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"Two-sided Strategic Information Transmission"

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Two-sided Strategic Information Transmission*

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Abstract

We study a cheap talk model in which a decision maker and an expert are both privately informed. Both players observe independent signals that jointly determine ideal actions for the players. Furthermore, in our model, the decision maker can send a cheap talk message to the expert, which is followed by the expert's cheap talk and then the decision maker's decision making. We show that the informed decision maker can informatively reveal her private information to the expert but her talk does not affect the quality of the expert's information transmission in models in which optimal actions are only additively or multiplicatively separable in the two players' information, and their preferences are represented by quadratic loss functions. We also apply our finding to a decision maker's information acquisition problem.

JEL Codes: D82, D83

Keywords: Cheap Talk, Two-Sided Asymmetric Information, Two-Way Communication

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1 Introduction

Decision-relevant information is often shared among multiple players whose interests do not always coincide. Hence, the organization may fail to aggregate this information in making decisions. This phenomenon has been investigated in numerous studies. We consider a unique situation in which a decision maker and an expert are privately informed, and they sequentially send cheap talk messages to each other before a decision is made. There is no monetary transfer and nether player makes commitment. To the best of our knowledge, this situation has not hitherto been analyzed in the literature.

The problem examined here arises when the expert does not fully reveal his information because his preference is not the same as that of the decision maker, while the decision maker benefits more if the expert is incentivized to transmit more accurate information. It is not clear under these circumstances how the extent of information revelation by the decision maker affects the expert's incentive. We investigate whether the decision maker benefits from revealing her private information to the expert.

To the best of our knowledge, this paper is the first to study informative communication from the decision maker.[1] Through our model of sequential cheap talk by two informed players, we show that the decision maker cannot improve the quality of the expert's information transmission by sending informative messages given the following two assumptions: optimal actions are only additively and/or multiplicatively separable in the two players' information, and their preferences can be represented by quadratic loss functions.

Consider the following example, which is an extension of a well-known example in Crawford and Sobel (1982) (hereinafter CS). There are two players, a decision maker and an expert. The decision maker chooses action y from real numbers. In this model, an ideal action for the decision maker depends on the state which depends on two signals. The signals are independent and are drawn from some continuous distribution. The decision maker privately observes one signal, denoted θ_D, and the

[1]Some literature such as Chen (2009) explored similar models but there were only babbling equilibra. We will discuss further details later.

expert privately observes the other signal, denoted θ_E. Each player's payoff is given by a quadratic loss function such that the more the action deviates from her/his ideal action, the more loss she/he incurs. The decision maker's ideal action is $y = \theta_D\theta_E$, while the expert's ideal action is $y = \theta_D\theta_E + b$, where $b > 0$. That is, their optimal actions are always different by b. Thus, parameter b quantifies conflicts of interest between the two players. Before the decision maker selects an action, the decision maker and the expert sequentially send cheap talk messages to each other. The decision maker talks first, and the expert talks next.[2]

As observed in models *à la* CS, the expert reveals information on θ_E in a partitional form. That is, the support of the signal is partitioned into finite intervals, and the expert reveals the interval to which his private information, θ_E, belongs. Given information revealed by the expert and the decision maker's own information, θ_D, the decision maker updates her belief and chooses an optimal action conditional on her updated belief. For the expert to reveal information in this way, the expert's information partition should satisfy incentive compatibility conditions. As b increases, the number of elements the expert's information partition can include decreases and hence coarser information on θ_E is transmitted from the expert to the decision maker. However, b is not the only factor that affects information transmission. The expert's inference concerning the decision maker's private information, θ_D, given the decision maker's message m^D, also matters in two ways. First, the expert's inference on θ_D determines the expert's direction of exaggeration. When the expert's inference on θ_D is positive (negative), he exaggerates his message upward (downward) because the decision maker chooses a large (small) action when the expert reports that θ_E is large. In addition, when the expert's inference on θ_D is 0, the expert does not have an incentive to curve the decision maker's decision because he is not sure which direction to mislead her decision. We refer to this as a *direction effect*. The second factor that influences information transmission is the decision maker's responsiveness to the expert's message. When $|\theta_D|$ is large, the expert's information is important for the decision maker's decision making and her decision is highly affected by the expert's message. When the expert's inference about $|\theta_D|$ given the decision maker's

[2]Trivially, models in which the expert talks first are equivalent to models in which the decision maker does not talk.

message m^D is large, the expert predicts that the decision maker responds well to his message and he can attain his self-interest with small noise. The expert chooses an informative message under these circumstances. This effect is called a *magnitude effect*.

Then, a combination of the two effects and parameter b provides a virtual conflict of interest, an *induced bias*, in our model and acts as parameter b in CS when the expert sends a message. As the conflict of interest measured by this term decreases, the information accuracy on θ_E that the expert can credibly reveal increases.

Consider when the decision maker is going to choose a cheap talk message. If a particular message serves to strictly decrease the conflict of interest term compared to other available messages, the decision maker always uses this message regardless of her private information. Hence, if information is transmitted from the decision maker to the expert, the decision maker's every message should lead to the same *induced bias* in an equilibrium. If this is the case, we have found that *induced biases* are the same in all equilibria that maximize the decision maker's ex-ante expected utility. In other words, the most informative information the expert can give the decision maker is independent of whatever information the decision maker gives the expert.[3]

This result contrasts sharply with the extant literature on informed decision makers. In the usual informed decision maker model, there are monetary transfers and a decision maker can make a commitment to her decision contingent on the message from an expert. Some studies have focused on the benefits to the decision maker of hiding her private information. Maskin and Tirole (1990) considered the implications of the decision maker's and expert's payoffs being private values. Cella (2008) explored the consequences of correlation between the decision maker's and expert's types. Unlike in the present study, they showed that the decision maker benefits from not revealing her private information in advance because of risk sharing between different the decision maker's type.

