c < 1 in  $F \setminus G$ . Then since  $\operatorname{Ker}(\tau) = \{1\}, \tau(c) < 1$ . Next, let  $d \in G$ . Then  $\tau(d) \in G$ , and for every n it cannot be  $\tau(d)^n \leq \tau(c)$ , otherwise  $\tau(c) \in G$ . Hence, for every  $n, \tau(c) < \tau(d)^n$ , and hence  $\tau(c)$  does not belong to the  $\tau$ -filter of  $\tau(\mathbf{M})$  generated by  $\tau(d)$ . By the minimality of F in  $\tau(\mathbf{M}), \tau(d) = 1$  and since  $\tau$  is faithful, we conclude that d = 1 and G is trivial, as desired.

Now suppose that  $\operatorname{Ker}(\tau)$  is nontrivial. By condition (2),  $\operatorname{Ker}(\tau)$  is subdirectly irreducible. Thus, let F be the least nontrivial filter of  $\operatorname{Ker}(\tau)$ . Then F is a non trivial  $\tau$ -filter, and we have to prove that F is the least nontrivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ . Let G be any non trivial  $\tau$ -filter of  $(\mathbf{M}, \tau)$ . If  $G \subseteq \operatorname{Ker}(\tau)$ , then it contains the least filter, F, of  $\operatorname{Ker}(\tau)$ , and  $F \subseteq G$ . Otherwise, G contains some  $x \notin \operatorname{Ker}(\tau)$ , and hence it contains  $\tau(x) < 1$ . Now by the disjunction property, for all y < 1in  $\operatorname{Ker}(\tau), \tau(x) \lor y < 1$  and  $\tau(x) \lor y \in \operatorname{Ker}(\tau) \cap G$ . Thus, G contains the filter generated by  $\tau(x) \lor y$ , which is a non trivial filter of the hoop  $\operatorname{Ker}(\tau)$ , and hence it contains F, the least nontrivial filter of  $\operatorname{Ker}(\tau)$ . This proves the claim.  $\Box$ 

By [13, Thm 3.5], conditions (1), (2), and (3) from Lemma 2.1 are independent ones even for state BL-algebras. In addition, Theorem 2.3 gives a characterization of subdirectly irreducible state BL-algebras. If  $(\mathbf{M}, \tau)$  is a state-morphism BLalgebra, combining [11, Thm 4.5] we can say more about subdirectly irreducible state-morphism BL-algebras. The following examples are from [11].

**Example 2.4.** Let **M** be a BL-algebra. On  $M \times M$  we define two operators,  $\tau_1$  and  $\tau_2$ , as follows

$$\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M.$$
 (2.0)

Then  $\tau_1$  and  $\tau_2$  are two state-morphism operators on  $\mathbf{M} \times \mathbf{M}$ . Moreover,  $(\mathbf{M} \times \mathbf{M}, \tau_1)$  and  $(\mathbf{M} \times \mathbf{M}, \tau_2)$  are isomorphic state BL-algebras under the isomorphism  $(a, b) \mapsto (b, a)$ .

We say that an element  $a \in M$  is *Boolean* if  $a^{--} = a$  and  $a \odot a = a$ . Let  $B(\mathbf{M})$  be the set of Boolean elements. Then  $0, 1 \in B(\mathbf{M}), B(\mathbf{M})$  is a subset of the MV-skeleton  $MV(\mathbf{M}) := \{x \in M : x^{--} = x\}$ , and  $a \in B(\mathbf{M})$  implies  $a^{-} \in B(\mathbf{M})$ . We recall that according to [26, Thm 2],  $MV(\mathbf{M})$  is an MV-algebra, therefore,  $B(\mathbf{M})$  is a Boolean subalgebra of  $MV(\mathbf{M})$ .

**Example 2.5.** Let **B** be a local MV-algebra such that  $\operatorname{Rad}_1(\mathbf{B}) \neq \{1\}$  is a unique nontrivial filter of *B*. Let **M** be a subalgebra of  $\mathbf{B} \times \mathbf{B}$  that is generated by  $\operatorname{Rad}_1(\mathbf{B}) \times \operatorname{Rad}_1(\mathbf{B})$ , that is  $M = (\operatorname{Rad}_1(\mathbf{B}) \times \operatorname{Rad}_1(\mathbf{B})) \cup (\operatorname{Rad}_1(\mathbf{B}) \times \operatorname{Rad}_1(\mathbf{B}))^-$ . Let  $\tau(x, y) := (x, x)$  for all  $x, y \in M$ . Then  $\tau$  is a state-morphism operator on **M**,  $\operatorname{Ker}(\tau) = \{1\} \times \operatorname{Rad}_1(\mathbf{B}) \subset \operatorname{Rad}_1(\mathbf{M}) = \operatorname{Rad}_1(\mathbf{B}) \times \operatorname{Rad}_1(\mathbf{B})$ , **M** has no Boolean nontrivial elements, and  $(\mathbf{M}, \tau)$  is a subdirectly irreducible state-morphism MV-algebra that is not linear.

**Example 2.6.** Let  $\mathbf{A}$  be a linear nontrivial BL-algebra and  $\mathbf{B}$  a nontrivial subdirectly irreducible BL-algebra with the smallest nontrivial BL-filter  $F_B$  and let  $h : \mathbf{A} \to \mathbf{B}$  be a BL-homomorphism. On  $M = \mathbf{A} \times \mathbf{B}$  we define a mapping  $\tau_h : M \to M$  by

$$\tau_h(a,b) = (a,h(a)), \quad (a,b) \in M.$$
 (2.2)

If we set y = (0, 1) and  $y^- = (1, 0)$ , then y and  $y^-$  are unique nontrivial Boolean elements.

Then  $\tau_h$  is a state-morphism operator on **M** and  $(\mathbf{M}, \tau_h)$  is a subdirectly irreducible state-morphism BL-algebra iff  $\operatorname{Ker}(h) = \{a \in A : h(a) = 1\} = \{1\}$ . In such a case,  $\operatorname{Ker}(\tau_h) = \{1\} \times B$  and  $F := \{1\} \times F_B$  is the least nontrivial state-morphism filter on **M**.

Now we present the main result on the complete characterization of subdirectly irreducible state-morphism BL-algebras which is a combination of [11, Thm 4.5] and Theorem 2.3.

**Theorem 2.7.** A state-morphism BL-algebra  $(\mathbf{M}, \tau)$  is subdirectly irreducible if and only if one of the following three possibilities holds.

- (i) **M** is linear,  $\tau = \text{Id}_M$  is the identity on *M*, and the *BL*-reduct **M** is a subdirectly irreducible *BL*-algebra.
- (ii) The state-morphism operator τ is not faithful, **M** has no nontrivial Boolean elements, and the BL-reduct **M** of (**M**, τ) is a local BL-algebra, Ker(τ) is a subdirectly irreducible irreducible hoop, and Ker(τ) and τ(**M**) have the disjunction property.

