On the Existence of Positive Equilibrium Profits in Competitive Screening Markets*

Yehuda John Levy† André Veiga‡

July 23, 2020

Abstract

Frictionless consumer choices and price competition are often associated with competitive markets and vanishing equilibrium profits. We discuss vanishing profits in competitive screening markets like insurance. We assume symmetric firms which exhibit constant returns to scale. Consumer heterogeneity creates the possibility of adverse selection. Firms can offer multiple contracts in equilibrium and (importantly) in any deviation. Nash equilibrium profits vanish if each consumer has a unique optimizing bundle at equilibrium prices or, more generally, if there exists a linear ordering of contracts that dictates the preferences of firms whenever consumers are indifferent between multiple optimal contracts. Of particular interest, equilibrium profits vanish if, for each agent, indifference curves are steeper than iso-profit curves. The results extend to Miyazaki-Wilson-Spence equilibria. We provide examples of economies where there exists an equilibrium with strictly positive profit and show that these examples are robust (hold for an open set of economies).

Keywords: Perfect Competition, Equilibrium, Screening
JEL Classification Codes: D41, C62, D82, G22

*We thank the editor and anonymous referees for their comments. All remaining mistakes are our own. Nathan Somerville provided excellent research assistance.

†Adam Smith Business School, University of Glasgow, Glasgow, UK. Email: john.levy@glasgow.ac.uk

‡Imperial College Business School, South Kensington Campus, Ayrton Rd, Kensington, SW7 2AZ, London. Email: a.veiga@imperial.ac.uk

1 Introduction

This article considers economies with a fixed number of symmetric firms, competition in prices, constant returns to scale, and frictionless consumer choices. More specifically, we consider screening markets (for instance, insurance) where firms offer multiple products, consumers have heterogeneous preferences over products and asymmetric information about their cost. This asymmetric information can cause adverse selection, as in Rothschild and Stiglitz (1976). Several articles studying these environments focus on equilibria where profits are zero. In this article, we derive conditions which imply vanishing equilibrium profits in screening markets and thus justify restriction attention to those equilibria. We also provide examples of screening markets where equilibria with positive profits exist and show that these examples are not knife-edge cases.

If free entry of firms is assumed, standard arguments imply that equilibrium profits are zero (e.g., Varian (2010, p. 433)). This argument was used in the context of screening markets by Rothschild and Stiglitz (1976, p. 634), and many others to justify focusing on equilibrium where firms breaking even; see, e.g., Fang and Wu (2018), who also use this argument, and the references within. However, there are also several frameworks where the number of firms is fixed and equilibrium profits are zero (this was called the *Bertrand paradox* by Tirole (1988, p. 10)). Conversely, there also exist environments where equilibrium profits are positive despite price competition and frictionless consumer choices. These include decreasing returns to scale, learning, imperfect information about rival's costs, etc, as we discuss further in Section 2. To our knowledge, there has been no known general condition that ensures zero equilibrium profits in screening markets.

This article considers markets where firms offer multiple products and consumers differ in their preferences over these products. Consumers also have asymmetric information regarding the costs that they would impose on firms following the purchase of each product. For instance, at a given price, unhealthy individuals are particularly costly to insurers when they buy high-coverage insurance plans. Firms must offer a given product at the same price to all consumers (due to regulation or informational asymmetries). This asymmetry of information can create adverse selection.

In most models of selection markets, in equilibrium, some types of consumers are indifferent between multiple optimal products. For instance, in Rothschild and Stiglitz (1976), in equilibrium, the high-cost type is indifferent between the full-insurance and partial-insurance contracts. However, even if consumers are indifferent between several optimal products, firms need not be indifferent about which product is chosen by each consumer type. We show that the nature of these preferences of firms over contracts is key to determining whether equilibrium profits can be positive.

For example, suppose that two firms ship items to consumers but are obliged to charge the same price to all consumers. Consumers are indifferent between buying from either firm, but each firm would prefer to sell to those consumers who live close-by to that firm.

In this case, equilibrium profits can be strictly positive. If a firm were to undercut these equilibrium prices, it would attract all buyers. Some of those buyers are costly to serve, thus making the deviation unprofitable. That is, deviations are unprofitable due to adverse self-selection of individuals into a product where they are unprofitable. This result can help explain positive markups in industries where the classic conditions for price competition are in place.

Our main result (Theorem 1) is that zero equilibrium profits are guaranteed if contracts offered in equilibrium can be "vertically differentiated" from the point of view of the firms. That is, contracts can be linearly ranked such that, whenever any consumer has several optimal choices, the firms prefer that he chooses the higher ranked option. This condition arises naturally in many frameworks of insurance markets like Rothschild and Stiglitz (1976), where the contracts are ordered in equilibrium by their level of coverage, and firms prefer that consumers buy the contract with higher coverage in cases of indifference. Theorem 3 shows that this ordering of contracts occurs in any equilibrium if indifference curves are steeper than iso-profit curves. This ordering also holds in any equilibrium if consumer utility is the sum of (minus) profit and a surplus (for instance, due to strictly positive risk aversion), and surplus increases more quickly with coverage than does profit. In these cases, any equilibrium has vanishing profit. Notice that, in the example with positive profits above, there does not exist such an ordering that is valid for all individuals: each firm prefers close-by consumers to buy, but far-away consumers not to buy.

To establish Theorem 1, we show that the above "vertical differentiate" condition implies that, if a firm was making a positive profit, another firm could lower prices ever so slightly and steal the entire market (or, at least, steal all those consumers who purchase in the market). Since the prices have only been lowered slightly, this firm would capture all but an infinitesimal amount of the total profit in the market. As a result, our Theorem 1 extends (Theorem 4) readily to the equilibria introduced and studied in Miyazaki (1977); Wilson (1977); Spence (1978). This notion of 'reactive equilibria' need only be robust against deviations which would be profitable for a firm if the other firms were to be allowed to react by withdrawing contracts from the market; indeed, since the deviations we present capture the entire market of active consumers, withdrawal of contracts on the parts of other firms would inconsequential.[1]

We further show that our examples of economies with positive equilibrium profit are not knife-edge cases (do not only occur in a negligible set of economies). A list of conditions that guarantee positive equilibrium profits (Lemma 6) are non-degenerates and hold in an open subset of economies (in a sense we make precise below, Theorem 5). We also show that, whenever positive profits do arise, a key ingredient is the presence of 'cycles' of consumers who are indifferent over two or more contracts. If such a cycle exists, in

[1] We are very grateful to an anonymous referee for pointing this extension out to us.

equilibrium, there are pairs of consumers indifferent between the contract purchased by himself and his 'neighbour', yet firms would prefer consumers not switch to the neighbor's contract and the 'string' of such pairs of consumers form a cycle (Proposition 1).

Our results show that an a priori focus only on equilibria with zero profits can be misleading, since equilibria with positive profits may exist. On the other hand, in some settings (such as some classes of insurance markets), our results show that *any* equilibrium must feature zero profit. In these settings, our results justify focusing on equilibria where firms break even and can be used to characterize these equilibria.

The paper is organized as follows. Section 2 contains a short literature review. Section 3 motivates the analysis with simple examples of screening markets where equilibrium features positive profits. Section 4 describes the model setup. Section 5 presents the results pertaining to zero profits. Section 6 discusses sources of positive profits and local equilibrium. All proofs, as well as some additional examples and details, are contained in the Appendix.

2 Literature Review

The Bertrand (1883) model of competition features firms choosing prices, undifferentiated products, constant returns to scale (constant marginal cost), and perfect information. In equilibrium, prices are equal to marginal cost and profits are zero. The "Bertrand paradox" refers to the fact that profits vanish even when there are only two firms competing.

Edgeworth (1925) pointed out that the Bertrand paradox requires constant returns and does not occur if firms have capacity constraints (a form of decreasing returns). A similar result is described by Levitan and Shubik (1972) and generalized by Kreps and Scheinkman (1983) to more general cost functions and asymmetric firms. Hotelling (1929) considers price competition in differentiated products, where equilibria with positive profits arise due to product differentiation which creates market power. Dos Santos Ferreira and Dufourt (2007) show that positive equilibrium profit is possible under free entry and decreasing average cost. Hurkens and Vulkan (2003) show that free entry does not result in zero profits when there is an (arbitrarily small) cost to learning demand. Alós-Ferrer, Ania and Schenk-Hoppé (2000) present an evolutionary model of Bertrand competition where, in the long run, firms make positive profits. Kaplan and Wettstein (2000) show that, with constant returns but unbounded revenue, it is possible to have mixed strategy equilibria with positive profits. Baye and Morgan (1999) show that, if monopoly profits are unbounded, then any feasible pay-off pair can be achieved in a Bertrand mixed strategy equilibrium. In Spulber (1995), if rivals' costs are unknown, firms price above marginal cost and have positive expected profit.

The articles listed in the paragraph above do not consider, as we do, markets where individuals differ in their cost, i.e., screening markets. In contrast, we assume price competition, undifferentiated products, symmetric firms, constant returns to scale, and frictionless consumer choices, but also allow individuals to differ in their preferences and in the cost that they impose on firms by buying each of the firm's products. It is this heterogeneity in cost which creates the possibility of positive equilibrium profits. In turn, this motivates our results which describe classes of competitive economies where positive profits in equilibrium can be ruled out.[2]

Several articles have argued that positive profits are possible in screening markets (as we do). To the best of our knowledge, those results rest crucially on a notion of equilibrium which restricts deviations to be of a single contract; we do not make any such restrictions. Wambach (2000) allows individuals to differ in both risk and initial wealth, and Villeneuve (2003) allows for privately known heterogeneity in risk aversion. In Smart (2000), individuals are heterogeneous in risk aversion and positive equilibrium profits are sustained by entry costs. Kubitza (2019) obtains positive equilibrium profits in an insurance setting with multi-dimensional types and single crossing, but also ignores multi-contract deviations. In contrast to these articles, we will allow firms to deviate by offering arbitrary menus of products. Moreover, we will consider settings that are significantly more general than the settings considered above.

In a related article, Snow (2009) considers insurance markets in which consumers have two possible levels of riskiness and two possible levels of risk aversion. Snow shows that positive profit equilibria do not exist if each firm is allowed to deviate by offering multiple contracts (as we do). Our main result (Theorem 1) applies to some classes of insurance markets where the space of types can be significantly more general than '2×2', and also to more general (non-insurance) screening markets. Therefore, the result can be seen as a significant generalization of Snow (2009). Moreover, this article also provides examples of economies where adverse selection does create the possibility of positive equilibrium profits in (non-insurance) screening markets where firms offer multiple contracts.

3 Motivating Examples

3.1 One Good

The following example, while much simpler than the settings we consider below, illustrates the possibility of positive equilibrium profits due to selection. Two firms offer a single product and compete in prices, p. Consumer types $\omega \in \{\omega_H, \omega_L\}$ capture the cost of serving each consumer. For low-cost consumers ($\omega_L = 0$), shipping costs are zero. For high-cost consumers ($\omega_H = 10$), shipping cost per consumer is 10. A given firm must charge the same price to all buyers (due to regulation or asymmetric information) and

[2] It is intuitive that, for instance, adding market power to a model of insurance would re-introduce the possibility of positive equilibrium profits. However, in this article we focus on competitive markets and examine whether asymmetric information might result in positive equilibrium profit.

cannot reject willing buyers. Firm profits per customer are given by

$$v(\omega, p) = p - \omega.$$

Consumers choose an option $x \in X = \{0, 1\}$, where $x = 1$ means buying the product, while $x = 0$ means not buying. If a consumer chooses to buy, she also chooses which firm to buy from. Utility is

$$u(\omega, x, p) = x(1 - p)$$

For either type, buying at a price of p yields utility $u = 1 - p$. Not buying yields zero. The mass of each type is given by $\lambda(\omega)$. For simplicity, let $\lambda(\omega_H) = \lambda(\omega_L) = 1/2$. Also for simplicity, production costs are zero. Firms simultaneously post prices, then consumers decide whether to buy and from which firm.

There exists a Nash equilibrium where both firms charge price $p = 1$, and only types ω_L buy, randomizing equally across firms. In this equilibrium, each firm makes a positive profit of $\pi = 1/2$. There is no profitable deviation. A firm that raises its price loses all customers and obtains profit $\pi = 0$. If a firm undercuts to any $p' < 1$, he would attract all buyers (high and low-cost), resulting in a profit of $\pi = p_j - 5 < -4$.

In this equilibrium, all consumers are indifferent between buying and not buying, but firms prefer that low-cost consumers buy but high-cost consumers do not. There does not exist, in this equilibrium, a linear ordering of contracts, which dictates preferences for the firms in cases of consumer indifference, that is valid for all types of buyers. This violates the assumptions required by our main result (Theorem 1) which guarantees vanishing equilibrium profits. Notice also that this example is not a "knife-edge" case. In any (sufficiently small) perturbation of the profits or utility functions, there exists an equilibrium with positive profit, as we discuss further in Section 6.2.

We do not claim profit is positive in *every* equilibrium. Indeed, there exists a mixed strategy equilibrium (mixed in the sense that not all consumers of the same type choose the same alternative) where a mass $\frac{1}{18}$ of types ω_H joins each firm and each firm obtains profit $\pi = \frac{1}{2}(1 - \omega_L) + \frac{1}{18}(1 - \omega_H) = 0$. Instead, our aim is to motivate the article by showing that an equilibrium with positive profit is possible. Below we will present conditions on primitives (Theorem 3) under which *all* equilibria feature vanishing profit.

3.2 Two Goods

Our second example better illustrates the generality of our setup, since firms offer multiple products. We now take as an example the health insurance industry. Consumer types are $\omega \in \{\omega_A, \omega_B\}$. The mass of each type is $\lambda(\omega_A) = \lambda(\omega_B) = 1/2$. There are $N \in \mathbb{N}$ firms, each offering two products, $x \in \{A, B\}$. We assume that the market is covered, so consumers buy one product or the other.[3] Let (ω, x) denote type ω purchasing contract x.

Fix $\alpha \in \mathbb{R}$. The utility of type ω purchasing x at a price p is

$$u(\omega, x, p) = \begin{cases} 2(\alpha - p) & \text{if } (\omega_B, A) \\ \alpha - p & \text{if } (\omega_B, B) \\ 6(\alpha - p) & \text{if } (\omega_A, A) \\ 4(\alpha - p) & \text{if } (\omega_A, B) \end{cases}$$

If both products are sold at the same price, it is more profitable for firms if types ω_A buy A and if ω_B to buy B. For instance, types ω_A might refer to patients with heart conditions, while ω_B refers to patients with cancer. Then, plans A, B might refer to hospitals which have a low cost of treating heart conditions and cancer, respectively. The products are otherwise symmetrical (in particular, patients will receive the same quality of care in either hospital). Firm payoffs are

$$v(\omega, x, p) = \begin{cases} p + 1 & \text{if } (\omega_A, A) \text{ or } (\omega_B, B) \\ p - 2 & \text{if } (\omega_A, B) \text{ or } (\omega_B, A) \end{cases} \tag{3.1}$$

The following is an equilibrium. Each firm sells each product at the same price, $p = \alpha$. At the symmetric price $p = \alpha$, every individual is indifferent between the two contracts. Each type chooses the contract where it is most cheaply served (types ω_A choose A and types ω_B choose B). Individuals randomize across firms. Equilibrium profit is $\pi = \frac{1}{N} > 0$.

There are no profitable deviations. Indeed, firms are allowed to deviate in the price of both contracts. If a firm raises both prices, he loses all consumers. Lowering both prices can increase market share but a deviating firm would wish to do so carefully, so as not to incur losses resulting from purchases by costly consumers; but in this case, it cannot be done in a way which preserves positive profits. Let p_A, p_B denotes the prices of contracts A, B at the deviation. Lowering p_B alone would attract types ω_A to buy B, which is unprofitable, so a drop in p_B must be accompanied by a drop in p_A that ensures ω_A still prefers A. However, a large drop in p_A would attract types ω_B to buy A, which is also unprofitable.

To show that this is an equilibrium, first notice the following ratio of price sensitivities, for each type, across contracts:

$$\frac{\frac{\partial u(\omega_B, B, p)}{\partial p}}{\frac{\partial u(\omega_B, A, p)}{\partial p}} = \frac{1}{2}, \qquad \frac{\frac{\partial u(\omega_A, A, p)}{\partial p}}{\frac{\partial u(\omega_A, B, p)}{\partial p}} = \frac{3}{2}.$$

Suppose a firm lowers p_B to some $p_B = \alpha - \delta$, for some $\delta > 0$. The accompanying drop in p_A must not attract types ω_B to buy A. Given the ratio of price sensitivities above, the

[3] This could be due to a mandate as in the United States Affordable Care Act. Alternatively, one could add a constant to all utilities to ensure all individuals buy one of the products in the relevant range of prices.

firm must choose $p_A \geq \alpha - \frac{1}{2}\delta$. Moreover, the drop in p_B cannot attract types ω_A to buy B. Given the ratio of price sensitivities above, p_B cannot fall by more than $(\frac{1}{2} \times \frac{3}{2})\delta = \frac{3}{4}\delta$, so the firm must choose $p_B \geq \alpha - \frac{3}{4}\delta$, which contradicts that p_B can be lowered by $\delta > 0$. This 2-product example is illustrated in Figure **??**.[4] In Appendix A we present an example with an arbitrary number of products.