In addition, the present paper adds to the large body of literature on communication among/between multiple informed parties. In the literature, researchers have

[3]In a general setting, there are equilibria such that the expert has different *induced biases* for different messages. If we restrict our attention to equilibria that maximize the decision maker's utility, these cases can be precluded.

explored communication from multiple informed experts to an uninformed decision maker. Wolinsky (2002) and McGee and Yang (2013) approached this problem in the context of multiple experts observing non-overlapped information. Multiple experts observe the same signals in Battaglini (2002), Krishna and Morgan (2001), and Miura (2014).

Other researchers have focused on communication between an expert and a decision maker where both are privately informed. The most similar of such studies to the present study are those of Harris and Raviv (2005) and Chen (2009).[4] In Harris and Raviv's model, a decision maker and an expert observe independent information and the decision maker's ideal action is the sum of the decision maker's own and the expert's information. However, their approach differs from ours in that they discussed the optimal allocation of a decision right without considering communication from the decision maker.[5] In fact, our results reveal that the decision maker's talk has no effect in their setting. Chen studied communication from the decision maker. In her model, the decision maker privately learns a signal about a distribution of the expert's private type, and she showed that informative communication by the decision maker fails in the equilibrium due to the binary signal. [6] On the other hand, in our model, the decision maker observes her private signal drawn from sets with many elements.. Hence, our decision maker can successfully reveal her information to the expert.

Moreno de Barreda (2013), Lai (2014), and Ishida and Shimizu (2016, 2018) configured a model in which a decision maker and an expert observe different correlated signals. They showed a tradeoff between the quality of the decision maker's information and the expert's message, but did not consider any communication from the decision maker.

Kolotilin, Mylovanov, Zapechelnyuk and Li (2017) showed equivalence between a persuasion mechanism that conditions information disclosure on the decision maker's report about her type and an experiment that discloses information independent of

[4]McGee (2013) investigated his model with an informed decision maker but restricted his attention to a babbling equilibrium.

[5]Aghion and Tirole (1997) also discussed allocation of a decision right when the information acquisition of a decision maker and an expert matters, rather than information transmission.

[6]Chen and Gordon (2015) also considered a setup of Chen (2009) and showed that the decision maker can fully reveal her information if her signal is verifiable (that is, the decision maker discloses her private signal or hides it).

the decision maker's type.

The remainder of this article is organized as follows. Section 2 introduces the model. Section 3 presents the main results. Section 4 discusses examples. Section 5 extends our basic model and evaluates the value of the decision maker's information acquisition. Finally, Section 6 presents our conclusions.

2 Model

There are two players, a decision maker (or D) and an expert (or E). The decision maker decides an action y from $\mathbb{R}$. There are two-dimensional signals $(\theta_D, \theta_E) \in \Theta_D \times \Theta_E = \left[\underline{\theta}_D, \bar{\theta}_D\right] \times \left[\underline{\theta}_E, \bar{\theta}_E\right] \subset \mathbb{R}^2$, which are independent. The signals are drawn according to cumulative distributions Φ^D and Φ^E. The decision maker privately and perfectly learns θ_D and the expert does the same for θ_E.

The payoff for each player is given by a quadratic loss function. The payoff functions of the decision maker and the expert are as follows:

$$
\begin{aligned}
U^D &= -\left\{f\left(\theta_D, \theta_E\right) - y\right\}^2, \\
U^E &= -\left\{f\left(\theta_D, \theta_E\right) + b - y\right\}^2,
\end{aligned}
$$

where $b \in \mathbb{R}$, and $f(\cdot, \cdot)$ is a bivariate function such that

$$
f\left(\theta_D, \theta_E\right) = g\left(\theta_D\right) h\left(\theta_E\right) + s\left(\theta_D\right)
$$

where $g(\cdot)$ and $s(\cdot)$ are continuous almost everywhere in θ_D and $h(\cdot)$ is continuous and strictly increasing in θ_E. Examples, which will be examined later, include $f(\theta_D, \theta_E) = \theta_D \theta_E$ and $f(\theta_D, \theta_E) = \theta_D + \theta_E$.

After observing (θ_D, θ_E), the decision maker sends a cheap talk message $m^D \in \Theta_D$ to the expert, and the expert sends a cheap talk message $m^E \in \Theta_E$ to the decision maker. Each player's message reaches the other player without any noise.

All aspects of the game except for (θ_D, θ_E) are common knowledge.

The timeline is as follows.

1. Nature chooses the two-dimensional signals θ_D and θ_E. The decision maker observes θ_D and the expert observes θ_E.

2. The decision maker sends a cheap talk message m^D to the expert.

3. The expert observes this message and then sends a cheap talk message m^E to the decision maker.

4. The decision maker selects an action y, and both players' payoffs are realized.

The solution concept is a Perfect Bayesian equilibrium.[7] In common with other cheap talk models, we face the issue of multiple equilibria. We focus on *the ex-ante optimal equilibrium*, the Perfect Bayesian equilibrium in which the decision maker's ex-ante expected utility is the highest.

Note that *the ex-ante optimal equilibrium* is ex-ante Pareto optimal when the decision maker chooses her ex-post optimal action.[8]

In our model, the decision maker does not have commitment power and chooses her ex-post optimal action conditional on the triplet $\left(\theta_D, m^D, m^E\right)$. Her problem of choosing her optimal action is

$$\underset{y\in\mathbb{R}}{max} E_{\theta_E}\left[-\left\{f\left(\theta_D,\theta_E\right)-y\right\}^2 \mid \theta_D, m^D, m^E\right],$$

where $E_r\left[\cdot\right]$ is expectation with a random variable r. The decision maker's optimal action is denoted by

$$y\left(\theta_D, m^D, m^E\right) \equiv E_{\theta_E}\left[f\left(\theta_D,\theta_E\right) \mid \theta_D, m^D, m^E\right].$$

Before proceeding to a general analysis, we consider the examples in the next section in order to clarify our problems.

[7]This includes mixed strategies.
[8]See Appendix for the proof.