Moreover,  $\mathbf{M}$  is linearly ordered if and only if  $\operatorname{Rad}_1(\mathbf{M})$  is linearly ordered, and in such a case,  $\mathbf{M}$  is a subdirectly irreducible BL-algebra such that if F is the smallest nontrivial state-filter for  $(\mathbf{M}, \tau)$ , then F is the smallest nontrivial BL-filter for  $\mathbf{M}$ .

If  $\operatorname{Rad}_1(\mathbf{M}) = \operatorname{Ker}(\tau)$ , then  $\mathbf{M}$  is linearly ordered.

(iii) The state-morphism operator  $\tau$  is not faithful, **M** has a nontrivial Boolean element. There are a linearly ordered BL-algebra **A**, a subdirectly irreducible BL-algebra **B**, and an injective BL-homomorphism  $h : \mathbf{A} \to \mathbf{B}$ such that  $(\mathbf{M}, \tau)$  is isomorphic as a state-morphism BL-algebra with the state-morphism BL-algebra  $(\mathbf{A} \times \mathbf{B}, \tau_h)$ , where  $\tau_h(x, y) = (x, h(x))$  for any  $(x, y) \in A \times B$ .

*Proof.* It follows from [11, Thm 4.5] and Theorem 2.3.

We recall that a *t*-norm is a function  $t : [0,1] \times [0,1] \rightarrow [0,1]$  such that (i) t is commutative, associative, (ii)  $t(x,1) = x, x \in [0,1]$ , and (iii) t is nondecreasing in both components. If t is continuous, we define  $x \odot_t y = t(x,y)$  and  $x \to_t y = \sup\{z \in [0,1] : t(z,x) \leq y\}$  for  $x, y \in [0,1]$ , then  $\mathbb{I}_t := ([0,1]; \min, \max, \odot_t, \to_t, 0, 1)$  is a BL-algebra. Moreover, according to [3, Thm 5.2], the variety of all BL-algebras is generated by all  $\mathbb{I}_t$  with a continuous t-norm t. Let  $\mathcal{T}$  denote the system of all BL-algebras  $\mathbb{I}_t$ , where t is any continuous t-norm.

The proof of the following result will follow from Theorem 5.2.

**Theorem 2.8.** The variety of all state-morphism BL-algebras is generated by the system  $\{D(\mathbb{I}_t) : t \in \mathcal{T}\}.$ 

## 3. General State-Morphism Algebras

In this section, we generalize the notion of state-morphism BL-algebras to an arbitrary variety of algebras of some type. It is interesting that many results known only for state-morphism MV-algebras or state-morphism BL-algebras have a very general presentation as state-morphism algebras. The main result of this section, Theorem 3.7, says that every subdirectly irreducible state-morphism algebra can be embedded into some diagonal one.

Let  $\mathbf{A}$  be any algebra of type F and let Con  $\mathbf{A}$  be the system of congruences on  $\mathbf{A}$  with the least congruence  $\Delta_{\mathbf{A}}$ . An endomorphisms  $\tau : \mathbf{A} \longrightarrow \mathbf{A}$  satisfying  $\tau \circ \tau = \tau$  is said to be a *state-morphism* on  $\mathbf{A}$  and a couple  $(\mathbf{A}, \tau)$  is said to be a *state-morphism algebra* or an algebra with internal state-morphism. Clearly, if  $\mathcal{K}$ is a variety of algebras of type F, then the class  $\mathcal{K}_{\tau}$  of all state-morphism algebras  $(\mathbf{A}, \tau)$ , where  $\mathbf{A} \in \mathcal{K}$  and  $\tau$  is any state-morphism on  $\mathbf{A}$ , forms a variety, too.

In the rest of the paper, we will assume that  $\mathbf{A}$  is an arbitrary algebra with a fixed type F; if  $\mathbf{A}$  is of a specific type, it will be said that and specified.

**Definition 3.1.** Let  $\mathbf{B} \in \mathcal{K}$ . Then an algebra  $D(\mathbf{D}) := (\mathbf{B} \times \mathbf{B}, \tau_B)$ , where  $\tau_B$  is defined by  $\tau_B(x, y) = (x, x), x, y \in B$ , is a state-morphism algebra (more precisely  $(\mathbf{B} \times \mathbf{B}, \tau_B) \in \mathcal{K}_{\tau}$ ); we call  $\tau_B$  also a *diagonal state-operator*. If a state-morphism algebra  $(\mathbf{C}, \tau)$  can be embedded into some diagonal state-morphism algebra,  $(\mathbf{B} \times \mathbf{B}, \tau_B), (\mathbf{C}, \tau)$  is said to be a *subdiagonal* state-morphism algebra, *or*, *more precisely*, **B**-*subdiagonal*.

Let  $(\mathbf{A}, \tau)$  be a state-morphism algebra. We introduce the following sets:

$$\theta_{\tau} = \{ (x, y) \in A \times A : \tau(x) = \tau(y) \}, \tag{3.1}$$

$$\tau(A) = \{\tau(x) : x \in A\}.$$

The subalgebra which is an image of  $\mathbf{A}$  by  $\tau$  is denoted by  $\tau(\mathbf{A})$  and thus  $\tau(\mathbf{A}) \in \mathcal{K}$  and  $(\tau(\mathbf{A}), \mathrm{Id}_{\tau(A)}) \in \mathcal{K}_{\tau}$ , where  $\mathrm{Id}_{\tau(A)}$  is the identity on  $\tau(A)$ ; we have also  $\tau | \tau(A) = \mathrm{Id}_{\tau(A)}$ .

If  $\phi \in \operatorname{Con} \tau(\mathbf{A})$ , we define

$$\theta_{\phi} := \{ (x, y) \in A \times A : (\tau(x), \tau(y)) \in \phi \}.$$

$$(3.2)$$

Finally, if  $\phi \subseteq A^2$  then the congruence on **A** generated by  $\phi$  is denoted by  $\Theta(\phi)$  and the congruence on  $(\mathbf{A}, \tau)$  generated by  $\phi$  is denoted by  $\Theta_{\tau}(\phi)$ . Clearly  $\operatorname{Con}(\mathbf{A}, \tau) \subseteq \operatorname{Con} \mathbf{A}$  and also  $\Theta(\phi) \subseteq \Theta_{\tau}(\phi)$ .

**Lemma 3.2.** Let  $(\mathbf{A}, \tau)$  be a state-morphism algebra. For any  $\phi \in \operatorname{Con} \tau(\mathbf{A})$ , we have  $\theta_{\phi} \in \operatorname{Con} (\mathbf{A}, \tau)$ , and  $\theta_{\phi} \cap \tau(A)^2 = \phi$ . In addition,  $\theta_{\tau} \in \operatorname{Con} (\mathbf{A}, \tau)$ ,  $\phi \subseteq \theta_{\phi}$ , and  $\Theta_{\tau}(\phi) \subseteq \theta_{\phi}$ .

*Proof.* Clearly,  $\theta_{\phi}$  is reflexive and symmetric. Moreover, if  $(x, y), (y, z) \in \theta_{\phi}$ , then  $(\tau(x), \tau(y)), (\tau(y), \tau(z)) \in \phi$  and thus  $(\tau(x), \tau(z)) \in \phi$  which gives  $(x, z) \in \theta_{\phi}$ .

