

Ordinal Motifs in Lattices

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Abstract Lattices are a commonly used structure for the representation and analysis of relational and ontological knowledge. In particular, the analysis of these requires a decomposition of a large and high-dimensional lattice into a set of understandably large parts. With the present work we propose /ordinal motifs/ as analytical units of meaning. We study these ordinal substructures (or standard scales) through order-embeddings and (full) scale-measures of formal contexts from the field of formal concept analysis. We show that the underlying decision problems are NP-complete and provide results on how one can incrementally identify ordinal motifs to save computational effort. Accompanying our theoretical results, we demonstrate how ordinal motifs can be leveraged to achieve textual explanations based on principles from human computer interaction.

Keywords: Ordered Sets, Explanations, Formal Concept Analysis, Closure System, Conceptual Structures

1 Introduction

The foundation of any formal analysis of data is the identification of unique and meaningful substructures and properties. The realm of ordinal structures, in particular lattices, is no exemption to that. The field of Formal Conceptual Analysis (FCA), which derives lattices from data tables, called formal contexts, is already very well equipped with tools and notions for identifying and analyzing important substructures. One essential tool of FCA is to provide a user a lattice diagram of meaningful size, which can be interpreted (or even explained). For obvious reasons, this approach defies any applicability to data sets as they are commonly used today, as the resulting lattices are comprised of thousands of elements. In addition, the lattice diagram itself, as the primary means of communication, presents a significant hurdle to interpretation for untrained users. Common approaches tackle the first problem by data set reductions within the data tables [10, 14] or within the resulting lattice structure [1, 2, 9, 15]. The second problem is treated, to some extent, by

improving order diagrams of lattices through locally optimal layouts [6] or by (interactively) collapsing [17]. These approaches are most often motivated from graph-theoretical points of view or apply statistical methods. In general, they do not explicitly account for identifying and employing basic ordinal sub-structures within the lattice, such as nominal scales, ordinal scales, or inter-ordinal scales. Even more, they do not allow the analysis of a lattice using arbitrary ordinal patterns.

With our work, we provide the theoretical foundations for analyzing (concept) lattices by means of ordinal substructures. We call this approach, in analogy to the notion established in network science [12, 13, 16], *ordinal motifs*. However, in contrast to network science, where motifs are recurrent and statistically significant subgraphs (or patterns), we understand motifs as user-defined set \mathcal{O} of ordered sets, usually represented as formal contexts [8]. The elements of this set can be of different sizes and (ordinal) complexities. They shall allow to analyze any lattice, or ordinal structure, by means of frequency and sizes of ordinal patterns. Thus, the set \mathcal{O} can be considered as an *ordinal tool-set*. In addition to the standard scales mentioned above, any pattern deemed relevant by a user lends itself to \mathcal{O} . However, we show in our work that already for standard scales the recognition of these motifs is a difficult problem.

In order to represent and compute ordinal motifs we employ recently developed methods from the realm of conceptual measurement, i.e., *scale-measures* [11]. These are continuous maps between closure systems and can be used to map an ordinal structure, or parts of it, to an ordinal motif. As these maps are continuous they ensure that the relation between objects and attributes in a motif is correct with respect to the underlying conceptual structure of the original data set.

In terms of theoretical results, we have shown the computational complexity of several decision problems for recognizing and finding scale-measures. In particular, we show that for finding a scale-measures for a given ordinal motif we have to solve an NP-complete problem. We show that motifs which have the special property of belonging to a hereditary class of scales offer many advantages in computation.

Finally, to demonstrate the applicability of the ordinal motif method we demonstrate how to find them and provide basic interpretations motifs based on standard scales in a medium sized data set, the spice planner data set [10].

Overall, our work proposes a new approach to the analysis of (large) lattices and, in particular, ordinal structures, in order to improve their human interpretability.

2 Ordinal Motifs

In the beginning of this section we recall all necessary basics from Formal Concept Analysis (FCA). After finishing this paragraph, readers who are familiar with FCA may skip directly to Section 2.2, in which we introduce our notion for *ordinal motifs*. To work with these, we draw from the notion of scale-measures, i.e., continuous maps between closure systems. An extension of these mappings with a local version allows us to prove the computational complexities for recognizing ordinal motifs.