3 Examples

3.1 Linear function

Let $f(\theta_D, \theta_E) = \theta_D + \theta_E$. The decision maker chooses

$$y\left(\theta_D, m^D, m^E\right) = E_{\theta_E}\left[\theta_E \mid m^D, m^E\right] + \theta_D,$$

and the expert's interim expected utility is

$$\begin{aligned} E_{\theta_D}\left[U^D \mid m^D\right] &= E_{\theta_D}\left[-\left(\theta_D + \theta_E + b - E_{\theta_E}\left[\theta_E | m^D, m^E\right] - \theta_D\right)^2 \mid m^D\right] \\ &= -\left(\theta_E + b - E_{\theta_E}\left[\theta_E \mid m^D, m^E\right]\right)^2. \end{aligned}$$

Since the expert's expected utility is independent of θ_D, his decision could be affected by the decision maker's message only through the decision maker's action, which is her inference about θ_E from $\left(m^D, m^E\right)$. However, m^D does not affect the expert's sequential rationality nor his strategy.

3.2 Cobb-Douglas function

Let $f(\theta_D, \theta_E) = \theta_D\theta_E$. Given a message m^E, the decision maker chooses

$$y\left(\theta_D, m^D, m^E\right) = \theta_D E_{\theta_E}\left[\theta_E \mid m^D, m^E\right].$$

Suppose that the expert observing θ_E is indifferent between sending messages $\hat{m}^E$ and $\tilde{m}^E$, then

$$\begin{aligned} &E_{\theta_D}\left[-\left(\theta_D\theta_E + b - \theta_D E_{\theta_E}\left[\theta_E \mid m^D, \hat{m}^E\right]\right)^2 \mid m^D\right] \\ &\qquad = E_{\theta_D}\left[-\left(\theta_D\theta_E + b - \theta_D E_{\theta_E}\left[\theta_E \mid m^D, \tilde{m}^E\right]\right)^2 \mid m^D\right], \end{aligned}$$

which implies

$$E_{\theta_E}\left[\theta_E \mid m^D, \hat{m}^E\right] + E_{\theta_E}\left[\theta_E \mid m^D, \tilde{m}^E\right] - 2\theta_E = 2b\frac{E_{\theta_D}\left[\theta_D \mid m^D\right]}{E_{\theta_D}\left[\theta_D^2 \mid m^D\right]}.$$

Again, m^D does not affect sequential rationality on the left-hand side. On the other hand, the right-hand side is affected by the decision maker's message through the last term, $b\frac{E_{\theta_D}\left[\theta_D|m^D\right]}{E_{\theta_D}\left[\theta_D^2|m^D\right]}$.

Further, suppose that each of θ_D and θ_E is drawn according to a uniform distribution with support $[0,1]$ and $b = \frac{1}{12}$. There are multiple equilibria as follows.

Babbling equilibrium The decision maker's strategy is independent of her signal. The expert uniquely partitions his signal space into $\left[0, \frac{1}{4}\right)$ and $\left[\frac{1}{4}, 0\right]$ when his message is informative.

Informative equilibrium The decision maker sends two different messages. She sends $\hat{m}^D$ when $\theta_D \in \left[0, \frac{3}{7}\right) \cup \left(\frac{6}{7}, 1\right]$ and $\tilde{m}^D$ when $\theta_D \in \left[\frac{3}{7}, \frac{6}{7}\right]$. Given each message, the expert uniquely partitions his signal space into $\left[0, \frac{1}{4}\right)$ and $\left[\frac{1}{4}, 0\right]$ when his message is informative.

In these examples, it turns out that the expert has the same information partitionin both equilibria. Besides these equilibria, there are other equilibria such as mixed equilibria. It is not known for us if there exists any equilibrium which the decision maker prefers to the babbling equilibrium.

4 General Results

In this section, we outline and explore our main results. First, we characterize the expert's equilibrium strategy.

Proposition 1. *The expert's message strategy is characterized by interval equilibria such that the expert partitions his information set into finite intervals and only reveals the interval to which his information θ_E belongs. Moreover, the expert behaves as if dissonance of the ideal action between the expert and decision maker is* $B\left(m^D\right) \equiv b\frac{E_{\theta_D}\left[g(\theta_D)|m^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m^D\right]}$.

Proof. See the Appendix. □

As in the CS model, the expert reveals information on θ_E in a partitional form. The partition depends on not only the expert's bias, b, but also the decision maker's message and the expert's inference on θ_D. Let $x = (x_0, x_1, \cdots, x_I)$ define partitions characterizing the expert's message strategy, where I is a positive integer. We let the expert's message be denoted by $m^E = m_i^E$ if $\theta_E \in (x_{i-1}, x_i)$ for $i \in \{1, \ldots, I\}$.

The expert's equilibrium partition can be obtained as follows. The decision maker's optimal action given θ_D and that the expert sends message $m^E = m_i^E$ is

$$y\left(\theta_D, m^D, m_i^E\right) = g\left(\theta_D\right) E_{\theta_E}\left[h\left(\theta_E\right) \mid m^D, m_i^E\right] + s\left(\theta_D\right), \tag{1}$$

where $E_{\theta_E}\left[h\left(\theta_E\right) | m^D, m_i^E\right]$ is the decision maker's inference on $h\left(\theta_E\right)$. Given the decision maker's action, the arbitrage condition for the expert observed $\theta_E = x_i$ is

$$\begin{aligned} &E_{\theta_D}\left[-\left\{f\left(\theta_D, x_i\right) + b - y\left(\theta_D, m^D, m_i^E\right)\right\}^2 \mid m^D\right] \\ &\qquad = E_{\theta_D}\left[-\left\{f\left(\theta_D, x_i\right) + b - y\left(\theta_D, m^D, m_{i+1}^E\right)\right\}^2 \mid m^D\right], \end{aligned}$$

which can be rewritten as

$$E_{\theta_E}\left[h\left(\theta_E\right) \mid m^D, m_{i+1}^E\right] + E_{\theta_E}\left[h\left(\theta_E\right) \mid m^D, m_i^E\right] - 2h\left(x_i\right) = 2b\frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m^D\right]}. \tag{2}$$

This equation clarifies how the decision maker's message affects the expert's behavior. The left-hand side of the equation is a function of the partitions, which are not affected by m^D under sequential rationality. These partitions are conditional on the right hand side, where we refer to $B\left(m^D\right) \equiv b\frac{E_{\theta_D}\left[g(\theta_D)|m^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m^D\right]}$ as an *induced bias*.