Let  $f^{\mathbf{A}}$  be an *n*-ary operation on  $\mathbf{A}$  and let  $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$  be such that  $(x_i, y_i) \in \theta_{\phi}$  for any  $i = 1, \ldots, n$ . Then  $(\tau(x_i), \tau(y_i)) \in \phi$  holds for any  $i = 1, \ldots, n$ . Due to  $\phi \in \operatorname{Con} \tau(\mathbf{A})$ , we obtain  $(f^{\tau(\mathbf{A})}(\tau(x_1), \ldots, \tau(x_n)), f^{\tau(\mathbf{A})}(\tau(y_1), \ldots, \tau(y_n))) \in \phi$ .

Because  $\tau$  is an endomorphism,  $\tau(f^{\mathbf{A}}(x_1,\ldots,x_n)) = f^{\tau(\mathbf{A})}(\tau(x_1),\ldots,\tau(x_n))$ and  $\tau(f^{\mathbf{A}}(y_1,\ldots,y_n)) = f^{\tau(\mathbf{A})}(\tau(y_1),\ldots,\tau(y_n))$  which gives  $(\tau(f^{\mathbf{A}}(x_1,\ldots,x_n)),$  $\tau(f^{\mathbf{A}}(y_1,\ldots,y_n))) \in \phi$  and finally also  $(f^{\mathbf{A}}(x_1,\ldots,x_n),f^{\mathbf{A}}(y_1,\ldots,y_n)) \in \theta_{\phi}$ .

Moreover, take an arbitrary  $(x, y) \in \theta_{\phi}$ . Then  $(\tau(\tau(x)), \tau(\tau(y))) = (\tau(x), \tau(y)) \in \phi$  which gives  $(\tau(x), \tau(y)) \in \theta_{\phi}$ .

Hence,  $\theta_{\phi} \in \text{Con}(\mathbf{A}, \tau)$  and if  $\phi = \Delta_{\tau(\mathbf{A})}$ , then  $\theta_{\phi} = \theta_{\tau}$ .

It is clear that  $\theta_{\phi} \cap \tau(A)^2 \supseteq \phi$ . Now let  $(x, y) \in \theta_{\phi} \cap \tau(A)^2$ . Then  $x, y \in \tau(A)$ ,  $(\tau(x), \tau(y)) \in \phi \subseteq \tau(A)^2$ , so that  $x = \tau(x) \in \tau(A)$ ,  $y = \tau(y) \in \tau(A)$ , and consequently,  $(x, y) \in \phi$ .

It is evident that  $\theta_{\tau}$  is a congruence on  $(\mathbf{A}, \tau)$ .

Finally, if  $(x, y) \in \phi$  then  $\tau(x) = x$  and  $\tau(y) = y$  which gives  $(\tau(x), \tau(y)) = (x, y) \in \phi$ . Thus  $(x, y) \in \theta_{\phi}$  which finishes the proof that  $\phi \subseteq \theta_{\phi}$  and  $\Theta_{\tau}(\phi) \subseteq \theta_{\phi}$ .

**Lemma 3.3.** Let  $\theta \in \text{Con } \mathbf{A}$  be such that  $\theta \subseteq \theta_{\tau}$ . Then  $\theta \in \text{Con } (\mathbf{A}, \tau)$  holds. Moreover, if  $x, y \in A$  are such that  $(x, y) \in \theta_{\tau}$ , then  $\Theta(x, y) = \Theta_{\tau}(x, y)$ .

*Proof.* If  $(x, y) \in \theta \subseteq \theta_{\tau}$ , then  $\tau(x) = \tau(y)$  and thus  $(\tau(x), \tau(y)) = (\tau(x), \tau(x)) \in \theta$  proves that  $\theta \in \text{Con}(\mathbf{A}, \tau)$ .

Moreover, if  $(x, y) \in \theta_{\tau}$ , then  $\Theta(x, y) \subseteq \theta_{\tau}$ . Due to the first part of Lemma, we obtain  $\Theta(x, y) \in \text{Con}(\mathbf{A}, \tau)$  and thus  $\Theta_{\tau}(x, y) \subseteq \Theta(x, y)$  holds. The second inclusion is trivial.

**Lemma 3.4.** If  $x, y \in \tau(\mathbf{A})$ , then  $\Theta(x, y) = \Theta_{\tau}(x, y)$ . Consequently,  $\Theta(\phi) = \Theta_{\tau}(\phi)$  whenever  $\phi \subseteq \tau(A)^2$ .

*Proof.* Let us denote by  $\phi$  the congruence on  $\tau(\mathbf{A})$  generated by (x, y). Clearly we obtain the chain of inclusions  $\phi \subseteq \Theta(x, y) \subseteq \Theta(\phi) \subseteq \theta_{\phi}$  (because  $(x, y) \in \phi$  and  $\phi \subseteq \theta_{\phi}$ , see Lemma 3.2).

Assume  $(a,b) \in \Theta(x,y)$ , then  $(a,b) \in \theta_{\phi}$  and thus  $(\tau(a),\tau(b)) \in \phi \subseteq \Theta(x,y)$ . Thus  $\Theta(x,y) \in \text{Con}(\mathbf{A},\tau)$  and  $\Theta_{\tau}(x,y) \subseteq \Theta(x,y)$  holds. The second inclusion is trivial.

Finally, let  $\phi \subseteq \tau(A)^2$ . By [2, Thm 5.3], the both congruence lattices of **A** and  $(\mathbf{A}, \tau)$  are complete sublattices of the lattice of equivalencies on **A**, and therefore, they have the same infinite suprema. Hence, by the first part of the lemma,

$$\Theta(\phi) = \bigvee_{(x,y)\in\phi} \Theta(x,y) = \bigvee_{(x,y)\in\phi} \Theta_{\tau}(x,y) = \Theta_{\tau}(\phi).$$

**Remark 3.5.** By Lemma 3.2, if  $\phi$  is a congruence on  $\tau(\mathbf{A})$ , then  $\theta_{\phi}$  is an extension of  $\phi$  on  $(\mathbf{A}, \tau)$  and  $\Theta(\phi) = \Theta_{\tau}(\phi) \subseteq \theta_{\phi}$ . There is a natural question whether  $\Theta(\phi) = \theta_{\phi}$ ? The answer is positive if and only if  $\tau$  is the identity on A. Indeed, if  $\tau$  is the identity on A, the statement is evident, in the opposite case, we have  $\theta_{\Delta_{\tau}(\mathbf{A})} = \theta_{\tau} \neq \Delta_{\mathbf{A}} = \Theta(\Delta_{\tau(\mathbf{A})}).$ 

**Theorem 3.6.** Let  $(\mathbf{A}, \tau)$  be a subdirectly irreducible state-morphism algebra such that  $\mathbf{A}$  is subdirectly reducible. Then there is a subdirectly irreducible algebra  $\mathbf{B}$  such that  $(\mathbf{A}, \tau)$  is  $\mathbf{B}$ -subdiagonal.