Notice that, in equilibrium, consumers are indifferent but firms obtain higher profit if ω_A types buy A rather than B, and vice-versa for ω_B types. There does not exist, a linear ordering of contracts, which dictates preferences for the firms in cases of consumer indifference, that is valid for all types of buyers. This violates the assumptions required by our main result (Theorem 1) that guarantees vanishing equilibrium profits. Notice also that this example is not a "knife-edge" case. In any (sufficiently small) perturbation of the profits or utility functions, there exists an equilibrium with positive profit, as we discuss further in Section 6.2.[5]

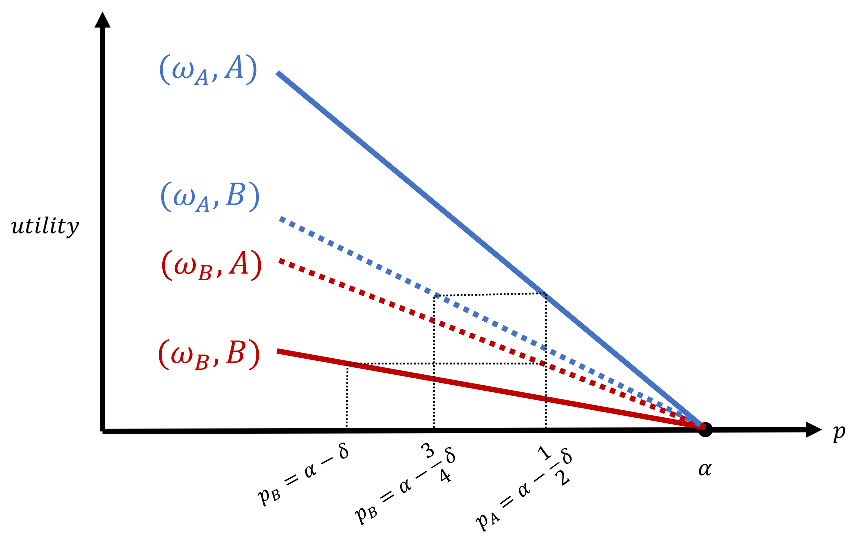

Figure 1: The graphic shows the level of utility, for different prices p, of each combination (ω, x). Equilibrium has $p_A = p_B = \alpha$ and $(\omega_A, A), (\omega_B, B)$. Full lines show utility at the contracts chosen in equilibrium. Dotted lines show utility when each type chooses the non-equilibrium contract. A deviation to $p_B = \alpha - \delta$ requires $p_A \geq \alpha - \frac{1}{2}\delta$ which, in turn, requires $p_B \geq \frac{3}{4}\delta$. There does not exist, for firms, a linear ordering of contracts that is valid for all types of buyers, which violates the assumptions of Theorem 1 which guarantees vanishing equilibrium profits.

[4] We thank an anonymous referee for the suggestion of this figure.

[5] Again, in this example, there exists a mixed strategy equilibrium with vanishing profit. If 1/3 of types ω_B buy B and 2/3 buy A, firms make zero profits. We thank an anonymous referee for pointing out this equilibrium.

4 Model Setup

4.1 Utility and Profit

Let X be the set of alternatives available to individuals. A typical option is denoted $x \in X$. We assume X is a locally compact Polish space[6] which may or may not include an outside option, denoted $x = 0$.[7] A typical type is denoted $\omega \in \Omega$. Let Ω also be a locally compact Polish space capturing the types of individuals. Types distribute via a measure λ on Ω, with $\lambda(\Omega) = 1$.

The utility that a consumer of type $\omega \in \Omega$ gains from obtaining item $x \in X$ at price $p \in \mathbb{R}_+$ is $u(\omega, x, p)$, assumed to be continuous and strictly decreasing in price. Individuals can choose only a single alternative from a single firm (or, possibly, not to purchase at all if $0 \in X$).

There are N symmetric firms indexed by $j \in \{1, ..., N\}$, each setting a price for each contract $x \in X$. With abuse of notation, let N also denote the set of all firms. All products are potentially available from all firms.[8]

The profit to a firm from providing contract x to type ω at price p is $v(\omega, x, p)$, assumed continuous and weakly increasing in p. If $0 \in X$, then $v(\cdot, 0, 0) \equiv 0$. That is, the outside option costs nothing to supply and yields no revenue. Further technical assumptions are given in Section 4.1.

The timing of the game is as follows. First, firms choose prices simultaneously. Then, consumers observe prices and choose to purchase one product from one firm (or, possibly, not buy if $0 \in X$).

Let p_x^j be the price charged by firm j for contract x. A price vector containing the price of each item is denoted $\overline{p} = (p_x)_{x \in X} \in \mathbb{R}_+^X$; if $0 \in X$, then $p(0) = 0$. The notation $\overline{p} < \overline{q}$ means $p(x) < q(x)$ for $x \neq 0$. We assume prices are non-negative but the paper could easily be modified to allow for negative prices.[9]

A consumer profile is a mapping $\sigma = (\sigma_N, \sigma_X) : \Omega \to N \times X$, specifying which firm and contract each consumer buys. A consumer profile $\sigma = (\sigma_N, \sigma_X) : \Omega \to N \times X$ is required to be λ-measurable. The profit of firm k under consumer profile σ when he sets price $\overline{p}^k$ is:

$$\pi^k(\sigma, \overline{p}^k) = \int_{\{\omega | \sigma_N(\omega) = k\}} v\big(\omega, \sigma_X(\omega), p^k(\sigma_X(\omega))\big) \cdot \lambda(d\omega)$$

[6] A Polish space is a complete, separable metric space.

[7] There need not be an outside option. For instance, the healthcare exchanges under the United States Affordable Care Act Exchanges, do not have a zero cost outside option since individuals are required to buy some insurance contract.

[8] We believe that the assumption of a finite number of firms is innocuous, and that a suitably re-stated version of our results would hold for the case of infinite firms. Indeed, we are not aware of any model of selection markets that models perfect competition by means of a continuum of firms.

[9] All the results in the paper would hold if we allowed for negative prices, but we show that they hold when prices (including prices at deviations) are constrained to be positive. Allowing for negative prices would make some of the arguments simpler: if prices can be negative, undercutting prices exist more easily.

The technical Assumption 3 below ensures that profit is always finite. We emphasize the difference between $v(\cdot)$, which gives the profit of a single interaction between a type purchasing an alternative at a given price, and the profile π^k of a firm, which gives profit over all interactions with Firm k as a function of prices and consumer choices.

The setup described above effectively allows for mixed strategies on the part of the consumers to be introduced by replacing the space (Ω, λ) of consumers with a space $(\Omega \times [0,1], \lambda \times m)$, where m is the Lebesgue measure. Hence, each consumer is replaced with a continuum of herself, allowing the consumer to effectively mix. This modeling technique is standard, going back to Aumann (1964).

We do not impose compactness on Ω, X. For instance, Levy and Veiga (2020) model competitive insurance markets with unbounded risk, hence unbounded Ω. In that setting, equilibrium existence may require an open contract space X. Hence, we should not impose a priori compactness on Ω, X.

4.2 Consumer Optimization and Equilibrium

A consumer profile $\sigma = (\sigma_N, \sigma_X)$ is *incentive compatible* w.r.t. a profile of prices $\overline{p}^1, \ldots, \overline{p}^N$ (where $\overline{p}^j = (p^j_x)_{x\in X}$) if

$$u(\omega, \sigma_X(\omega), p_{\sigma(\omega)}) \geq u(\omega, y, p^j_y),\ \forall \omega \in \Omega, \forall y \in X, \forall j \in N$$

where $p_{\sigma(\omega)}$ is the price paid by type ω for the product and firm that this type chooses under σ. If $\overline{p} = (p_x)_{x\in X}$ is a price function, by mildly abusive notation, $\overline{p}$ will also denote the profile of prices in which all firms choose prices $\overline{p}$.

We follow the literature in considering Nash equilibria in pure strategies, symmetric on the part of the firms.[10] Formally, a price vector $\overline{p} = (p_x)_{x\in X}$ and an incentive compatible consumer profile σ w.r.t. $\overline{p}$ is an *equilibrium* if it yields each firm maximal profit against any deviation for *some* incentive compatible reaction. That is if, for each firm k and each deviation $\overline{q} = (q_x)_{x\in X}$ of firm k, there is an incentive compatible consumer profile τ w.r.t $(\overline{q}, \overline{p}^{-k})$ s.t. $\pi^k(\tau, \overline{q}) \leq \pi^k(\sigma, \overline{p})$, where $\overline{p}^{-k}$ denotes the profile of firms other than k each playing $\overline{p}$. Notice that we allow for deviations in the prices of all contracts, whereas Wambach (2000); Smart (2000); Villeneuve (2003); Kubitza (2019) allow only deviations in a single price.[11][12]

[10]Technically, standard Nash equilibrium would have required the consumer profiles to be a function of both type *and* the price profile of the firm, specifying what each consumer should choose after each possible price profile. To avoid such overly cumbersome objects, we take as Nash equilibrium, on the consumers' part, to be just the realization of such a function on the path of play, i.e., the realized consumer choices when facing the selected prices.

[11]In the proofs, we will several times implicitly use the fact that, if at some prices for each consumer there is an optimal contact, then there is an incentive compatible consumer profile. This jump makes use of a measurable selection theorem like Aumann's theorem, see e.g., Theorem 5.2 of Himmelberg (1975).

[12]We note that there is a technical point that equilibrium or local equilibrium prices require that every consumer has a utility maximizing contract at those prices.

4.3 Technical Assumptions

We now introduce the model's main technical assumptions. These assumptions are innocuous and satisfied in all models of selection markets that we are aware of. For instance, Assumptions 2 and 3 hold whenever $v(\cdot,\cdot,\cdot)$ is bounded.

Assumption 1. *The utility function $u : \Omega \times X \times \mathbb{R}_+ \to \mathbb{R}$ is jointly measurable, is strictly decreasing in price, and for each fixed ω, $u(\omega,\cdot,\cdot)$ is continuous and strictly decreasing in price. The profit function $v : \Omega \times X \times \mathbb{R}_+ \to \mathbb{R}$ is jointly measurable, is weakly increasing in price, and for each fixed ω, $v(\omega,\cdot,\cdot)$ is continuous.*

Assumption 2. *For each fixed $p_0 \geq 0$,*

$$\int_\Omega \sup_{x\in X} |v(\omega,x,p_0)| \cdot \lambda(d\omega) < \infty \tag{4.1}$$

Assumption 3. *For each lower semi-continuous price function $\overline{p} = (p_x)_{x\in X}$ and each incentive compatible consumer profile $\sigma = (\sigma_N, \sigma_X)$,*

$$\int_\Omega |v(\omega,\sigma_X(\omega),p(\sigma_X(\omega)))| \cdot \lambda(d\omega) < \infty \tag{4.2}$$

i.e., total firm profit is integrable.

The following is not an assumption on primitives although, due to its importance, we list it alongside the technical assumptions.

Assumption 4. *Price vectors $\overline{p} : X \to \mathbb{R}_+$ satisfy $p(0) = 0$ if $0 \in X$, and are assumed to be lower semi-continuous. That is, for each $x_0 \in X$, we have* $\liminf_{x\to x_0} p(x) \geq p(x_0)$.[13]

Assumption 4 serves to guarantee that, for each vector of prices $\overline{p}$ and each type ω, the mapping $x \to u(\omega,x,p(x))$ attains a maximum on any compact set, as u is decreasing in price.[14]

5 Zero Profits

We now present our main results: conditions under which profit vanishes in any equilibrium. We first present a condition that, while more general, imposes conditions on the equilibrium in question. In Section 5.2, we describe conditions on primitives that ensure vanishing profit for any equilibrium.

[13] Equivalently, we require that, for each $\alpha \in \mathbb{R}$, the set $\{x \in X \mid p(x) > \alpha\}$ is open.

[14] Lower semi-continuity is an innocuous technical assumption. Had we had Borel prices $\overline{p}$ and consumer profile (defined ahead) $\sigma = (\sigma_N, \sigma_X)$, fixing a firm j and letting μ denote the measure induced on X by λ and σ_X of those consumers purchasing from firm j, Lusin's theorem guarantees that we can take a compact set $K \subseteq X$ with measure as close to $\mu(X)$ as we please on which $\overline{p}$ is continuous. Defining $\overline{q}$ to coincide with $\overline{p}$ on K and be very large outside of K, $\overline{q}$ would be l.s.c. and the resulting profit for firm j under $\overline{q}$ - after those consumers who had been purchasing outside of K adjust their choices - would be arbitrarily close to the profits of firm j under $\overline{p}$.

5.1 Zero Profits Conditions

Recall that, by a continuous ordering $\preceq$ on X, we mean that the set $\{(x, y) \mid x \preceq y\}$ is closed in $X \times X$.

Theorem 1. *Let $\overline{p} = (p_x)_{x \in X} > 0$ be a price vector with well-defined profit, and let $\overline{p}, \sigma$ be an equilibrium. Suppose there is a continuous (weak) linear ordering $\preceq$ on it, such that*

$$u(\omega, x, p_x) = u(\omega, y, p_y) = \max_z u(\omega, z, p_z) \text{ and } x \preceq y$$
$$\to v(\omega, x, p_x) \leq v(\omega, y, p_y), \qquad \text{for } \lambda\text{-a.e. } \omega \in \Omega, \forall x, y \in X \tag{5.1}$$

(If $0 \in X$, 0 is required to be $\preceq$-minimal.) Then profits to all firms are zero.

Theorem 1 can be stated verbally as follows. Suppose that, when a consumer is indifferent between alternatives which are among the best choices for him, firms weakly prefer the alternative ranked higher under the ordering $\preceq$ (if there is an outside option 0, $\preceq$ ranks it last). If such an ordering exists, then firms make zero profit in any equilibrium. We emphasize that the same ranking $\preceq$ must apply to all consumers (and all firms, since firms are symmetric). An example of Theorem 1 (via Theorem 3 below) is given in Section 5.3 in the context of insurance markets. Note the importance of the order of the quantifiers in (5.1) when there are uncountably many alternatives.

Proof. We provide here a sketch of the proof and include the details in the Appendix. First, we consider the case where X, Θ are finite (Theorem 7 in the Appendix). If there were positive profits in equilibrium, one firm could slightly undercut prices, in a specific way we describe below, to capture the whole market. We construct a deviation that would not cause consumers to buy contracts that give lower profits to the firm, using the ordering $\preceq$. In particular, if a consumer buys a different contract in the deviation than in the equilibrium, it will be a $\preceq$-higher ranked contract. The firm begins by lowering the price of the $\preceq$-maximal contract. This attracts from other firms any consumer who was buying this top option (if there were any) as well as possibly some who were buying other options. Since these latter consumers are 'moving up' the $\preceq$ ranking, the deviating firm does not lose as a result of this transition. The firm then lowers the price of the $\preceq$-second-maximal contract besides the top one, but only slightly, so as not to cause anyone purchasing the $\preceq$-maximal contract to change their choice. This allows the deviating firm to capture from other firms those consumers purchasing the $\preceq$-second-maximal contract (and possibly others who 'move up' the $\preceq$ ranking). And so forth, for every contract. This deviation is illustrated in Figure 2. For any number of firms in the market N, there exists a small enough change in the prices for which the increase in market share more than compensates for the lower prices, resulting in a profitable deviation from the candidate equilibrium.

Theorem 1 extends this result to general contract (X) and type (Ω) spaces. The generalization consists of two steps. First, we generalize Theorem 7 to allow the space of types Ω and contracts X to each be a compact continuum, but the latter consists of finitely many 'small connected clusters' of contracts. The proof of this step is similar to that of Theorem 7, with some alterations. Secondly, we make a reduction to the case described in the first step. Here, we make repeated use of Lusin's theorem to restrict types and contracts to spaces on which all relevant functions are continuous. We also make repeated use of the inner-regularity of λ to guarantee a compact type space. Both such applications only involve disregarding sets of types of arbitrarily small measure, and hence if a profitable deviation exists in a potential positive-profit equilibria after such modifications (by the first step), a profitable deviation exists in the original economy with general X, Ω as well. $\square$

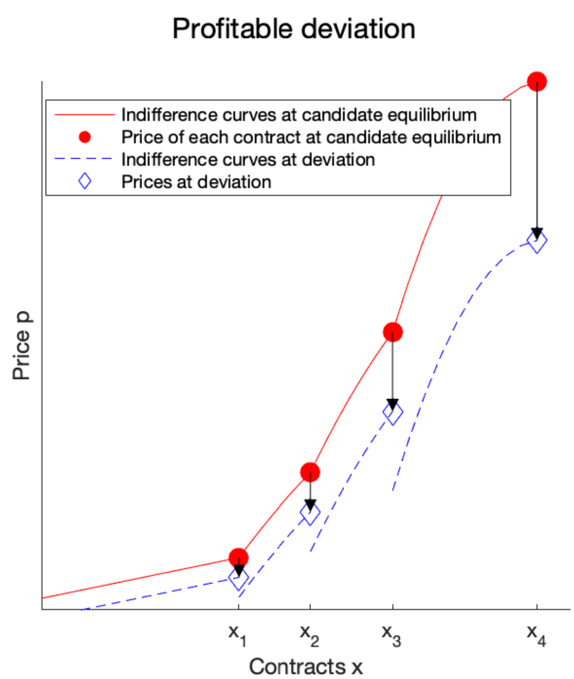

Figure 2: Example of a profitable deviation used in the proof of Theorem 1. Here, we consider an economy with finitely X, Ω. In this example, we show only 4 contracts ($x_1, ..., x_4$). We have drawn indifference curves as if intermediate contracts did exist. In red are contracts and prices at a candidate equilibrium with positive profit, and the indifference curve of the consumer choosing each contract. In blue are the deviation contracts and prices, and the corresponding indifference curves.

In the examples of Section 3, such a linear ordering does not exist. In the first example (Subsection 3.1), in equilibrium, consumers are indifferent between buying and not buying, but firms prefer that low-cost consumers buy and high-cost consumers do not. There does not exist, for firms, a linear ordering of contracts that is valid for all types of buyers. In the second example (Subsection 3.2), in equilibrium, consumers are indifferent but firms obtain higher profit if ω_A types buy A rather than B, and vice-versa for ω_B types.

A few remarks are in order. First, Theorem 1 allows for an economy to possess some equilibria with zero profit and others with positive profit. However, in Section 5.2, we discuss classes of economies for which *all* equilibria satisfy Theorem 1, and hence *all* equilibria make zero profits. Second, Theorem 1 does not guarantee equilibrium existence. It states only that any equilibrium satisfying (5.1) must feature vanishing profit. Third, the result says nothing on whether equilibrium contracts are cross-subsidizing or not.

In all screening models that we are aware of, most consumer types are indifferent between multiple (typically two) contracts in equilibrium. For instance, in Rothschild and Stiglitz (1976), high-cost individuals are indifferent between full insurance and partial insurance. This is also the case in screening models like Mussa and Rosen (1978). Our next result shows that, if this is not the case, equilibrium profits are also guaranteed to vanish.

Theorem 2. *Let $\overline{p} = (p_x)_{x\in X} > 0$ be a price vector, and let $\overline{p}, \sigma$ be an equilibrium. Suppose for a.e. consumer ω, $\sigma_X(\omega)$ is the unique bundle which maximizes type ω's utility at prices $\overline{p}$. Then profits to all firms are zero.*

Proof. One may take $\preceq$ to be any continuous ordering on X (e.g., $x \succeq y$ for all $x, y \in X$) and the condition given in Theorem 1 will hold vacuously. $\square$

5.2 Zero-Profit Conditions on Primitives

A drawback of Theorem 1 is that it does not impose conditions directly on model primitives. We now present conditions on primitives (the profit and utility functions) that guarantee zero profits. Among other assumptions, in this section we take the alternative space X to be one-dimensional. Recall the notation $u_\omega = u(\omega, x, p)$ and $v_\omega = v(\omega, x, p)$.