2.1 Formal Concept Analysis

Throughout this paper we use the notation as introduced by Ganter and Wille [8]. That is, in the following $\mathbb{K} := (G, M, I)$ denote a *formal context*. The sets G and M are respectively called object and attribute set, and the binary relation $I \subseteq G \times M$, called *incidence*, indicates if an object $g \in G$ has an attribute $m \in M$ by $(g, m) \in I$. The incidence relation I gives rise to two important maps, called derivation operators, $\cdot' : \mathcal{P}(G) \rightarrow \mathcal{P}(M)$, $A \mapsto A' := \{m \in M \mid \forall g \in A : (g, m) \in I\}$, and (the dual) $\cdot' : \mathcal{P}(M) \rightarrow \mathcal{P}(G)$, $B \mapsto B' := \{g \in G \mid \forall m \in B : (g, m) \in I\}$. There are situations where multiple formal contexts are used, in these cases will explicitly note which incidence relation is applied, i.e., we write A^I instead of A' .

The namesake for FCA are the *formal concepts*, i.e., pairs $(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M)$ where $A' = B$ and $B' = A$. The sets A, B are called *extent* and *intent* respectively. The set of all concepts of a formal context \mathbb{K} is denoted by $\mathfrak{B}(\mathbb{K})$, which is a lattice ordered set, called *concept lattice*, given the following relation: $(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C$. We denote by $\text{Ext}(\mathbb{K})$ the set of all extents and by $\text{Int}(\mathbb{K})$ the set of all intents. Both sets each form a closure system and there is an isomorphism between them. The corresponding closure operators are the respective compositions of the derivations.

For a closure system \mathcal{C} on G we call \mathcal{D} a *finer* closure system (denoted by $\mathcal{C} \leq \mathcal{D}$) iff \mathcal{D} is a closure system on G and $\mathcal{C} \subseteq \mathcal{D}$. Conversely we say \mathcal{C} is *coarser* than \mathcal{D} . In the particular case where \mathcal{C} is a closure system on $H \subseteq G$ and $\mathcal{C} = \{H \cap D \mid D \in \mathcal{D}\}$ we call \mathcal{C} a *sub-closure system* of \mathcal{D} .

Our work uses in particular *scale-measures* (Definition 2.1), for which we needs maps between different object sets of different formal contexts. For such a map $\sigma : G_1 \rightarrow G_2$ we remind the reader that the image of a set $A \subseteq G_1$ is $\sigma(A) := \bigcup_{g \in A} \sigma(g)$. Moreover, for any $\mathcal{A} \subseteq \mathcal{P}(G)$ we set $\sigma(\mathcal{A}) := \{\sigma(A) \mid A \in \mathcal{A}\}$. Essential for scale-measures will be the pre-image of sets $A \in G_2$, i.e., $\sigma^{-1}(A) := \{\sigma^{-1}(g) \mid g \in A\}$.

2.2 Mapping and Representation

The overall goal for ordinal motifs is to identify frequent recurring ordinal patterns that allow for analyzing large and complex ordinal structures. Thus, ordinal motifs are themselves ordered structures.

There are various ways for representing ordinal structures. To draw from the powerful theoretical and algorithmic tool-set of FCA, we consider any ordered set (P, \leq) represented as context, i.e., (P, P, \leq) . This context is called the *general ordinal scale* and its concept lattice $\mathfrak{B}(P, P, \leq)$ is the smallest complete lattice in which (P, \leq) can be order-embedded [8, Theorem 4].

Definition 2.1 (full scale-measure (Definition 91 [8])). *For two formal contexts \mathbb{K}, \mathbb{S} a map $\sigma : G_{\mathbb{K}} \rightarrow G_{\mathbb{S}}$ is a scale-measure iff for all $A \in \text{Ext}(\mathbb{S})$ the pre-image $\sigma^{-1}(A)$ is in $\text{Ext}(\mathbb{K})$. A scale-measure is full iff $\text{Ext}(\mathbb{K}) = \sigma^{-1}(\text{Ext}(\mathbb{S}))$.*

The formal context \mathbb{S} in the definition above is called *scale*, hence the name scale-measure. However, there is no restriction on what can be used as a scale context. Given this tool of continuous maps we want to express ordinal motifs in the

language of formal contexts. In doing so, we want to consider the following aspects: *scope* and *coverage*. We will first give an informal explanation of the two properties and then derive the mathematical tools and a precise problem definition.