*Proof.* First, if  $\theta_{\tau} = \Delta_{\mathbf{A}}$ , then for any  $x \in A$ , the equality  $\tau(x) = x$  holds and thus Con  $\mathbf{A} = \text{Con}(\mathbf{A}, \tau)$  which is absurd because  $\mathbf{A}$  is subdirectly irreducible and  $(\mathbf{A}, \tau)$  is not subdirectly irreducible.

The subdirect irreducibility of  $(\mathbf{A}, \tau)$  implies that there is a least proper congruence  $\theta_{\min} \in \text{Con}(\mathbf{A}, \tau)$ . Moreover, due to Lemma 3.3, the congruence  $\theta_{\min}$  is also a least proper congruence  $\theta$  on  $\mathbf{A}$  with  $\theta \subseteq \theta_{\tau}$  and thus  $\theta_{\min}$  is an atom in Con  $\mathbf{A}$ . Let us denote

$$\theta_{\tau}^{\perp} = \{ \theta \in \operatorname{Con} \mathbf{A} : \theta \cap \theta_{\tau} = \Delta_{\mathbf{A}} \}.$$

First, we prove that there exists proper  $\theta \in \theta_{\tau}^{\perp}$ . The subdirect reducibility of **A** shows that there exists proper  $\theta \in \text{Con } \mathbf{A}$  with  $\theta_{\min} \not\subseteq \theta$ . Hence,  $\theta_{\tau} \cap \theta = \Delta_{\mathbf{A}}$  holds (because if  $\theta_{\tau} \cap \theta \neq \Delta_{\mathbf{A}}$ , then  $\theta_{\tau} \cap \theta$  is a proper congruence contained in  $\theta_{\tau}$  and minimality of  $\theta_{\min}$  yields  $\theta_{\min} \subseteq \theta \cap \theta_{\tau} \subseteq \theta$ , which is absurd).

Moreover, let us have  $\theta_n \in \theta_{\tau}^{\perp}$  for any  $n \in \mathbb{N}$  with  $\theta_n \subseteq \theta_{n+1}$ , then clearly  $\bigvee_{n \in \mathbb{N}} \theta_n = \bigcup_{n \in \mathbb{N}} \theta_n \in \theta_{\tau}^{\perp}$ . Due to Zorn's Lemma, there is maximal  $\theta^* \in \theta_{\tau}^{\perp}$ .

We have proved that both  $\theta_{\tau}$  and  $\theta^*$  are proper congruences on  $\mathbf{A}$  with  $\theta_{\tau} \cap \theta^* = \Delta_{\mathbf{A}}$ . By the Birkhoff Theorem about subdirect reducibility,  $\mathbf{A}$  is a subdirect product of two algebras  $\mathbf{A}/\theta_{\tau}$  and  $\mathbf{A}/\theta^*$  with an embedding  $h : \mathbf{A} \longrightarrow \mathbf{A}/\theta_{\tau} \times \mathbf{A}/\theta^*$  defined by  $h(x) = (x/\theta_{\tau}, x/\theta^*)$ .

Now we define the mapping  $\psi : A/\theta_{\tau} \longrightarrow A/\theta^*$  by  $\psi(x/\theta_{\tau}) = \tau(x)/\theta^*$ . Clearly  $\psi$  is well-defined because  $x/\theta_{\tau} = y/\theta_{\tau}$  yields  $\tau(x) = \tau(y)$  and thus  $\psi(x/\theta_{\tau}) = \tau(x)/\theta^* = \tau(y)/\theta^* = \psi(y/\theta_{\tau})$ . Let us suppose that  $\psi(x/\theta_{\tau}) = \psi(y/\theta_{\tau})$ . Then  $\tau(x)/\theta^* = \tau(y)/\theta^*$  and  $(\tau(x), \tau(y)) \in \theta^*$ . Hence,  $\Theta(\tau(x), \tau(y)) \subseteq \theta^*$  holds. Finally, if  $\tau(x) \neq \tau(y)$  (thus  $\Theta(\tau(x), \tau(y))$  is a proper congruence), then  $\tau(x), \tau(y) \in \tau(\mathbf{A})$  and Lemma 3.4 yields  $\Theta(\tau(x), \tau(y)) \in \operatorname{Con}(\mathbf{A}, \tau)$  and thus  $\theta_{\min} \subseteq \Theta(\tau(x), \tau(y)) \subseteq \theta^*$  which is absurd  $(\theta_{\min} \subseteq \theta_{\tau} \cap \theta^* = \Delta_{\mathbf{A}})$ . Therefore, the mapping  $\psi$  is injective.

We shall prove that  $\psi$  is a homomorphism (and thus an embedding). If  $f^{\mathbf{A}}$  is an *n*-ary operation and  $x_1/\theta_{\tau}, \ldots, x_n/\theta_{\tau} \in \mathbf{A}/\theta_{\tau}$ , then

$$\psi(f^{\mathbf{A}/\theta_{\tau}}(x_{1}/\theta_{\tau},\ldots,x_{n}/\theta_{\tau})) = \psi(f^{\mathbf{A}}(x_{1},\ldots,x_{n})/\theta_{\tau})$$

$$= \tau(f^{\mathbf{A}}(x_{1},\ldots,x_{n}))/\theta^{*}$$

$$= f^{\mathbf{A}}(\tau(x_{1}),\ldots,\tau(x_{n}))/\theta^{*}$$

$$= f^{\mathbf{A}/\theta^{*}}(\tau(x_{1})/\theta^{*},\ldots,\tau(x_{n})/\theta^{*})$$

$$= f^{\mathbf{A}/\theta^{*}}(\psi(x_{1}/\theta_{\tau}),\ldots,\psi(x_{n}/\theta_{\tau})).$$

Now we prove that **A** is  $\mathbf{A}/\theta^*$ -diagonal. Let  $g : A \longrightarrow (A/\theta^*)^2$  be defined via  $g(x) = (\psi(x/\theta_\tau), x/\theta^*) = (\tau(x)/\theta^*, x/\theta^*)$ . Because the mapping g is the composition of two functions h and  $\psi$  which are embeddings, g is also an embedding of **A** 

into  $(\mathbf{A}/\theta^*)^2$ . Now we can compute:

$$g(\tau(x)) = (\tau(\tau(x))/\theta^*, \tau(x)/\theta^*)$$
  
=  $(\tau(x)/\theta^*, \tau(x)/\theta^*)$   
=  $\tau_{\mathbf{A}/\theta^*}(\tau(x)/\theta^*, x/\theta^*)$   
=  $\tau_{\mathbf{A}/\theta^*}(g(x)),$ 

where  $\tau_{\mathbf{A}/\theta^*}$  is the diagonal state-morphism on the product  $\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*$ . Therefore,  $g: (\mathbf{A}, \tau) \longrightarrow (\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*, \tau_{\mathbf{A}/\theta^*})$  is an embedding and  $(\mathbf{A}, \tau)$  is  $\mathbf{A}/\theta^*$ -diagonal.