Theorem 3. *Suppose the alternatives space is an interval (bounded or unbounded) $X \subseteq \mathbb{R}_+$, utilities and profits are continuously differentiable in price and alternatives, with*

$$\frac{\partial u_\omega}{\partial p} < 0 < \frac{\partial v_\omega}{\partial p}, \qquad \frac{\partial u_\omega}{\partial x} > 0 > \frac{\partial v_\omega}{\partial x} \tag{5.2}$$

and furthermore

$$-\frac{\frac{\partial u_\omega}{\partial x}}{\frac{\partial u_\omega}{\partial p}} > -\frac{\frac{\partial v_\omega}{\partial x}}{\frac{\partial v_\omega}{\partial p}} \tag{5.3}$$

for a.e.-$\omega \in \Omega$, throughout $(x, p) \in int(X) \times [0, \infty)$.[15] *Then in any equilibrium, profits to all firms are zero.*

Proof. By Theorem 1, it suffices to show that, for those ω which satisfy the given inequalities, whenever (x_1, p_1), (x_2, p_2) are such that $u_\omega(x_1, p_1) = u_\omega(x_2, p_2)$ and $x_2 > x_1$, then

[15] $int(X)$ denotes the interior of X. As we see later in the case of insurance markets, we may wish to allow equality to hold on the boundary of X.

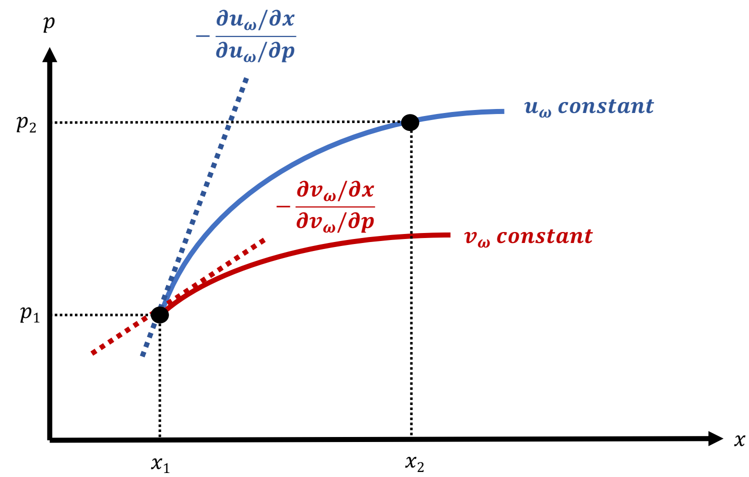

Figure 3: Illustration of the Proof of Theorem 3. The iso-utility and iso-profit curves intersect at the contract (x_1, p_1).

$v_\omega(x_2, p_2) \geq v_\omega(x_1, p_1)$. A formal proof is given in the appendix, but the intuition is: fixing ω and denoting the indifference curves of u_ω, v_ω through (x_1, p_1) by ϕ, ψ, we have $\phi(x_1) = \psi(x_1)$, and by the implicit function theorem and (5.3), we have $\phi'(\cdot) > \psi'(\cdot)$ in a neighborhood of any point x at which $\phi(x) = \psi(x)$. Together, these imply the desired result. We illustrate the proof in Figure 3. $\square$

Notice that $-\frac{\partial u_\omega}{\partial x} / \frac{\partial u_\omega}{\partial p}$ is the marginal rate of substitution of quality for price for type ω (the rate at which p changes with x, along the indifference curve of type ω). Therefore, the intuition for the conditions stated in Theorem 3 is that indifference curves are steeper than iso-profit curves for each type. This is a feature of many markets. For instance, in a class of insurance markets we will consider in Section 5.3, these conditions hold if individuals are strictly risk averse. In this case, as coverage increases at an actuarily fair price for a given consumer, firms' profits remain constant but that consumer is made strictly better off. This is also the intuition for Corollary 1 below. If the condition holds, whenever individuals are indifferent between two optimal contracts, firms necessarily prefer that individuals choose the contract with higher x.

The intuition for Theorem 1 holds also for Theorem 3. If some prices lead to positive profit for some firm, another firm could deviate and capture all consumers who had been purchasing from any firm, and in particular those consumers who had been fueling the positive profits. Recall that, when individuals are indifferent between contracts, the firms prefer that agents go with the "higher" contract. This is the key condition which drives the success of this construction. The deviator would begin by lowering the prices of higher contracts, and then of slightly lower contracts, and so forth, doing so in small deviations tailored inductively so that most consumers who would change their choice would change to a higher contract, not a lower one (see Figure 2). Such changes in contracts could only help the deviator, who in this way is able to capture the entire purchasing market, with some consumers perhaps generating higher firm profits than they had previously. We again emphasize, the linear ranking of contracts is key for the existence of such a deviation, since it allows the the deviating firm to offer a menu of contracts which attracts the entire market without incurring loses from the revised contract selections of the consumers.

In many markets, there are natural parameterizations that satisfy the conditions of Theorem 3. For instance, in insurance markets, it is common to assume that certainty equivalents are the sum of (minus) the expected net cost to the insurer (denoted C_ω below) and a surplus (denoted S_ω) that arises, for instance, due to risk aversion. This parameterization is used, for instance, in Veiga and Weyl (2016); Levy and Veiga (2020).

Corollary 1. *Suppose the alternatives space is an interval $X \subseteq \mathbb{R}_+$. Suppose that profit is $v_\omega(x,p) = -C_\omega(x,p)$, and that the preferences of type ω can be represented by the utility function*

$$u_\omega(x,p) = C_\omega(x,p) + S_\omega(x,p)$$

where C_ω, S_ω are continuously differentiable with

$$\frac{\partial C_\omega}{\partial x} > 0,\ \frac{\partial S_\omega}{\partial x} > 0,\ \frac{\partial S_\omega}{\partial p} \leq 0,\ \frac{\partial C_\omega}{\partial p} < 0.$$

Finally, suppose that

$$-\frac{\partial C_\omega}{\partial p}\frac{\partial S_\omega}{\partial x} > -\frac{\partial C_\omega}{\partial x}\frac{\partial S_\omega}{\partial p}$$

for a.e.-$\omega \in \Omega$, throughout $(x,p) \in int(X) \times [0,\infty)$. Then in any equilibrium, profits to all firms are zero.

Proof. The conditions imply (5.2) holds. We show that (5.3) holds as well. The assumptions imply $(-\frac{\partial C_\omega}{\partial p})\frac{\partial S_\omega}{\partial x} > \frac{\partial C_\omega}{\partial x}(-\frac{\partial S_\omega}{\partial p})$. We add $\frac{\partial C_\omega}{\partial x}(-\frac{\partial C_\omega}{\partial p})$ to both sides of the inequality. Some additional algebra yields

$$\frac{\frac{\partial C_\omega}{\partial x} + \frac{\partial S_\omega}{\partial x}}{(-\frac{\partial C_\omega}{\partial p}) + (-\frac{\partial S_\omega}{\partial p})} > \frac{\frac{\partial C_\omega}{\partial x}}{(-\frac{\partial C_\omega}{\partial p})},$$

which is equivalent to (5.3). □

Notice that we only require the weak inequality $\frac{\partial S_\omega}{\partial p} \leq 0$. This will be relevant in Section 5.3, where we present a model of insurance markets in which $\frac{\partial S_\omega}{\partial p} = 0$ and $\frac{\partial S_\omega}{\partial x} > 0$, so Corollary 1 holds. Intuitively, the condition given in Corollary 1 corresponds to constant surplus curves increasing more sharply than constant cost curves with coverage x.

The conditions required by Corollary 1 are related (but not equivalent) to Assumption 4 in Fang and Wu (2018), which amounts to assuming that individuals obtain higher surplus from more generous insurance, at fair prices. This is captured, in our setting, by $\frac{\partial S_\omega}{\partial x} > 0$.

5.3 Application to Insurance Markets

Theorem 1 applies to a broad range of insurance markets, from Rothschild and Stiglitz (1976) to some of the recent models with multi-dimensional types such as Levy and Veiga (2016); Kubitza (2019); Wambach (2000). In these models, Theorem 1 holds independently of the equilibrium prices, so *every* equilibrium has zero profit.[16]

To illustrate, consider the following explicit model of insurance. For simplicity, all individuals have initial wealth W. An individual's type is defined by their riskiness, a measurable function $\xi : \Omega \to [0,1]$, which denotes the probability of losing Z from initial wealth W for each type ω. Let $\mu_\omega = \xi_\omega Z$ denote the expected loss of type ω. Rothschild and Stiglitz (1976) considered environments with two types, so $|\Omega| = 2$, but we allow for any number of types.

Assume each consumer $\omega \in \Omega$ has a strictly increasing and strictly concave utility of money $U_\omega(\cdot)$. Let the space of alternatives be $X = [0,1]$, so that $x \in X$ is the level of insurance coverage ($x = 0$ corresponds to 0% and $x = 1$ to 100% coverage). Expected utility is

$$u(\omega, x, p) = \xi_\omega \cdot U_\omega(W - (1-x)Z - p) + (1 - \xi_\omega) U_\omega(W - p)$$

The expected cost to the insurer of selling coverage x to type ω is $\mu_\omega \cdot x$.[17] Profit is

$$v(\omega, x, p) = p - \mu_\omega \cdot x.$$

To show that Theorem 3 holds, notice that

$$-\frac{\frac{\partial v(\omega,x,p)}{\partial x}}{\frac{\partial v(\omega,x,p)}{\partial p}} = \mu_\omega \leq -\frac{\frac{\partial u(\omega,x,p)}{\partial x}}{\frac{\partial u(\omega,x,p)}{\partial p}}, \forall p \in \mathbb{R}_+, x \in [0,1], \textit{with equality iff } x = 1 \tag{5.4}$$

See Figure 4 for an illustration of the proof of Condition (5.4). Therefore, profits vanish in this setting.

Figure 4 provides an illustration of Condition (5.4). The figure shows the unique pure-strategies Nash equilibrium in the Rothschild and Stiglitz (1976) model with two types. In this equilibrium, the high (riskier) types μ_H are indifferent between the two contracts, but all high types choose the contract with higher coverage x (otherwise the profits would be negative). In this equilibrium, the profit of each firm is 0 (in fact, the profit of each

[16] Veiga and Weyl (2016) consider a model of insurance with multi-dimensional types, but firms choose a single price-quality pair from a continuum, so that model does not fit our framework. We are also grateful to an anonymous referee for pointing out that, for instance, Example 3 of Azevedo and Gottlieb (2017), as well as the setting in Kubitza (2019), do not satisfy our assumptions (hence our claim that our results imply that *some* insurance market models yield zero profits).

[17] Notice that utilities satisfies single-crossing (Spence (1978)) since $\frac{\partial u}{\partial \xi_\omega \partial x} > 0$ since $U'(\cdot) > 0$.

contract is 0). We have also included the break-even (zero profit) lines for each type. The blue line consists of the coverage and price pairs that, if bought by the high type μ_H, yield zero profit. This line has slope μ_H. The red line consists of the contract and price pairs that, if bought by the low type μ_L, yield zero profit. This line has slope μ_L. For each type, at their chosen contract, the slope of their indifference curve is greater than the slope of the break-even line, thereby providing a graphical proof of condition (5.4) (the slopes are equal for the riskiest type, who purchases full insurance).

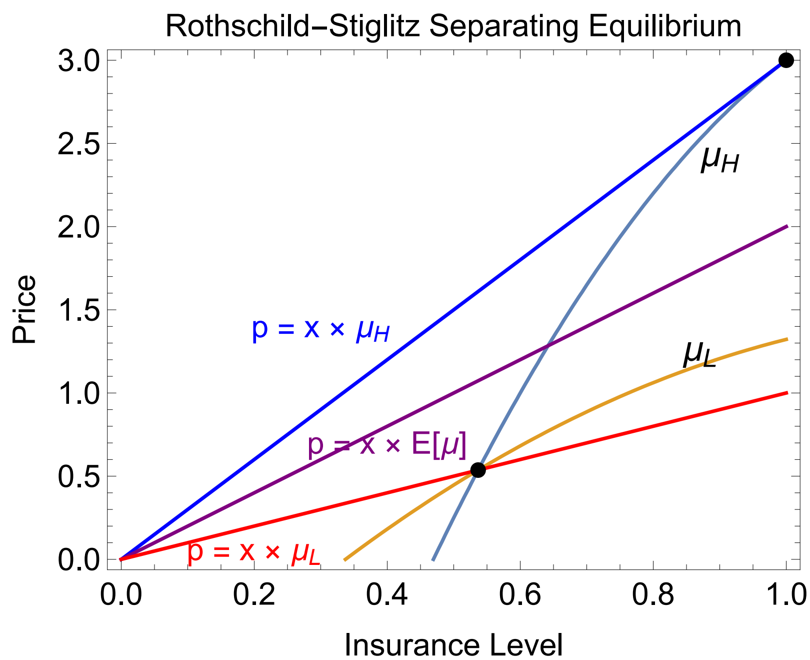

Figure 4: The two contracts constitute a separating equilibrium as long as the break-even line for the economy, $p = x \cdot E[\mu]$, does not intersect the indifference curve of μ_L, as in the case pictured.

To illustrate the usefulness of Corollary 1, we consider the commonly used CARA-Gaussian framework (Levy and Veiga (2016), Veiga and Weyl (2016)). An individual with type ω and wealth c obtains utility $U_\omega(c) = -e^{-a_\omega \cdot c}$, where the coefficient of constant absolute risk aversion is a_ω. Initial wealth is w_ω and individuals are subject to wealth shocks $-Z_\omega$, where $Z_\omega \sim \mathcal{N}(\mu_\omega, \sigma_\omega^2)$, with $\mu_\omega > 0$. A contract $x \in [0,1]$ implies that a share x of the individual's shock is absorbed by the insurer, so the individual is exposed to a change in wealth of $-(1-x)\, Z_\omega$. Price is p. Notice that utility depends directly on type ω, and the loss distribution is non-binary. Individual certainty equivalents for a contract (x, p) are a monotonic transformation of

$$u_\omega(x,p) = \mu_\omega x - \frac{(1-x)^2}{2}\rho_\omega - p$$

where we denote $\rho_\omega = a_\omega \sigma_\omega^2 > 0$. Furthermore, we denote $C_\omega = \mu_\omega \cdot x - p$ and $S_\omega = -\frac{(1-x)^2}{2}\rho_\omega^2$. Then utility is $u_\omega(x,p) = C_\omega + S_\omega$ and profit is $v_\omega(x,p) = -C_\omega$. Since

$$\frac{\partial C_\omega}{\partial x} = \mu_\omega > 0,\ \frac{\partial S_\omega}{\partial x} = \rho_\omega(1-x) > 0,\ \frac{\partial S_\omega}{\partial p} = 0,\ \frac{\partial C_\omega}{\partial p} = -1 < 0,$$

we have

$$\frac{\partial C_\omega}{\partial p}\frac{\partial S_\omega}{\partial x} < 0 = \frac{\partial C_\omega}{\partial x}\frac{\partial S_\omega}{\partial p},$$

so the conditions of Corollary 1 holds: all equilibria have zero profits.

5.4 Miyazaki–Wilson–Spence Equilibrium

In insurance markets, the notion of equilibrium used by this article and Rothschild and Stiglitz (1976) (and motivated by Nash equilibrium) is often viewed as overly simplistic since it does not incorporate natural dynamics that would follow a deviation in prices or contracts by a single firm. Several authors have have proposed equilibrium concepts that account for such dynamics.

Perhaps the most appealing such concept is the 'anticipatory equilibrium' of Wilson (1977), Miyazaki (1977) and Spence (1978) (henceforth, MWS). Here, 'each firm assumes that any policy will be immediately withdrawn which becomes unprofitable after that firm makes its own policy offer' Wilson (1977, p1). The ability of other firms to withdraw contracts is viewed by the deviator a threats that discourages 'cream skimming deviations' meant to grab a small share of profitable consumers. After a cream skimming deviation, the remaining firms would withdraw their then non-profitable contracts. The costly buyers of those contracts would then flock to the deviator, rendering the deviation non-profitable.

Several recent articles have formalized the dynamics underlying such an equilibrium, leading to a range of closely related (although not completely identical) equilibrium concepts. In particular, it is well known that prices which constitute a MWS equilibrium need not constitute equilibria under concept used by this article and Rothschild and Stiglitz (1976). Here, we follow the survey of MWS equilibria by Mimra and Wambach (2019). That paper describes the following dynamics, appropriately generalized here to a general screening market like those considered in this paper:

- Stage 0: The type of each individual is chosen by nature. (In this paper, this is incorporated in the distribution λ over the type space.)

- Stage 1: Each firm offers a set of initial contracts. (In this paper, this corresponds to pricing of each alternative.) The offered sets are observed by all firms before the beginning of the next stage.

- Stage 2: Consists of $t = 1, 2, \ldots$ rounds. In each round, each firm can withdraw a set from its remaining contracts. (In our paper, this would correspond to raising the price of alternatives above competitors; an option of an infinite price could be also introduced.) After each round, firms observe the remaining contract offered by all firms.

- Stage 3: Consumers choose among the remaining contracts. (In this paper, this corresponds to consumers choosing at the prices set.)

Prices $(p^1, \ldots, p^N)$ of the N firms are MWS equilibrium prices if there is a subgame-perfect equilibrium of the above game resulting in these prices. For simplicity, we will restrict ourselves to symmetric equilibrium prices, which further require naturally that, during Stage 2, no contracts are withdrawn (i.e., all firms are satisfied with the prices initially set). We will also, in keeping with the rest of the paper, refer to an MWS equilibria as the pair of MWS equilibrium prices and the consumer profile σ of resulting purchases, specifying from which firm and which alternative each agent purchases.

We then obtain the following result.

