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Coalitional Bargaining Games with Random Proposers: Theory and Application

by

Akira Okada

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Abstract

We consider a noncooperative coalitional bargaining game with random proposers. In a general case that the recognition probability is arbitrary and players have different discount factors for future payoffs, the existence of a stationary subgame perfect equilibrium (SSPE) is proved, and the condition for the grand coalition to be formed is provided. We also prove that the grand-coalition SSPE is a unique symmetric SSPE for any discount factor in a symmetric game with nonempty core. In the last part of the paper, we apply the bargaining model to a production economy with one employer and multiple workers. When players are sufficiently patient, the economy has a unique SSPE payoff. The equilibrium allocation is compared with cooperative solutions such as the core, the Shapley value and the nucleolus. The SSPE payoff and the nucleolus have similar distributional properties.

Key words: noncooperative coalitional bargaining, random proposers, Nash program, production economy, core, nucleolus

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†Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601 JAPAN. Phone: +81 42 580 8599, Fax: +81 42 580 8748, E-mail: aokada@econ.hit-u.ac.jp

1 Introduction

Game theory is traditionally divided into two branches, noncooperative game theory and cooperative game theory. Both theories can be applied to the same problem of multilateral bargaining which has been one of the central theme in game theory since the classic work of von Neumann and Morgenstern (1944). It is desirable to explore similarities and differences of the two theories for our better understanding of multilateral bargaining problems. Nash (1951) proposed a research program, now called the Nash program, to unify these two branches by formulating preplay negotiations in cooperative games as moves in a larger noncooperative game in extensive form. Since the seminal work of Rubinstein's (1982) two-person alternating-offers model, several extensions of it to an n-person game in coalitional form have been studied in the literature. Among them is the random-proposer model in which a proposer is randomly selected in the beginning of every bargaining round. The aim of this paper is to present some theoretical results of the bargaining model and to apply it to a production economy of one employer and multiple workers.

The random-proposer model has the following negotiation rule. In the beginning of every bargaining round, one player is randomly selected as a proposer among "active" players who have not joined any coalitions in previous rounds. The probability for each player to be selected as a proposer is called her recognition probability. The selected player proposes a coalition of active players and a payoff allocation in the coalition. Thereafter, all other members in the coalition either accept or reject the proposal sequentially. The proposal is agreed by the unanimous rule. If it is agreed, then the players outside the coalition can continue their negotiations in the next round under the same rule. If the proposal is rejected, then the bargaining round is repeated with the same set of players. Negotiations stop if all players belong to any coalitions. Players discount future payoffs.

The results of the paper are as follows. In the first part, we prove the existence of a behavior-strategy stationary subgame perfect equilibrium (SSPE) in a general situation where the recognition probability for players is arbitrary and players may have different discount factors for future payoffs. We next consider the efficiency of an SSPE. It is shown that the grand coalition is formed in an SSPE if and only if the expected payoffs of players belong to an enlarged set of the core (called an ε-core in the literature). This enlarged core shrinks to the usual core in the limit that players' discount factors for future payoffs go to one. The expected payoff of every player i in the grand coalition is proportional to the ratio $p_i/(1-\delta_i)$ where p_i is player i's recognition probability and δ_i is her discount factor. The ratio represents the bargaining power of the player. The uniqueness of an SSPE payoff for a general coalitional game is an open question. We prove that the grand-coalition SSPE is a unique symmetric SSPE in a symmetric game with nonempty core for all discount factors for future payoffs.

In the last part of the paper, the bargaining model is applied to a production economy with one employer and multiple workers. All players have the same discount factors for future payoffs, and the recognition probability is uniform. The economy has a non-empty core. We prove that the economy has a unique SSPE payoff when all players are sufficiently patient. The SSPE payoff allocation is efficient in the limit that the discount factor is almost equal to one. The equal distribution is attained if and only if the full-employment has the highest productivity per capita. The latter condition is equivalent to that the equity allocation is in the core. If the condition does not hold, the employer receives the smallest payoff in the core. We compare the SSPE payoffs with cooperative solutions of the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969). We show that the SSPE payoff and the nucleolus have similar distributional properties.

The literature on the random-proposer model for coalitional bargaining is growing. We here review only those which we think are closely related to the present paper. The random-proposer model has been extensively studied for legislative bargaining since a seminal paper of Baron and Ferejohn (1989). Baron and Ferejohn (1989) characterized a unique SSPE payoff of the model (called a closed rule) for a majority voting game when voters are identical in recognition probability and discount factors for future payoffs. Eraslan (2002) proved the existence of an SSPE and the uniqueness of an SSPE payoff in a general case that voters may be different in recognition probability and discount factors. Eraslan and McLennan (2006) extended the uniqueness result to a general voting game where a voting rule is given by a class of winning coalitions. Norman (2002) studied the finite-horizon version of the Baron-Ferejohn model. Banks and Duggan (2000) generalized the analysis of Baron and Ferejohn (1989) to the multidimensional voting game with a general voting rule. They proved the existence of an SSPE and provide a sufficient condition on preferences for no-delay of an SSPE in the case that voters may be different in recognition probability and discount factors. When there exist veto players, they also established the equivalence between the core of the voting game and no-delay SSPE payoffs when voters are perfectly patient. Montero (2002) considered a special type of weighted majority voting game (called apex games) and characterized the kernel as a unique SSPE payoff under two (egalitarian and proportional) recognition rules. Montero (2006) characterized the nucleolus as a self-confirming power index for a majority voting game in the case that voters have the same discount factors. Eraslan and Merlo (2002) extended the analysis of Baron and Ferejohn to a stochastic environment where the total surplus is a stochastic variable. They proved the existence of an SSPE when voters have the same discount factors, and showed that the SSPE payoffs need not be unique under majority rules.

The random-proposer model was applied to an n-person supper-additive coalitional game with transferable utility in Okada (1996). We showed no-delay of agreement in an SSPE and characterized a necessary and sufficient condition for the grand-coalition SSPE to exist in the case that players are identical in recognition probability and discount factors. Okada (2005) extended the random-proposer model to an n-person cooperative game in strategic form and characterized a generalized Nash bargaining solution as the grand-coalition SSPE payoff in the case of the common discount factor. Okada (2000) incorporated the possibility of renegotiations into the random-proposer model, and showed that successive renegotiations necessarily lead to an efficient agreement, while the equity of the agreement may be distorted. Gomes (2005) generalized this result to coalitional games with externalities (described in a partition function form). He showed that, if the grand coalition is not efficient, bargaining delay may arise in positive-externality games. Gomes and Jehiel (2004) considered a similar model with finite states where coalitions may break up. Yan (2002, 2005) proved the uniqueness of an SSPE in the restricted case that the game stops after one coalition forms (one-stage property) and that players have the same discount factors. Yan (2002) proved that every core allocation can be sustained as a unique SSPE payoff if (with normalization) it is employed as the recognition probability. Yan (2005) proved the uniqueness result for a symmetric game when players are identical in recognition probability. Finally, Montero and Okada (2007) showed non-uniqueness of SSPE payoffs in a three-person coalitional game with discrete feasible payoffs.

The paper is organized as follows. Section 2 presents the random-proposer model for an n-person coalitional game with transferable utility. Section 3 proves the existence of a behavior-strategy SSPE. Section 4 characterizes the grand-coalition SSPE in a general game, and proves the uniqueness of a symmetric SSPE in a symmetric game with nonempty core. Section 5 analyzes an production economy with one employer and multiple workers. Proofs are given in the Appendix. Section 6 concludes the paper.

2 Definitions

We consider an n-person game (N, v) in coalitional form with transferable utility. $N = \{1, 2, \cdots, n\}$ is the set of players. A nonempty subset S of N is called a *coalition* of players. Let $\mathcal{C}(N)$ be the set of all coalitions of N. The *characteristic function* v is a real-valued function on $\mathcal{C}(N)$ satisfying (1) (zero-normalized) $v(\{i\}) = 0$ for all $i \in N$, (2) (super-additive) $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalitions S and T, and (3) (essential) $v(N) > 0$. For coalition S, $v(S)$ is interpreted to be a sum of money that the members of S can distribute in any way if they agree to it.

A payoff allocation for coalition S is a vector $x^S = (x_i^S)_{i \in S}$ of real numbers where x_i^S represents a payoff to player $i \in S$. A payoff allocation x^S for S is *feasible* if $\sum_{i \in S} x_i^S \leq v(S)$. Let X^S denote the set of all feasible payoff allocations for S, and let X_+^S denote the set of all elements in X^S with non-negative components. For a finite set T, the notation $\Delta(T)$ denotes the set of all probability distributions on T

As a non-cooperative bargaining procedure for a game (N, v), we consider the random proposer model presented in Okada (1996). Let p be a function which assigns to every coalition $A \subset N$ a probability distribution $p^A \in \Delta(A)$. The interpretation of p is that the distribution p^A selects a proposer $i \in A$ randomly when A is the set of all active players in negotiations. Following the literature on legislative bargaining, we will call p the *recognition probability.*

The bargaining model has the following rule. Negotiations take place over a (possibly) infinite number of bargaining rounds $t \; (= 1, 2, \cdots)$. Let $N_t (\subset N)$ be the set of all "active" players who do not belong to any coalitions in round t. In the initial round, we put $N_1 = N$. The bargaining process in round t runs as follows. In the beginning, a player $i \in N_t$ is randomly selected as a proposer according to the probability distribution $p^{N_t} \in \Delta(N_t)$. The selected player i proposes a coalition S with $i \in S \subset N_t$ and a payoff allocation $x^S \in X_+^S$. Then, all other members in S either accept or reject the proposal (S, x^S) sequentially. The order of responders do not affect the result in any critical way. If all responders accept the proposal, then it is agreed-upon, and negotiations go to the next round $t + 1$ with $N^{t+1} = N^t - S$. If any one responder rejects the proposal, then negotiations continue in the next round

$t+1$ with $N^{t+1} = N^t$. The bargaining process ends when every player in N joins some coalition.

The payoffs of players are defined as follows. When a proposal (S, x^S) is agreed in round t, the payoff of every player $i \in S$ is $\delta_i^{t-1} x_i^S$ where $\delta_i (0 \leq \delta_i < 1)$ is player i's discount factor for future payoffs. When the bargaining does not stop, all players who fail to join any coalitions receive zero payoffs. In the model, all players have perfect information about game play when they make decisions.

The bargaining model above is denoted by $\Gamma(N, p, \delta)$ where N is the initial set of players, p the random rule of selecting proposers, and $\delta = (\delta_1, \cdots, \delta_n)$ players' discount factors for future payoffs. Formally, the bargaining model $\Gamma(N, p, \delta)$ is represented as an infinite-length extensive game with perfect information and with chance moves. The rule of the game is the common knowledge of players.