Finally, we prove the subdirect irreducibility of  $\mathbf{A}/\theta^*$ . Of course,  $\theta_{\min} \cap \theta^* = \Delta_{\mathbf{A}}$ yields  $\theta_{\min} \not\subseteq \theta^*$  and thus  $\theta^* \subset \theta^* \lor \theta_{\min}$ . Moreover, if  $\theta^* \subset \theta$ , from maximality of  $\theta^*$  we obtain  $\theta \cap \theta_\tau \neq \Delta_{\mathbf{A}}$  and thus  $\theta_{\min} \subseteq \theta_\tau \cap \theta$ . Finally,  $\theta_{\min} \lor \theta^* \subseteq (\theta_\tau \cap \theta) \lor \theta^* \subseteq (\theta_\tau \cap \theta) \lor \theta = \theta$  holds. Hence, for any congruence  $\theta \in \text{Con } \mathbf{A}$ , the inequality  $\theta^* \subset \theta^* \cap \theta_{\min} \subseteq \theta$  holds. Due to the Birkhoff's Theorem and the Second Homomorphism Theorem, an algebra  $\mathbf{A}/\theta^*$  is subdirectly irreducible.

Theorem 3.6 can be extended as follows.

**Theorem 3.7.** For every subdirectly irreducible state-morphism algebra  $(\mathbf{A}, \tau)$ , there is a subdirectly irreducible algebra  $\mathbf{B}$  such that  $(\mathbf{A}, \tau)$  is  $\mathbf{B}$ -subdiagonal.

*Proof.* There are two cases: (1)  $(\mathbf{A}, \tau)$  and  $\mathbf{A}$  are subdirectly irreducible, and (2)  $(\mathbf{A}, \tau)$  is a subdirectly irreducible state-morphism algebra and  $\mathbf{A}$  is a subdirectly reducible algebra

(1) Assume that  $(\mathbf{A}, \tau)$  and  $\mathbf{A}$  are subdirectly irreducible. Define two statemorphism algebras  $(\tau(\mathbf{A}) \times \mathbf{A}, \tau_1)$  and  $(\mathbf{A} \times \mathbf{A}, \tau_2)$ , where  $\tau_1(a, b) = (a, a), (a, b) \in \tau(A) \times A$ , and  $\tau_2(a, b) = (a, a), a, b \in A$ . Then the first one is a subalgebra of the second one.

Define a mapping  $\phi : A \to \tau(A) \times A$  defined by  $\phi(a) = (\tau(a), a), a \in A$ . Then  $\phi$  is injective because if  $\phi(a) = \phi(b)$  then  $(\tau(a), a) = (\tau(b), b)$  and a = b. We show that  $\phi$  is a homomorphism. Let  $f^{\mathbf{A}}$  be an *n*-ary operation on  $\mathbf{A}$  and let  $a_1, \ldots, a_n \in A$ . Then

$$\phi(f^{\mathbf{A}}(a_1,\ldots,a_n)) = (\tau(f^{\mathbf{A}}(a_1,\ldots,a_n)), f^{\mathbf{A}}(a_1,\ldots,a_n))$$
$$= (f^{\mathbf{A}}(\tau(a_1),\ldots,\tau(a_n)), f^{\mathbf{A}}(a_1,\ldots,a_n))$$
$$= f^{\tau(\mathbf{A})\times\mathbf{A}}((\tau(a_1),a_1),\ldots,(\tau(a_n),a_n))$$
$$= f^{\tau(\mathbf{A})\times\mathbf{A}}(\phi(a_1),\ldots,\phi(a_n)).$$

Since  $\phi : \mathbf{A} \to \tau(\mathbf{A}) \times \mathbf{A} \subseteq \mathbf{A} \times \mathbf{A}$ ,  $\phi$  can be assumed also as an injective homomorphism from the state-morphism algebra  $(\mathbf{A}, \tau)$  into the state-morphism algebra  $D(\mathbf{B})$ , where  $\mathbf{B} := \mathbf{A}$  is a subdirectly irreducible algebra.

(2) This case was proved in Theorem 3.6.

For example, a state-morphism algebra  $(\mathbf{A}, \mathrm{Id}_A)$ , where  $\mathrm{Id}_A$  is the identity on A, is subdirectly irreducible if and only if  $\mathbf{A}$  is subdirectly irreducible. Therefore,  $(\mathbf{A}, \mathrm{Id}_A)$  can be embedded into  $(\mathbf{A} \times \mathbf{A}, \tau_A)$  under the mapping  $a \mapsto (a, a), a \in A$ . Consequently, every subdirectly irreducible state-morphism algebra  $(\mathbf{A}, \mathrm{Id}_A)$  is  $\mathbf{A}$ -subdiagonal with  $\mathbf{A}$  subdirectly irreducible.

We note that in the same way as in [13, Lem 6.1], it is possible to show that the class of subdiagonal state-morphism algebras is closed under subalgebras and ultraproducts, and not closed under homomorphic images, see [13, Lem 6.6].

4. VARIETIES OF STATE-MORPHISM ALGEBRAS AND THEIR GENERATORS

In this section, we study varieties of state-morphism algebras and their generators. It is interesting to note that some similar results proved for state-morphism MV-algebras in [13] can be obtained in an analogous way also for a general variety of algebras.

Let  $\tau$  be a state-morphism operator on an algebra **A**. We set

$$\operatorname{Ker}(\tau) := \{ (x, y) \in A \times A : \tau(x) = \tau(y) \},\$$

the kernel of  $\tau$ . We say that  $\tau$  is faithful if  $\text{Ker}(\tau) = \Delta_{\mathbf{A}}$ . It is evident that  $\tau$  is faithful iff  $\tau(x) = x$  for each  $x \in A$ . In addition,  $\tau$  is faithful iff  $\tau$  is injective.

For every class  $\mathcal{K}$  of same type algebras, we set  $\mathsf{D}(\mathcal{K}) = \{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$ , where  $D(\mathbf{A}) = (\mathbf{A} \times \mathbf{A}, \tau_A)$ .

As usual, given a class  $\mathcal{K}$  of algebras of the same type,  $I(\mathcal{K})$ ,  $H(\mathcal{K})$ ,  $S(\mathcal{K})$  and  $P(\mathcal{K})$  and  $P_U(\mathcal{K})$  will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from  $\mathcal{K}$ , respectively. Moreover,  $V(\mathcal{K})$  will denote the variety generated by  $\mathcal{K}$ .

**Lemma 4.1.** (1) Let  $\mathcal{K}$  be a class of algebras of the same type F. Then  $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{\tau}$ .

(2) Let  $\mathcal{V}$  be any variety. Then  $\mathcal{V}_{\tau} = \mathsf{ISD}(\mathcal{V})$ .