The numerator explains the direction in which the expert has an incentive to mislead the decision maker, which we refer to as a *direction effect*. Assume $b > 0$. When $g\left(\theta_D\right)$ is positive, the decision maker takes large action for large $h\left(\theta_E\right)$, and the expert wants to mislead the decision maker's belief on $h\left(\theta_E\right)$ upward. Since θ_D is the decision maker's private information, this effect is through the expert's expectation on $g\left(\theta_D\right)$ given the decision maker's message, that is, when $E_{\theta_D}\left[g\left(\theta_D\right) \mid m^D\right]$ is positive (negative), the expert exaggerates so that the decision maker believes $h(\theta_E)$ is large (small), and the expert has an incentive to mislead her belief upward (downward). When $E_{\theta_D}\left[g(\theta_D) \mid m^D\right]$ is close to 0, the expert is uncertain whether $g(\theta_D)$ is positive or negative and is uncertain as to which direction he should mislead the decision maker. The expert has a small incentive to mislead her.

The denominator, $\frac{1}{E_{\theta_D}\left[g(\theta_D)^2 | m^D\right]}$, is referred to as a *magnitude effect*, which indicates how important the expert's information is for the decision making. When $|g(\theta_D)|$ is large, the expert's information, $h(\theta_E)$, is significantly important to the decision maker. The decision maker's action is sensitive to the expert's message, and the expert can attain his self-interest by adding small noise to his message. The decision maker's message thus affects the expert through this *induced bias*.

Given $h(\theta_E)$ is strictly increasing, partitions that satisfy (2) are $\underline{\theta}_E = x_0 < x_1 < \cdots < x_I = \bar{\theta}_E$ when $B \geq 0$ and $\underline{\theta}_E = x_I < \cdots < x_1 < x_0 = \bar{\theta}_E$ when $B < 0$. Analogous to the condition in CS, we define our monotonicity condition as follows.

Definition 1. Given each m^D, the M condition is satisfied if for any two solutions to (2), $\hat{x}$ and $\tilde{x}$ with $\hat{x}_0 = \tilde{x}_0$ and $\hat{x}_1 > \tilde{x}_1$, then $\hat{x}_i > \tilde{x}_i$ for all $i \geq 2$.

For expository purposes, we now make the following assumption.

Assumption 1. *The M condition holds for the expert's information partition.*

This condition is satisfied, for example, when $f(\theta_D, \theta_E) = \theta_D \theta_E$ and θ_D and θ_E are uniformly distributed. When the M condition is satisfied, the decision maker is better off if the expert sends a message according to a partition with more intervals. This is because the expert's utility is represented by a quadratic loss function. Hence, in *the ex-ante optimal equilibrium*, the expert's information partition must include the maximum number of intervals among the perfect Bayesian equilibria for some *induced bias*.[9]

Now, we consider the decision maker's message. When the M condition holds, the decision maker's interim expected utility can be denoted by a function of an *induced bias*,

$$EU^D\left(B\left(m^D\right) \mid \theta_D\right).$$

[9] This expert's action is supported by the NITS condition (Chen *et al.*, 2008).

Note that this function is unimodal with a peak at zero *induced bias*.[10] The decision maker sends m^D to maximize her expected utility. We show that the decision maker can send informative messages in Perfect Bayesian equilibria. Unlike the expert's strategy, the decision maker's equilibrium strategy is not a simple partition of Θ_D. Suppose that the decision maker sends J messages, $\left\{m_1^D, m_2^D, \cdots, m_J^D\right\}$, on the equilibrium path where J is any positive integer. Among these messages, if two messages m_j^D and m_k^D result in different *induced biases* and $EU^D\left(B\left(m_j^D\right) \mid \theta_D\right) > EU^D\left(B\left(m_k^D\right) \mid \theta_D\right)$ for some $j, k \in \{1, 2, \ldots, J\}$, the decision maker does not send message m_k^D. Therefore, the decision maker should be indifferent for all messages on this equilibrium:

$$EU^D\left(B\left(m_j^D\right) \mid \theta_D\right) = EU^D\left(B\left(m_k^D\right) \mid \theta_D\right) \qquad (\forall j, k \in \{1, 2, \ldots, J\}) \quad (3)$$

From this condition, we can further induce the following result for the *induced bias* in *the ex-ante optimal equilibrium.*

Proposition 2. *The decision maker's message strategy is the ex-ante optimal equilibrium only if it induces*

$$\frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_1^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_1^D\right]} = \frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_j^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_j^D\right]} \quad (\forall j \in \{1, 2, \ldots, J\}).$$

Proof. See the Appendix. □

When $\frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} \frac{E_{\theta_D}\left[g(\theta_D) \mid m_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_k^D\right]} \geq 0$, the above condition is immediate from condition (3). When $\frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} \frac{E_{\theta_D}\left[g(\theta_D) \mid m_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_k^D\right]} < 0$ for some $j, k \in \{1, 2, \ldots, J\}$, then the decision maker is better off hiding her private information rather than sending an informative message and this message strategy cannot be *the ex-ante optimal equilibrium.*

Now we compare two different Perfect Bayesian equilibria that satisfy Proposition 1. Let $\hat{m}^D = \left\{\hat{m}_1^D, \cdots, \hat{m}_{\hat{J}}^D\right\}$ be the decision maker's messages in one equilib-

[10] Apply CS's Theorems 3-4 for the *induced bias* B instead of b.