A *(behavior) strategy* for player i in $\Gamma(N, p, \delta)$ is defined in a standard manner. A history h_i^t before player i's move in round t is a sequence of all past actions in $\Gamma(N, p, \delta)$ including the selections of proposers. Roughly, a strategy σ_i of player i is a function which assigns her (randomized) action $\sigma_i(h_i^t)$ to every possible history h_i^t. Specifically, when player i is a proposer in round t, $\sigma_i(h_i^t)$ is a probability distribution (with a finite support) on the set of all possible proposals (S, x^S) with $i \in S \subset N^t$ and $x^S \in X_+^S$. When player i is a responder in round t, $\sigma_i(h_i^t)$ is in $\Delta(\{accept, reject\})$. For a strategy combination $\sigma = (\sigma_1, \cdots, \sigma_n)$, the expected (discounted) payoff for player i in $\Gamma(N, p, \delta)$ can be defined in a usual way.

For every coalition $S \in \mathcal{C}(N)$, a subgame of the extensive game $\Gamma(N, p, \delta)$ which starts after the coalition S has formed is identical to the bargaining model $\Gamma(N - S, p, \delta)$. A strategy σ_i for every player i naturally induces her strategy in the subgame $\Gamma(N - S, p, \delta)$. A strategy σ_i for player i in $\Gamma(N, p, \delta)$ is called *stationary* if player i's action depends only on payoff-relevant history, not on a whole part of the history. A payoff-relevant history for player i consists of the set N_t of active players in negotiations when she is a proposer, and it also includes a proposal made in the present period when she is a responder.

The solution concept that we apply to the bargaining model $\Gamma(N, p, \delta)$ is a stationary subgame perfect equilibrium.

Definition 2.1. A strategy combination $\sigma^* = (\sigma_1^*, \cdots, \sigma_n^*)$ of $\Gamma(N, p, \delta)$ is called a *stationary subgame perfect equilibrium (SSPE)* if σ^* is a subgame perfect equilibrium of $\Gamma(N, p, \delta)$ and every player i's strategy σ_i^* is stationary

It is well-known that in a broad class of Rubinstein-type sequential multilateral bargaining games including our model $\Gamma(N, p, \delta)$, there is a large multiplicity of (non-stationary) subgame perfect equilibria when the discount factor of future payoffs is sufficiently close to one (see Sutton 1986 and Osborne and Rubinstein 1990). The multiplicity of subgame perfect equilibria holds even in the n-person pure bargaining game where no subcoalitions are allowed. An SSPE is the simplest type of a subgame perfect equilibrium and thus it may be easier for players to coordinate their mutual expectations on it (see Baron and Kalai 1993 and Chatterjee and Sabourian 2000). The SSPE is a natural reference point of the analysis in multilateral bargaining models.

3 Equilibrium configuration

For an SSPE $\sigma = (\sigma_1, \cdots, \sigma_n)$ of the game $\Gamma(N, p, \delta)$ and a coalition S, let v_i^S be the expected payoff of player i under σ in the subgame $\Gamma(S, p, \delta)$, and let q_i^S be the probability distribution with which player i chooses coalitions T with $i \in T \subset S$ under σ in $\Gamma(S, p, \delta)$. Let $v^S = (v_i^S)_{i \in S}$ and $q^S = (q_i^S)_{i \in S}$. We call the collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ the *configuration* of an SSPE σ.

Definition 3.1. A collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ of players' expected payoffs v^S and their randomized choices q^S of coalitions for all sets S of active players is called an *equilibrium configuration* for $\Gamma(N,p,\delta)$ if it is a configuration of some SSPE σ for $\Gamma(N,p,\delta)$.

The next lemma plays a critical role in characterizing an SSPE. The lemma was first proved in Okada (1996) in the case that all players are identical in recognition probability and discount factors.

Lemma 3.1. In every SSPE $\sigma = (\sigma_1, \cdots, \sigma_n)$ for $\Gamma(N,p,\delta)$, every player's proposal is accepted in the initial round of negotiations. In the proposal, all other members j in the coalition are offered their discounted expected payoffs $\delta_j v_j^N$.

Proof. For every $i \in N$, let v_i be player i's expected payoff for σ in $\Gamma(N,p,\delta)$. By the rule of $\Gamma(N,p,\delta)$, the super-additivity of v yields

$$\sum_{i \in N} v_i \leq v(N) \quad \text{and} \quad v_i \geq 0 \quad \text{for all } i \in N.$$

Consider the maximization problem

$$\begin{aligned} \max_{S,y} \quad & v(S) - \sum_{j \in S, j \neq i} y_j \\ \text{s.t.} \quad & \text{(i) } i \in S \subset N, \quad y \in X_+^S \\ & \text{(ii) } y_j \geq \delta_j v_j \quad \text{for all} \quad j \in S, \ j \neq i. \end{aligned} \tag{1}$$

Let m^i be the optimal value of (1). Since $(N, (\delta_1 v_1, \cdots, \delta_n v_n))$ is a feasible solution, we have

$$m^i \geq v(N) - \sum_{j \in N, j \neq i} \delta_j v_j \geq \delta_i v_i. \tag{2}$$

If $m^i = \delta_i v_i$, then (2) implies

$$v(N) = \sum_{j \in N} \delta_j v_j. \tag{3}$$

Since

$$v(N) \geq \sum_{j \in N} v_j \geq \sum_{j \in N} \delta_j v_j,$$

(3) implies $\sum_{j \in N} v_j = \sum_{j \in N} \delta_j v_j$. Since $\delta_j < 1$ for all j, we must have $v_j = 0$ for all $j \in N$. This yields $v(N) = 0$ from (3), which contradicts $v(N) > 0$. Therefore, we obtain $m^i > \delta_i v_i$.

Let (S, y^S) be the optimal solution of (1). Then, (S, y^S) satisfies

$$\begin{aligned} m^i &= v(S) - \sum_{j \in S, j \neq i} y_j^S \\ y_j^S &= \delta_j v_j, \quad \forall j \in S, j \neq i. \end{aligned}$$

For a sufficiently small $\epsilon > 0$, define a payoff allocation $z^S \in X_+^S$ such that

$$z_i^S = m^i - \epsilon, \quad z_j^S = \delta_j v_j + \frac{\epsilon}{s-1}, \quad \forall j \in S, j \neq i,$$

where s is the number of all members in S. If player i proposes (S, z^S), then this is accepted in σ since all responders j receive only the discounted payoffs $\delta_j v_j$ if they reject the proposal. Since $z_i^S = m^i - \epsilon > \delta_i v_i$ for any sufficiently small $\epsilon > 0$, the conditional expected payoff for player i in σ when she is a proposer is strictly greater than $\delta_i v_i$, which is her continuation value when her proposal is rejected. This fact implies that player i's proposal must be accepted in σ. QED.

An important implication of this lemma is that an SSPE σ of the bargaining model $\Gamma(N, p, \delta)$ is determined uniquely (up to responses off equilibrium path) by its configuration $(v^S, q^S)_{S \in \mathcal{C}(N)}$. When player i chooses a coalition S in σ, she proposes the payoff allocation $x^S = (x_j^S)_{j \in S}$ for coalition S such that

$$x_i^S = v(S) - \sum_{j \in S, j \neq i} \delta_j v_j^N, \quad x_j^S = \delta_j v_j^N, \quad \forall j \in S, j \neq i,$$

and all other members of S accept it. Another implication of the lemma is that an agreement of a coalition (possibly inefficient) is made in the first round in the bargaining game $\Gamma(N, p, \delta)$, regardless of a proposer, if the characteristic function is super-additive. That is, the bargaining game has no delay of agreements. This result does not hold in other bargaining games where the first rejector becomes the next proposer (see Chatterjee, Dutta. Ray and Sengupta 1993, and Ray and Vohra 1999).

The next step of our analysis is to characterize an equilibrium configuration in $\Gamma(N, p, \delta)$.

Lemma 3.2. In a subgame $\Gamma(S, p, \delta)$, let p_i^S be player i's recognition probability, v_i^S player i's expected payoff, and q_i^S player i's randomized choice of coalitions where a coalition $T \subset S$ is chosen with probability $q_i^S(T)$. A collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ where $v^S = (v_i^S)_{i \in N}$ and $q^S = (q_i^S)_{i \in N}$ is an equilibrium configuration of $\Gamma(N, p, \delta)$ if and only if the following conditions hold for every coalition $S \in \mathcal{C}(N)$ and every $i \in S$:
(i) $q_i^S(\hat{S}) > 0$ implies that $\hat{S}$ is a solution of

$$\max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j v_j^S), \tag{4}$$

(ii) $v_i^S \in R_+$ satisfies

$$\begin{aligned} v_i^S &= p_i^S \max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j v_j^S) \\ &\quad + \sum_{j \in S, j \neq i} p_j^S \delta_i \, (\sum_{j \in T \subset S, i \in T} q_j^S(T) v_i^S + \sum_{j \in T \subset S, i \notin T} q_j^S(T) v_i^{S-T}). \end{aligned} \tag{5}$$

Proof. It follows from Lemma 3.1 that, in an SSPE σ of $\Gamma(N, p, \delta)$, player i receives (4) when she is a proposer, and receives $\delta_i v_i^S$ and $\delta_i v_i^{S-T}$ as a responder, when player $j (\neq i)$ proposes coalitions T including i and T excluding i, respectively. Noting this fact, we can see that condition (i) means that player i's randomized choice q_i^S of coalitions composes her locally optimal choice when she is selected as a proposer, and that condition (ii) defines the expected payoffs $(v_i^S)_{S\in\mathcal{C}(N)}$ of player i recursively. The if-part can be proved by a well-known fact (sometimes called the single-period deviation property) that the local optimality of a strategy implies the global optimality in an infinite-length extensive game with perfect information such as the bargaining game $\Gamma(N,p,\delta)$ (see Selten (1981, p.137)). QED

In Lemma 3.2, we call condition (i) *the optimality condition* and condition (ii) *the payoff equation* for an equilibrium configuration in $\Gamma(N,p,\delta)$. With help of these two conditions, we will characterize an SSPE of the bargaining model $\Gamma(N,p,\delta)$ in the next section.

With help of Lemma 3.2, the existence of an SSPE in the bargaining model $\Gamma(N,p,\delta)$ can be proved by Kakutani's fixed point theorem in a standard manner. The existence of an SSPE in related models are proved in the literature (Ray and Vohra 1999, Eraslan 2002, Gomes 2005 among others).

Proposition 3.1. There exists an SSPE of the bargaining model $\Gamma(N,p,\delta)$.