Starting from a given ordinal data set $\mathbb{D} := (G_{\mathbb{D}}, M_{\mathbb{D}}, I_{\mathbb{D}})$ and an ordinal motif \mathbb{S} , both in the form of a formal context, the scope of the ordinal motif is

- **global**, if it covers the entire data, i.e., all objects $G_{\mathbb{D}}$, or
- **local**, if it covers only parts of $G_{\mathbb{K}}$.

Since it is very difficult to find a motif that captures the complex structure of a given data set, one usually relies on local motifs. However, scale-measures are incapable of capturing an ordinal motif only locally, i.e., only on a part of the data. We will therefore introduce scale-measures based on partial maps $\sigma : H \subseteq G_1 \rightarrow G_2$ in a few moments. The coverage of an ordinal motif concerns the portion of the ordinal structure that is captured by the motif. We say an ordinal motif

- has **full coverage**, if every element of the ordinal structure of \mathbb{D} , i.e., of the concept lattice, has a correspondence in the ordinal structure of the motif, or
- has **partial coverage**, otherwise.

For example, the latter case exists if there are concepts of \mathbb{D} that are not the pre-image of an extent of \mathbb{S} . In case there is a full scale-measure from a context \mathbb{D} to a context \mathbb{S} , we can infer that the closure system of \mathbb{D} on $G_{\mathbb{D}}$ is, except for relabeling, identical to that of \mathbb{S} . A scale-measure from \mathbb{D} to \mathbb{S} , on the other hand, only guarantees that the closure system of \mathbb{D} on $G_{\mathbb{D}}$ has at least all closed sets that the context \mathbb{S} has, up to relabeling.

Remark 2.1 (Surjective Scale-Measures). *It is reasonable to consider only surjective maps when using scale contexts for ordinal motifs. Since objects that are not contained in the image of the scale-measure σ do not contribute to the set of reflected extents, dropping surjectivity would allow for trivial maps into ordinal motifs.*

In order to introduce a local variant of scale-measures, we need to fix some notation. Given a formal context $\mathbb{K} = (G, M, I)$, by $\mathbb{K}[H, N]$ we refer to the *induced subcontext* of \mathbb{K} on $H \subseteq G$ and $N \subseteq M$, i.e., $(H, N, I \cap H \times N)$.

Definition 2.2 (local scale-measures). *For $\mathbb{K} = (G_{\mathbb{K}}, M_{\mathbb{K}}, I_{\mathbb{K}})$ and scale context \mathbb{S} a map $\sigma : H \rightarrow G_{\mathbb{S}}$ is a local scale-measure, if*

1. $H \subseteq G_{\mathbb{K}}$ and
2. σ is a scale-measure from $\mathbb{K}[H, M_{\mathbb{K}}]$ to \mathbb{S} .

We say a local scale-measure is full, iff σ is a full scale-measure from $\mathbb{K}[H, M_{\mathbb{K}}]$ to \mathbb{S} .

For local and full scale-measures the relation between the respective concept lattices is captured by the following proposition. In it the relation symbol \cong is used to indicate that two ordered sets are isomorphic and $(\mathcal{A}, \subseteq) \leq_{\text{Ext}} (\mathcal{B}, \subseteq)$ denotes and (\mathcal{A}, \subseteq) is a sub-closure system of (\mathcal{B}, \subseteq) .

Proposition 2.1 (local and full scale-measure). *For contexts \mathbb{K}, \mathbb{S} , a surjective full scale-measure σ from \mathbb{K} to \mathbb{S} , and the closure operator $\varphi_{\mathbb{K}}$ on $\text{Ext}(\mathbb{K})$ we find that*

$$(\text{Ext}(\mathbb{K}), \subseteq) \cong (\sigma(\text{Ext}(\mathbb{K})), \subseteq) \cong (\text{Ext}(\mathbb{S}), \subseteq) \cong (\sigma^{-1}(\text{Ext}(\mathbb{S})), \subseteq). \quad (1)$$

For a local surjective scale-measure σ with $H \subseteq G_{\mathbb{K}}$ we find that

$$(\text{Ext}(\mathbb{S}), \subseteq) \cong (\sigma^{-1}(\text{Ext}(\mathbb{S})), \subseteq) \leq_{\text{Ext}} (\text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}]), \subseteq) \quad (2)$$

and that

$$(\text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}]), \subseteq) \cong (\{\varphi_{\mathbb{K}}(E) \mid E \in \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}])\}, \subseteq). \quad (3)$$