Theorem 4. *Let $\overline{p} = (p_x)_{x \in X} > 0$ be a price vector with well-defined profit, and let $(\overline{p}, \sigma)$ be an MWS equilibrium. Suppose there is a continuous (weak) linear ordering $\preceq$ on it, such that (5.1) holds. Then profits to all firms are zero.*

Proof. The proof of Theorem 1 shows that (5.1) implies that, if some firm had positive profits, then some other firm would possess a profitable deviation of prices $\overline{q} = (q_x)_{x \in X} > 0$ s.t. $\overline{q} < \overline{p}$. In particular, this firm would capture the entire market (among those consumers who choose to enter), so no consumer would purchase from any other firm. As a result, the withdrawal of contracts on the part of other firms would not serve as a threat in this case, since whether or not they withdraw, no one would purchase from them in any case. □

6 Positive Profits and Local Equilibria

6.1 Positive Profit Equilibria: Properties

We have shown, in Section 3, examples of economies with positive equilibrium profits; further examples appear in Section A. The following result illustrates a key aspect of those economies. Recall Theorem 1: if $\overline{p}$ are equilibrium prices yielding strictly positive profit, the alternatives in X cannot be ordered in such a way that, when two or more types are indifferent among optimal choices, all firms prefer that all such types purchase the "higher" contract at prices $\overline{p}$. Proposition 1 below shows that, if there is an equilibrium with positive profit, then there exists a "cycle" of pairs contracts such that, for each pair, there is an individual indifferent between the two contracts and the firms prefer him to choose the "upper" contract. Importantly, the linear ordering of contracts required by Theorem 1, prevents the existence of such cycles. Hence, Proposition 1 is a partial converse to Theorem 1.

Proposition 1. *Suppose Ω, X are finite, and $\overline{p}$ are equilibrium prices yielding positive profit. Then, there is an integer $K \in \mathbb{N}$, contracts $x_1, \ldots, x_K \in X$, and types $\omega_1, \ldots, , \omega_K \in \Omega$ such that, for each $j = 1, \ldots, K$, we have $\lambda(\omega_j) > 0$,*

$$x_{j-1}, x_j \in argmax(u(\omega_j, \cdot, p)).$$

and

$$v(\omega_j, x_{j-1}, p_{x_{j-1}}) \leq v(\omega_j, x_j, p_{x_j}) \tag{6.1}$$

with strict inequality for at least one $j = 1, \ldots, K$. Here, $u(\omega_j, \cdot, p)$ is understood to be the map $x \to u(\omega_j, x, p_x)$, and subscript arithmetic is modular in $\{1, \ldots, K\}$ (i.e., $x_0 = x_K$),

Proof. WLOG assume λ gives positive mass to all of the finitely many types in Ω. For simplicity we deal here with the case that for each $\omega \in \Omega$, the mapping $x \to v(\omega, x, p_x)$ is injective. In this case, we show (6.1) holds strictly for all $j = 1, \ldots, K$. The general case is proved in the Appendix. Define the directed graph[18] G on X by:

$$G = \{(x, y) \mid \exists \omega \in \Omega, [x, y \in argmax(u(\omega, \cdot, p)) \wedge v(\omega, x, p_x) < v(\omega, y, p_y)]\}$$

Since equilibrium yields positive profit, the assumption of Theorem 1 does not hold, and in particular G cannot be completed to a strict linear ordering (indeed, if it could be completed to $\prec$, $\prec$ would satisfy the conditions of Theorem 1 due to our injectiveness assumption on v).[19] From Cormen et al. (2001, Section 22.4), if a directed graph cannot be completed to a linear ordering, then it must possess a cycle, i.e., there must be $K \in \mathbb{N}$ and $x_1, \ldots, x_K \in \Omega$ with $x_1 \, G \, x_2 \, G \, \ldots \, x_K \, G \, x_1$, which gives the conclusion. □

Proposition 1 serves to show where positive profits come from. It also serves to show how one may construct economies with positive profit equilibria (although the presence of such cycles is not sufficient to guarantee positive profits). See Appendices A and C.2.

6.2 Positive Profit Equilibria: Genericity

Building on Proposition 1, we now show that positive equilibrium profits are not a knife-edge case (i.e., occur in a negligible set of economies). To this end, we isolate five properties that hold in the economies of Section 3 and their generalization in Appendix A, and also in all sufficiently similar economies (in a sense we make precise below). One such condition is the existence of cycles, as described in Proposition 1. For any sufficiently small perturbation of these economies, we show that there exists an equilibrium with positive profit.

[18] By a directed graph, we mean an irreflexive relation on X.

[19] Without the injectiveness assumption, problems may arise if there are types ω_1, ω_2, indifferent between x, y at prices $\overline{p}$, but $v(\omega_1, x, p_x) < v(\omega_1, y, p_y)$ while $v(\omega_2, x, p_x) = v(\omega_2, y, p_y)$.

We define an economy $\mathcal{E}$ as a tuple $\mathcal{E} = [\Omega, X, \lambda(\cdot), u(\cdot), v(\cdot)]$, consisting of the type and contract space, the distribution of types, utility and profit functions. In this section, we assume that Ω, X are both *finite*. We will consider perturbations of an economy where the utility $u(\cdot)$ and profit $v(\cdot)$ functions, as well as λ, are subject to a small perturbation. Call a tuple $x_1, \ldots, x_K \in X, \omega_1, \ldots, , \omega_K \in \Omega$ as described in Proposition 1 a cycle of length K if there is no strict subset $y_1, \ldots, y_L$ of the $x_1, \ldots, x_K$ such that for some types $\theta_1, \ldots, , \theta_L \in \Omega$, the tuple $y_1, \ldots, y_L, \theta_1, \ldots, , \theta_L$ satisfies the conclusion of Proposition 1 as well.

Theorem 5. *In the space of continuously differentiable utility functions and continuous profit functions on $\Omega \times X \times \mathbb{R}_+$, for each integer $K \in \mathbb{N}$, the set of economies for which there exist positive prices with cycles of length K is non-empty and open, and furtheremore, the set of economies possessing an equilibrium with strictly positive profits contains a non-empty open set.*

Proof. See Appendix C.5. □

Theorem 5 uses the regular Euclidean topology on $\mathbb{R}^\Omega$ (for the distributions of types), topology of uniform convergence on compact sets (for profit functions), and uniform convergence on compact sets of functions and derivatives (for utility functions). See precise definitions in Appendix C.1 (in particular Remark 1 there).

To prove Theorem 5, we first identify several properties of the examples of Section 3 and Appendix A (the latter having cycles of arbitrary length) which drive the positive profits (in addition to the cycles). These conditions make undercutting unprofitable and are presented in Lemma 6. We then show that the properties given in Lemma 6 hold for small perturbations. The main difficulty in proving that Theorem 5 follows from Lemma 6 is in verifying how the vector of prices which satisfies the existence of cycles changes if the economy is slightly perturbed. (A small perturbation of the utilities and their derivatives, of the profit functions, and of the prices will easily be seen to preserve the other properties of Lemma 6.) To demonstrate that the required prices yielding a cycle change only slightly when the functions change, we use a form of the implicit mapping theorem.

A few remarks are in order. First, notice that multidimensional type are *not necessary* for an economy to have positive profits in equilibrium, as in the examples of Section 3. Second, we depart from the existing literature by allowing firms to deviate by offering a menu of contracts, instead of a single contract. This is in contrast to Wambach (2000) and Kubitza (2019), where multidimensional types are required and firms can offer only a single contract in a deviation. However, we do not allow for firms to offer *mechanisms* as in Maskin and Tirole (1990, 1992). Such mechanisms could, for instance, account for a renegotiation stage after an initial purchase by consumers. In this case, it is possible that zero profits can be restored, as we discuss further in the conclusion.

6.3 Local Equilibria

We now briefly show that the results above hold also under a weaker equilibrium concept: local equilibrium. The deviation exploited in the proof of Theorem 1 is a local deviation, so the arguments apply intuitively also to local equilibria. Moreover, local arguments are commonly used to discuss equilibria and deviations in screening markets (e.g., Rothschild and Stiglitz (1976) use local arguments to show the non-existence of pooling equilibria). Local arguments also commonly used to show the existence of zero profits in equilibrium by considering deviations where one firm undercuts price to just below other firms' prices.

A price vector $\overline{p} = (p_x)_{x \in X}$ and an incentive compatible consumer profile σ w.r.t. $\overline{p}$ is a *local equilibrium* if it yields each firm maximal profit against any local deviation for *some* incentive compatible reaction. That is, if there exists $\delta > 0$ such that for each firm k and each deviation $\overline{q} = (q_x)_{x \in X}$ of firm k satisfying $\forall x \in X, |q_x - p_x| < \delta$, there is an incentive compatible consumer profile τ w.r.t $(\overline{q}, \overline{p}^{-k})$ s.t. $\pi^k(\tau, \overline{q}) \leq \pi^k(\sigma, \overline{p})$. Every equilibrium is a local equilibrium.

Theorem 6. *Theorems 1, 2, 3, 5 hold for local equilibria as well.*

That is, these theorems hold if the term 'equilibrium' is replaced with the term 'local equilibrium' in their statements. Theorem 5 holds for local equilibria because every equilibrium is a local equilibrium. Theorem 1 (and therefore Theorems 3 and 2) hold for local equilibria since its proofs show that, if profits were non-zero, a profitable *local* deviation would exist.

7 Conclusion

Studies of screening markets typically assume price competition, constant returns to scale, and frictionless choices on the part of consumers, as we do. A number of recent articles define equilibrium using, as their point of departure, the condition that each firm (and, sometimes, each contract) breaks even.

This article shows that focusing only on zero profit equilibria can be misleading. We show that it is possible for equilibria with positive profits to emerge in screening markets. These equilibria are sustained by the presence of adverse selection: undercutting by firms attracts high-cost individuals which makes deviations unprofitable. We provide a simple condition which guarantees that firms break even. The condition allows us to establish conditions on primitives guaranteeing zero profits. We deduce in particular that if a.e. consumer possesses a unique optimizing contract in equilibrium, then firms break even. Equilibrium profits also vanish if utility increases with coverage more steeply than does profit. Moreover, positive profits implies the existence of 'cycles', as described in Proposition 1. Finally, positive profits do not occur only for a negligible set of economies.

There remain several avenues for future work. First, Theorem 5 shows genericity for the case of finite types and contracts. A generalization of this theorem for the case of general linear orderings is left for future research. Second, our analysis focuses primarily on Nash equilibria. A pure-strategies Nash equilibrium often does not exist in screening markets (see Farinha Luz (2017) for a discussion of mixed strategies by firms). We have also discussed how our results extend to MWS equilibria, which exist in a wider range of economies. However, several alternative equilibrium concepts have been explored in the literature (e.g., Riley (1979), Dosis (2018), Azevedo and Gottlieb (2017)). A generalization of the analysis in this article to these other equilibrium concepts is outside the scope of the present article and left for future research. Third, we have allowed firms to compete in menus, but not in mechanisms as in Maskin and Tirole (1990, 1992). Adding a renegotiation stage to the game may result in zero equilibrium profits in a wider range of settings than we have considered. Fourth, the examples of Section 3 features mixed strategy equilibria with vanishing profit. Whether such equilibria always exist, possibly alongside equilibria with positive profits, is an interesting open question.

References

Aliprantis, C.D., and K.C. Border. 2007. *Infinite Dimensional Analysis: A Hitchhiker's Guide.* Springer. B.2.1, B.2.3

Alós-Ferrer, Carlos, Ana B Ania, and Klaus Reiner Schenk-Hoppé. 2000. "An evolutionary model of Bertrand oligopoly." *Games and Economic Behavior*, 33(1): 1–19. 2

Anderson, Robert M., and William R. Zame. 2001. "Genericity with Infinitely Many Parameters." *The B.E. Journal of Theoretical Economics*, 1(1): 1–64. 2

Aumann, Robert. 1964. "Mixed and Behavior Strategies in Infinite Extensive Games." *Advances in Game Theory*, 627–650. 4.1

Azevedo, Eduardo M, and Daniel Gottlieb. 2017. "Perfect competition in markets with adverse selection." *Econometrica*, 85(1): 67–105. 16, 7

Baye, Michael R, and John Morgan. 1999. "A folk theorem for one-shot Bertrand games." *Economics Letters*, 65(1): 59–65. 2

Bertrand, Joseph. 1883. "Théorie mathématique de la richesse sociale." *Journal des savants*, 67(1883): 499–508. 2

Cormen, Thomas H., Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2001. *Introduction to Algorithms.* . 2 ed., The MIT Press. 6.1, C.4

Dosis, Anastasios. 2018. "On signalling and screening in markets with asymmetric information." *Journal of Mathematical Economics*, 75: 140–149. 7

Dos Santos Ferreira, Rodolphe, and Frédéric Dufourt. 2007. "Free entry equilibria with positive profits: A unified approach to quantity and price competition games." *International Journal of Economic Theory*, 3(2): 75–94. 2

Edgeworth, Francis Y. 1925. "The pure theory of monopoly." *Papers relating to political economy*, 1: 111–142. 2

Fang, Hanming, and Zenan Wu. 2018. "Multidimensional private information, market structure, and insurance markets." *The RAND Journal of Economics*, 49(3): 751–787. 1, 5.2

Farinha Luz, Vitor. 2017. "Characterization and uniqueness of equilibrium in competitive insurance." *Theoretical Economics*, 12(3): 1349–1391. 7

Himmelberg, C. 1975. "Measurable relations." *Fundamenta Mathematicae*, 87(1): 53–72. 11

Hotelling, Harold. 1929. "Stability in Competition." *Economic Journal*, 39(153): 41–57. 2

Hurkens, Sjaak, and Nir Vulkan. 2003. "Free entry does not imply zero profits." *Economics Letters*, 81(3): 285–290. 2

Kaplan, Todd R, and David Wettstein. 2000. "The possibility of mixed-strategy equilibria with constant-returns-to-scale technology under Bertrand competition." *Spanish Economic Review*, 2(1): 65–71. 2

Kreps, David M., and José A. Scheinkman. 1983. "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes." *Bell Journal of Economics*, 14(2): 326–337. 2

Kubitza, Christian. 2019. "Heterogeneous preferences in competitive selection markets." *Working Paper*. 2, 4.2, 5.3, 16, 6.2

Levitan, Richard, and Martin Shubik. 1972. "Price duopoly and capacity constraints." *International Economic Review*, 111–122. 2

Levy, Yehuda, and Andre Veiga. 2016. "Frictions and Equilibria in Insurance Markets." *Working Paper*. 5.3, 5.3

Levy, Yehuda, and Andre Veiga. 2020. "Competitive Insurance Markets with Unbounded Cost." *Working Paper*. 4.1, 5.2

Maskin, Eric, and Jean Tirole. 1990. "The principal-agent relationship with an informed principal: The case of private values." *Econometrica: Journal of the Econometric Society*, 379–409. 6.2, 7

Maskin, Eric, and Jean Tirole. 1992. "The principal-agent relationship with an informed principal, II: Common values." *Econometrica: Journal of the Econometric Society*, 1–42. 6.2, 7

McLennan, A. 2018. *Advanced Fixed Point Theory for Economics.* Springer Singapore. C.5

Mimra, Wanda, and Achim Wambach. 2019. "Contract withdrawals and equilibrium in competitive markets with adverse selection." *Economic Theory*, 67(4): 875–907. 5.4

Miyazaki, Hajime. 1977. "The Rat Race and Internal Labor Markets." *The Bell Journal of Economics*, 8(2): 394–418. 1, 5.4

Mussa, Michael, and Sherwin Rosen. 1978. "Monopoly and Product Quality." *Journal of Economic Theory*, 18(2): 301–317. 5.1

Parthasarathy, K.R. 1976. *Probability Measures on Metric Spaces. AMS Chelsea Publishing Series*, Academic Press. B.2.1

Riley, John G. 1979. "Informational Equilibrium." *Econometrica*, 47(2): 331–359. 7

Rothschild, Michael, and Joseph E. Stiglitz. 1976. "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information." *Quarterly Journal of Economics*, 90(4): 629–649. 1, 5.1, 5.3, 5.3, 5.4, 6.3

Rudin, W. 1976. *Principles of Mathematical Analysis. International series in pure and applied mathematics*, McGraw-Hill. 1

Rudin, W. 1991. *Functional Analysis. International series in pure and applied mathematics*, McGraw-Hill. C.1

Rudin, Walter. 1987. *Real and complex analysis.* Tata McGraw-Hill Education. B.2.1

Smart, Michael. 2000. "Competitive Insurance Markets with Two Unobservables." *International Economic Review*, 41(1): 153–170. 2, 4.2

Snow, Arthur. 2009. "On the possibility of profitable self-selection contracts in competitive insurance markets." *Journal of Risk and Insurance*, 76(2): 249–259. 2

Spence, Michael. 1978. "Product differentiation and performance in insurance markets." *Journal of Public Economics*, 10(3): 427–447. 1, 17, 5.4

Spulber, Daniel F. 1995. "Bertrand competition when rivals' costs are unknown." *The Journal of Industrial Economics*, 1–11. 2

Tirole, Jean. 1988. *The Theory of Industrial Organization.* Cambridge, Mass. 1

Varian, Hal R. 2010. *Intermediate Microeconomics: A Modern Approach.* W.W. Norton & Company. 1

Veiga, Andre, and E Glen Weyl. 2016. "Product Design in Selection Markets." *Quarterly Journal of Economics*. 5.2, 16, 5.3

Villeneuve, Bertrand. 2003. "Concurrence et antisélection multidimensionnelle en assurance." *Annales d'Economie et de Statistique*, 119–142. 2, 4.2

Wambach, Achim. 2000. "Introducing Heterogeneity in the Rothschild-Stiglitz Model." *Journal of Risk and Insurance*, 67(4): 579–591. 2, 4.2, 5.3, 6.2

Wilson, Charles. 1977. "A Model of Insurance Markets with Incomplete Information." *Journal of Economic Theory*, 16(2): 167–207. 1, 5.4

Appendix

A Positive Profits with Many Goods

We now generalize the examples of Section 3 to an arbitrary number of products. We assume $N \in \mathbb{N}$ firms. The type space is $\Omega = (\omega_k)_{k=1}^K \times \Theta$, for some set Θ, with a set of K alternatives $X = (X_k)_{k=1}^K$. (Θ can be thought of as the 'mass' of each type. It can be taken to be $[0,1]$ WLOG). Let λ be a measure on Ω s.t. $\lambda(\{\omega_k\} \times \Theta) = \frac{1}{K}$ for each $k = 1, \ldots, K$.

The utilities of the types satisfy

$$u((\omega_j, \theta), X_l, p) \begin{cases} = \alpha - p & \text{if } j = l \\ = \beta(\alpha - p) & \text{if } l = j - 1 \\ < \beta(\alpha - p) & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $\beta > 1$, and where the arithmetic on the indices is modular (so $\omega_{K+1} = \omega_1$, etc). (Below we will write $u(\omega_j, X_l, p)$, i.e., we will drop reference to $\theta \in \Theta$.) Effectively, an individual of type ω_j chooses between products j and $j-1$.