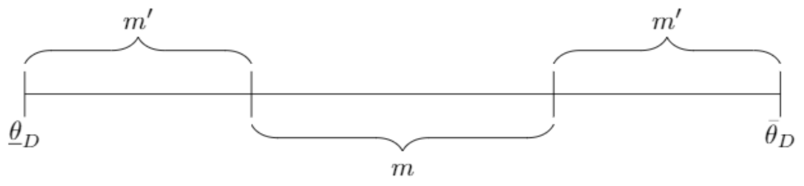

Figure 1: Example: *the ex-ante optimal equilibrium* such as $\frac{E_{\theta_D}[g(\theta_D)|m]}{E_{\theta_D}\left[g(\theta_D)^2|m\right]}\frac{E_{\theta_D}[g(\theta_D)|m']}{E_{\theta_D}\left[g(\theta_D)^2|m'\right]} > 0$.

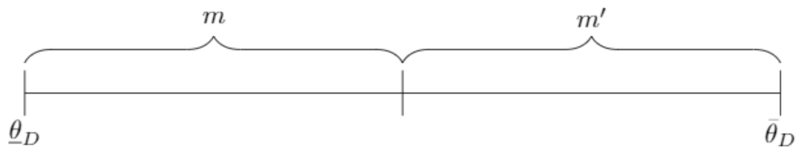

Figure 2: Example: a Perfect Bayesian equilibrium such as $\frac{E_{\theta_D}[g(\theta_D)|m]}{E_{\theta_D}\left[g(\theta_D)^2|m\right]}\frac{E_{\theta_D}[g(\theta_D)|m']}{E_{\theta_D}\left[g(\theta_D)^2|m'\right]} < 0$.

rium and $\tilde{m}^D = \left\{\tilde{m}_1^D, \cdots, \tilde{m}_{\tilde{J}}^D\right\}$ be the messages in another equilibrium. We have $\frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_j^D\right]}$ $\left(\forall j \in \left\{1, 2, \ldots, \hat{J}\right\}\right)$ and $\frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_k^D\right]}$ $\left(\forall k \in \left\{1, 2, \ldots, \tilde{J}\right\}\right)$. However we do not know whether $\frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_1^D\right]}$ and $\frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_1^D\right]}$ are different or not. If they are different, then the decision maker may be better off by choosing the message strategy that induces an *induced bias* which she prefers. Hence, the remaining question is whether *induced biases* can be varied by the decision maker's message strategies.

Lemma 1. *Consider the decision maker's strategy in which he sends* $\left\{m_1^D, \cdots, m_J^D\right\}$. *If*

$$\frac{E_{\theta_D}\left[g(\theta_D) \mid m_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} \quad (\forall j \in \{1, 2, \ldots, J\})$$

then the following relation holds;

$$\frac{E_{\theta_D}\left[g(\theta_D)\right]}{E_{\theta_D}\left[g(\theta_D)^2\right]} = \frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} \quad (\forall j \in \{1, 2, \ldots, J\}).$$

Proof. See the Appendix. □

The lemma reveals that for the decision maker's two different strategies, if $\frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_j^D\right]}$ and $\frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_k^D\right]}$, then $\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\hat{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\hat{m}_1^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|\tilde{m}_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|\tilde{m}_1^D\right]}$. In other words, although the expert's belief on the decision maker's type depends on the decision maker's messages and there are multiple equilibria, including an equilibrium in which no information is revealed by the decision maker, the expert's *induced biases* are the same in all equilibria. This result has an important implication for model analysis. Our main results are presented here.

Proposition 3. *The induced biases are the same for all the ex-ante optimal equilibria. Hence, the expert's information partition is independent of the decision maker's messages in the ex-ante optimal equilibria.*

The next corollary immediately follows from Proposition 3.

Corollary 1. *The upper bound of the decision maker's ex-ante expected payoff can be calculated assuming that the decision maker sends babbling messages.*

Hence, when we need to solve for the upper bound of the decision maker's expected utility, we can conduct our analysis by assuming that the decision maker cannot send any message to the expert.

5 Value of the Decision Maker's Information

This section applies the above results and explores the value of the decision maker's information acquisition. The question is whether the decision maker's information acquisition of θ_D can facilitate communication from the expert. We compare the cases in which the decision maker learns and does not learn realization of θ_D. Whether the decision maker knows realization of θ_D is common knowledge.

First, when the decision maker learns θ_D, she can send some messages to the expert. When we restrict our attention to the outcome that maximizes the decision maker's ex-ante expected utility, our result in the previous section indicates that we can restrict our attention to the case in which the decision maker commits to not sending any message. In this case, the expert behaves as if his bias is $b\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]}$.

Next, when the decision maker does not acquire any information about θ_D, there is no communication from the decision maker. The decision maker's optimal action given m^E is

$$y\left(m^E\right) \equiv E_{\theta_D}\left[g\left(\theta_D\right)\right] E_{\theta_E}\left[h\left(\theta_E\right) \mid m^E\right] + E_{\theta_D}\left[s\left(\theta_D\right)\right],$$

and the expert's arbitrage condition sending different messages m_{i-1}^E and m_i^E when he observes x_i is

$$E_{\theta_D}\left[-\left\{f\left(\theta_D, x_i\right) + b - y\left(m_{i-1}^E\right)\right\}^2\right] = E_{\theta_D}\left[-\left\{f\left(\theta_D, x_i\right) + b - y\left(m_i^E\right)\right\}^2\right].$$

This condition can be rewritten as

$$E_{\theta_E}\left[h\left(\theta_E\right) \mid m_{i-1}^E\right] + E_{\theta_E}\left[h\left(\theta_E\right) \mid m_i^E\right] - 2h\left(x_i\right) = 2b\frac{1}{E_{\theta_D}\left[g\left(\theta_D\right)\right]},$$

and the expert behaves as if his bias is $b\frac{1}{E_{\theta_D}[g(\theta_D)]}$.