Proof. Equation (1): From the surjectivity of σ we can deduce via [8, Proposition 118] that the map σ^{-1} exists, which is injective. This also means that every extent $E \in \text{Ext}(\mathbb{S})$ is mapped to a unique extent $\hat{E} \in \text{Ext}(\mathbb{K})$. Moreover, since σ is a full scale-measure, every extent of \mathbb{K} is also a pre-image of an extent of \mathbb{S} . From this it follows that σ^{-1} bijectively maps the extents from \mathbb{S} and \mathbb{K} . Finally, since there is no $g \in G_{\mathbb{S}}$ with $\sigma^{-1}(g) = \emptyset$, for every $E, \hat{E} \in \text{Ext}(\mathbb{S})$ with $E \subseteq \hat{E}$ it is true that $\sigma^{-1}(E) \subseteq \sigma^{-1}(\hat{E})$.

Equation (2): From left to right, the first \cong -relation can be inferred from Equation (1). The second, i.e., \leq_{Ext} , follows by definition of scale measures.

Equation (3): For the final \cong -relation we can note that for $A \in \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}])$ the difference $\varphi_{\mathbb{K}}(A) \setminus A$ is in $G \setminus H$. This means, the closure of A in $\text{Ext}(\mathbb{K})$ adds only elements from $G \setminus H$. Thus, since $\varphi_{\mathbb{K}}$ is a closure operator we find that for $A, C \in \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}])$ with $A \subset C$ we have $\varphi_{\mathbb{K}}(A) \subset \varphi_{\mathbb{K}}(C)$. Hence, $\varphi_{\mathbb{K}} : \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}]) \rightarrow \text{Ext}(\mathbb{K})$ is an injective map and by restricting the codomain we find a bijective map $\hat{\varphi}_{\mathbb{K}} : \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}]) \rightarrow \{\varphi_{\mathbb{K}}(E) \mid E \in \text{Ext}(\mathbb{K}[H, M_{\mathbb{K}}])\}$. \square

Proposition 2.1 reveals the relations between a context \mathbb{K} and an ordinal motif \mathbb{S} . It summarizes known results and shows new equivalences. For the case of full scale-measures we now know that the closure systems of \mathbb{K} and \mathbb{S} are equal up to relabeling. Hence, to analyze an ordinal structure \mathbb{D} via ordinal motifs in the full scale-measure setting would mean to simply speak about \mathbb{D} with different labels. For local case we find that scale-measures reflect a coarser closure system.

The following problem summarizes the technical observations so far and (finally) states all notions for ordinal motif.

Problem 1 (Finding Ordinal Motifs). *Given a formal context \mathbb{K} and an ordinal motif \mathbb{S} find a surjective map from \mathbb{K} into \mathbb{S} that is a:*

	global	local
partial	scale-measure	local scale-measure
full	full scale-measure	local full scale-measure

In the next section, we employ those maps to substitute elements in ordinal motif explanations by the real world objects. The result is then an explanation of the data set.

2.3 Recognizing Scale-Measures

Recognizing scale-measures is the problem for deciding if for a given formal context \mathbb{K} a scale \mathbb{S} and a map σ is a scale-measure of \mathbb{K} . This problem has been studied in Hanika and Hirth [11] and the time complexity was found to be in $O(|\mathbb{K}| \cdot |\mathbb{S}|)$. On top of that one has to check for full scale-measures if for each meet-irreducible extent A of \mathbb{K} that $\sigma(A) \in \text{Ext}(\mathbb{S})$ and $\sigma^{-1}(\sigma(A)) = A$. However, this problem is dual to the original scale-measure recognition problem. Thus verifying full scale-measure can be done in time $O(|\mathbb{K}| \cdot |\mathbb{S}|)$. The check for local (full) scale-measures has the same cost, since it is the same check but for subcontext $\mathbb{K}[H, M_{\mathbb{K}}] \leq \mathbb{K}$.

Corollary 2.1 (Recognizing Ordinal Motifs). *Given two formal contexts \mathbb{K}, \mathbb{S} and a map $\sigma: G_{\mathbb{K}} \rightarrow G_{\mathbb{S}}$, deciding if σ is (local) (full) scale-measure is in $O(|\mathbb{K}| \cdot |\mathbb{S}|)$.*

Scale-Measures and Implicational Theories Before we now turn to finding ordinal motifs in ordinal data, i.e., finding scale-measures, we want to point out one more practical relevant observation with the proposition at the end of this subsection. In practice, context like data sets are large, however, mostly only in one dimension. The usual case is that the number of objects in a formal context is many times larger than the number of attributes. The reverse case, of course, also occurs. The most expensive computation for context and scales is the derivation, in particular in the direction of the larger dimension, i.e., objects or attributes. We therefore want to present an alternative representation using implications in contexts.