Proof. By Lemma 3.2, it suffices to prove that there exists a collection $(v^S,q^S)_{S\in\mathcal{C}(N)}$ of players' expected payoffs $v^S=(v_i^S)_{i\in S}$ and their randomized choices $q^S=(q_i^S)_{i\in S}$ of coalitions in all subgames $\Gamma(S,p,\delta)$ such that (4) and (5) hold for every coalition $S\in\mathcal{C}(N)$ and every $i\in S$. We prove this claim by the induction regarding the number s of players in coalition S. When $s=1$ where $S=\{i\}$, the claim trivially holds by putting $v_i^{\{i\}}=0$ and $q_i^{\{i\}}(\{i\})=1$. For any $2\le s\le n$, suppose that the claim holds for all $t=1,\cdots,s-1$. Let $S\in\mathcal{C}(N)$ be any coalition with s members. For all proper subsets T of S, let $v^T=(v_j^T)_{j\in T}$ be expected payoffs for members in T and let $q^T=(q_i^T)_{i\in T}$ be their randomized choices of coalitions in the subgame $\Gamma(T,p,\delta)$ such that (4) and (5) hold. The existence of such v^T and q^T is guaranteed by the supposition of the induction.

For every $i \in S$, let $\Delta_i^S = \Delta(\{T \mid i \in T \subset S\})$. Recall that $\Delta(A)$ is the set of all probability distributions on a finite set A. Δ_i^S is the set of player i's all randomized choices of coalitions when she is selected as a proposer. Define a multi-valued mapping F from a compact and convex set $X_+^S \times \prod_{i \in S} \Delta_i^S$ to itself as follows. For $(x,q) \in X_+^S \times \prod_{i \in S} \Delta_i^S$, $F(x,q)$ is the set of all $(y,r) \in X_+^S \times \prod_{i \in S} \Delta_i^S$ which satisfy for all $i \in S$

(i) $r_i \in \Delta(\{\hat{S} \mid i \in \hat{S} \subset S$ and $\hat{S}$ is a solution of

$$\max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j x_j)\},$$

and

(ii) $y_i \in R_+$ satisfies

$$\begin{aligned} y_i &= p_i^S \max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j x_j) \\ &\quad + \sum_{j \in S, j \neq i} p_j^S \delta_i \left(\sum_{j \in T \subset S, i \in T} r_j(T) x_i + \sum_{j \in T \subset S, i \notin T} r_j(T) v_i^{S-T} \right), \end{aligned}$$

where $r_j(T)$ is the probability which r_j assigns to $T \subset S$.

We can show without much difficulty that $F(x,q)$ is a non-empty convex set in $X_+^S \times \prod_{i \in S} \Delta_i^S$, and that F is upper-semicontinuous. By Kakutani's fixed point theorem, there exists a fixed point (x^*, q^*) of F with $(x^*, q^*) \in F(x^*, q^*)$. Now, we define $v_i^S = x_i^*$ and $q_i^S = q_i^*$ for all $i \in S$. Then, the fact that $(v^S, q^S) \in F(v^S, q^S)$ implies that $v^S = (v_i^S)_{i \in S}$ and $q^S = (q_i^S)_{i \in S}$, together with (v^T, q^T) for all proper subsets T of S, satisfy (4) and (5) in Lemma 3.2 for every $i \in S$. QED

We remark that Proposition 3.1 does not hold for a pure-strategy SSPE. To obtain the existence of an SSPE, one needs to allow a randomized choice of coalitions by proposers. As we will show in the next section, a pure-strategy SSPE with an agreement of the grand coalition N exists if the expected equilibrium payoff allocation is in the (enlarged) core of the underlying game.

4 Characterizations

Let $p_i = p_i^N$ for every $i \in N$. The payoff equation (5) for an equilibrium configuration $(v^S, q^S)_{S\in\mathcal{C}(N)}$ in $\Gamma(N,p,\delta)$ can be rewritten as

$$\begin{aligned}
&(1-\delta_i \sum_{j\in N, j\neq i} p_j \sum_{i,j\in T\subset N} q_j^N(T))v_i^N + p_i \sum_{j\in N, j\neq i} (\sum_{i,j\in S\subset N} q_i^N(S))\delta_j v_j^N \\
= \;& p_i \sum_{i\in S\subset N} q_i^N(S)v(S) + \delta_i \sum_{j\in N, j\neq i} p_j \sum_{j\in T\subset N, i\notin T} q_j^N(T)v_i^{N-T}, \qquad (6)
\end{aligned}$$

and, in a matrix form, as

$$\begin{pmatrix}
1-\delta_1 \sum_{j\neq 1} p_j \sum_{1,j\in T} q_j^N(T) & \delta_2 p_1 \sum_{1,2\in T} q_1^N(T) & \cdots & \delta_n p_1 \sum_{1,n\in T} q_1^N(T) \\
\delta_1 p_2 \sum_{1,2\in T} q_2^N(T) & 1-\delta_2 \sum_{j\neq 2} p_j \sum_{2,j\in T} q_j^N(T) & \cdots & \delta_n p_2 \sum_{2,n\in T} q_2^N(T)) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_1 p_n \sum_{1,n\in T} q_n^N(T) & \delta_2 p_n \sum_{2,n\in T} q_n^N(T) & \cdots & 1-\delta_n \sum_{j\neq n} p_j \sum_{n,j\in T} q_j^N(T)
\end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix}
p_1 \sum_{1\in S} q_1^N(S)v(S) + \delta_1\alpha_1 \\
p_2 \sum_{2\in S} q_2^N(S)v(S) + \delta_2\alpha_2 \\
\vdots \\
p_n \sum_{n\in S} q_n^N(S)v(S) + \delta_n\alpha_n
\end{pmatrix}$$

where $\alpha_i = \sum_{j\neq i} p_j \sum_{i\notin T, j\in T} q_j^N(T)v_i^{N-T}$.

We first consider under which conditions the grand coalition N is formed, independent of a proposer.

Definition 4.1. A behavior strategy σ for $\Gamma(N,p,\delta)$ is called the *grand-coalition* SSPE if it is an SSPE of $\Gamma(N,p,\delta)$ and the grand coalition N forms, independent of a proposer.

The next theorem characterizes the grand-coalition SSPE.

Theorem 4.1. The grand-coalition SSPE of $\Gamma(N, p, \delta)$ is characterized as follows.
(i) The expected payoff v_i for player i is given by

$$v_i = \frac{\frac{p_i}{1-\delta_i}}{\sum_{j \in N} \frac{p_j}{1-\delta_j}} v(N). \tag{7}$$

The grand-coalition SSPE exists if and only if

$$\sum_{i \in S} v_i + \sum_{j \in N-S} v_j (1-\delta_j) \geq v(S) \quad \text{for all} \quad S \subset N. \tag{8}$$

(ii) If all players have the common discount factor δ, then the expected payoff v_i for player i is given by $p_i v(N)$. For any δ almost close to one, the grand-coalition SSPE exists if and only if its expected payoff vector $(p_1 v(N), \cdots, p_n v(N))$ is in the core of (N, v). Every player proposes the payoff vector $(p_1 v(N), \cdots, p_n v(N))$ in the limit that δ goes to one.

Proof. (i) The payoff equation of the grand-coalition SSPE is given by

$$\begin{pmatrix} 1-\delta_1(1-p_1) & \delta_2 p_1 & \dots & \delta_n p_1 \\ \delta_1 p_2 & 1-\delta_2(1-p_2) & \dots & \delta_n p_2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 p_n & \delta_2 p_n & \dots & 1-\delta_n(1-p_n) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} p_1 v(N) \\ p_2 v(N) \\ \vdots \\ p_n v(N) \end{pmatrix}.$$

Calculating the i-th row in the equation above, we obtain

$$(1-\delta_i) v_i + p_i \sum_{j \in N} \delta_j v_j = p_i v(N), \quad \forall i \in N. \tag{9}$$

Summing up both sides of (9) for all $i \in N$ yields

$$\sum_{i \in N} v_i = v(N). \tag{10}$$

Letting $k = \sum_{i \in N} \delta_i v_i$, (9) is rewritten as

$$v_i = \frac{p_i}{1 - \delta_i}(v(N) - k), \quad \forall i \in N. \tag{11}$$

It follows from (11) that

$$k = \sum_{i \in N} \delta_i v_i = (v(N) - k) \sum_{i \in N} \frac{\delta_i}{1 - \delta_i} p_i.$$

This yields

$$k = \frac{\sum_{j \in N} \frac{\delta_j}{1-\delta_j} p_j}{1 + \sum_{j \in N} \frac{\delta_j}{1-\delta_j} p_j} v(N) = \frac{\sum_{j \in N} \frac{\delta_j}{1-\delta_j} p_j}{\sum_{j \in N} \frac{1}{1-\delta_j} p_j} v(N). \tag{12}$$

Substituting (12) into (11), we prove (7). The optimality condition (4) is given by

$$v(N) - \sum_{j \in N, j \neq i} \delta_j v_j \geq v(S) - \sum_{j \in S, j \neq i} \delta_j v_j, \quad \forall S \subset N.$$

In view of (10), we can see that the optimality condition is equivalent to (8). By Lemma 3.2, we can prove the last part of (1).
(ii) When $\delta_1 = \cdots = \delta_n = \delta$, it can be easily shown that (7) implies $v_i = p_i v(N)$. Note that v_i is independent of δ. Suppose first that there exists the grand-coalition SSPE for any δ almost close to one. Taking $\delta \to 1$ in (8), we obtain $\sum_{i \in S} v_i \geq v(S)$ for all $S \subset N$. This means that the expected payoff vector $(p_1 v(N), \cdots, p_n v(N))$ is in the core of (N, v). Conversely, suppose that $\sum_{i \in S} v_i = \sum_{i \in S} p_i v(N) \geq v(S)$ for all $S \subset N$. Since $v_j > 0$ for all $j \in N - S$, we can show that (8) holds for any $\delta < 1$. By Lemma 3.2, this implies that there exists the grand-coalition SSPE of $\Gamma(N, p, \delta)$ for any δ, and it has the expected payoff vector $(p_1 v(N), \cdots, p_n v(N))$. In the grand-coalition SSPE, player i

proposes a payoff allocation $(\delta p_1 v(N), \cdots, (1-\delta\sum_{j\neq i} p_j)v(N), \cdots, \delta p_n v(N))$. This proposal converges to $(p_1 v(N), \cdots, p_n v(N))$ as $\delta \to 1$. QED

The theorem has several implications. First, it shows that when players form the grand coalition in the bargaining model $\Gamma(N, p, \delta)$, their expected payoffs are proportional to the ratio $\frac{p_i}{1-\delta_i}$. This fact suggests that the ratio represents the bargaining power of each player. Players who have higher recognition probability and are more patient can receive higher expected payoffs. The random-proposer model can explain the bargaining power of a player from two factors, procedural rules and players' time preferences.[1]

Second, the theorem shows that the existence of the grand-coalition SSPE for $\Gamma(N, p, \delta)$ is closely related to the non-emptiness of the core of the cooperative game (N, v). (8) requires that the expected payoffs v_i for players satisfy, for all $S \subset N$,

$$\sum_{i\in S} v_i \geq v(S) - \epsilon, \quad \text{where} \quad \epsilon = \sum_{j \in N-S} v_j(1-\delta_j).$$

This means that the expected payoffs for players in the grand-coalition SSPE belongs to a larger set of the core of (N, v), which is called the ϵ-core in the literature. As δ goes to one, this set converges to the core of (N, v). Namely, the grand-coalition SSPE exists for any δ almost close to one if and only if its expected payoff vector is in the core of the game (N, v). An immediate corollary of this result is that if the core of the game is empty, then the grand-coalition SSPE never exists, whatever recognition probability distribution is, when players are sufficiently patient.