*Proof.* (1) If  $D(\mathbf{A}) \in \mathsf{D}(\mathcal{K})$  (thus  $\mathbf{A} \in \mathcal{K}$ ), then the *F*-reduct of the algebra  $D(\mathbf{A})$  is the algebra  $\mathbf{A} \times \mathbf{A}$  which belongs to the variety  $\mathsf{V}(\mathcal{K})$ . Due to definition of  $\mathsf{V}(\mathcal{K})_{\tau}$ , we obtain also  $D(\mathbf{A}) \in \mathsf{V}(\mathcal{K})_{\tau}$ . We have proved that  $\mathsf{D}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{\tau}$ . Because  $\mathsf{V}(\mathcal{K})_{\tau}$  is a variety then also  $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{\tau}$ 

(2) Let  $(\mathbf{A}, \tau) \in \mathcal{V}_{\tau}$ . As we have seen in the proof of Theorem 3.7, the map  $\phi : a \mapsto (\tau(a), a)$  is an injective homomorphism of  $(\mathbf{A}, \tau)$  into  $D(\mathbf{A})$ . Hence,  $\phi$  is compatible with  $\tau$ , and  $(\mathbf{A}, \tau) \in \mathsf{ISD}(\mathcal{V})$ . Conversely, the *F*-reduct of any algebra in  $\mathsf{D}(\mathcal{V})$  is in  $\mathcal{V}$ , (being a direct product of algebras in  $\mathcal{V}$ ), and hence the *F*-reduct of any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathsf{ISD}(\mathcal{V}) = \mathcal{V}$ . Hence, any member of  $\mathsf{ISD}(\mathcal{V})$  is in  $\mathcal{V}_{\tau}$ .  $\Box$ 

**Lemma 4.2.** Let  $\mathcal{K}$  be a class of algebras of the same type F. Then:

- (1)  $\mathsf{DH}(\mathcal{K}) \subseteq \mathsf{HD}(\mathcal{K}).$ (2)  $\mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K}).$
- (3)  $\mathsf{DP}(\mathcal{K}) \subseteq \mathsf{IPD}(\mathcal{K})$ .
- (4)  $VD(\mathcal{K}) = ISD(V(\mathcal{K})).$

*Proof.* (1) Let  $D(\mathbf{C}) \in \mathsf{DH}(\mathcal{K})$ . Then there are  $\mathbf{A} \in \mathcal{K}$  and a homomorphism h from  $\mathbf{A}$  onto  $\mathbf{C}$ . Let for all  $a, b \in A$ ,  $h^*(a, b) = (h(a), h(b))$ . We claim that  $h^*$  is a homomorphism from  $D(\mathbf{A})$  onto  $D(\mathbf{C})$ . That  $h^*$  is a homomorphism is clear. We verify that  $h^*$  is compatible with  $\tau_A$ . We have  $h^*(\tau_A(a, b)) = h^*(a, a) = (h(a), h(a)) = \tau_C(h(a), h(b)) = \tau_C(h^*(a, b))$ . Finally, since h is onto, given  $(c, d) \in C \times C$ , there are  $a, b \in A$  such that h(a) = c and h(b) = d. Hence,  $h^*(a, b) = (c, d)$ ,  $h^*$  is onto, and  $D(\mathbf{C}) \in \mathsf{HD}(\mathcal{K})$ .

(2) It is trivial.

(3) Let  $\mathbf{A} = \prod_{i \in I} (\mathbf{A}_i) \in \mathsf{P}(\mathcal{K})$ , where each  $\mathbf{A}_i$  is in  $\mathcal{K}$ . Then the map

$$\Phi: ((a_i: i \in I), (b_i: i \in I)) \mapsto ((a_i, b_i): i \in I)$$

is an isomorphism from  $D(\mathbf{A})$  onto  $\prod_{i \in I} D(\mathbf{A}_i)$ . Indeed, it is clear that  $\Phi$  is an *F*-isomorphism. Moreover, denoting the state-morphism of  $\prod_{i \in I} D(\mathbf{A}_i)$  by  $\tau^*$ , we

$$\Phi(\tau_A((a_i:i\in I),(b_i:i\in I))) = \Phi((a_i:i\in I),(a_i:i\in I)) = \\ = ((a_i,a_i):i\in I) = (\tau_{\mathbf{A}_i}(a_i,b_i):i\in I) = \tau^*(\Phi((a_i:i\in I),(b_i:i\in I))),$$

and hence  $\Phi$  is an isomorphism.

(4) By (1), (2) and (3),  $\mathsf{DV}(\mathcal{K}) = \mathsf{DHSP}(\mathcal{K}) \subseteq \mathsf{HSPD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$ , and hence  $\mathsf{ISDV}(\mathcal{K}) \subseteq \mathsf{ISVD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$ . Conversely, by Lemma 4.1(1),  $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{\tau}$ , and by Lemma 4.1(2),  $\mathsf{V}(\mathcal{K})_{\tau} = \mathsf{ISDV}(\mathcal{K})$ . This proves the claim.

**Theorem 4.3.** (1) For every class  $\mathcal{K}$  of algebras of the same type F,  $V(D(\mathcal{K})) = V(\mathcal{K})_{\tau}$ .

(2) Let  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be two classes of same type algebras. Then  $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$  if and only if  $V(\mathcal{K}_1) = V(\mathcal{K}_2)$ .

*Proof.* (1) By Lemma 4.2(4),  $VD(\mathcal{K}) = ISD(V(\mathcal{K}))$ . Moreover, by Lemma 4.1(2),  $V(\mathcal{K})_{\tau} = ISDV(\mathcal{K})$ . Hence,  $V(D(\mathcal{K})) = V(\mathcal{K})_{\tau}$ .

(2) We have  $V(D(\mathcal{K}_1)) = V(\mathcal{K}_1)_{\tau}$  and  $V(D(\mathcal{K}_2)) = V(\mathcal{K}_2)_{\tau}$ . Clearly,  $V(\mathcal{K}_1) = V(\mathcal{K}_2)$  implies  $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$ , and hence  $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$ . Conversely,  $V(D(\mathcal{K}_1)) = V(D(\mathcal{K}_2))$  implies  $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$ . But any algebra  $\mathbf{A} \in V(\mathcal{K}_1)$  is the *F*-reduct of a state-morphism algebra in  $V(\mathcal{K}_1)_{\tau}$ , namely of  $(\mathbf{A}, \mathrm{Id}_A)$ .

It follows that, if  $V(\mathcal{K}_1)_{\tau} = V(\mathcal{K}_2)_{\tau}$ , then the classes of *F*-reducts of  $V(\mathcal{K}_1)_{\tau}$  and of  $V(\mathcal{K}_2)_{\tau}$  coincide, and hence  $V(\mathcal{K}_1) = V(\mathcal{K}_2)$ .

As a direct corollary of Theorem 4.3, we have:

**Theorem 4.4.** If a system  $\mathcal{K}$  of algebras of the same type F generates the whole variety  $\mathcal{V}(F)$  of all algebras of type F, then the variety  $\mathcal{V}(F)_{\tau}$  of all state-morphism algebras  $(\mathbf{A}, \tau)$ , where  $\mathbf{A} \in \mathcal{V}(F)$ , is generated by the class  $\{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$ .

Some applications of the latter theorem for different varieties of algebras will be done in Section 5.

**Theorem 4.5.** If **A** is a subdirectly irreducible algebra, then any state-morphism algebra  $(\mathbf{A}, \tau)$  is subdirectly irreducible.