Firms are symmetric, with payoffs given (again, dropping reference to $\theta \in \Theta$) by

$$v(\omega_j, l, p) = \begin{cases} p & \text{if } j = l \\ p - \gamma & \text{if } j \neq l \end{cases}.$$

Individuals of type ω_j are, from the perspective of the firms, more cheaply served in product j than in any other. Importantly, we assume that the cost of mismatches is sufficiently large: $\gamma > K\alpha$.

Suppose that the price of each good is $p_l = \alpha$ and the consumer choice profile is $\sigma_X(\omega_k, \theta) = X_k$ for each $k = 1, \ldots, K, \theta \in \Theta$. This consumer profile is incentive compatible given the prices $p_l = \alpha$. We contend that these prices and consumer profile constitute an equilibrium. Notice that at least one firm's profits is positive (any firm with a positive mass of clients).

To show that this is an equilibrium, we first claim that, if $\overline{q} \leq \alpha$, with $q(x) < \alpha$ for some $x \in X$, then in any incentive compatible consumer profile τ, there is some $k = 1, \ldots, K$ s.t. $q(x_{k-1}) < \alpha$ and $\tau_X(\omega_k, \theta) \neq X_k$ for a.e. $\theta \in \Theta$. Let k_0 be s.t. $q(X_{k_0}) < \alpha$; denote $\delta_1 = \alpha - q(X_{k_0}) > 0$. If $\tau_X(\omega_{k_0+1}, \theta) \neq X_{k_0+1}$ for a.e. $\theta \in \Theta$, we are done; $k = k_0 + 1$ is the desired k. Otherwise, $\tau_X(\omega_{k_0+1}, \theta) = X_{k_0+1}$ with λ-positive measure, then it must be that $u(\omega_{k_0+1}, X_{k_0+1}, q(X_{k_0+1})) \geq u(\omega_{k_0+1}, X_{k_0}, q(X_{k_0}))$, i.e.,

$$\delta_2 := \alpha - q(X_{k_0+1}) \geq \beta(\alpha - q(X_{k_0})) = \beta\delta_1$$

as otherwise all ω_{k_0+1} would prefer to switch to X_{k_0}. Either $\tau_X(\omega_{k_0+2}, \theta) \neq X_{k_0+2}$ for all $\theta \in \Theta$, or we continue in the same way, eventually either finding a desired k, or - by way of contradiction - constructing a chain going over all alternatives until we come back to the one we began with, s.t. denoting $\delta_j = \alpha - q(X_{k_0+j-1})$, we have $\delta_{j+1} \geq \beta\delta_j$ for each $j = 1, \ldots, K$ (recall, subscript arithmetic is modular). Hence $\delta_1 = \delta_{K+1} \geq \beta^K \delta_1 > \delta_1$, a contradiction.

Now, suppose a firm j^* attempts a profitable deviation by offer some $\overline{p} \neq \alpha$. We have that $\overline{q} \geq \alpha$ cannot result in an increase in profit, so suppose $\overline{q}$ has a strictly lower price for some good other than α; by the claim, there is some $k = 1, \ldots, K$ s.t. $q(x_{k-1}) < \alpha$ and $\tau_X(\omega_k, \theta) \neq X_k$ for all $\theta \in \Theta$. In particular, since ω_k weakly preferred X_{k-1} at prices α over any other option, and $q(X_{k-1}) < \alpha$, ω_k-types must now be purchasing entirely from the deviating firm j^*. Firm j^* will make profit at most α from attracting all consumers, but will lose $\gamma \cdot \frac{1}{K}$ from attracting those of type ω_k. Since $\gamma > K\alpha$, this deviation is not profitable.

B Zero Profits: Proofs

B.1 Proof of Theorem 1 for Finite Model

The following is Theorem 1 for a finite model, i.e., Ω, X finite (in which case, continuity of an ordering is meaningless, and one can simply require the main condition over all types, not just a.e. type.)

Theorem 7. *Let $\overline{p} = (p_x)_{x \in X} > 0$ be a price vector, and let $(\overline{p}, \sigma)$ be an equilibrium. Suppose there is a (weak) linear ordering $\preceq$ on X such that*

$$u(\omega, x, p_x) = u(\omega, y, p_y) = \max_z u(\omega, z, p_z) \text{ and } x \preceq y$$
$$\to v(\omega, x, p_x) \leq v(\omega, y, p_y), \ \forall \omega \in \Omega, \forall x, y \in X \tag{B.1}$$

(If $0 \in X$, require $0 \preceq x$, $\forall x \in X$.) Then the profit for all firms in this equilibrium is 0.

As stated in the main text, the intuition for the proof is that, if there exists a candidate equilibrium with positive profit and the required ordering on contracts, there exists a profitable deviation. In this deviation, each contract's price is lowered in an appropriate way so that the deviating firm captures the entire market. Although serving clients at lower prices, we show that there always exists a small enough deviation where the gain in market share more than compensates for these lower prices. This deviation is a local deviation, so we focus on a local equilibrium in what follows.

First, in a local equilibrium $(\overline{p}, \sigma)$ all firms have non-negative profit. A firm making strictly negative profit would deviate to some prices $\overline{q} = (q_x)_{x \in X}$ with $q_x > p_x$ for all $x \in X$, which would yield zero profit as all consumers would buy elsewhere.

Suppose that, in local equilibrium, firm k_0 has positive profit $\pi^* := \pi^{k_0} > 0$. Fix another firm j_0. Define a new consumer profile $\eta : \Omega \to \{j_0\} \times X \subseteq N \times X$ by

$$\eta(\omega) = (j_0, x), \ \text{ if for some } k, \ \sigma(\omega) = (k, x) \tag{B.2}$$

i.e., consumers buy the same alternative under η as under σ, but from firm j_0.

The profit of firm j_0 under η satisfies

$$\begin{aligned}\pi^{j_0}(\eta,\overline{p}) &= \sum_{x\in X}\int_{\{\omega|\eta(\omega)=(j_0,x)\}} v(\omega,x,p_x)d\lambda(\omega)\\ &= \sum_{x\in X}\sum_{k\in N}\int_{\{\omega|\sigma(\omega)=(k,x)\}} v(\omega,x,p_x)d\lambda(\omega) = \sum_{k\in N}\pi^k(\sigma,\overline{p}) = \sum_{k\in N}\pi^k > \pi^{j_0}\end{aligned}$$

Fix $0<\varepsilon<\sum_{k\neq j_0}\pi^k$. Enumerate $X=\{x_1,\dots,x_K\}$ with $x_1 \preceq \cdots \preceq x_K$. (If $0\in X$, let $x_1=0$.) Let $\delta_0>0$, and let $\Delta>0$ be s.t.

$$\sum_{\omega\in\Omega}\max_{x\in X}|v(\omega,x,p_x)-v(\omega,x,q_x)|<\varepsilon \tag{B.3}$$

whenever $\overline{q}\in\prod_{x\in X}[p_x-\Delta,p_x+\Delta]$. Let $p_m=\min_{x\in X,x\neq 0}p_x>0$, the lowest price among (non-zero) contracts.

We claim that there exists $0=\delta_1<\delta_2,\dots,\delta_K<\min[\delta_0,\Delta,p_m]$ such that

$$u(\omega,x_j,p_j)<u(\omega,x_k,p_k)\to u(\omega,x_j,p_j-\delta_j)<u(\omega,x_k,p_k-\delta_k),\ \forall\omega\in\Omega,\forall j\neq k \tag{B.4}$$

and

$$u(\omega,x_j,p_j)=u(\omega,x_k,p_k)\to u(\omega,x_j,p_j-\delta_j)<u(\omega,x_k,p_k-\delta_k),\ \forall\omega\in\Omega,\forall j<k \tag{B.5}$$

(B.4) means that if ω prefers x_k over x_j at the original prices, he will continue to do at modified prices; and B.5 means that if ω is indifferent between x_j,x_k ($j>k$) at original prices, he prefers the latter at new prices.

(B.4) can be guaranteed - using u's continuity - by requiring that $\delta_0>0$ be small enough and by requiring $\delta_2,\dots,\delta_K<\delta_0$. To guarantee the other conditions we proceed inductively backwards: Let $0<\delta_K<\min[\delta_0,\Delta,p_m]$ be arbitrary. Henceforth, for $J>1$, given such $\delta_{J+1},\dots,\delta_K$ which satisfy (B.5) for $k,j>J$, choose $\delta_J<\min[\delta_0,\Delta,p_m]$ small enough so that

$$u(\omega,x_J,p_J)<u(\omega,x_k,p_k-\delta_k)\to u(\omega,x_J,p_J-\delta_J)<u(\omega,x_k,p_k-\delta_k),\forall\omega,\forall k>J$$

which can also be guaranteed by continuity, and recalling that u is strictly decreasing in price. Define $q_k=p_k-\delta_k$ for $k=1,\dots,k$, and let τ be a incentive compatible reaction to $\overline{q}=(q_x)_{x\in X}$.

Lemma 1. *If $\omega\in\Omega$ with $\eta(\omega)=x_j\neq\tau(\omega)=x_k$ for some j,k, then $v(\omega,x_j,p_j)\leq v(\omega,x_k,p_k)$.*

In other words, if when prices change from $\overline{p}$ to $\overline{q}$, a consumer changes their choice, it is to a choice which is more profitable (at least under $\overline{p}$) for the firms.

Proof. Suppose ω, j, k are such. Since $\tau(\omega) = x_k$, it must be then that

$$u(\omega, x_j, p_j - \delta_j) \leq u(\omega, x_k, p_k - \delta_k)$$

Our construction implies,

$$u(\omega, x_j, p_j - \delta_j) \neq u(\omega, x_k, p_k - \delta_k),\ \forall j, k$$

and therefore

$$u(\omega, x_j, p_j - \delta_j) < u(\omega, x_k, p_k - \delta_k) \tag{B.6}$$

which by (B.4) and (B.5) implies

$$u(\omega, x_j, p_j) \leq u(\omega, x_k, p_k)$$

But since $\eta(\omega) = x_j$,

$$u(\omega, x_j, p_j) \geq u(\omega, x_k, p_k)$$

and therefore

$$u(\omega, x_j, p_j) = u(\omega, x_k, p_k) \tag{B.7}$$

Combining (B.6) with (B.7) and (B.5) shows $j < k$. Hence, combining this with (B.7) and the condition (B.1) assumed in Theorem 7 gives the desired conclusion. □

To complete the proof, notice that the deviator now owns the market (except possibly the outside option); at original prices, this would give the deviation profits $\sum_k \pi_k$. Since prices have been slightly lowered, the profit to the deviator is just below $\sum_k \pi_k$. However,

$$\begin{aligned}
\pi^{j_0}(\tau, \overline{q}) &= \sum_{j=1}^{K} \sum_{\{\omega | \tau(\omega) = x_j\}} \lambda_j \cdot v(\omega, x_j, p_j - \delta_j) \\
&\geq \sum_{j=1}^{K} \sum_{\{\omega | \tau(\omega) = x_j\}} \lambda_j \cdot v(\omega, x_j, p_j) - \sum_{\omega \in \Omega} \sup_j |v(\omega, x_j, p_j) - v(\omega, x_j, p_j - \delta_j)| \lambda_j \\
&\geq \sum_{j=1}^{K} \sum_{\{\omega | \tau(\omega) = x_j\}} \lambda_j \cdot v(\omega, x_j, p_j) - \varepsilon \\
&\geq \sum_{j=1}^{K} \sum_{\{\omega | \eta(\omega) = x\}} \lambda_j \cdot v(\omega, x_j, p_j) - \varepsilon = \sum_k \pi^k - \varepsilon > \pi^{j_0}
\end{aligned}$$

where the first equality is by definition, the first inequality from the triangle inequality, the second inequality by (B.3) since $\overline{p} - \Delta < \overline{p} - \delta < \overline{p}$, the third inequality by Lemma 1, the last equality by definition, and the final inequality by choice of ε. Hence, we have a profitable deviation, a contradiction to the assumption that $\overline{p}$ is a local equilibrium.

B.2 Proof of Theorem 1 in the General Model

As discussed briefly in Section 5.1,the proof consists of two steps: Firstly, we generalize the proof of the case of the finite model (Section B.1) to allow the space of types Ω and contracts X to each be a compact continuum, but consisting of finitely many 'small connected clusters' of contracts; this step is Proposition 2. The proof of this step is similar to that of Theorem 7, with some alterations. Secondly, we make a reduction to the case described in the first step; this is Lemma 5. Here, we make repeated use of Lusin's theorem to restrict types and contracts to spaces on which all relevant functions are continuous. We also make repeated use of the inner-regularity of λ to guarantee a compact type space. Both such applications only involve disregarding sets of types of arbitrarily small measure, and hence if a profitable deviation exists in a potential positive-profit equilibria after such modifications (by the first step), a profitable deviation exists in the original economy as well.

B.2.1 Function Spaces, Compactness, and Regularity

Recall that a Polish space is a complete, superable metric space. Recall that a Borel measure μ on a metric space X is called *inner regular* (or *tight*) if for all Borel $B \subseteq X$,

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B \text{ is compact }\}$$

It is known that every finite measure on a Polish space is inner regular, e.g., (Parthasarathy, 1976, Thm 3.2).

If Z is a compact metric space, $C(Z)$ denotes the Banach space of real-valued continuous functions on Z, with norm $||f||_\infty = \max_{z\in Z} |f(z)|$. If Ω is a measurable space and $\phi : \Omega \times Z \to \mathbb{R}$ is measurable, and continuous in Z, then the mapping $\Omega \to C(Z)$ given by $\omega \to \phi(\omega, \cdot)$ is measurable (e.g., (Aliprantis and Border, 2007, Thm. 4.55)).

Fix a metric d_Z on a compact Z. The Arzela-Ascoli theorem (e.g., (Rudin, 1987, Thm 11.28)) states that a subset $K \subseteq C(Z)$ is compact if and only if it is closed, bounded, and uniformly equicontinuous; the latter condition means that

$$\forall \varepsilon > 0, \exists \delta > 0, \forall f \in K, \forall z, w \in Z, d_Z(z, w) < \delta \to |f(z) - f(w)| < \varepsilon$$

We also recall Lusin's theorem, (e.g., (Aliprantis and Border, 2007, Thm. 12.8)): Let X, Y be Polish space, $\varepsilon > 0$, let μ be a finite Borel measure on X, and let $f : X \to Y$ be Borel. Then there is $Z \subseteq X$ compact with $\mu(X \backslash Z) < \varepsilon$ s.t. the restriction $f|_Z : Z \to Y$ is continuous.

B.2.2 Preliminaries Lemmas

Lemma B.1. Let Z be a compact space, let $[a,b]$ be an interval, and let $g : Z \times [a,b] \to \mathbb{R}$ be strictly decreasing (in second variable) and continuous. Let $p : Z \to [a+\delta_0, b]$ for some $\delta_0 > 0$ be continuous. Then the mapping

$$\delta \to \sup_{z\in Z} g(z, p(z) - \delta)$$

is strictly increasing in $[0, \delta_0]$.

Proof. It suffices to show that $\sup_{z\in Z} g(z, p(z) - \delta) > \sup_{z\in Z} g(z, p(z))$. Observe that, for every z, for $0 < \delta < \delta_0$,

$$g(z, p(z) - \delta) - g(z, p(z)) > 0$$

and therefore, due to the continuity of u and the compactness of Z,

$$\Delta := \inf_{z\in Z}(g(z, p(z) - \delta) - g(z, p(z))) > 0$$

Hence, letting z^* be such that

$$g(z^*, p(z^*)) > \sup_{z\in Z} g(z, p(z)) - \frac{\Delta}{2}$$

Then

$$\sup_{z\in Z} g(z, p(z) - \delta) \geq g(z^*, p(z^*) - \delta) \geq g(z^*, p(z^*)) + \Delta > \sup_{z\in Z} g(z, p(z)) + \frac{\Delta}{2}$$

□

The following lemmas follow from our continuity and integrability assumptions, and the dominated convergence theorem.

Lemma B.2. For each $\varepsilon > 0$ and $p_M \geq 0$, there is $\zeta > 0$ such that if $\Omega' \subseteq \Omega$ satisfies $\lambda(\Omega') < \zeta$, then

$$\int_{\Omega'} \sup_{x\in X} |v(\omega, x, p_M)| \lambda(d\omega) < \frac{\varepsilon}{6}$$

Lemma B.3. If $\overline{p} = (p_x)_{x\in X}$ is bounded, then for each $\varepsilon > 0$, there is $\Delta > 0$ such that if $\overline{q} \in D_0' := \prod_{x\in X}[p_x - \Delta, p_x + \Delta]$,

$$\int_{\Omega} \sup_{x\in X} |v(\omega, x, p_x) - v(\omega, x, q_x)| d\lambda(\omega) < \frac{\varepsilon}{6}$$

Lemma B.4. Fix $\varepsilon > 0$, prices $\overline{p}$ and a consumer profile σ. There is $\zeta > 0$ such that if $\Omega' \subseteq \Omega$

satisfies $\lambda(\Omega') < \zeta$, then

$$\int_{\Omega'} |v(\omega, \sigma_X(\omega), p(\sigma_X(\omega)))| d\lambda(\omega) < \frac{\varepsilon}{6}$$

B.2.3 Preliminary Result

In this section, we prove a variation Theorem 1 under additional assumptions, and in subsequent sections show that these assumptions can be removed. Fix metric d_X on X. First, we establish a sort of continuity property of (5.1):[20]

Lemma 2. *Assume state and alternative spaces, Ω and X, are compact, and both the utility and profit functions $u(\cdot)$, $v(\cdot)$ are continuous on $\Omega \times X \times \mathbb{R}_+$. Suppose $\overline{p} : X \to \mathbb{R}_+$ is continuous, and $\varepsilon > 0$. Assume the assumptions of Theorem 1, but with (5.1) holding for all Ω (not just λ-a.e.). Then there is $\delta^\circ > 0$, such that for all $\omega \in \Omega$, all $u, v, w, z \in X$, if*

$$u(\omega, u, p(u)) = u(\omega, v, p(v)), \;\; w \preceq z, \;\; d_X(u, w) < \delta^\circ, \;\; d_X(v, z) < \delta^\circ$$

then

$$v(\omega, u, p(u)) < v(\omega, v, p(v)) + \frac{\varepsilon}{4}$$

Proof. Suppose not. Then there are sequences $\delta_n \to 0$, $(u_n), (v_n), (z_n), (w_n), (\omega_n)$ such that

$$u(\omega_n, u_n, p(u_n)) = u(\omega_n, v_n, p(v_n)), \; w_n \preceq z_n, \; d_X(u_n, w_n) < \delta_n, d_X(v_n, z_n) < \delta_n$$

and yet

$$v(\omega_n, u_n, p(u_n)) \geq v(\omega_n, v_n, p(v_n)) + \frac{\varepsilon}{4}$$

W.l.o.g., since Ω, X are compact, we may assume $u_n \to u, v_n \to v$, which implies $z_n \to u, w_n \to v$; we may also assume $\omega_n \to \omega$. For such ω we deduce, since $\preceq$ is a continuous ordering and $\overline{p}$ is continuous, that

$$u(\omega, u, p(u)) = u(\omega, v, p(v)), \; u \preceq v$$

but

$$v(\omega, u, p(u)) \geq v(\omega, v, p(v)) + \frac{\varepsilon}{4}$$

which contradicts (5.1). □

We then obtain the following result.