Thirdly, when players have the common discount factors almost close to one, the proposal of every player in the grand-coalition SSPE is equal to a generalized Nash bargaining solution with weights $(p_1, \cdots, p_n)$, which is a solution

[1] One may interpret this result negatively, arguing that the equilibrium outcome of a noncooperative bargaining model is sensitive to unimportant procedural details. In our view, the recognition probability for players should not be regarded as unimportant details.

of

$$\max_x \prod_{i \in N} x_i^{p_i} \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N), \ x_i \geq 0, \forall i \in N$$

where the disagreement point is given by players' payoffs $v(\{i\}) = 0, \ i = 1, \cdots, n$ in the case of no coalitions.[2] This relation between the grand-coalition SSPE payoff and the generalized Nash bargaining solution is extended by Okada (2005) to a general n-person cooperative game, derived from a strategic-form game, in which utility may not be transferable and externality prevails.

As we have reviewed in the introduction, the uniqueness of an SSPE of the random-proposal model is an open question for a general cooperative game. Eraslan (2002) proves the uniqueness of an SSPE payoff in a majority voting game. Yan (2000) proves that the grand-coalition SSPE is a unique SSPE for a game with nonempty core when the recognition probability is equal to some (normalized) core payoff in a restricted case that the bargaining stops after one coalition forms and that players have the common discount factors.

We consider the uniqueness of an SSPE in an n-person symmetric game which includes a majority voting game as a special case.

Definition 4.2.
(i) A game (N, v) is called *symmetric* if the value $v(S)$ of coalition S depends only on the size s of S. Whenever no confusion arises, $v(S)$ is denoted by $v(s)$.
(ii) Let (N, v) be a symmetric game. An SSPE σ of $\Gamma(N, p, \delta)$ is called *symmetric* if the following conditions hold: if any player i chooses a coalition S with positive probability in any round t, then every player $j \in N$ also proposes all coalitions (including herself) of the same size as S with (possibly unequal) positive probability in the same round.

A symmetric SSPE requires that all proposers' choices of coalitions are in-

[2]Note that, if players have different discount factors, the expected payoffs in (7) does not have a (unique) limit as players' discount factors go to one.

variant to renaming of players. That is, all proposers treat equally all other players as their coalition partners. A symmetric SSPE is a natural criterion of equilibrium selection in a symmetric game. It includes the grand-coalition SSPE and the Baron-Ferejohn equilibrium of a majority voting game in which every proposer chooses equally all minimal winning coalitions (including herself). In a general setup where coalitions form sequentially, the Baron-Ferejohn equilibrium can be generalized to an equilibrium in which all proposers choose randomly all coalitions of the same size in each round until all players join any coalition. Remark that a symmetric SSPE does not rule out the possibility that a proposer chooses randomly coalitions of different sizes in the same round. In what follows in this section, we consider only a symmetric SSPE.

Lemma 4.1. Let (N, v) be a symmetric game. In every symmetric SSPE of $\Gamma(N, p, \delta)$ except the grand-coalition SSPE, all players $i \in N$ receive the same discounted expected payoffs $\delta_i v_i$.

Proof. Let σ be a symmetric SSPE of $\Gamma(N, p, \delta)$ other than the grand-coalition SSPE. In σ, there exists some player i who proposes a subcoalition $S \neq N$ with positive probability. By the optimality condition of σ, it must hold that

$$v(S) - \sum_{j \in S, j \neq i} \delta_j v_j \geq v(T) - \sum_{j \in T, j \neq i} \delta_j v_j, \quad \forall T \subseteq N.$$

Pick any $j \in S, j \neq i$, and any $k \notin S$. Substituting $T = (S - \{j\}) \cup \{k\}$ into the condition above yields $\delta_k v_k \geq \delta_j v_j$. By the definition of a symmetric SSPE, player i proposes the coalition $(S - \{j\}) \cup \{k\}$ with positive probability. Replacing S with $(S - \{j\}) \cup \{k\}$ in the arguments above, we can obtain $\delta_j v_j \geq \delta_k v_k$. Thus, $\delta_j v_j = \delta_k v_k$. Finally, since σ is symmetric, player j chooses S with positive probability. By repeating the same arguments as above, we can show that $\delta_i v_i = \delta_k v_k$. Thus, $\delta_i v_i = \delta_j v_j = \delta_k v_k$. QED

The lemma shows that when all players have a risk not to be invited to coalitions in a symmetric game, their discounted expected payoffs are identical in equilibrium. This result is caused by a competition among players in coalition formation. Every player accepts to join any coalition if she is offered her discounted expected payoff. In this sense, the discounted expected payoff (called a continuation payoff in the bargaining literature) can be interpreted as a "price" which all proposers should pay when they want her as their coalitional partners. When a subcoalition is formed, any "expensive" player is excluded from the coalition. A mechanism similar to price competition makes all players' prices equal.

Lemma 4.2. For an SSPE σ of $\Gamma(N, p, \delta)$ with a configuration $(v^S, q^S)_{S\in\mathcal{C}(N)}$, let v_i be the expected equilibrium payoff for player i, and let $V_\sigma = \sum_{i\in N} v_i$. Then,

$$V_\sigma = \sum_{i\in N} p_i \sum_{S: i\in S\in\mathcal{C}(N)} q_i^N(S)v(S) + \sum_{i\in N} \delta_i\alpha_i \tag{13}$$

where $\alpha_i = \sum_{j\neq i} p_j \sum_{i\notin T, j\in T} q_j^N(T)v_i^{N-T}$.

Proof. Summing up both sides of the payoff equation (6) for all $i \in N$, we obtain (13). QED

The RHS of (13) represents the discounted sum of coalitional values realized in equilibrium. In a majority voting game, V_σ is simply equal to the value of a winning coalition for all σ.

Proposition 4.2. Let (N, v) be a symmetric game, and σ be a symmetric SSPE other than the grand-coalition SSPE. Then, the expected payoff v_i for player i in σ is given by

$$v_i = \frac{V_\sigma}{\delta_i \sum_{j\in N} \frac{1}{\delta_j}} \tag{14}$$

where V_σ is the discounted sum of coalitional values in σ defined by (13). An

SSPE σ exists only if

$$v(s_1) - v(t) \geq \frac{V_\sigma}{\sum_{j \in N} \frac{1}{\delta_j}}(s_1 - t) \quad \text{for all} \quad t = 1, \cdots, n. \tag{15}$$

where s_1 is the size of coalitions formed in the initial round.[3]

Proof. (14) follows from Lemma 4.1 and $V_\sigma = \sum_{i \in N} v_i$. The optimality condition for σ implies

$$v(s_1) - \sum_{j \in S_1, j \neq i} \delta_j v_j \geq v(T) - \sum_{k \in T, k \neq i} \delta_k v_k, \quad \forall T \subseteq N$$

where S_1 is a coalition formed in the initial round. With (14), this yields (15). QED

The proposition shows that the expected payoffs for players are inversely proportional to their discount factors in any symmetric SSPE other than the grand-coalition SSPE.[4] This result is in stark contrast to Theorem 4.1. The conflict of coalition formation changes drastically the payoff distribution among players. Players with high discount factors receive less expected payoffs. Any "expensive" player who is patient is unlikely invited by other players to coalitions, and thus the player's expected payoff decreases.

The next theorem provides a uniqueness result of the grand-coalition SSPE.

Theorem 4.2. In a symmetric game (N, v) with non-empty core, the grand-coalition SSPE is a unique symmetric SSPE of $\Gamma(N, p, \delta)$ for any discount factors $\delta = (\delta_1, \cdots, \delta_n)$.

[3]The conditions similar to (15) also hold in all other rounds if subcoalitions form in these rounds. Since the proof is the same, we omit it.

[4]Kawamori (2005) also derives (14) in a majority voting game in a restricted case that voters are sufficiently similar in recognition probability and discount factors.

Proof. Suppose that there exists a symmetric SSPE σ of $\Gamma(N, p, \delta)$ other than the grand-coalition SSPE. Let $s(< n)$ be the size of a coalition which may form with positive probability in the initial round in σ. Then, it follows from (15) that

$$v(s) - v(t) \geq \frac{V_\sigma}{\sum_{j \in N} \frac{1}{\delta_j}}(s - t), \quad \text{for all} \quad t = 1, \cdots, n.$$

Putting $t = n$ in the condition above, we obtain

$$v(s) \geq (1 - \frac{n - s}{\sum_{j \in N} \frac{1}{\delta_j}})v(n), \tag{16}$$

since $V_\sigma \leq v(n)$ and $s < n$. Since $\sum_{j \in N} \frac{1}{\delta_j} > n$, (16) implies

$$v(s) > (1 - \frac{n - s}{n})v(n) = \frac{s}{n}v(n).$$

This contradicts the assumption that the core of (N, v) is non-empty. Thus, we can prove that the grand-coalition SSPE is a unique symmetric SSPE. QED

Together with Theorem 4.1, the theorem shows that if the proportional distribution of the total value $v(N)$ under the ratio $(p_i/(1 - \delta_i))_{i \in N}$ is in the core, then the grand-coalition SSPE exists and it is a unique symmetric SSPE. The expected payoffs for players are given by this proportional distribution. To this extent, we can say that the random-proposer model is consistent with the core theory. If the recognition probability p is an extreme one such that the proportional distribution above is not in the core, then the model does not capture a bargaining process which is implicitly assumed in the core theory.