*Proof.* Let **A** be a subdirectly irreducible algebra and let  $\tau$  be a state-morphism operator on **A**. If  $\tau$  is the identity on A, then Con  $\mathbf{A} = \text{Con}(\mathbf{A}, \tau)$  and, consequently,  $(\mathbf{A}, \tau)$  is subdirectly irreducible. If  $\tau$  is not the identity on A, then  $\theta_{\tau}$ , defined by (3.1), is a nontrivial congruence on **A**, and thus  $\theta_{\min} \subseteq \theta_{\tau}$ , where  $\theta_{\min} \in \text{Con } \mathbf{A}$ is the least nontrivial congruence. Hence,  $\theta_{\min}$  belongs to the set Con $(\mathbf{A}, \tau)$ , see Lemma 3.3. Therefore, Con $(\mathbf{A}, \tau) \subseteq$  Con **A** yields the subdirect irreducibility of the algebra  $(\mathbf{A}, \tau)$ , more precisely,  $\theta_{\min}$  is also the least proper congruence in Con $(\mathbf{A}, \tau)$ .

We remind the following Mal'cev Theorem, [2, Lem 3.1].

**Theorem 4.6.** Let **A** be an algebra and  $\phi \subseteq A^2$ . Then  $(a,b) \in \Theta(\phi)$  if and only if there exist two finite sequences of terms  $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$  and pairs  $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$  with

$$a = t_1(\overline{x}_1, a_1), t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1}) \text{ and } t_n(\overline{x}_n, b_n) = b$$

for some  $\overline{x}_1, \ldots, \overline{x}_n \in A$ .

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We say that an algebra **B** has the Congruence Extension Property (CEP for short) if, for any algebra **A** such that **B** is a subalgebra of **A** and for any congruence  $\theta \in \text{Con } \mathbf{B}$ , there is a congruence  $\phi \in \text{Con } \mathbf{A}$  such that  $\theta = (B \times B) \cap \phi$ . A variety  $\mathcal{K}$  has the CEP if every algebra in  $\mathcal{K}$  has the CEP. For example, the variety of MValgebra, or the variety of BL-algebras or the variety of state-morphism MV-algebras (see [13, Lem 6.1]) satisfies the CEP.

### **Theorem 4.7.** A variety $\mathcal{V}_{\tau}$ satisfy the CEP if and only if $\mathcal{V}$ satisfies the CEP.

*Proof.* Let us have a variety  $\mathcal{V}$  with the CEP. If  $\mathbf{A} \in \mathcal{V}$  is such that  $(\mathbf{A}, \tau)$  is an algebra with state-morphism, for any subalgebra  $(\mathbf{B}, \tau) \subseteq (\mathbf{A}, \tau)$  and any  $\phi \in \text{Con}(\mathbf{B}, \tau)$ , the condition  $\phi = B^2 \cap \Theta(\phi)$  holds.

Now we prove  $\Theta(\phi) = \Theta_{\tau}(\phi)$ . To show that, assume  $(a, b) \in \Theta(\phi)$ . Mal'cev's Theorem shows the existence of finite sequences of terms  $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$  and pairs  $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$  with

$$a = t_1(\overline{x}_1, a_1), t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1}) \text{ and } t_n(\overline{x}_n, b_n) = b$$

for some  $\overline{x}_1, \ldots, \overline{x}_n \in A$ . Because  $\tau$  is an endomorphism, we obtain also equalities

$$\tau(a) = t_1(\tau(\overline{x}_1), \tau(a_1)), \ t_i(\tau(\overline{x}_i), \tau(b_i)) = t_{i+1}(\tau(\overline{x}_{i+1}), \tau(a_{i+1}))$$

and

$$t_n(\tau(\overline{x}_n), \tau(b_n)) = \tau(b).$$

We have assumed that  $\phi \in \text{Con}(\mathbf{B}, \tau)$ , thus  $(a_i, b_i) \in \phi$  yields  $(\tau(a_i), \tau(b_i)) \in \phi$ for any i = 1, ..., n. Now, we have obtained  $(\tau(a), \tau(b)) \in \Theta(\phi)$ . In other words,  $\Theta(\phi) \in \text{Con}(\mathbf{A}, \tau)$  and thus  $\Theta(\phi) = \Theta_{\tau}(\phi)$ .

If  $\mathcal{V}_{\tau}$  has the CEP, then for any  $\mathbf{A} \in \mathcal{V}$ , we have  $\operatorname{Con} \mathbf{A} = \operatorname{Con}(\mathbf{A}, \operatorname{Id}_A)$ . Clearly, the CEP on  $(\mathbf{A}, \operatorname{Id}_A)$  yields the CEP on  $\mathbf{A}$ .

### 5. Applications to Special Types of Algebras

In this section, we apply a general result concerning generators of some varieties of state-morphism algebras, Theorem 4.3, to the variety of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BLalgebras, and state-morphism pseudo MV-algebras, when we use different systems of t-norms on the real interval [0, 1] and a special type of pseudo MV-algebras, respectively.

Algebras for which the logic MTL is sound are called MTL-algebras. They can be characterized as prelinear commutative bounded integral residuated lattices. In more detail, according to [15], an algebraic structure  $\mathbf{A} = (A; \land, \lor, *, \rightarrow, 0, 1)$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  is an *MTL-algebra* if

- (M1)  $(A; \land, \lor, 0, 1)$  is a bounded lattice with the top element 0 and bottom element 1,
- (M2) (A; \*, 1) is a commutative monoid,
- (M3) \* and  $\rightarrow$  form an adjoint pair, that is,  $z * x \leq y$  if and only if  $z \leq x \rightarrow y$ , where  $\leq$  is the lattice order of  $(A; \land, \lor)$  for all  $x, y, z \in A$ , (the residuation condition),
- (M4)  $(x \to y) \lor (y \to x) = 1$  holds for all  $x, y \in A$  (the prelinearity condition).

If t is any left-continuous t-norm on [0, 1], we define two binary operations  $*_t \to_t$ on [0, 1] via  $x *_t y = t(x, y)$  and  $x \to_t y = \sup\{z \in [0, 1] : t(z, x) \le y\}$  for  $x, y \in [0, 1]$ , then  $\mathbb{I}_t = ([0, 1]; \min, \max, *_t, \rightarrow_t, 0, 1)$  is an example of an MTL-algebra. An MTL-algebra  $\mathbb{I}_t$  is a BL-algebra iff t is continuous.

Due to [15], the class  $\mathcal{T}_{lc}$ , which denotes the system of all BL-algebras  $\mathbb{I}_t$ , where t is a left-continuous t-norm on the interval [0, 1], generates the variety of MTL-algebras. This result was strengthened in [27] who introduced the class of regular left-continuous t-norms which is strictly smaller than the class of left-continuous t-norms, but they generate the variety of MTL-algebras.

According to [1], we say that an algebra  $\mathbf{A} = (A; \lor, \land, \cdot, \rightarrow, 0, 1)$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  is a *non-associative BL-algebra* (naBL-algebra in short) if

- (A1)  $(A; \lor, \land, 0, 1)$  is a bounded lattice,
- (A2)  $(A; \cdot, 1)$  is a commutative groupoid with the neutral element 1,
- (A3) any  $x, y, z \in A$  satisfy  $x \cdot y \leq z$  if and only if  $x \leq y \rightarrow z$ ,
- (A4) algebra satisfy the divisibility axiom  $(x \cdot (x \to y) = x \land y)$ ,
- (A5) algebra satisfy the  $\alpha$ -prelinearity and  $\beta$ -prelinearity  $(x \to y \lor \alpha_b^a(y \to x) = x \to y \lor \beta_b^a(y \to x) = 1)$ , where  $\alpha_b^a(x) = (a \cdot b) \to (a \cdot (b \cdot x))$  and  $\beta_b^a(x) = b \to (a \to ((a \cdot b) \cdot x))$ .