Proposition 2. *Fix $\varepsilon > 0$, $\delta_0 > 0$, and prices $\overline{p} = (p_x)_{x \in X}$. Let $(\pi^j)_{j=1}^N$ denote the profits of the firms at these prices for some incentive compatible reaction of consumers. Suppose that, in addition to the assumptions of Theorem 1, the following conditions hold:*

[20] It is in this lemma that we use the fact that $\preceq$ is a *continuous* ordering.

1. *([5.1](#)) holds for all Ω (not just λ-a.e.).*

2. *X and Ω are compact.*

3. *$\overline{p}$ is continuous.*

4. *If $0 \in X$, then it is an isolated point of X. Denote*

$$p_M := \max_{x \in X} p(x) < \infty, \; p_m := \min_{x \in X, x \neq 0} p(x)$$

   Assume $p_m > 0$.

5. *Both the utility and profit functions $u(\cdot), v(\cdot)$ are continuous on $\Omega \times X \times \mathbb{R}_+$. (As a result, the families $(u_\omega(\cdot,\cdot))_{\omega \in \Omega}$ and $(v_\omega(\cdot,\cdot))_{\omega \in \Omega}$ are compact in $C(X \times [0, p_M])$.) Fix $\delta^* > 0$ such that*

$$\text{If } d_X(x,y) < \delta^* \text{ and } |p - q| < \delta^* \text{ then } |v_\omega(x,p) - v_\omega(y,q)| < \frac{\varepsilon}{6} \tag{B.8}$$

   and we can choose $\delta^* \leq \delta^o$, with δ^o being as in Lemma 2.

6. *The alternatives space can be partitioned finitely $X = \cup_{j=1}^K X_j$, with each X_j compact and radius $\leq \delta^*$, and such that for each j and each $y, z \in X_j$, $|p(y) - p(z)| < \delta^*$.*

Then there is a continuous deviation $\overline{q}$ with $\overline{p} - \delta_0 \leq \overline{q} < \overline{p}$ which, if taken by a firm, gives profit at least $\sum_{j=1}^N \pi^j - \varepsilon$ (for any incentive compatible reaction).

Essentially, the proof involves re-doing the proof of Theorem 7, where the sets $X_1, \ldots, X_k$ partitioning X replace the individual alternatives.

Proof. Fix a firm $j_0, \varepsilon, \delta_0 > 0$, and prices $\overline{p}$ satisfying the conditions. W.l.o.g., $\lambda(\Omega) = 1$. Let $\zeta > 0$ such that if $B \subseteq \Omega$ satisfies $\lambda(B) < \zeta$, then

$$\int_B \sup_{x \in X} |v(\omega, x, p_M)| \lambda(d\omega) < \frac{\varepsilon}{6} \tag{B.9}$$

(Lemma B.2) and let $\Delta > 0$ be s.t.if $\overline{q} \in D_0' := \prod_{x \in X} [p_x - \Delta, p_x + \Delta]$,

$$\int_\Omega \sup_{x \in X} |v(\omega, x, p_x) - v(\omega, x, q_x)| < \frac{\varepsilon}{6} \tag{B.10}$$

(Lemma B.3). Enumerate[21] the partition of X by $X_1, \ldots, X_K$, in such a way that if $i < j$, then there is some $x \in X_i$ and some $y \in X_j$ with $x \preceq y$. (If $0 \in X$, then take $X_1 = \{0\}$; 0 is $\preceq$-minimal.) Let 1_j denote the indicator function of X_j. Define for convenience, for each

[21] E.g., choose one point from each component, and number according to the induced order on these points.

continuous price vector $\overline{\rho}$, a mapping $U(\cdot,\cdot,\overline{\rho}) : \Omega \times \{1,\ldots,K\} \to \mathbb{R}$ by

$$U(\theta, j, \overline{\rho}) = \max_{x \in X_j} u(\theta, x, \overline{\rho}(x))$$

i.e., the highest utility in X_j for consumer θ at prices $\overline{\rho}$. Note that the maximum is obtained, since $\overline{\rho}$ and U are continuous and each X_j is compact; furthermore, for each $\theta \in \Omega$, the mapping $\overline{\rho} \to U(\theta, j, \overline{\rho})$ on $C(X)$ is continuous by the Berge maximum theorem (e.g., (Aliprantis and Border, 2007, Thm. 17.31)). □

Hence, the following lemma follows.

Lemma 3. *There is $\delta^\ell > 0$ such that if $\overline{q}', \overline{q}''$ are price vectors with $\overline{q} - \delta^\ell < \overline{q}', \overline{q}'' \leq \overline{q}$,*

$$\lambda(\{\omega \mid \exists j,k, U(\omega, j, \overline{q}') < U(\omega, k, \overline{q}') \text{ and } U(\omega, j, \overline{q}'') \geq U(\omega, k, \overline{q}'')\}) < \frac{\zeta}{2} \tag{B.11}$$

i.e., very few types would 'flip' their choices when prices change from $\overline{q}'$ and $\overline{q}''$, as long as both are very close (and below) to $\overline{p}$.

Fix such δ^ℓ. We claim that there exists a set $\Omega_0 \subseteq \Omega$ with $\lambda(\Omega_0) > 1 - \zeta$, and $0 \leq \delta_1 < \delta_2, \ldots, \delta_K < \min[\delta_0, \delta^\ell, \Delta, p_m]$ (with $\delta_1 = 0$ iff $0 \in X$) such that it holds that

$$U(\omega, j, \overline{p}) < U(\omega, k, \overline{p}) \to U(\omega, j, \overline{p} - \overline{\delta}) < U(\omega, k, \overline{p} - \overline{\delta}),\ \forall \omega \in \Omega_0, \forall j \neq k \tag{B.12}$$

and

$$U(\omega, j, \overline{p}) = U(\omega, k, \overline{p}), j < k \to U(\omega, j, \overline{p} - \overline{\delta}) < U(\omega, k, \overline{p} - \overline{\delta}),\ \forall \omega \in \Omega_0, \forall j \neq k \tag{B.13}$$

where $\overline{\delta} = \sum_{j=1}^K \delta_j \cdot 1_j$, which we observe is continuous. (These equations parallel (B.4) and (B.5) from the proof of Theorem 7.)

Indeed, requiring $\delta_2, \ldots, \delta_K < \delta^\ell$ of Lemma 3 guarantees (B.12) if we construct Ω_0 to exclude the set not satisfying (B.11), which is of measure at most $\frac{\zeta}{2}$. To guarantee the construction satisfies (B.13), repeat the corresponding step in the proof of Theorem 7 (which can be done, thanks to Lemma B.1), such that at each stage the measure of the types in violation of (B.13) is of measure $< \frac{\zeta}{2K}$.

Now, define $\overline{q}$ by $\overline{q} = \overline{p} - \overline{\delta}$. Like in the proof of Theorem 7, let τ be any incentive compatible consumer profile when j_0 chooses prices $\overline{q}$ and the other firms stick with $\overline{p}$. Observe that τ always chooses firm j_0 (except possibly when purchasing 0). The parallel of Lemma 1 is:

Lemma 4. *If $\omega \in \Omega_0$ with $\eta_X(\omega) = x_\eta \in X_j$, $\tau_X(\omega) = x_\tau \in X_k$ for some j,k, then $v(\omega, x_\eta, p(x_\eta)) \leq v(\omega, x_\tau, p(x_\tau)) + \frac{5\varepsilon}{12}$.*

Proof. If $j = k$, then by assumption,

$$|v(\omega, x_\eta, p(x_\eta)) - v(\omega, x_\tau, p(x_\tau))| \leq \frac{\varepsilon}{6}$$

so we concentrate on the case $j \neq k$. As in the proof of Lemma 1, it is established verbatim that for such ω, j, k, it must hold that $j < k$ and that

$$U(\omega, j, \overline{p}) = U(\omega, k, \overline{p}) \tag{B.14}$$

(This is the parallel of (B.7).) Hence, since

$$U(\omega, j, \overline{p}) = u(\omega, x_\eta, p(x_\eta))$$

there is $z \in X_k$ with

$$u(\omega, x_\eta, p(x_\eta)) = u(\omega, z, p(z)) \tag{B.15}$$

By our assumption, each component is of radius $\leq \delta^* \leq \delta^o$ defined in Lemma 2. Also note (by definition) the existence of $u \in X_j, v \in X_k$ with $u \preceq v$. Putting this together with (B.15) and with the conclusion of Lemma 2 gives.

$$v(\omega, x_\eta, p(x_\eta)) \leq v(\omega, z, p(z)) + \frac{\varepsilon}{4} \tag{B.16}$$

Since $z, x_\tau \in X_k$, $d_X(z, x_\tau) < \delta^*$ and $|p(z) - p(x_\tau)| < \delta^*$, by (B.8),

$$|v(\omega, z, p(z)) - v(\omega, x_\tau, p(x_\tau))| < \frac{\varepsilon}{6} \tag{B.17}$$

and hence the conclusion follows from (B.16) and (B.17).

$\square$

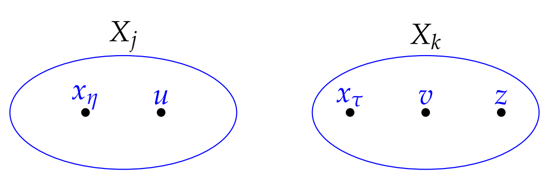

Figure 5: Points in Proof of Lemma 4
$u \preceq v$, $u(\omega, x_\eta, p(x_\eta)) = u(\omega, z, p(z))$, $\eta_X(\omega) = x_\eta$, $\tau_X(\omega) = x_\tau$,

Now, to complete the proof. Using (B.9), (B.10), and Lemma 4,

$$
\begin{aligned}
\pi^{j_0}(\tau,\overline{q}) &= \int_\Omega v(\omega,\tau(\omega),\overline{q}(\tau(\omega))) \geq \int_{\Omega_0} v(\omega,\tau(\omega),\overline{q}(\tau(\omega))) - \frac{\varepsilon}{6} \\
&\geq \int_{\Omega_0} v(\omega,\tau(\omega),\overline{p}(\tau(\omega))) - \frac{\varepsilon}{6} - \frac{\varepsilon}{6} \geq \int_{\Omega_0} v(\omega,\eta(\omega),\overline{p}(\eta(\omega))) - \frac{\varepsilon}{3} - \frac{5\varepsilon}{12} \\
&\geq \int_{\Omega} v(\omega,\eta(\omega),\overline{p}(\eta(\omega))) - \frac{\varepsilon}{3} - \frac{5\varepsilon}{12} - \frac{\varepsilon}{6} > \sum_k \pi^k - \varepsilon
\end{aligned}
$$

as required.

B.2.4 Completion of Proof

Recall that $f|_A$ denotes the restriction of a function f to domain A.

Lemma B.5. Let X be a metric space, $Z \subseteq X$ closed, $f : X \to \mathbb{R}$ l.s.c., and $g : X \to \mathbb{R}$ s.t. $f|_{X\setminus Z} = g|_{X\setminus Z}$, $g \leq f$, and $g|_Z$ is l.s.c.. Then g is l.s.c..

Proof. Let $\alpha \in \mathbb{R}$ and $x_0 \in X$ with $g(x_0) > \alpha$, we must show that there is open neighborhood U of x_0 in which $g > \alpha$. If $x_0 \notin Z$,since Z is closed and $f|_{X\setminus Z} = g|_{X\setminus Z}$, this is immediate as f is l.s.c.. If $x_0 \in Z$, then let U be an open neighborhood of x_0 in which $f > \alpha$ (such exists as $f(x_0) \geq g(x_0) > \alpha$ and f is l.s.c.), and let $V \subseteq Z$ be a set relatively open in Z s.t. $x_0 \in V$ and $g > \alpha$ in V; w.l.o.g. $V \subseteq U$. Hence there is an open set $x_0 \in W \subseteq U$ s.t. $V \cap Z = W$; $g > \alpha$ in W, as required. ☐

The following lemma will allow us to link between the general economy (after we make a series of reductions) and economies as dealt with in Proposition 2.

Lemma 5. *Fix an economy satisfying the assumptions of Sec 4.1, an l.s.c. price function $\overline{p}$ and incentive compatible consumer profile σ. Suppose there is a continuous (weak) linear ordering $\preceq$ on it, such that (5.1) holds.*[22] *Fix $\varepsilon > 0$. Then there is $\zeta > 0$, such that if:*

- *$Z \subseteq X$ (if $0 \in X$, assume $0 \in Z$) is a compact subset.*
- *$\Omega_0 \subseteq \Omega$ is a subset satisfying*

$$\sigma_X(\Omega_0) \subseteq Z \text{ and } \lambda(\Omega_0) > \lambda(\Omega) - \zeta \tag{B.18}$$

- *In the reduced game with types Ω_0 and contract space Z, at prices $\overline{p}|_Z$, some firm can increase profit by least $\Delta\Pi$ with an l.s.c. deviation $\overline{q}$ satisfying $\overline{q} < \overline{p}|_Z$.*

Then that firm can increase profit by at least $\Delta\Pi - \varepsilon$ in the original game.

[22] We actually do not need the full strength of (5.1) for this lemma; only that if $0 \in X$, for λ-a.e. $\omega \in \Omega$, whenever $u(\omega, 0, 0) = u(\omega, x, p(x))$ for some $x \in X$, then $v(\omega, x, p(x)) \geq 0$.

By "increase profit at least x", we mean by choosing an l.s.c. price to which there is at least one incentive compatible reaction of consumers, and in *any* incentive compatible reaction, the profit is least x higher than under the pair $(\overline{p}, \sigma)$.

Proof. First, let $\zeta_1 > 0$ correspond to ε and $p_M = 0$ as in Lemma B.2, and $\zeta_2 > 0$ correspond to $\varepsilon > 0$, the prices $\overline{p}$ and the profile σ as in Lemma B.4, and take $\zeta = \min[\zeta_1, \zeta_2]$; specifically then if $\Omega' \subseteq \Omega$ is s.t. $\lambda(\Omega') < \zeta$,

$$\int_{\Omega'} \inf_{x \in X} v(\omega, x, 0) d\lambda(\omega) \geq -\frac{\varepsilon}{6} \text{ and } \int_{\Omega'} |v(\omega, \sigma_X(x), p(\sigma_X(x)))| d\lambda(\omega) \leq \frac{\varepsilon}{6} \tag{B.19}$$

Let $\Omega_0 \subseteq \Omega$, $Z \subseteq X$, and deviation $\overline{q}$ be as in the conditions of the lemma, with firm k_0 increasing profit at least $\Delta\Pi$. Extend $\overline{q}$ to all of X by $\overline{q}|_{X \setminus Z} = \overline{p}|_{X \setminus Z}$; $\overline{q}$ is l.s.c. by Lemma B.5. Furthermore, since Z was compact one sees that for each consumer there is an incentive compatible contract. Let τ be a incentive compatible consumer profile w.r.t. $\overline{q}$; for simplicity, assume that if $\sigma_X(\omega) = 0 \in X$, then $\tau_X(\omega) \in Z$; since $0 \in Z$ if $0 \in X$, the only other option for such an ω purchasing 0 under σ would be switch to some $(x, q(x)) = (x, p(x))$ for $x \notin Z$ with $u(\omega, 0, 0) = u(\omega, x, p(x))$, which λ-a.s. could only add to the profit.

We contend that the restriction $\tau' := \tau|_{\Omega_0}$ is then a incentive compatible consumer profile in Ω_0 w.r.t. $\overline{q}$, making selections in Z and from firm k_0. Indeed, let $\omega \in \Omega_0$. Intuitively, the options in Z have become more attractive relative to the options outside of Z. Formally, by assumption $\sigma_X(\omega) \in Z$, $argmax_{x \in X} u(\omega, x, p(x)) \cap Z \neq \emptyset$. Since $\overline{q}|_Z < \overline{p}|_Z$ and $\overline{q}|_{X \setminus Z} = \overline{p}|_{X \setminus Z}$ outside of Z, we have either $argmax_{x \in X} u(\omega, x, q(x)) \subseteq Z$ (recall $0 \in Z$ if $0 \in X$) or $\tau_X(\omega) = 0$, as required.

Finally, by (B.19), if Π' (resp. Π) denotes the profit of firm k_0 after (resp. before) deviation,

$$\begin{aligned}
\Pi' = \int_{\Omega} 1_{\tau_N(\omega)=k_0} \cdot v(\omega, \tau_X(\omega), q(\tau_X(\omega)) d\lambda(\omega) &\geq \int_{\Omega_0} 1_{\tau_N(\omega)=k_0} \cdot v(\omega, \tau_X(\omega), q(\tau_X(\omega)) d\lambda(\omega) \\
&+ \int_{\Omega \setminus \Omega_0} 1_{\tau_N(\omega)=k_0} \cdot v(\omega, \tau_X(\omega), 0) d\lambda(\omega) \\
&\geq \int_{\Omega_0} 1_{\tau_N(\omega)=k_0} \cdot v(\omega, \tau_X(\omega), p(\tau_X(\omega)) d\lambda(\omega) - \frac{\varepsilon}{6} \\
&\geq \int_{\Omega_0} 1_{\sigma_N(\omega)=k_0} \cdot v(\omega, \sigma_X(\omega), p(\sigma_X(\omega)) d\lambda(\omega) + \Delta\Pi - \frac{\varepsilon}{6} \\
&\geq \int_{\Omega} 1_{\sigma_N(\omega)=k_0} \cdot v(\omega, \sigma_X(\omega), p(\sigma_X(\omega)) d\lambda(\omega) + \Delta\Pi - \frac{\varepsilon}{3} = \Pi + \Delta\Pi - \frac{\varepsilon}{3}
\end{aligned}$$

□

Now to complete the proof of Theorem 1, suppose some firm j_0 had profit $\pi^{j_0} > 0$. W.l.o.g., assume $\lambda(\Omega) = 1$. Fix $0 < \varepsilon < \frac{\pi^{j_0}}{N+1}$, and $\delta_0 > 0$. Let $\zeta > 0$ correspond to ε as in both Lemma 5 and Lemma B.4, and assume $\zeta \leq \varepsilon$.