When the core is empty, the uniqueness of a symmetric SSPE payoff is an open question except a special case of majority voting games. In every majority game (except the unanimous game), every proposer chooses a minimal winning coalition in every SSPE since the expected payoffs (acceptance level) of all other voters are strictly positive (Lemma 3.1). That is, an SSPE of the majority game is not the grand-coalition SSPE. In the game, the sum of all players' expected payoffs in every SSPE is equal to the value of winning. Thus, Proposition 4.2 implies that a symmetric SSPE payoff of a majority voting game is unique for any discount factor.

To conclude this section, we summarize the SSPE payoff in a three-person symmetric game.

Example 4.1. Three-person symmetric games

The player set is $\{1, 2, 3\}$. The value of a coalition S is represented by $v(s)$ where s is the number of players in S. We normalize v as $v(3) = 1$ and $v(1) = 0$ for $i = 1, 2, 3$. Let $v(2) = a$ where $0 \leq a \leq 1$. The core of the game is non-empty if and only if $0 \leq a \leq 2/3$. Other cooperative solutions such as the Shapley value and the nucleolus are equal to the equity allocation $(1/3, 1/3, 1/3)$, regardless of the value a. For simplicity of analysis, we assume the equal recognition probability ($p_1 = p_2 = p_3 = 1/3$) and the common discount factor ($\delta_1 = \delta_2 = \delta_3 = \delta$).

First, consider the grand-coalition SSPE. By Theorem 4.1, every player receives the expected payoff $1/3$, and thus the grand-coalition SSPE exists if and only if $a \leq \frac{3-\delta}{3}$ (see (8)). Next, consider a Baron-Ferejohn equilibrium in which every player chooses every two-person coalition including herself at random. It follows from Proposition 4.2 that every player receives the expected payoff $a/3$, and that the Baron-Ferejohn equilibrium exists if and only if $\frac{3}{3+\delta} \leq a \leq 1$ (putting $t = 3$ in (15)).

For the range $\frac{3-\delta}{3} < a < \frac{3}{3+\delta}$, we will show that there exists a symmetric SSPE in which every player chooses both the grand coalition and all two-person coalitions including herself at random. Suppose that every player chooses the grand coalition with probability p $(0 < p < 1)$ and each of two-person coalitions including herself with probability q $(0 < q < 1/2)$ where $p = 1 - 2q$. Lemma 4.1 shows that every player receives the same expected payoff, denoted by v.

The payoff equation (5) of the SSPE is given by

$$v = \frac{1}{3}(p(1 - 2\delta v) + 2q(a - \delta v)) + \frac{2}{3}(p + q)\delta v.$$

This solves $p = \frac{3v-a}{1-a}$. On the other hand, the optimality condition (4) of the SSPE requires that $1 - 2\delta v = a - \delta v$, which implies $v = \frac{1-a}{\delta}$. Thus, we have $p = \frac{3-(3+\delta)a}{(1-a)\delta}$. The constraint $0 < p < 1$ means that $\frac{3-\delta}{3} < a < \frac{3}{3+\delta}$.

The expected payoff v of every player is illustrated in Figure 4.1 when the discount factor for future payoffs is almost equal to one. The bargaining outcome depends on the value a of a two-person coalition as follows. When $0 \leq a \leq \frac{3-\delta}{3}$. the grand coalition forms and v is constant at 1/3. When $\frac{3-\delta}{3} \leq a \leq \frac{3}{3+\delta}$, it is interesting to notice that the expected payoff decreases as the value of two-person coalitions increases. In this region, every player proposes all coalitions including herself at random, and the probability of the grand coalition decreases as two-person coalitions becomes more valuable. When $a = \frac{3}{3+\delta}$, the grand coalition is never chosen, and thereafter v increases as the value of two-person coalitions increases, and becomes 1/3 again when $a = 1$ (the majority game).

5 Application: one employer and workers

Consider a production economy $\mathcal{E}$ with one employer (player 1) and $n - 1$ identical workers i $(= 2, \cdots, n)$. Let $N = \{1, \cdots, n\}$. If a coalition $S \subset N$ consists of the employer and $s-1$ $(s \geq 1)$ workers, then the benefit of S is given by a real-valued function $f(s)$ which is monotonically increasing in $s = 1, \cdots, n$ with $f(1) = 0$. Otherwise, the benefits of coalitions are assumed to be zero. Shapley and Shubik (1967) investigated this economy by the cooperative game theory. The core of the economy is always non-empty since the allocation with

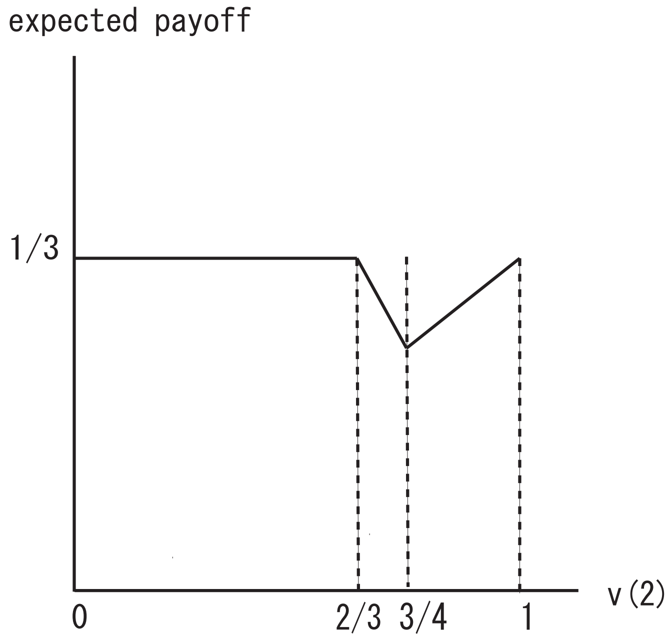

Figure 4.1 three-person symmetric game

the employer exploiting the total benefit $f(n)$ is in the core. The Shapley value and the nucleolus in the economy will be given in Proposition 5.4.

We will apply the random proposer model $\Gamma(N, p, \delta)$ to the production economy $\mathcal{E}$ and will characterize an SSPE of the economy. For simplicity of analysis, we will assume that all players have the common discount factor δ and the equal recognition probability $p = (1/n, \cdots, 1/n)$. Let v_i be the expected payoff of player i $(= 1, \cdots, n)$ in an SSPE. The grand-coalition SSPE will be called the *full-employment equilibrium*, and any other SSPE a *partial-employment equilibrium.*

The first result regarding the full employment is derived by Theorem 4.1.

Proposition 5.1. The full-employment equilibrium of the production economy $\mathcal{E}$ is characterized as follows.

(i) The full-employment equilibrium exists if and only if

$$\frac{n-(n-s)\delta}{n}f(n) \geq f(s) \quad \text{for all} \quad s = 1, \cdots, n. \tag{17}$$

(ii) The expected payoff v_i of every player i ($i = 1, \cdots, n$) is given by $f(n)/n$. Every proposer offers $\delta f(n)/n$ to all other players, and it is accepted.

When the discount factor δ is almost equal to one, (17) is equivalent to $f(n)/n \geq f(s)/s$ for all $s = 1, \cdots, n$. This condition means that the full employment has the highest average productivity.

We next characterize a partial-employment equilibrium where there exist some workers unemployed. Proposition 5.1 shows that all players receive the same expected payoffs in the full employment equilibrium. The next lemma says that this is also true for all workers in every SSPE.

Lemma 5.1. For all workers i and j, $v_i = v_j$ for every SSPE.

The proof of the lemma is given in the Appendix. We define the following two types of partial-employment equilibria.

Definition 5.1. Let σ be a partial-employment equilibrium of the production economy $\mathcal{E}$.
(i) For $2 \leq s < n$, σ is called an *s-equilibrium* if only coalitions with s members form with positive probability in σ.
(ii) For $2 \leq s < t \leq n$, σ is called an *(s,t)-equilibrium* if only coalitions with s and t members form with positive probability in σ.

In an s-equilibrium, the employer hires only $s-1$ workers. In an (s,t)-equilibrium, the employer hires either $s-1$ or $t-1$ workers. We will show that except a degenerate class of the economy $\mathcal{E}$, there exists no other types of partial-employment equilibria.

Proposition 5.2. For $2 \leq s < n$, an s-equilibrium of the production economy $\mathcal{E}$ is characterized as follows.

(i) Let $v^*(\delta)$ be the expected payoff of the employer, and let $w^*(\delta)$ be the expected payoff of every worker. Then,

$$v^*(\delta) = \frac{n-1-(s-1)\delta}{(n-1)\{n-(n-1)\delta\}-(s-1)\delta} f(s) \quad (18)$$

$$w^*(\delta) = \frac{(n-1)(1-\delta)}{(n-1)\{n-(n-1)\delta\}-(s-1)\delta} f(s). \quad (19)$$

Every worker receives an offer with probability $\frac{(s-2)n+1}{n(n-1)}$.

(ii) There exists an s-equilibrium if and only if $f(s) - f(t) \geq (s-t)\delta w^*(\delta)$ for all $t \neq s$.

(iii) An s-equilibrium exists for any δ sufficiently close to one if and only if $f(s) > f(t)$ for all $t < s$ and $f(s) = f(t)$ for all $t > s$. As δ goes to one, the agreement in the s-equilibrium converges to a unique core-allocation of the economy $\mathcal{E}$ where the employer exploits the total payoff $f(n)$.

The proposition provides the condition (ii) for an s-equilibrium to exist. The condition is rewritten as

$$\frac{f(s)-f(t)}{s-t} \geq \delta w^*(\delta) \quad \text{for} \quad t < s$$

$$\frac{f(t)-f(s)}{t-s} \leq \delta w^*(\delta) \quad \text{for} \quad t > s.$$

This means that the continuation value $a = \delta w^*(\delta)$ of each worker provides a "supporting hyperplane" $y = ak+b$ (b is a constant) of the production function $y = f(k)$ at $k = s$. Since the worker's equilibrium wage must be equal to the continuation value $a = \delta w^*(\delta)$, this fact means that the equilibrium coalition size s is the optimal solution of the employer's profit maximization

$$\max f(k) - (\delta w^*(\delta))(k-1).$$

where $k-1$ is the number of workers hired by the employer. The existence condition of an s-equilibrium in our discrete case corresponds to the competitive equilibrium condition $f'(s) = \delta w^*(\delta)$ in the continuous case.

When the discount factor for future payoffs is almost close to one, the expected payoffs w^* for workers converge to zero (and thus the equilibrium wages, too). Therefore, the production function $y = f(k)$ attains the maximum at the equilibrium coalition size s, and the marginal productivity is zero beyond s. The employer exploits the total production, that is, her expected payoff v^* is equal to $f(n)$.