A function  $t : [0,1] \times [0,1] \rightarrow [0,1]$  on the interval [0,1] of reals is said to be a *non-associative* t-norm (nat-norm briefly) if

(nat1) ([0,1];t,1) is a commutative groupoid with the neutral element 1,

(nat2) t is continuous in the usual sense,

(nat3) if  $x, y, z \in [0, 1]$  are such that  $x \leq y$ , then  $t(x, z) \leq t(y, z)$ .

According to [1, Thm 5], for any nat-norm there is a unique binary operation  $\rightarrow_t$  satisfying the adjointness condition, i.e.  $t(x,y) \leq z$  if and only if  $x \leq y \rightarrow_t z$ . Moreover, an algebra  $\mathbb{I}_t^{na} := ([0,1]; \min, \max, t, \rightarrow_t, 0, 1)$  is an naBL-algebra.

The class of all naBL-algebras is denoted by  $na\mathcal{BL}$  and  $na\mathcal{T}$  denotes the class of all naBL-algebras  $\mathbb{I}_t^{na}$  for any non-associative t-norm. The main result on nonassociative BL-algebras says that  $na\mathcal{T}$  is the generating class for the variety  $na\mathcal{BL}$ , [1, Thm 8]:

Theorem 5.1. There hods

$$na\mathcal{BL} = \mathsf{IP}_{\mathsf{S}}\mathsf{SP}_{\mathsf{U}}(na\mathcal{T}).$$

Finally, we recall that a noncommutative generalization of MV-algebras was introduced in [17] as pseudo MV-algebras or in [25] as generalized MV-algebras. According to [10], every pseudo MV-algebra  $(M; \oplus, \neg, \sim, 0, 1)$  of type  $\langle 2, 1, 1, 0, 0 \rangle$ is an interval in a unital  $\ell$ -group (G, u) with strong unit u, i.e.  $M \cong \Gamma(G, u) := [0, u]$ , where  $x \oplus y = (x + y) \land$ ,  $x^{\neg} = u - x$ ,  $x^{\sim} = -x + u$ , 0 = 0, and 1 = u. If (G, u)is double transitive (for definitions and details see [12]), then  $\Gamma(G, u)$  generates the variety of pseudo MV-algebras, [12, Thm 4.8]. For example, if Aut( $\mathbb{R}$ ) is the set of all automorphisms of the real line  $\mathbb{R}$  preserving the natural order in  $\mathbb{R}$  and u(t) := t + 1,  $t \in \mathbb{R}$ , let Aut<sub>u</sub>( $\mathbb{R}$ ) = { $g \in Aut(\mathbb{R}) : g \leq nu$  for some integer  $n \geq 1$ }. Then  $\Gamma(Aut_u(\mathbb{R}), u)$  is double transitive and it generates the variety of pseudo MV-algebras, see [12, Ex 5.3].

Now we apply the general statement, Theorem 4.4, on generators to different types of state-morphism algebras. We recall that  $\mathcal{T}$  was defined as the class of all BL-algebras  $\mathbb{I}_t$ , where t is a continuous t-norm on [0, 1].

**Theorem 5.2.** (1) The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra  $D([0,1]_{MV})$ .

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(2) The variety of all state-morphism BL-algebras is generated by the class  $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}.$ 

(3) The variety of all state-morphism MTL-algebras is generated by the class  $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc}\}.$ 

(4) The variety of all state-morphism naBL-algebras is generated by the class  $\{D(\mathbb{I}_t^{na}) : \mathbb{I}_t \in na\mathcal{T}\}.$ 

(5) If a unital  $\ell$ -group (G, u) is double transitive, then  $D(\Gamma(G, u))$  generates the variety of state-morphism pseudo MV-algebras.

*Proof.* (1) It follows from the fact that the MV-algebra of the real interval [0,1] generates the variety of MV-algebras, see e.g. [4, Prop 8.1.1], and then apply Theorem 4.4.

(2) The statement follows from the fact that  $V(\mathcal{T})$  is by [3, Thm 5.2] the variety  $\mathcal{BL}$  of all BL-algebras. Now it suffices to apply Theorem 4.4.

(3) By [15], the class  $\mathcal{T}_{lc}$  of all  $\mathbb{I}_t$ , where t is any left-continuous t-norms on the interval [0, 1], generates the variety of MTL-algebras; then apply Theorem 4.4.

(4) By [1, Thm 8] or Theorem 5.1, the class  $na\mathcal{T}$  of all  $\mathbb{I}_t$ , where t is any non-associative t-norms on the interval [0, 1], generates the variety of non-associative BL-algebras; then apply again Theorem 4.4.

(5) By the above,  $\Gamma(G, u)$  generates the variety of pseudo MV-algebras, see also [12, Thm 4.8]; then apply Theorem 4.4.

We note that the case (1) in Theorem 4.4 was an open problem posed in [7] and was positively solved in [13, Thm 5.4(3)].

### 6. CONCLUSION

In the paper, we have presented a general approach to theory of state-morphism algebras which generalizes state-morphism MV-algebras and state-morphism BL-algebras as pairs  $(\mathbf{A}, \tau)$ , where  $\mathbf{A}$  is an algebra of type F and  $\tau$  is an endomorphism of  $\mathbf{A}$  such that  $\tau \circ \tau = \tau$ .

This enables us to present complete characterizations of subdirectly irreducible state BL-algebras and subdirectly irreducible state-morphism BL-algebras, Theorem 2.7, which generalizes the results from [7, 9, 11, 13].

A general approach is studied in the third section where the main result, Theorem 3.7, says that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one.

The fourth section describes some generators of the varieties of state-morphism algebras, and Theorem 4.4 shows that if a class  $\mathcal{K}$  generates a variety  $\mathcal{V}$  of algebras of the same type F, then the variety of state-morphism algebras whose F-reduct belongs to the class  $\mathcal{K}$  is generated by the class of diagonal state-morphism algebras  $D(\mathbf{A})$ , where  $\mathbf{A} \in \mathcal{K}$ . In addition, Theorem 4.7 deals with the CEP for the variety of state-morphism algebras.

In Theorem 5.2, Theorem 4.4 was applied to the special class of algebras: MValgebras, BL-algebras, MTL-algebras, non-associative BL-algebras, and pseudo MValgebras to obtain the generators of the corresponding varieties of state-morphism algebras.

During the study on this paper, we found some interesting open problems like: (1) find a characterization of an analogue of a state-operator that is not necessarily a state-morphism operator, (2) if the lattice of varieties of some variety is countable,

how big is the lattice of corresponding state-morphism algebras, e.g. in the case of MV-algebras, the lattice under question is uncountable [13], (3) decidability of the variety of state-morphism algebras.

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