We will define $\Omega_0 \subseteq \Omega_5 \subseteq \Omega_4 \subseteq \Omega_3 \subseteq \Omega_2 \subseteq \Omega_1 \subseteq \Omega$, via six reductions: 1) To bounded prices. 2) To compact alternative space. 3) To compact consumer space and continuous utility and profit functions. 4) To a smaller alternative space with the structure required for Proposition 2. 5) To continuous prices. 6) Re-compactifying consumer space. Then we will apply Proposition 2 to complete the proof.

Henceforth, let μ denote the measure induced on X by λ and σ - formally $\mu = \lambda \circ \sigma_X^{-1}$.

First, let $0 < p_m < p_M$ be such that $p_m \leq \overline{p} \circ \sigma \leq p_M$ or $\overline{p} \circ \sigma = 0$ with probability $> 1 - \frac{\zeta}{6}$; formally, such that if

$$\Omega_1 := \{\omega \mid p(\sigma_X(\omega)) \in \{0\} \cup [p_m, p_M]\}$$

then

$$\lambda(\Omega_1) > 1 - \frac{\zeta}{6}$$

Denote $D = \{0\} \cup [p_m, p_M]$.

Next, let $\hat{X} \subseteq X$ be compact such that $\mu(\hat{X}) > \mu(X) - \frac{\zeta}{6}$; denoting $\Omega_2 = \sigma_X^{-1}(\hat{X})$, we have

$$\lambda(\Omega_2) > \lambda(\Omega_1) - \frac{\zeta}{6}$$

Next, recall from Section B.2.1 that since $u, v : \Omega_2 \times \hat{X} \times D \to \mathbb{R}$ are jointly Borel measurable, the induced mappings $\phi_u, \phi_v : \Omega_2 \to C(\hat{X} \times D)$ are Borel, and hence, Lusin's theorem shows that there is a compact subset $\Omega_3 \subseteq \Omega_2$ s.t. the induced mappings ϕ_v,ϕ_u restricted to Ω_3 are continuous, and such that

$$\lambda(\Omega_3) > \lambda(\Omega_2) - \frac{1}{6}\zeta$$

By the Arzela-Ascoli theorem, the families $(v_\theta)_{\theta \in \Omega_3}$, $(u_\theta)_{\theta \in \Omega_3}$ are uniformly equicontinuous on $\hat{X} \times [0, p_M]$.

Next, let δ^* be as in Proposition 2; i.e., fix $\delta^* > 0$ such that if $d_X(x, y) < \delta^*$ and $|p - q| < \delta^*$, such that for all $\omega \in \Omega_3$, $|v_\omega(x, p) - v_\omega(y, q)| < \frac{\varepsilon}{6}$. Such exists by the previous reduction to Ω_3. Also require $\delta^* \leq \delta_0$. Now, partition $\hat{X}$ with finitely many Borel sets $Z_1, \ldots, Z_L$ such that:

- Each Z_i is of radius $< \delta^*$.
- For each i, $\sup_{x \in X_i} p(x) - \inf_{x \in X_i} p(x) < \delta^*$.
- If $0 \in \hat{X}$, $Z_1 = \{0\}$.

This is easily seen to be possible by the compactness of X and the fact $p_m \leq \overline{p} \leq p_M$. For each $i = 1, \ldots, L$, let $X_i \subseteq Z_i$ be compact such that $\mu(X_i) > \mu(Z_i) - \frac{\zeta}{6L}$; such exists by inner regularity. Let $\tilde{X} = \cup_i X_i$ and $\Omega_4 = \sigma^{-1})_X(\tilde{X}) \cap \Omega_3$. Then

$$\lambda(\Omega_4) > \lambda(\Omega_3) - \frac{\zeta}{6}$$

The space Ω_4 and sets $X_1, \dots, X_L$ which partition $\tilde{X}$ satisfies the conditions of Proposition 2 for the ε, δ_0 chosen.

Let $X_0 \subseteq \tilde{X}$ be compact such that $\overline{p}|_{X_0}$ is continuous and such that $\mu(X_0) > \mu(\tilde{X}) - \frac{\zeta}{6}$. Such exists by another application of Lusin's theorem. Set $\Omega_5 = \sigma_X^{-1}(X_0) \cap \Omega_4$. Then,

$$\lambda(\Omega_5) > \lambda(\Omega_4) - \frac{\zeta}{6}$$

Finally, choose $\Omega_0 \subseteq \Omega_5$ compact with

$$\lambda(\Omega_0) > \lambda(\Omega_5) - \frac{\zeta}{6} > 1 - \zeta$$

In the original game, each firm j made profit π^j. Let $\tilde{\pi}^j$ denote the profit of firm j in the game restricted to types Ω_0 and alternative space $X_0 := \cup_\ell X_\ell$ under the profile $\sigma|_{\Omega_0}$. Then since ζ satisfies Lemma B.4 w.r.t $\varepsilon > 0$ and prices $\overline{p}$,

$$\tilde{\pi}^j \geq \pi^j - \varepsilon, j = 1, \dots, K$$

Recall j_0 is such that $0 < (N+1)\varepsilon < \pi_{j_0}$, and fix another firm k_0. Hence, by Proposition 2, there is $\overline{q}|_{X_0} < \overline{p}|_{X_0}$ that would give the deviating firm k_0 an increase in profit of at least

$$\Delta\Pi \geq \sum_{j \neq k_0} \tilde{\pi}^j - \varepsilon \geq \sum_{j \neq k_0} (\pi^j - \varepsilon) - \varepsilon > \pi^{j_0} - N\varepsilon$$

Hence, in the original game (since ζ was chosen by Lemma 5) firm k_0 can increase his profit $\Delta\Pi - \varepsilon \geq \pi^{j_0} - (N+1)\varepsilon > 0$, a contradiction.

B.3 Proof of Theorem 3

By Theorem 1, it suffices to show that for those ω which satisfy the given inequalities, whenever (x,p), (y,q) are such that $u_\omega(x,p) = u_\omega(y,q)$ and $y > x$, then $v_\omega(y,q) \geq v_\omega(x,p)$. Fix such $\omega, (x,p), (y,q)$; denote $f = u_\omega, g = v_\omega$. Denote $C = f(x,p) (= f(y,q))$ and $C' = g(x,p)$. Since $f(u,v), g(u,v)$ are continuously differentiable with $\frac{\partial f}{\partial v} \neq 0$, $\frac{\partial g}{\partial v} \neq 0$, the implicit function theorem shows that we may parametrize the curves $f \equiv C$ and $g \equiv C'$ by functions $\phi : [x,y] \to \mathbb{R}$ and $\psi : [x,y] \to \mathbb{R}$, respectively - that is $f(u, \phi(u)) = C$ and $g(u, \psi(u)) = C'$ for all $u \in [x,y]$ - s.t. for all $r \in [x,y]$,[23]

$$\phi'(r) = -\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}}(r, \phi(r)),\ \psi'(r) = -\frac{\frac{\partial g}{\partial u}}{\frac{\partial g}{\partial v}}(r, \psi(r))$$

In light of the assumption $\frac{\partial g}{\partial v}(u,v) > 0$, it suffices to show that $\phi(y) > \psi(y)$. If $w \in [x,y]$ is s.t. $\phi(w) = \psi(w)$, then by continuous differentiability of f, g, and by the continuity of ϕ, ψ,

[23]At x, y, the derivatives are one-sided.

$\phi'(r) > \psi'(r)$ for all r in a neighborhood of w. (Note that it need not be the case that $\phi' > \psi'$ everywhere.) By assumption $\phi(x) = \psi(x) = p$; since $\phi' > \psi'$ in a (right-)neighborhood of x, we get $\phi(r) > \psi(r)$ in such a neighborhood. Suppose by way of contradiction, there is $s \in (r, y]$ for which $\phi(s) = \psi(s)$, and in particular let s be the smallest such s; such minimum would exist by continuity of ϕ, ψ, and $s > x$. There is $\delta > 0$ s.t. in $(s-\delta, s)$, $\phi'(r) > \psi'(r)$; we may take δ s.t. $r < s - \delta$. By assumption, $\phi(s-\delta) > \psi(s-\delta)$; but since $\phi'(r) > \psi'(r)$ in $(s-\delta, s)$, also $\phi(s) > \psi(s)$, a contradiction to $\phi(s) = \psi(s)$.

Hence, $\psi(y) < q$, and since v is strictly increasing in price, the result follows.

C Positive Profits: Proofs

C.1 Topology on Function Spaces

See, e.g., Rudin (1991) for more thorough treatment for topologies on function spaces. We make use of these in stating and proving Theorem 5, and in proving Theorem 1.

Let $\mathcal{C} = C(\mathbb{R}_+)$ be the space of continuous functions $\mathbb{R}_+ \to \mathbb{R}$, with the topology of uniform convergence on compact sets; this is a completely metrizable separable topology, i.e., a Polish space. Explicitly, let $K_1 \subseteq K_2 \subseteq \cdots \mathbb{R}_+$ be a sequence of compact sets with K_j contained in the interior of K_{j+1}. Then $\mathcal{C}$ is completely metrizable, e.g., by the metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \min[\sup_{K_n} |f-g|, 1]$$

(The topology induced is independent of the sequence (K_j) chosen.) Let $\mathcal{C}_{\uparrow}, \mathcal{C}_{\nearrow}, \mathcal{C}_{\downarrow}, \mathcal{C}_{\searrow}$ denote the subspaces of strictly increasing, weakly increasing, strictly decreasing, and weakly decreasing functions, respectively. The spaces can be shown to be Polish as well. Specifically,

$$\mathcal{C}_{\uparrow} = \cap_{p > q \in \mathbb{R}_+} \{f \in \mathcal{C}_{\uparrow} \mid f(p) > f(q)\}$$

is G_δ as the countable intersection of open sets, and G_δ sets are Polish, while

$$\mathcal{C}_{\nearrow} = \cap_{x \geq y \in \mathbb{R}_+} \{f \in \mathcal{C}_{\uparrow} \mid f(x) \geq f(y)\}$$

is closed as the intersection of closed sets.

Similarly, let $\mathcal{C}^1 = C^1(\mathbb{R}_+)$ be the space of continuously differentiable functions $\mathbb{R}_+ \to \mathbb{R}$, with the topology of uniform convergence of both the function and its derivative on compact sets. The space can similarly be complete metrized, using $K_1 \subseteq K_2 \subseteq \cdots \mathbb{R}_+$ as above, by, e.g., the metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \min\left[\sup_{K_n} \max[|f-g|, |f'-g'|], 1\right]$$

The subspaces $\mathcal{C}^1_{\uparrow}, \mathcal{C}^1_{\nearrow}, \mathcal{C}^1_{\downarrow}, \mathcal{C}^1_{\searrow}$ are similarly defined and Polish.

Remark 1. The space of functions Theorem 5 refers to is utility functions in $\mathcal{C}^1_{\downarrow}(\mathbb{R}_+)$ and profit functions in $\mathcal{C}_{\nearrow}(\mathbb{R}_+)$.

A subset Y of a Polish space X is said to be *meagre* (a.k.a., of the first category) if Y can be written $Y = \cup_{n=1}^{\infty} Y_n$, where each Y_n's closure has empty interior. A property is said to hold *generically* if the set on which it does not hold is meagre.

Remark 2. In Anderson and Zame (2001), two notions with measure-theoretical motivations are presented to describe genericity in infinite-dimensional spaces (e.g., function spaces) by attempting to define what a 'small' (in their terminology, 'shy') set would be. In this paper, in Theorems 5 and 6, we state that certain properties leading to (local) equilibria with positive profit hold throughout open sets in function space, which as a result in particular excludes them from being 'shy', as 'shy' sets cannot contain open sets.

C.2 Perturbable Economies with Positive Profits

Remark 3. To demonstrate Theorem 5, we identify the key properties of the economies described in Section 3 and Appendix A, which guarantee the existence of an equilibrium with positive profit. After establishing the sufficient conditions given in the lemma below, we show (Appendix C.5) that any economy with fundamentals sufficiently close to an economy that satisfies these conditions, will also satisfy them.

We emphasize that Lemma 6 is not intended as general conditions under which economies have equilibria with positive profits. The conditions required by this result are, admittedly, quite demanding. Instead, it is meant as a tool that will allow us to demonstrate that economies with positive profit equilibria are not "knife-edge" cases.

In the examples we have presented, it is straightforward to verify that the prices constitute positive profit equilibria. However, the perturbations may be given by functions for which finding equilibrium and studying perturbations may not be tractable.

Lemma 6. *Suppose $\overline{p} = (p_x)_{x \in X}$ is a price vector such there are types $\omega_1, \ldots, \omega_m$ and different contracts $x_1, \ldots, x_m \in X$ (if $0 \in X$, $x_i \neq 0$ for each i), N firms, and a probability vector $(\lambda_k)_{k=1}^m$, for which:*

1.

$$0 < v(\omega_k, x_k, p_{x_k}), \ \forall k = 1, \ldots, m \tag{C.1}$$

2.

$$argmax_{(x,p_x), x \in X} u(\omega_k, \cdot) = \{(x_k, p_{x_k}), (x_{k-1}, p_{x_{k-1}})\}, \forall k = 1, \ldots, m \tag{C.2}$$

3. *For any prices $\overline{q} \leq \overline{p}$,*[24]

$$argmax_{(x,q_x),x\in X} u(\omega_k, \cdot) \subseteq \{(x_k, q_{x_k}), (x_{k-1}, q_{x_{k-1}})\}, \forall k = 1, \ldots, m \tag{C.3}$$

4.

$$\sum_{j=1}^{m} v(\omega_j, x_j, p_{x_j})\lambda_j < \frac{N}{N-1}\lambda_k(v(\omega_k, x_k, 0) - v(\omega_k, x, p_x)),\ \forall k = 1, \ldots, m, \forall x \in X\backslash\{0\} \tag{C.4}$$

where we interpret $\omega_0 = \omega_m$, $x_0 = x_m$.

5. *Utility $u(\cdot)$ is differentiable in price and*

$$\prod_{k=1}^{m} \frac{\sup\{\frac{\partial u(\omega_k, x_k, q)}{\partial q} \mid q \leq p_{x_k}\}}{\inf\{\frac{\partial u(\omega_k, x_{k-1}, q)}{\partial q} \mid q \leq p_{x_{k-1}}\}} < 1 \tag{C.5}$$

Then, for $\tilde{\Omega} := \{\omega_1, \ldots, \omega_m\} \times \{1, \ldots, N\}$ with distribution $\lambda(\{\omega_k, j\}) = \frac{1}{N}\lambda_k$, and the incentive compatible consumer profile $\sigma(\omega_k, j) = (x_k, j)$, we have that $(\overline{p}, \sigma)$ is an equilibrium of $(\tilde{\Omega}, \lambda)$ with strictly positive profit.

We now provide an intuition for the conditions of Lemma 6. Condition 1 means that firms are making a positive profit on each type. Condition 2 means that, for type ω_k, the two most preferred contracts are $(x_k, p_{x_k}), (x_{k-1}, p_{x_{k-1}})$, and he is indifferent between them (we interpret $k-1$ as m for $k = 1$). Condition 3 means that, if prices are (weakly) lowered, the most preferred contracts of type ω_k are one or both of the contracts that type preferred at the original prices.[25] Condition 4 means that it is sufficiently costly for firms to provide to type ω_k any option other than x_k (the left-hand side of (C.4) is the total profit in the market.).

Formally, the final assertion of equilibrium is on $\tilde{\Omega}$ and utility functions and profit functions extend from Ω to $\tilde{\Omega}$ naturally, $u((\omega, k), x, p) := u(\omega, x, p)$, $v((\omega, k), x, p) := v(\omega, x, p)$.

The equilibrium described above is one where all consumers of type ω_k buy alternative x_k and randomize uniformly across firms. Observe that (C.1) and (C.4) together imply that

$$v(\omega_k, x_k, p_{x_k}) > v(\omega_k, x_{k-1}, p_{x_{k-1}}),\ k = 1, \ldots, m \tag{C.6}$$

which, together with (C.2) implies that there *cannot* be an order $\preceq$ such that (B.1) is satis-

[24] The requirement that this hold for all $\overline{q} \leq \overline{p}$ is quite strict, but emphasise we are not aiming for generality, but rather identifying conditions which can be shown to hold on an open set of economies.

[25] An equivalent formulation, since utility is weakly decreasing in price, is

$$u(\omega_k, x_j, p_j) > u(\omega_k, x_i, 0),\ \forall k = 1, \ldots, m,\ \forall j = k-1, k.\ \forall i \neq k-1, k$$

fied, hence the conditions of Theorem 1 cannot hold. In fact, as we show in Lemma C.1 below, Condition (C.5) guarantees that the prices cannot all be lowered in a way such that each consumer ω_j chooses x_j, which is the preferred option for the firms between x_{j-1} and x_j at prices at or close to $\overline{p}$.

We can now see how this result relates to the example of Section 3. In the example, we had utilities linear in price, and

$$\frac{\frac{\partial u(\omega_B,U,p)}{\partial p}}{\frac{\partial u(\omega_B,V,p)}{\partial p}} \times \frac{\frac{\partial u(\omega_A,V,p)}{\partial p}}{\frac{\partial u(\omega_A,U,p)}{\partial p}} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4} < 1$$

Therefore, (C.5) is satisfied. Moreover, (C.2), (C.1) and (C.4) are also satisfied. Condition (C.3) is vacuously satisfied. Hence, the conditions of Lemma 6 hold. (A similar verification is carried out for the example in Appendix A.) We can also see how the conditions apply to Section A, as

$$\frac{\frac{\partial u(\omega_j,X_j,p)}{\partial p}}{\frac{\partial u(\omega_j,X_{j-1},p)}{\partial p}} = \frac{1}{\beta} < 1$$

for each consumer j.

C.3 Proof of Lemma 6

The proof generalizes the examples of Section 3.2 and Section A. The main idea is that the conditions required by the theorem imply that attempting to undercut would attract some type to buy a contract which is is very expensive to supply to this type.