The intuitive reason for this result is as follows. A "wage-cut" competition among workers equalizes all workers' wages. If any worker's wage is higher than others, then the worker is not hired by the employer. When the discount factors for future payoffs are close to one, the workers' continuation payoffs are equal to their expected payoffs in a stationary equilibrium. Also, the expected payoff $v^*(\delta)$ of the employer satisfies the equation

$$v^*(\delta) = \frac{1}{n}(f(s) - (s-1)\delta w^*(\delta)) + \frac{n-1}{n}\delta v^*(\delta).$$

In the limit that δ goes to one, the equality $v^* + (s-1)w^* = f(s)$ holds. On the other hand, since an s-member coalition forms with probability one in an s-equilibrium, it must hold that $v^* + (n-1)w^* = f(s)$. These two equalities imply $w^* = 0$. Since $f(s) = f(n)$, the core of the economy consists uniquely of the equilibrium allocation $(f(n), 0, \cdots, 0)$.

Proposition 5.3. For $2 \leq s < t \leq n$, an (s,t)-equilibrium of the production economy $\mathcal{E}$ is characterized as follows.

(i) Let $v(\delta)$ be the expected payoff of the employer, $w(\delta)$ be the expected payoff of every worker, and $p(\delta)$ be the probability that an s-member coalition is formed. Then,

$$v(\delta) = \frac{(t-1)f(s)-(s-1)f(t)}{\{n-(n-1)\delta\}(t-s)} \tag{20}$$

$$w(\delta) = \frac{f(t)-f(s)}{(t-s)\delta} \tag{21}$$

$$p(\delta) = \frac{f(t)-v(\delta)}{f(t)-f(s)} - \frac{n-1}{(t-s)\delta}. \tag{22}$$

(ii) There exists an (s,t)-equilibrium if and only if $0 < p(\delta) < 1$, and $f(s) - f(k) \geq (s-k)\delta w(\delta)$ for all $k \neq s$ with the equality for $k = t$.

(iii) Every (s,t)-equilibrium and every (s',t')-equilibrium are payoff-equivalent if $s, s' < t, t'$.

(iv) Assume that $t < n$. If an (s,t)-equilibrium exists for any δ sufficiently close to one, then $f(k) = f(n)$ for all $s \leq k \leq n$. Moreover, $v(\delta)$ and $w(\delta)$ converge to $f(n)$ and 0, respectively, as δ goes to one.

(v) Assume that $t = n$. If an (s,n)-equilibrium exists for any δ sufficiently close to one, then $\frac{f(n)}{n} \leq \frac{f(s)}{s}$. Moreover, $v(\delta)$ and $w(\delta)$ converge to $v^* = \frac{(n-1)f(s)-(s-1)f(n)}{n-s}$ and $w^* = \frac{f(n)-f(s)}{n-s}$, respectively, as δ goes to one. If $f(s) < f(n)$, then $p(\delta)$ converges to zero as δ goes to one. The expected payoff vector $(v^*, w^*, \cdots, w^*)$ is in the core of the production economy $\mathcal{E}$, and the employer receives the minimum payoff v^* in the core.

The proposition characterizes a partial-employment equilibrium where the employer may hire two different numbers of workers. In a similar manner to the case of an s-equilibrium, the existence condition (ii) can be interpreted as that the two equilibrium coalition sizes s and t $(s < t)$ maximize the employer's profit $f(k) - (\delta w^*(\delta))(k-1)$ where k is the number of hired workers and $\delta w^*(\delta)$ is the worker's wage (continuation value). When the employer does not hire all workers ($t < n$), an (s,t)-equilibrium has the same property as an s-equilibrium. Namely, when the discount factor δ is close to one, the equilibrium wage converges to zero, and the production function $f(k)$ attains the maximum at the equilibrium coalition sizes $k = s, t$. The employer exploits the total benefit $f(n)$. The intuition for this result is similar to the case of an s-equilibrium.

When the employer may hire all workers ($t = n$), the workers receive positive wage $w^* = \frac{f(n)-f(s)}{n-s}$, provided that players are sufficiently patient. The fact that the employer maximizes the profit $f(k) - w^*(k-1)$ at $k = s, n$ implies that the equilibrium wage $w^* = \frac{f(n)-f(s)}{n-s}$ is the solution of

$$\min_{1 \leq k \leq n-1} \frac{f(n) - f(k)}{n - k}. \tag{23}$$

Unlike the case of $t < n$, the production function $y = f(k)$ does not necessarily attain the maximum at $k = s$, but the production of an s-member coalition per capita is greater than or equal to that of the full employment. The distribution $(v^*, w^*, \cdots, w^*)$ among all players is in the core of the economy and the employer receives the smallest payoff in the core.

When the discount factor is close to one, the residual claim of the employer is equal to her expected payoff v^* in all coalitions which she may propose. Since the employer is indifferent to proposing an s-member coalition and the grand coalition, it must hold that $v^* = f(s) - (s-1)w^* = f(n) - (n-1)w^*$. This gives $w^* = \frac{f(n)-f(s)}{n-s}$. The condition $\frac{f(n)}{n} \leq \frac{f(s)}{s}$ means that the employer's expected payoff is greater than or equal to those of workers. Let k ($2 \leq k \leq n$) be any number of workers. The payoff maximization of the employer implies that $v^* \geq f(k) - (k-1)w^*$ for any k. That is, $v^* + (k-1)w^* \geq f(k)$. On the other hand, since the employer may propose either s-member coalition or the grand coalition, it holds that $v^* + (s-1)w^* = f(s)$ and $v^* + (n-1)w^* = f(n)$. These three conditions imply that the equilibrium distribution $(v^*, w^*, \cdots, w^*)$ is in the core of the economy, and that the employer receives the smallest payoff in the core.

We illustrate by the following three-person economy how the type of an SSPE is determined by the production function and the discount factor.

Example 5.1. One employer and two workers

Consider an economy with one employer and two workers. The production function is given by $f(1) = 0, f(2) = a$ $(0 \leq a \leq 1), f(3) = 1$. Figure 5.1 illustrates the regions for three types of equilibria in the economy, depending on the productivity a of a two-member coalition and the discount factor δ. Proposition 5.1 shows that the full-employment equilibrium exists if and only if $\frac{3-\delta}{3} \geq a$ (region (A)). In region A, all players receive the same expected payoffs 1/3. Proposition 5.2 shows that a 2-equilibrium exists if and only if $a \geq \frac{6-5\delta}{6-3\delta-2\delta^2}$ (region (C)). In region C, the employer receives the expected payoff $\frac{2-\delta}{6-5\delta}a$, and the workers receive $\frac{2-2\delta}{6-5\delta}a$. A (2, 3)-equilibrium exists in the region (B), where the employer receives the expected payoff $\frac{2a-1}{3-2\delta}$, and the workers receive the expected payoff $\frac{1-a}{\delta}$. In region (B), the probability of full employment converges to one as the discount factor goes to one. In the limit, the payoff to the employer is $2a - 1$ if $2/3 < a \leq 1$.

We summarize the equilibrium of the production economy $\mathcal{E}$ when the employer and all workers are sufficiently patient.

Theorem 5.1. The production economy $\mathcal{E}$ has a unique SSPE in terms of expected payoffs when the employer and all workers are sufficiently patient. In the limit that the discount factor δ goes to one, the expected payoffs for players are characterized as follows.

(i) If $f(n)/n \geq f(s)/s$ for all s, then all players receive the equal payoff $f(n)/n$. The payoff distribution is in the core of the economy.

(ii) Otherwise, the workers receive the wage $w^* = \frac{f(n)-f(s)}{n-s}$ where s is the

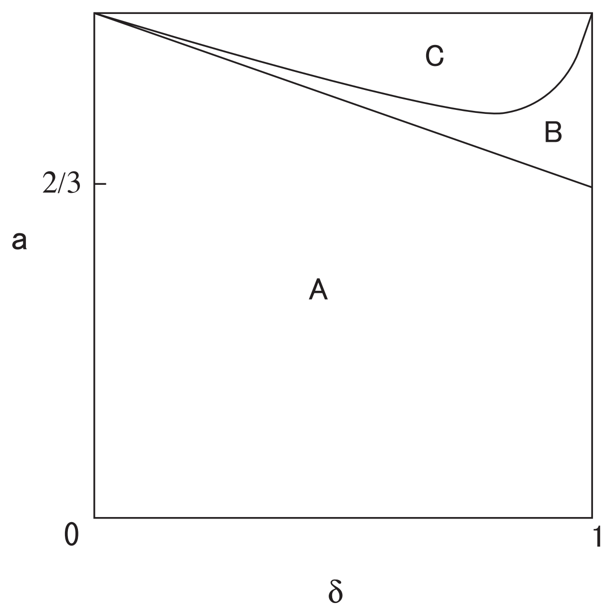

Figure 5.1 One employer and two workers

solution of $\min_{1 \leq k \leq n-1} \frac{f(n)-f(k)}{n-k}$. The employer maximizes the profit by hiring $s-1$ workers. The payoff distribution is in the core of the economy, and the employer receives the smallest payoff in the core.

The theorem shows that when all players are sufficiently patient, they agree to an efficient payoff distribution in the bargaining model. The payoff distribution, however, depends on the production function. The equal division is realized if and only if the full employment attains the highest productivity per capita. It is interesting to see how the SSPE payoff is related to cooperative solutions. We consider the Shapley value and the nucleolus in the production economy $\mathcal{E}$. Since the definitions of these cooperative solutions are standard, we omit them.

Proposition 5.4. The Shapley value and the nucleolus in the production economy $\mathcal{E}$ are characterized as follows.

(1) The Shapley value of the employer is

$$\phi = \sum_{s=1}^{n} \frac{1}{n} f(s). \tag{24}$$

(2) The wage w for workers in the nucleolus is

$$\min_{1 \le s \le n-1} \frac{f(n) - f(s)}{n - s + 1}. \tag{25}$$

The payoff of the employer is

$$\mu = \frac{(n-1)f(s^*) - (s^* - 2)f(n)}{n - s^* + 1}$$

where s^* is the solution of (25).

The intuition for the Shapley value is as follows. When the employer may enter a randomly forming coalition, she joins any s-member coalition with equal probability ($\binom{n-1}{s-1}\frac{(s-1)!(n-s)!}{n!} = \frac{1}{n}$). Since the marginal contribution of the employer to the coalition is $f(s)$, the Shapley value allocation to the employer is given by (24).