From Jensen's inequality,

$$E_{\theta_D}\left[g\left(\theta_D\right)^2\right] \geq E_{\theta_D}\left[g\left(\theta_D\right)\right]^2 \quad \Leftrightarrow \quad \frac{1}{E_{\theta_D}\left[g\left(\theta_D\right)\right]} \geq \frac{E_{\theta_D}\left[g\left(\theta_D\right)\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2\right]},$$

holds if θ_D is degenerate. The expert's *induced bias* is smaller when the decision maker learns realization of θ_D, as compared to the case in which the decision maker does not learn realization of θ_D.

This shows that the decision maker's information acquisition on θ_D facilitates communication from the expert. Moreover, the decision maker can take an action close to her own ideal action when she knows θ_D. Thus the decision maker benefits from her information acquisition.

Proposition 4. *The upper bound of the decision maker's expected utility increases if the decision maker acquires information.*

In contrast with this result, Moreno de Barreda (2013), Lai (2014), and Ishida and

Shimizu (2016) found that the decision maker's private information harms communication from the expert in environments in which the decision maker observes a noisy signal of the expert's private information. According to Moreno de Barreda, their results are due to a tradeoff between the *information effect* and the *risk effect.* When the decision maker has accurate information, her action is not sensitive to the expert's message because its importance is small. The expert needs to exaggerate his message more to achieve his self-interest. Hence, the *information effect* impedes information transmission from the expert. At the same time, when the decision maker has her private information, the decision maker's action is now a lottery for the expert. The expert's incentive to exaggerate is weakened by a lottery over actions. Hence, the decision maker's private information facilitates information transmission through the *risk effect.* Since the *information effect* overwhelms the *risk effect,* communication from the expert is accurate when the decision maker has private information.

In our model, the *information effect* is null because the decision maker and the expert observe independent signals, whereas the *risk effect* persists. When the decision maker does not observe θ_D, the *induced bias* is $b\frac{1}{E_{\theta_D}[g(\theta_D)]}$ because the *direction effect* is $E_{\theta_D}\left[g\left(\theta_D\right)\right]$ and the *magnitude effect* is $\frac{1}{E_{\theta_D}[g(\theta_D)]^2}$. This *magnitude effect* is smaller than that when the decision maker observes θ_D, $\frac{1}{E_{\theta_D}\left[g(\theta_D)^2\right]}$, due to the *risk effect.* In summary, the decision maker's information acquisition facilitates communication from the expert.[11]

6 When Degree of the Loss Function is Higher

In this section, we extend our analysis to the case in which the expert's utility is

$$U^E = -\left\{f\left(\theta_D, \theta_E\right) + b - y\right\}^{2n},$$

[11] Ishida and Shimizu (2018) found another effect whereby the decision maker's information acquisition facilitates communication from the expert. When the decision maker and the expert observe correlated signals, the expert's message provides information not only about the true state but also about the reliability of the decision maker's private information.

where $n \in \mathbb{N}$. Given the decision maker's reaction (1), the expert's interim expected utility can be rewritten as

$$
\begin{aligned}
&- E_{\theta_D}\left[g\left(\theta_D\right)^{2n} \mid m^D\right] \\
&\sum_{k=0}^{2n}(-1)^{\alpha}\begin{pmatrix} 2n \\ \alpha \end{pmatrix} b^{2n-\alpha}\frac{E_{\theta_D}\left[g\left(\theta_D\right)^{\alpha} \mid m^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^{2n} \mid m^D\right]}\left[E_{\theta_E}\left[h\left(\theta_E\right) \mid m^D, m^E\right] - h\left(\theta_E\right)\right]^{\alpha}.
\end{aligned}
$$

Hence, a series of $\left\{b^{2n-\alpha}\frac{E_{\theta_D}\left[g(\theta_D)^{\alpha}|m^D\right]}{E_{\theta_D}\left[g(\theta_D)^{2n}|m^D\right]}\right\}_{\alpha=\{0,\ldots,2n\}}$ is an *induced bias* of this expert. If $b^{2n-\alpha}\frac{E_{\theta_D}\left[g(\theta_D)^{\alpha}|m_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^{2n}|m_1^D\right]} = b^{2n-\alpha}\frac{E_{\theta_D}\left[g(\theta_D)^{\alpha}|m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^{2n}|m_j^D\right]}$ for all $\alpha \in \{0, \ldots, 2n\}$ and $j \in \{1, 2, \ldots, J\}$, we can show that $b^{2n-\alpha}\frac{E_{\theta_D}[g(\theta_D)^{\alpha}]}{E_{\theta_D}\left[g(\theta_D)^{2n}\right]} = b^{2n-\alpha}\frac{E_{\theta_D}\left[g(\theta_D)^{\alpha}|m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^{2n}|m_j^D\right]}$ for all α in the same way as Lemma 2. However, we know little about a condition corresponding to the M condition when $n \geq 2$, and it is an open question whether the independence result shown in Proposition 2 holds for this case.

7 Conclusion

The present paper studies how the decision maker's message affects communication from the expert when both the decision maker and the expert are privately informed. When ideal actions are multiplicatively separable in the two players' information, the decision maker's message affects the expert's behavior through his belief on the decision maker's type and the importance of his information to the decision maker's decision. Although the decision maker's message has an effect on the expert's belief, the decision maker can only configure and send messages that give the expert the same *induced bias* in equilibria. Furthermore, the decision maker cannot improve the upper bound of her own ex-ante expected utility by managing her message. In addition, when ideal actions are additively separable in the two players' information, the decision maker's message has no effect at all on the expert's belief. In summary, the decision maker's message does not affect the upper bound of her expected utility in our model.

This finding contributes to knowledge in organizational economics. A number of studies have considered one-sided (the expert's) private information. However, it was hitherto not clear whether these results still hold when the decision maker has private information. Our results suggest that even when the decision maker can send a message to the expert, the optimal outcome for the decision maker can be obtained by assuming that the decision maker cannot send a message to the expert.