Let $\overline{p} = (p_x)_{x\in X}$ be a price vector and let types $\omega_1,\ldots,\omega_m$ and different items $x_1,\ldots,x_m \in X$ satisfy the conditions of Lemma 6, with weights $\lambda_1,\ldots,\lambda_m$. Let $c_1,\ldots,c_m,d_1,\ldots,d_m > 0$ with $\prod_{j=1}^m \frac{c_j}{d_j} < 1$ such that if $\overline{q} = (q_x)_{x\in X}$ satisfies $\overline{q} \leq \overline{p}$, then

$$|u(\omega_k,x_k,p_k) - u(\omega_k,x_k,q_k)| \leq c_k|p_k - q_k|, \forall k = 1,\ldots,m \tag{C.7}$$

$$|u(\omega_k,x_{k-1},p_{k-1}) - u(\omega_k,x_{k-1},q_{k-1})| \geq d_k|p_{k-1} - q_{k-1}|, \forall k = 1,\ldots,m \tag{C.8}$$

We note that by (C.4), since v is weakly increasing in price, if $\overline{q} = (q_x)_{x\in X}$ satisfies $\overline{q} \leq \overline{p}$,

$$\sum_{j=1}^m v(\omega_j,x_j,p_{x_j})\lambda_j < \frac{N}{N-1}\lambda_k(v(\omega_k,x_k,q_{x_k}) - v(\omega_k,x_{k-1},q_{x_{k-1}})),\ \forall k = 1,\ldots,m \tag{C.9}$$

and by (C.3),

$$argmax_{(x,q_x)} u(\omega_k,\cdot) \subseteq \{(x_k,q_{x_k}),(x_{k+1},q_{x_{k+1}})\}, \forall k = 1,\ldots,m \tag{C.10}$$

Let $\tilde{\Omega} := \{\omega_1,\ldots,\omega_m\} \times \{1,\ldots,N\}$ with distribution $\lambda(\{\omega_k,j\}) = \frac{1}{N}\lambda_k$, and let σ be the incentive compatible consumer profile defined by $\sigma(\omega_k, j) = (x_k, j)$. We need to show that $(\overline{p}, \sigma)$ is an equilibrium.

Let $\overline{q} = (q_x)_{x\in X} \neq \overline{p}$ be such a deviation of some firm k_0, and let τ be a reaction, which is *consistent* in the following sense: a *consistent reaction* to a deviation $\overline{q} = (q_x)_{x\in X}$ of a firm k is an incentive compatible consumer profile τ such that if $x \in X$ is such that $q_x \geq p_x$ and $\tau(\theta) = (j, x)$ for some $\theta \in \tilde{\Omega}$ and $(j, x) \in N \times X$,, and if $x \in argmax_X(x \to u(\theta, x, p_x))$, then $\sigma(\theta) = (j, x)$. I.e., if the contract x chosen now from some firm post deviation was also an optimal choice pre-deviation, and its price has not lowered, then it must also have been chosen before and from the same firm.

We show that $\pi^{k_0}(\tau, \overline{q}) \leq \pi^{k_0}(\sigma, \overline{p})$. Assume for at least one j, $q_j \leq p_j$; otherwise, $\pi^{k_0}(\tau, \overline{q}) = 0 \leq \pi^{k_0}(\sigma, \overline{p})$.

Lemma C.1. If for some $j = 1, \ldots, m$, $q_j < p_j$, then there is some $i = 1, \ldots, m$ such that for all $z \in \{1, \ldots, N\}$, $\tau(\omega_i, z) = (k_0, x_{i-1})$.

In other words, if the price of some contract is decreased in the deviation $\overline{q}$ of firm k_0 relative to $\overline{p}$, then for some i, consumers of type ω_i buy x_{i-1} from firm k_0.

Proof. Suppose for each $i = 1, \ldots, m$ it holds for some $z \in \{1, \ldots, N\}$, $\tau(\omega_i, z) \neq (k_0, x_{i-1})$.

First we claim that for all j, $q_j < p_j$. If not, for some j, $q_j < p_j$ but $q_{j+1} \geq p_{j+1}$, we would have

$$u(\omega_{j+1}, x_j, q_j) > u(\omega_{j+1}, x_j, p_j) = u(\omega_{j+1}, x_{j+1}, p_{j+1}) \geq u(\omega_{j+1}, x_{j+1}, q_{j+1})$$

which would imply $\tau(\omega_j, z) = (k_0, x_{j-1})$ by (C.10), a contradiction.

So we must have $q_j < p_j$ for all j. Hence, no one purchases under τ from anyone but k_0; since types ω_i are not purchasing x_{i-1}, it must hold that $u(\omega_i, x_{i-1}, q_{i-1}) \leq u(\omega, x_i, q_i)$. Hence,

$$\begin{aligned}
u(\omega_{j+1}, &x_{j+1}, q_{j+1}) \geq u(\omega_{j+1}, x_j, q_j) \text{ and } u(\omega_{j+1}, x_{j+1}, p_{j+1}) = u(\omega_{j+1}, x_j, p_j) \\
&\to u(\omega_{j+1}, x_{j+1}, q_{j+1}) - u(\omega_{j+1}, x_{j+1}, p_{j+1}) \geq u(\omega_{j+1}, x_j, q_j) - u(\omega_{j+1}, x_j, p_j) \\
&\to c_{j+1}(p_{j+1} - q_{j+1}) \geq d_{j+1}(p_j - q_j) \to \frac{p_j - q_j}{p_{j+1} - q_{j+1}} \geq \frac{c_j}{d_j}
\end{aligned}$$

Hence,

$$1 > \prod_{j=1}^{m} \frac{c_j}{d_j} \geq \prod_{j=1}^{m} \frac{p_j - q_j}{p_{j+1} - q_{j+1}} = 1 \tag{C.11}$$

a contradiction. □

Observe that under τ, by (C.10), since v is increasing in prices and p_j is the most any consumer would pay for x_j,

$$\pi^{k_0}(\tau,\overline{q}) \leq \sum_{j=1}^{m}\Big[\lambda\Big(\big\{(\omega,z)\mid \omega=\omega_j,\tau(\omega,z)=(k_0,x_{j-1})\big\}\Big)v(\omega_j,x_{j-1},p_{j-1}) \\ +\lambda\Big(\big\{(\omega,z)\mid \omega=\omega_j,\tau(\omega,z)=(k_0,x_j)\big\}\Big)v(\omega_j,x_j,p_j)\Big] \tag{C.12}$$

We consider two cases. First suppose $\forall j, q_j \geq p_j$. Since τ is a *consistent* reaction (as defined above) no one switches from another firm to firm k_0, and among any consumer of type ω_j - who were all purchasing x_j under σ - some may be buying x_j and the rest, by (C.10), will be buying x_{j-1}. By (C.4), $v(\omega_k,x_k,p_{x_k}) \geq v(\omega_k,x_{k-1},p_{x_{k-1}})$, so $\pi^{k_0}(\sigma,\overline{p}) \geq \pi^{k_0}(\tau,\overline{q},\overline{p}^{-k_0})$.

In the second case, if for some j, $q_j < p_j$, then by Lemma C.1 there exists some i_0 such that for all $z\in\{1,\ldots,N\}$, $\tau(\omega_{i_0},z)=(k_0,x_{i_0-1})$. In this case, by use of (C.4) and (C.9), (C.12) implies

$$\begin{aligned}\pi^{k_0}(\tau,\overline{q}) &\leq \lambda_{i_0}\cdot v(\omega_{i_0},x_{i_0-1},q_{i_0-1}) + \sum_{j\neq i_0}\lambda_j\cdot v(\omega_j,x_j,p_j)\\ &= \lambda_{i_0}\cdot\left(v(\omega_{i_0},x_{i_0-1},q_{i_0-1}) - v(\omega_{i_0},x_{i_0},q_{i_0})\right) + \sum_j \lambda_j\cdot v(\omega_j,x_j,p_j)\\ &\leq \frac{1}{N}\sum_j\lambda_j\cdot v(\omega_j,x_j,p_j) = \pi^{k_0}(\sigma,\overline{p})\end{aligned}$$

as required.

C.4 Proof of Proposition 1

Before proving Proposition 1, we prove a preliminary result, which may be of independent interest: Let E be reflexive and symmetric, and let G be a directed graph, s.t. for any $x,y\in X$, at most one of the following is true: either $(x,y)\in G \vee (y,x)\in G$, or $(x,y),(y,x)\in E$. (Since E is reflexive, this condition can be written $E\cap G=\emptyset$). We denote a cycle as a K-tuple of different $x_1,\ldots,x_K$ s.t. for each $j=1,\ldots,K$, either $(x_j,x_{j+1})\in G$ or $\in E$, where $x_{K+1}:=x_1$. We call a cycle strict if there is at least one j s.t. $(x_j,x_{j+1})\notin E$. For a weak ordering $\preceq$ on X, $x\prec y$ denotes $x\preceq y$ but not $y\preceq x$, while $x\sim y$ denotes $x\preceq y$ and $y\preceq x$.

Lemma 7. *There are no strict cycles for the pair G,E iff there exists a weak ordering $\preceq$ on X s.t. for every $(x,y)\in G\cup E$, it holds that $(x,y)\in G$ iff $x\prec y$ (equivalently, since $G\cap E$, $(x,y)\in E$ iff $x\sim y$).*

Proof. The converse is clear. Suppose there are no strict cycles for the pair G,E. Let E_0 denote the transitive closure of E, which is therefore an equivalence relation on X. Let X' denote the collection of equivalence classes of E_0, and define a relation G' on X': for classes

$C, D \in X'$,

$$(C, D) \in G' \leftrightarrow \exists x \in C, y \in D, (x, y) \in G$$

We claim that G' is indeed a directed graph (i.e. irreflexive, that is, no 'loops' of the form $C\, G'\, C$ for $C \in X'$) and has no cycles, that is, no $C_1, \ldots, C_K$ s.t. $C_1\, G'\, C_2 G' \cdots G' C_K\, G'\, C_1$. Since a loop is a particular case of the latter with $K = 1$, we simply show the non-existence of the latter. By definition of G', we can find $x_1, \ldots, x_K, y_1, \ldots, y_K$ s.t. $x_1\, E\, y_1\, G\, x_2\, E\, y_2\, G \ldots G\, x_K\, E\, y_K\, G\, x_1$, resulting in a strict cycle for the pair E_0, G, which is easily shown to result in a strict cycle for the pair E, G a contradiction. Hence, by standard results (e.g. Cormen et al. (2001, Sec. 22.4)), there is a complete strict linear ordering $\preceq'$ on X' s.t. $C \prec' D$ iff $(C, D) \in G'$. Define a weak ordering $\preceq$ on X by $x \sim y$ iff $(x, y) \in E'$ and $x \prec y$ iff $[x] \prec [y]$, where $[\cdot]$ denotes the E_0 equivalence class.

Now, fix $(x, y) \in G \cup E$. If $(x, y) \in G$, then we have $x \prec y$ by definition. Conversely, if $x \prec y$, but $(x, y) \notin G$, then $(x, y) \in E$, so there are $x_1, x_2, \ldots, x_M$ in the same E_0 class as x, and $y_1, \ldots, y_L$ in the same E_0 class as y,s.t. $x\, x_1\, E \cdots E\, x_M\, G\, y_L\, E \cdots E\, y_1\, Ey\, x$, contradicting that the pair G, E has no strict cycles. □

We now turn to the proof of Proposition 1; Define the directed graph G on X in the same way; for brevity, denote $A_\omega = argmax(u(\omega, \cdot, p))$:

$$G = \{(x, y) \mid \exists \omega \in \Omega, [x, y \in A_\omega \wedge v(\omega, x, p_x) < v(\omega, y, p_y)]\}$$

and also denote

$$E' = \{(x, y) \mid (\exists \omega \in \Omega, x, y \in A_\omega) \wedge (\forall \omega \in \Omega, x, y \in A_\omega \rightarrow v(\omega, x, p_x) = v(\omega, y, p_y))\}$$

and let E be the reflexive closure of E'. E is reflexive and symmetric, and G is irreflexive. Then, if $(x, y) \in G$ or $(y, x) \in G$, we do not have $(x, y) \in E$. Suppose there does not exists a K-tuple of different $x_1, \ldots, x_K$ s.t. which constitutes a strict cycle. By the lemma, there exists a weak ordering $\preceq$ on X s.t. for every $(x, y) \in G \cup E$, it holds that $(x, y) \in G$ iff $x \prec y$. Such a weak ordering, however, satisfies the conditions of Theorem 1, a contradiction to the assumption in Proposition 1 of positive profits. Hence, there does exist a strict cycle, which is exactly the required conclusion of the proposition.

C.5 Proof of Theorem 5

First, recall that, $||f|| = \sup |f|$ and $||f||_1 = \max[\sup |f|, \sup |f'|]$, the supremum being taken over the domain of f.

Primarily, we need to show that if the pair of utility and profit functions (u_0, v_0) is an economy satisfying the conditions of Lemma 6 for some prices $\overline{p}^0$ and some distribution of types λ_0, then there exists $\xi > 0$ such that if (u, v) is an economy (with u continuously differentiable) and

$$||u - u_0||_1 < \xi,\ ||v - v_0|| < \xi,\ ||\lambda - \lambda_0|| < \xi \tag{C.13}$$

then there are prices $\overline{p}$ for which (u, v) with λ satisfies the conditions of Lemma 6 as $\overline{p}^0$ does for (u_0, v_0).If ξ is small enough and $\overline{p}$ is close enough to $\overline{p}^0$, then (C.1), (C.3), (C.4), and (C.5) will be satisfied. The crux is to show that there exists $\overline{p}$ close enough to $\overline{p}^0$ and which satisfies (C.2).

Proposition C.2. Let $U_1, \ldots, U_m$ be open intervals, and $p_j^0 \in U_j$ for each j. Let $f_j : U_j \to \mathbb{R}$, $g_j : U_{j-1} \to \mathbb{R}$ be strictly increasing, and differentiable at p_j^0 with positive derivative, with (where $j-1$ is interpreted as m if $j = 1$, and $j+1$ is interpreted as 1 if $j = m$):

$$\prod_{j=1}^{m} \frac{f_j'(p_j^0)}{g_j'(p_{j-1}^0)} \neq 1 \tag{C.14}$$

Suppose $f_j(p_j^0) = g_j(p_{j-1}^0)$ for each $j = 1, \ldots, m$. Fix $\delta > 0$. Then there is $\varepsilon > 0$ s.t. if $\tilde{f}_j : U_j \to \mathbb{R}$, $\tilde{g}_j : U_{j-1} \to \mathbb{R}$ for $j = 1, \ldots, m$, with

$$|\tilde{f}_j(p) - f_j(p)| < \varepsilon \text{ and } |\tilde{g}_j(p') - g_j(p')| < \varepsilon \text{ for each } p \in U_j, p' \in U_{j-1} \tag{C.15}$$

then are $p_1 \in U_1, \ldots, p_m \in U_m$ with $|p_j - p_j^0| < \delta$ for each $j = 1, \ldots, m$ s.t. $f_j(p_j) = g_j(p_{j-1})$ for each $j = 1, \ldots, m$.

The proof of Theorem 5 follows from Proposition C.2, by taking $f^j, g^j : \mathbb{R}_{++} \to \mathbb{R}$, $j = 1, \ldots, m$, by

$$f_j(p) = u_0(\omega_j, x_j, p),\ g_j(p) = u_0(\omega_j, x_{j-1}, p)$$

since (C.2) for $\overline{p}^0 = (\overline{p}_j)$ can be written

$$f_j(p_j^0) = g_j(p_{j-1}^0),\ \forall j = 1, \ldots, m$$

and (C.14) holds by (C.5). To prove Proposition C.2:

Proposition C.3. Let $U \subseteq \mathbb{R}^m$ be open, $F : U \to \mathbb{R}^m$, $x_0 \in U$ s.t. $F(x_0) = 0$ and F is differentiable at x_0 with $DF(x_0)$ non-singular. Let $\delta > 0$. Then there is $\varepsilon > 0$ s.t. if $V \subseteq U$ and $G : V \to \mathbb{R}^m$ is continuous with $||G - F|_V|| < \varepsilon$, then there is $y_0 \in V$ with $||x_0 - y_0|| < \delta$ and $G(y_0) = 0$.

Remark 1. Proposition C.3 actually does not require that F be differentiable at x_0 with $DF(x_0)$ non-singular, rather only that for every open neighborhood U of x_0, $F(U)$ contains an open neighborhood of 0. The former follows from the latter by standard techniques (e.g., first part of proof of the Inverse Mapping Theorem in Rudin (1976)).

The proof of Proposition C.3 follows standard techniques, e.g. McLennan (2018); we include a sketch:

Proof. Fix $\delta > 0$. As per the remark, we can may assume δ is small enough s.t. if $B_\delta(x_0) \subseteq \mathbb{R}^N$ is the closed δ-ball at x_0 and $S_\delta(x_0)$ is the boundary of $B_\delta(x_0)$, $0 \notin F(S_\delta(x_0))$. Let $\varepsilon > 0$ be such that $B_{2\varepsilon}(0) \cap F(S_\delta) = \emptyset$. If $||F|_V - G|| < \varepsilon$, then denoting $H(\alpha, x) = (1-\alpha)F(x) + \alpha G(x)$, $0 \notin H(\alpha, S_\delta(x_0))$ for any α. A standard application of Brouwer's theorem (or the Jordan-Brouwer separation theorem) completes the proof. □

This implies Proposition C.2, as if we denote

$$F_j(p_1, \ldots, p_m) = f_j(p_j) - g_j(p_{j-1})$$

then the Jacobian of F at $p^0 = (p_1^0, \ldots, p_1^m)$ is

$$\begin{aligned}
\det(F(p^0)) &= \begin{vmatrix} f_1'(p_1^0) & 0 & 0 & \cdots & -g_1'(p_m^0) \\ -g_2'(p_1^0) & f_2'(p_2^0) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & -g_m'(p_{m-1}^0) & f_m'(p_m^0) \end{vmatrix} \\
&= (-1)^{m+1} \prod_{j=1}^m f_j'(p_j^0) + (-1)^{m+m} \prod_{j=1}^m (-g_j'(p_{j-1}^0)) \\
&= (-1)^{m+1} \Big(\prod_{j=1}^m f_j'(p_j^0) - \prod_{j=1}^m g_j'(p_{j-1}^0) \Big) \neq 0
\end{aligned}$$

where the penultimate equality can be seen be expansion by the first row, and the final $\neq 0$ is by (C.14).