The nucleolus and the SSPE payoff have similar properties with respect to payoff distribution. The equal division is realized in the nucleolus if and only

$$\frac{f(n)}{n} = \min_{1 \le s \le n-1} \frac{f(n) - f(s)}{n - s + 1}.$$

This condition implies that, for $s = 2, \cdots, n-1$, $\frac{f(s)}{s-1} \le \frac{f(n)}{n}$, which is slightly stronger than the corresponding one $\frac{f(s)}{s} \le \frac{f(n)}{n}$ for the full-employment equilibrium. When the full-employment equilibrium does not prevail, the wage formula (25) in the nucleolus is almost the same as that (23) in the partial-employment SSPE, while the ratio $\frac{f(n)-f(s)}{n-s+1}$ is replaced with $\frac{f(n)-f(s)}{n-s}$ in (23). From these observations, we can say that the difference between the nucleolus and the SSPE payoff is marginal in a large economy. These two solutions, however, yield different payoff distributions when the number of players is small. The following example shows this point.

Example 5.2. Comparison of three solutions

Consider again the economy with one employer and two workers in Example 5.1. The production function is given by $f(1) = 0, f(2) = a$ $(0 \leq a \leq 1), f(3) = 1$. The discount factor δ is almost equal to one. The nucleolus has the following simple formula. The wage of workers is given by $w = \min(\frac{1}{3}, \frac{1-a}{2})$. The payoff to the employer is given by $v = 1 - 2w$. Figure 5.2 illustrates the employer's payoff in the SSPE, the Shapley value and the nucleolus. The SSPE payoff and the nucleolus, both included in the core, give the equal division when the productivity of a two-member coalition of the employer and one worker satisfies $0 \leq a \leq 1/3$. When the two-member coalition becomes more productive $(1/3 \leq a \leq 2/3)$, the nucleolus wage is $(1 - a)/2$ but the SSPE wage is still $1/3$. When the two-person coalition becomes highly productive $(2/3 < a)$, the SSPE wage becomes $1 - a$. Note that the nucleolus wage is one half of the SSPE wage. The SSPE payoff allocation is equal to the equity allocation $(1/3.1/3, 1/3)$ as long as the latter belongs to the core. The nucleolus, however, departs earlier from the equity allocation as the two-member coalition becomes productive $(1/3 < a)$. The Shapley value is different from the SSPE payoff and the nucleolus. It does not produce the equity allocation if a coalition of the employer and one worker is productive. The wage by the Shapley value decreases as the two-member coalition becomes productive, but the magnitude of the decrease is not as steep as the nucleolus and the SSPE. It gives each worker the positive payoff $1/6$ even when the employer can make the largest production with only one worker. However, in this case, the core consists of a single allocation where the employer exploits the total production. The SSPE payoff and the nucleolus predict the employer's exploitation.

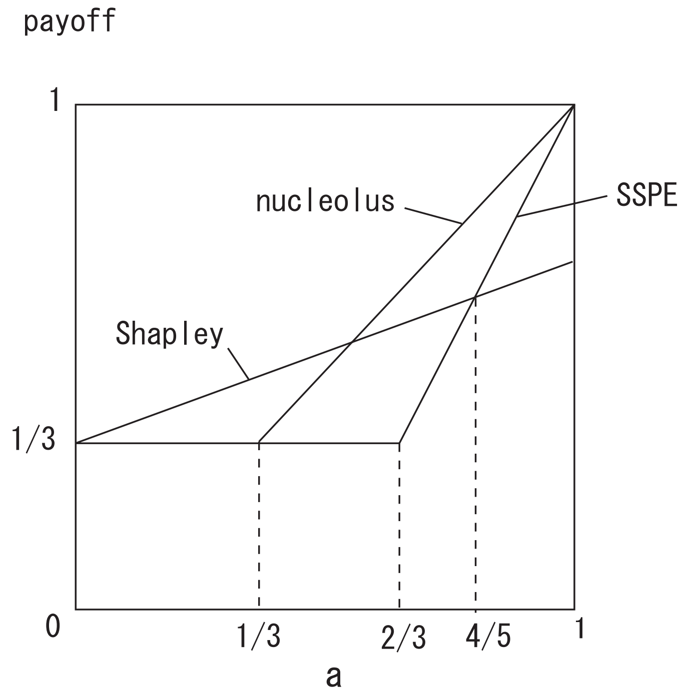

Figure 5.2 The payoff of the employer in three solutions

Figure 5.3 illustrates different solutions, the SSPE payoff, the core, the Shapley value and the nucleolus, in the set of payoff distributions $(x_1, x_2.x_3)$ which satisfy $x_1 + x_2 + x_3 = 1$. In the figure, the equity allocation (1/3, 1/3, 1/3) is outside the core. The SSPE payoff of the employer is the smallest payoff in the core.

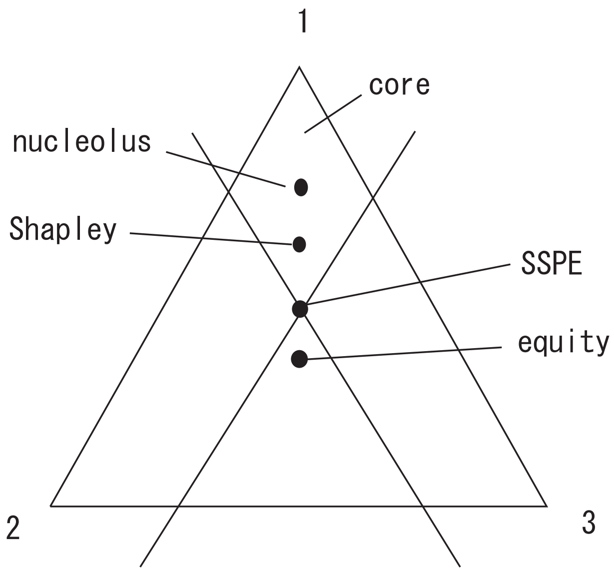

Figure 5.3 Comparison of different solutions

6 Conclusion

We have considered the random-proposer model as a noncooperative bargaining procedure for an n-person coalitional game. By analyzing an SSPE of the bargaining model, we have examined to what extent two approaches, noncooperative game theory and cooperative game theory are complementary in the problem of coalitional bargaining. We have shown that the non-emptiness of the core in a game critically affects an equilibrium of the bargaining model. The grand coalition forms in equilibrium only if the expected payoffs of players lie in an enlarged set of the core, which shrinks to the core as players become sufficiently patient. The bargaining power of a player is determined by the ratio of a recognition probability to a payoff loss by discounting. In the last part, we have applied the bargaining model to a production economy of one employer and multiple workers. We have shown that the SSPE payoff and the nucleolus have similar distributional properties: both solutions lie in the core and predict the equal division when the full employment has a high productivity per capita. The SSPE payoff and the nucleolus have similar wage formula. The Shapley value yields a different distribution. This discrepancy among solutions is caused by the fact that the SSPE and the nucleolus capture strategic conflict of coalition formation in multilateral bargaining, but that the Shapley value does not. It may be interesting for future studies to examine how the results of the production economy can be extended to a general cooperative game.

7 Appendix

Proof of Lemma 5.1. The lemma is proved by Yan (2005, Proposition 5). For convenience of readers, we provide her proof here. Let $\mathcal{C}_i$ be the set of coalitions which player i proposes with positive probability in an SSPE, and let q_i^j be the probability that player i receives an offer when player j is selected as a proposer. For any S including i, define $w(S) = v(S) - \sum_{j \in S} \delta v_j$. Then, it must hold from Lemma 3.2 that

$$v_i = \frac{1}{n}(w(S) + \delta v_i) + \frac{1}{n} \sum_{j \neq i} q_i^j \delta v_i \tag{26}$$

for all $S \in \mathcal{C}_i$, and

$$w(S) \geq w(T) \tag{27}$$

for all $S \in \mathcal{C}_i$ and all $T \subset N$ with $i \in T$. (26) implies

$$v_i = \frac{w(S)}{n - \delta(1 + \sum_{j \neq i} q_i^j)} \quad \text{for all} \quad S \in \mathcal{C}_i. \tag{28}$$

Remark from (27) that $w(S) = w(S')$ for all $S, S' \in \mathcal{C}_i$.

Suppose that $v_i \neq v_j$. With no loss of generality, we assume that $v_i > v_j$.

Claim 1. $w(S^j) \geq w(S^i)$ for all $S^j \in \mathcal{C}_j$ and all $S^i \in \mathcal{C}_i$.

Proof of claim 1. Take any $S^j \in \mathcal{C}_j$ and any $S^i \in \mathcal{C}_i$. If $j \in S^i$, then the claim follows from (27). Otherwise, we have $w(S^j) \geq w((S^i - \{i\}) \cup \{j\}) = w(S^i) + \delta v_i - \delta v_j > w(S^i)$.

Claim 2. $\sum_{k \neq j} q_j^k \geq \sum_{k \neq i} q_i^k$.

Proof of claim 2. Consider any $k \neq i, j$. For any $S \in \mathcal{C}_k$, $i \in S$ implies $j \in S$ since $\delta v_i > \delta v_j$. Hence, $q_j^k \geq q_i^k$. We next prove $q_j^i \geq q_i^j$. It suffices us to show that $q_j^i < 1$ implies $q_i^j = 0$, that is, if there exists some $S^i \in \mathcal{C}_i$ with $j \notin S^i$, then any $S^j \in \mathcal{C}_j$ does not include i. Suppose not. Then, there exists some $S^i \in \mathcal{C}_i$ with $j \notin S^i$ and some $S^j \in \mathcal{C}_j$ with $i \in S^j$. Since $S^i \in \mathcal{C}_i$ and $i \in S^j$, we have $w(S^i) \geq w(S^j) > w(S^j) + \delta v_j - \delta v_i$. On the other hand, since $S^j \in \mathcal{C}_j$, we have $w(S^j) \geq w((S^i - \{i\}) \cup \{j\}) = w(S^i) + \delta v_i - \delta v_j$. A contradiction.

By (28), the two claims imply $v_j \geq v_i$. This contradicts the supposition. QED

Proof of Proposition 5.2.

(i) It follows from Lemma 3.2 that

$$\begin{aligned} v^*(\delta) &= \frac{1}{n}(f(s) - (s-1)\delta w^*(\delta)) + \frac{n-1}{n}\delta v^*(\delta) \\ f(s) &= v^*(\delta) + (n-1)w^*(\delta). \end{aligned}$$

It can be easily shown that this system has a unique solution (18) and (19). Let q be the probability that every worker receives an offer. Again, from Lemma 3.2, we have

$$w^*(\delta) = \frac{1}{n}(f(s) - \delta v^*(\delta) - (s-2)\delta w^*(\delta)) + q\delta w^*(\delta).$$

Together with (18) and (19), we can prove that this equation has a solution $q = \frac{(s-2)n+1}{n(n-1)}$.