Discussion In our game, multiple equilibria exist and some are less informative than others. We focus on *the ex-ante optimal equilibrium*. Although the expert's equilibrium strategy profile can be justified with the NITS condition (Chen *et al.*, 2008), an equivalent condition for justifying the decision maker's equilibrium strategy is unknown.

Another unclarified issue is whether the current results hold for more general settings, namely for the case in which $f(\theta_D, \theta_E)$ is neither additively nor multiplicatively separable, two stochastic variables are correlated, and the utility function is not simply quadratic.

These issues are left for future research.

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A Proofs

A.1 Proof of Pareto Optimality of *the Ex-ante Optimal Equilibrium*

Proof. When the decision maker chooses her ex-post optimal action, $y\left(\theta_D, m^E\right) \equiv E_{\theta_E}\left[f\left(\theta_D, \theta_E\right) \mid m^E\right]$, the decision maker's ex-ante expected payoff is

$$EU^D = -E\left[f\left(\theta_D, \theta_E\right) - y\left(\theta_D, m^E\right)\right]^2,$$

where $E\left[\cdot\right]$ is the expectation with both θ_D and θ_E. The expert's ex-ante expected payoff is

$$\begin{aligned} EU^E &= -E\left[f\left(\theta_D, \theta_E\right) + b - y\left(\theta_D, m^E\right)\right]^2 \\ &= -E\left[f\left(\theta_D, \theta_E\right) - y\left(\theta_D, m^E\right)\right]^2 + 2bE\left[f\left(\theta_D, \theta_E\right) - y\left(\theta_D, m^E\right)\right] - b^2. \end{aligned}$$

Since $E\left[f\left(\theta_D, \theta_E\right)\right] = E\left[y\left(\theta_D, m^E\right)\right]$, we have

$$EU^D = EU^E + b^2.$$

When a perfect Bayesian equilibrium maximizes the decision maker's ex-ante expected utility, that of the expert is also maximized. □

A.2 Proof of Proposition 1

Proof. Through the proof, rather than specifying the decision maker's talk strategy, we suppose that the decision maker already sent some message m^D according to some strategy, and analyze the decision maker's action strategy and the expert's talk strategy both of which are sequentially rational. First, the decision maker's optimal action (i.e., her equilibrium action strategy) is given by (1) in Section 4.

Next, consider the expert's interim expected utility right after the decision maker sends a message,

$$E_{\theta_D}\left[-\left\{f\left(\theta_D,\theta_E\right)+b-y\right\}^2 \mid m^D,m^E\right]$$

which is strictly concave with respect to action y. Let $\bar{h}$ denote the decision maker's belief:

$$\bar{h}\equiv E_{\theta_E}\left[h\left(\theta_E\right)\mid m^D,m^E\right].$$

Then, the expert's interim expected utility is described as:

$$E_{\theta_D}\left[-\left\{f\left(\theta_D,\theta_E\right)+b-\left(g\left(\theta_D\right)\bar{h}+s\left(\theta_D\right)\right)\right\}^2\mid m^D\right]$$

$$=-E_{\theta_D}\left[g\left(\theta_D\right)^2\mid m^D\right]\left(h\left(\theta_E\right)-\bar{h}\right)^2-2bE_{\theta_D}\left[g\left(\theta_D\right)\mid m^D\right]\left(h\left(\theta_E\right)-\bar{h}\right)-b^2$$

which is strictly concave with respect to the decision maker's belief $\bar{h}$.

For the expert, the ideal decision maker's belief $\bar{h}$ is

$$h\left(\theta_E\right)+B\left(m^D\right),$$

where $B\left(m^D\right)\equiv b\frac{E_{\theta_D}\left[g(\theta_D)\mid m^D\right]}{E_{\theta_D}\left[g(\theta_D)^2\mid m^D\right]}$, while the decision maker wants to know the truth $h\left(\theta_E\right)$. That is, at the interim stage, the ideal decision maker's belief is strictly increasing in θ_E for each player since $h\left(\cdot\right)$ is strictly increasing. Moreover, the ideal beliefs always differ by the induced bias $B\left(m^D\right)$, which is constant given m^D .

Therefore, our induced bias $B\left(m^D\right)$ (difference in the ideal interim belief on $h\left(\theta_E\right)$ between the players) is comparable to bias b in CS's model. We can make claims like CS's Lemma 1 and Theorem 1 and show that the expert's equilibrium talk strategies are characterized by monotone partitions. □

A.3 Proof of Proposition 2

Proof. In the following proofs, we use the following notation:

$$G_j \equiv \int_{\theta_D \in \Theta_D} g(\theta_D)\, d\Phi^D \left(\theta_D \mid m_j^D\right),$$

and

$$L_j \equiv \int_{\theta_D \in \Theta_D} g(\theta_D)^2\, d\Phi^D \left(\theta_D \mid m_j^D\right).$$

Note that,

$$\frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} = \frac{\int_{\theta_D \in \Theta_D} g(\theta_D)\, d\Phi^D \left(\theta_D \mid m_j^D\right) / \int_{\theta_D \in \Theta_D} d\Phi^D \left(\theta_D \mid m_j^D\right)}{\int_{\theta_D \in \Theta_D} g(\theta_D)^2\, d\Phi^D \left(\theta_D \mid m_j^D\right) / \int_{\theta_D \in \Theta_D} d\Phi^D \left(\theta_D \mid m_j^D\right)} = \frac{G_j}{L_j}.$$

Furthermore,

$$E_{\theta_D}\left[g(\theta_D)\right] = \sum_{j=1}^{J} \int_{\theta_D \in \Theta_D} g(\theta_D)\, d\Phi^D \left(\theta_D \mid m_j^D\right) = \sum_{i=1}^{J} G_j,$$

and

$$E_{\theta_D}\left[g(\theta_D)^2\right] = \sum_{j=1}^{J} \int_{\theta_D \in \Theta_D} g(\theta_D)^2\, d\Phi^D \left(\theta_D \mid m_j^D\right) = \sum_{j=1}^{J} L_j.$$

First, suppose $G_j G_k \geq 0$ for all $j, k \in \{1, 2, \ldots, J\}$. Then $\frac{G_j}{L_j} = \frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D) \mid m_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_k^D\right]} = \frac{G_k}{L_k}$ when $EU^D\left(b\frac{E_{\theta_D}\left[g(\theta_D) \mid m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_j^D\right]} \,\middle|\, \theta_D\right) = EU^D\left(b\frac{E_{\theta_D}\left[g(\theta_D) \mid m_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_k^D\right]} \,\middle|\, \theta_D\right)$.

Now, suppose $G_j G_k < 0$ for some $j, k \in \{1, 2, \ldots, J\}$ in *the ex-ante optimal equilibrium*. Define sets of integers $J^+ \equiv \{j \mid G_j \geq 0, j \in \{1, 2, \ldots, J\}\}$ and $J^- \equiv \{j \mid G_j < 0, j \in \{1, 2, \ldots, J\}\}$. Then, both J^+ and J^- are non-empty. Without loss of generality, let $G_1 > 0 > G_J$. When $\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]}$ is positive,

$$\begin{aligned} 0 < \frac{E_{\theta_D}\left[g(\theta_D)\right]}{E_{\theta_D}\left[g(\theta_D)^2\right]} < \frac{E_{\theta_D}\left[g(\theta_D) \mid m_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2 \mid m_1^D\right]} &\Leftrightarrow \frac{\sum_{j \in J^+} G_j + \sum_{j \in J^-} G_j}{\sum_{j=1}^{J} L_j} < \frac{G_1}{L_1} \\ &\Leftrightarrow L_1 \left(\sum_{j \in J^+} G_j + \sum_{j \in J^-} G_j\right) < G_1 \sum_{j=1}^{J} L_j. \end{aligned}$$

From $\frac{G_1}{L_1} = \frac{G_j}{L_j}$ for $j \in J^+$ and $L_1 \sum_{j \in J^+} G_j = G_1 \sum_{j \in J^+} L_j$, the above condition is equivalent to

$$L_1 \sum_{j\in J^-} G_j < G_1 \sum_{j\in J^-} L_j.$$

Since $G_1 > 0$, $\sum_{j\in J^-} G_j < 0$ and $L_j > 0$ for all $j \in \{1,\ldots,J\}$, this inequality is true. This contradicts the ex-ante optimality because $EU^D\left(\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]}\right) > EU^D\left(b\frac{E_{\theta_D}\left[g(\theta_D)|m_1^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m_1^D\right]}\,\middle|\,\theta_D\right)$.

Similarly, when $\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]}$ is negative,

$$\begin{aligned}
0 > \frac{E_{\theta_D}\left[g\left(\theta_D\right)\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2\right]} > \frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_J^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_J^D\right]} &\Leftrightarrow \frac{\sum_{j\in J^+} G_j + \sum_{j\in J^-} G_j}{\sum_{j=1}^J L_j} > \frac{G_J}{L_J} \\
&\Leftrightarrow L_J\left(\sum_{j\in J^+} G_j + \sum_{j\in J^-} G_j\right) > G_J \sum_{j=1}^J L_j \\
&\Leftrightarrow L_J \sum_{j\in J^+} G_j > G_J \sum_{j\in J^+} L_j,
\end{aligned}$$

from $\frac{G_J}{L_J} = \frac{G_j}{L_j}$ for $j \in J^-$ and $L_J \sum_{j\in J^-} G_j = G_J \sum_{j\in J^-} L_j$. The inequality is true because $G_J < 0$, $\sum_{j\in J^+} G_j \geq 0$ and $L_j > 0$. Again, this is a contradiction because $EU^D\left(\frac{E_{\theta_D}[g(\theta_D)]}{E_{\theta_D}\left[g(\theta_D)^2\right]}\right) > EU^D\left(b\frac{E_{\theta_D}\left[g(\theta_D)|m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m_j^D\right]}\,\middle|\,\theta_D\right)$.

Thus, $G_j G_k \geq 0$ and $\frac{E_{\theta_D}\left[g(\theta_D)|m_j^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m_j^D\right]} = \frac{E_{\theta_D}\left[g(\theta_D)|m_k^D\right]}{E_{\theta_D}\left[g(\theta_D)^2|m_k^D\right]}$ for all $j,k \in \{1,2,\ldots,J\}$, and we have proved our claim. □

A.4 Proof of Lemma 1

Proof. We can rewrite our condition as

$$\frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_j^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_j^D\right]} = \frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_k^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_k^D\right]} \Leftrightarrow \frac{G_j}{L_j} = \frac{G_k}{L_k} \Leftrightarrow G_j L_k = G_k L_j, \qquad (4)$$

for all $j,k \in \{1,2,\cdots,J\}$.

Observe

$$\frac{G_1}{L_1} = \frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_1^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_1^D\right]} = \frac{E_{\theta_D}\left[g\left(\theta_D\right)\right] - \sum_{j=2}^{J} G_j}{E_{\theta_D}\left[g\left(\theta_D\right)^2\right] - \sum_{j=2}^{J} L_j}$$
$$\Leftrightarrow G_1 E_{\theta_D}\left[g\left(\theta_D\right)^2\right] - G_1 \sum_{j=2}^{J} L_j = L_1 E_{\theta_D}\left[g\left(\theta_D\right)\right] - L_1 \sum_{j=2}^{J} G_j.$$

According to (4), the second terms on both sides offset. Therefore,

$$G_1 E_{\theta_D}\left[g\left(\theta_D\right)^2\right] = L_1 E_{\theta_D}\left[g\left(\theta_D\right)\right] \Leftrightarrow \frac{E_{\theta_D}\left[g\left(\theta_D\right)\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2\right]} = \frac{E_{\theta_D}\left[g\left(\theta_D\right) \mid m_1^D\right]}{E_{\theta_D}\left[g\left(\theta_D\right)^2 \mid m_1^D\right]}.$$

□