(ii) The optimality condition (4) for the employer is given by

$$f(s) - (s-1)\delta w^*(\delta) \geq f(t) - (t-1)\delta w^*(\delta) \quad \text{for all} \quad t \neq s.$$

This is equivalent to $f(s) - f(t) \geq (s-t)\delta w^*(\delta)$ for all $t \neq s$. The optimality conditions for workers give the same inequality. Conversely, we can construct an s-equilibrium as follows. The employer selects every $s-1$ workers with equal probability, and offers $\delta w^*(\delta)$ to them. Every worker selects every s-member coalition including the employer and herself with equal probability, and offers $\delta v^*(\delta)$ to the employer, and $\delta w^*(\delta)$ to other workers in the coalition. When this strategy is employed, every worker receives an offer with probability $\frac{(s-2)n+1}{n(n-1)}$. If the condition in the proposition holds, then we can prove from Lemma 3.2 that the constructed strategy is an SSPE.

(iii) In view of (19) and $s < n$, $w^*(\delta)$ converges to zero as δ goes to one. Noting this fact, we can show that for all t,

$$f(s) - f(t) \geq (s-t)\delta w^*(\delta) \quad \text{for any } \delta \text{ sufficiently close to one}$$

if and only if

$$f(s) \geq f(t) \quad \text{for all} \quad t = 1, \cdots, n$$

with the strict inequality for all $t < s$. Since $f(s)$ is a monotonically increasing function, this proves the first part. By (18) and (19), we have

$$\lim_{\delta \to 1} v^*(\delta) = f(n) \quad \text{and} \quad \lim_{\delta \to 1} w^*(\delta) = 0.$$

Hence, the employer offers zero payoff to every worker in a coalition in the limit that δ goes to one. When $f(s) = f(t)$ for all $t \geq s$, the core of the production economy $\mathcal{E}$ consists of a unique allocation $(f(n), 0, \cdots, 0)$. This proves the second part. QED

Proof of Proposition 5.3.

(i) By Lemma 3.2, $v = v(\delta)$ and $w = w(\delta)$ satisfy the following conditions.

$$\begin{aligned} v &= \frac{1}{n}(f(s) - (s-1)\delta w) + \frac{n-1}{n}\delta v && (29)\\ v + (n-1)w &= pf(s) + (1-p)f(t) && (30)\\ f(s) - (s-1)\delta w &\geq f(k) - (k-1)\delta w \ \text{ for all } \ k \neq s && (31)\\ f(t) - (t-1)\delta w &\geq f(k) - (k-1)\delta w \ \text{ for all } \ k \neq t && (32)\end{aligned}$$

By (31) and (32), we have $f(s) - (s-1)\delta w = f(t) - (t-1)\delta w$, which yields (21). By substituting (21) into (29), we have (20). Finally, we obtain (22) from (21) and (30).

(ii) (31) and (32) are equivalently reduced to $f(s) - f(k) \geq (s-k)\delta w(\delta)$ for all $k \neq s, t$ and $f(s) - f(t) = (s-t)\delta w(\delta)$. The proposition can be proved by Lemma 3.2.

(iii) Let v and w are the expected payoffs for the employer and workers, respectively, in an (s,t)-equilibrium, and let v' and w' are the expected payoffs for the employer and workers, respectively, in an (s',t')-equilibrium. In view of (21), the optimality conditions for an (s,t)-equilibrium are

$$f(s) - f(k) \geq (s-k)\frac{f(t) - f(s)}{t-s} \tag{33}$$

$$f(t) - f(k) \geq (t-k)\frac{f(t) - f(s)}{t-s}. \tag{34}$$

for any $k \neq s, t$. Similarly, the optimality conditions for an (s',t')-equilibrium are

$$f(s') - f(k) \geq (s'-k)\frac{f(t') - f(s')}{t'-s'} \tag{35}$$

$$f(t') - f(k) \geq (t'-k)\frac{f(t') - f(s')}{t'-s'} \tag{36}$$

for any $k \neq s, t$. Putting $k = s'$ in (34) and $k = t$ in (35), we obtain

$$\frac{f(t) - f(s)}{t-s} \leq \frac{f(t') - f(s')}{t'-s'}$$

since $s' < t$. Similarly, putting $k = t'$ in (33) and $k = s$ in (36), we obtain

$$\frac{f(t) - f(s)}{t - s} \geq \frac{f(t') - f(s')}{t' - s'}$$

since $s < t'$. Thus, we have $w = w'$. If $s = s'$, then $v = v'$ easily follows from (29) and $w = w'$. Suppose that $s > s'$, without loss of generality. Putting $k = s'$ in (33), we have

$$\frac{f(s) - f(s')}{s - s'} \geq \delta w.$$

Similarly, putting $k = s$ in (35), we have

$$\frac{f(s) - f(s')}{s - s'} \leq \delta w'.$$

Hence, we have

$$\frac{f(s) - f(s')}{s - s'} = \delta w = \delta w'.$$

This equality with (29) yields $v = v'$.

(iv) From (20) and (21), we can see that $v(\delta)$, $w(\delta)$ converge to

$$v^* = \frac{(t-1)f(s) - (s-1)f(t)}{t - s}, \quad w^* = \frac{f(t) - f(s)}{t - s}, \tag{37}$$

respectively, as δ goes to one. Let p^* be any accumulation point of $\{p(\delta)\}$. By taking $\delta \to 1$ in (30), we have $v^* + (n-1)w^* = p^* f(s) + (1 - p^*) f(t)$. Substituting (37) into this equation, we obtain

$$(p^* + \frac{n - t}{t - s}) f(s) = (p^* + \frac{n - t}{t - s}) f(t). \tag{38}$$

Since $t < n$, $p^* + \frac{n-t}{t-s} > 0$. Then, it follows from (38) that $f(s) = f(t)$. Thus, $v^* = f(s)$ and $w^* = 0$ from (37). Finally, $f(s) = f(t)$ implies $f(s) = f(n)$ from (33) with $k = n$.

(v) Putting $t = n$ in (37) yields $v^* = \frac{(n-1)f(s)-(s-1)f(n)}{n-s}$ and $w^* = \frac{f(n)-f(s)}{n-s}$. Since $t = n$, (38) implies that $p^* f(s) = p^* f(n)$ for any accumulation point p^*

of $\{p(\delta)\}$. If $f(s) < f(n)$, then we obtain $p^* = 0$. Hence, $\{p(\delta)\}$ converges to 0 as δ goes to one. Hence, whichever $f(s) < f(n)$ or $f(s) = f(n)$ holds, we obtain

$$v^* + (n-1)w^* = f(n). \tag{39}$$

Since $p \geq 0$ and $f(s) \leq f(n)$, (30) implies $v(\delta) + (n-1)w(\delta) \leq f(n)$. Substituting (20) and (21) into this inequality, a tedious calculation yields

$$\frac{f(s)}{n-(n-s)\delta} \geq \frac{f(n)}{n}.$$

By taking $\delta \to 1$, we obtain $\frac{f(s)}{s} \geq \frac{f(n)}{n}$. We will prove the last part. It follows from (37) that

$$v^* + (k-1)w^* = \frac{(n-k)f(s) - (s-k)f(n)}{n-s} \tag{40}$$

for all $k \geq 2$. On the other hand, (34) yields

$$f(n) - f(k) \geq (n-k)\frac{f(n)-f(s)}{n-s}. \tag{41}$$

(40) and (41) imply that $v^* + (k-1)w^* \geq f(k)$ for all $k \geq 2$. Together with (39), this means that the payoff allocation $(v^*, w^*, \cdots, w^*)$ is in the core of the production economy $\mathcal{E}$. Any core allocation $(v_1, w, \cdots, w)$ in which all workers receive the same payoffs w satisfies $v + (n-1)w = f(n)$ and $v + (s-1)w \geq f(s)$. These conditions imply $v \geq v^*$. Finally, for any core allocation $(v_1, w_2, \cdots, w_n)$, there exists some value w such that the allocation $(v_1, w, \cdots, w)$ is in the core. Thus, the employer's payoff v^* is her minimum payoff in the core. QED

Proof of Theorem 5.1. We first note that even if there exists an SSPE with more than two different sizes of coalitions, the same proof as in Proposition 5.3 can be applied, and thus the economy has the same payoff distribution as in an

(s,t)-equilibrium in the limit that the discount factor δ goes to one. The ratio $\frac{f(s)-f(t)}{s-t}$ is constant for any two coalition sizes s and t in the equilibrium. The equilibrium wage is equal to this ratio. From this argument, it follows that the economy has a unique SSPE payoff when the discount factor δ is close to one. (i) is proved by Proposition 5.1. If (i) does not hold, there exists some k such that $f(n)/n < f(k)/k$, which implies $\frac{f(n)-f(k)}{n-k} < \frac{f(n)}{n}$. Let s be the solution of $\min_{1\leq k\leq n-1} \frac{f(n)-f(k)}{n-k}$ and let $w = \frac{f(n)-f(s)}{n-s}$. Geometrically, s satisfies the property that all points $(k, f(k))$ are included in the half-space determined by the line connecting points $(s, f(s))$ and $(n, f(n))$. The employer's profit $f(k)-(k-1)w$ is maximized at $k = s, n$. Then, (ii) is proved by Propositions 5.2 and 5.3. QED.

Proof of Proposition 5.4.

(1) By the definition of the Shapley value, we have

$$\phi_1 = \sum_{s=1}^{n} \binom{n-1}{s-1} \frac{(s-1)!(n-s)!}{n!} f(s) = \sum_{s=1}^{n} \frac{1}{n} f(s). \tag{42}$$

(2) Since the nucleolus satisfies the axiom of symmetry, all (identical) workers receive the same payoffs. Let v be the employer's payoff and w be the wage for workers. Then, the nucleolus is the solution of the following program:

$$\begin{aligned} \min \quad & \varepsilon \\ s.t. \quad & v + (s-1)w + \varepsilon \geq f(s), \quad s = 1, \cdots, n-1 \\ & sw + \varepsilon \geq 0, \quad s = 1, \cdots, n-1 \\ & v + (n-1)w = f(n) \end{aligned}$$

It can be easily seen that the optimal solution of w satisfies $w^* \geq 0$. This means that constraints $sw + \varepsilon \geq 0$ for $s = 2, \cdots, n-1$ are redundant. We can show that the remaining constraints imply

$$\varepsilon \geq -\frac{f(n) - f(s)}{n - s + 1}.$$

Thus, the optimal solution $(v^*, w^*, \varepsilon^*)$ satisfies

$$w^* = -\varepsilon^* = \min_{1 \leq s \leq n-1} \frac{f(n) - f(s)}{n - s + 1}, \ v^* = f(n) - (n-1)w^*.$$

This proves the proposition. QED

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