In a formal context $\mathbb{K} = (G, M, I)$, we say a pair $(A, B) \in \mathcal{P}(M) \times \mathcal{P}(M)$ is a *valid attribute implication*, usually denoted by $A \rightarrow B$, iff $A' \subseteq B'$. In other words, all objects having the attribute set A do also have the attributes B . The set of all valid attribute implications is commonly denoted by $\text{Th}(\mathbb{K})$. Of course, one may analogously define and use *object implications*, as we will do in the following. Hence, $\text{Th}(\mathbb{K})$ refers to the set of valid object implications in \mathbb{K} .

To syntactically link implications with scale measures, we use the short hand $\sigma^{-1}(A \rightarrow B) := \sigma^{-1}(A) \rightarrow \sigma^{-1}(B)$. For the theory $\text{Th}(\mathbb{K})$ we define $\sigma^{-1}(\text{Th}(\mathbb{K})) := \{\sigma^{-1}(A \rightarrow B) \mid A \rightarrow B \in \text{Th}(\mathbb{K})\}$.

Proposition 2.2 (Recognizing (full) Scale-Measures using Implications).

For a context \mathbb{K} a scale \mathbb{S} and a map $\sigma: G_{\mathbb{K}} \rightarrow G_{\mathbb{S}}$ we find that

- i) σ is a scale-measure $\iff \sigma^{-1}(\text{Th}(\mathbb{S})) \vdash \text{Th}(\mathbb{K})$
- ii) σ is a full scale-measure $\iff \text{Th}(\mathbb{K}) \equiv \sigma^{-1}(\text{Th}(\mathbb{S}))$.

Proof. First, we note that for two implicational theories Th_1, Th_2 , i.e., transitive closures of implication sets, it holds that $\text{Th}_1 \subseteq \text{Th}_2 \iff \text{Th}_2 \vdash \text{Th}_1$. Secondly, we note that there is a Galois connection between the lattice of all implicational theories and the lattice of all closure systems [3, Theorem 57] to which the hierarchy of scale-measures is isomorph [11, Proposition 11].

- i) The map σ is a scale-measure iff the closure system $\sigma^{-1}(\text{Ext}(\mathbb{S}))$ is a sub-closure system of $\text{Ext}(\mathbb{K})$ on $G_{\mathbb{K}}$. Given our preliminary considerations this is the case if and only if the theory of $\text{Th}(\mathbb{K})$ is entailed in $\sigma^{-1}(\text{Ext}(\mathbb{S}))$, i.e., $\sigma^{-1}(\text{Th}(\mathbb{S})) \vdash \text{Th}(\mathbb{K})$.

- ii) The map σ is a full scale-measure iff the closure system $\sigma^{-1}(\text{Ext}(\mathbb{S}))$ is equal to $\text{Ext}(\mathbb{K})$ (cf. Proposition 2.1). Given our preliminary considerations this is the case if and only if their theories are equal.

□

With the help of Proposition 2.2 one may use already existent logical inference checkers for the verification of (local) (full) scale-measures.

2.4 Ordinal Motif Problems

Starting from Problem 1, we now want to formulate a decision problem to investigate the complexity of Problem 1. In the following we refer by *DSM* to the decision problem, if for two formal contexts \mathbb{K} and \mathbb{S} there exists a surjective scale-measure from \mathbb{K} to \mathbb{S} , the *Deciding Surjective Scale-Measures* problem. Analogously, we refer by *DfSM* to the decision problem, if for two formal contexts there exists a full surjective scale-measure. As remarked before, considering scale-measures that are not surjective is not meaningful for ordinal motifs. In particular for the problem definition, there is a always trivial scale-measure that maps all objects onto a single object of \mathbb{S} .

Theorem 2.1 (Ordinal Motif Problems). *For two formal contexts \mathbb{K} and \mathbb{S} , *DSM* and *DfSM* are NP-complete.*

Proof. To avoid any peculiarities, we consider in the following reductions graphs of size at least three.

- a) **hardness:** To show NP-hardness of the DSM problem, we reduce the subgraph isomorphism (SI) problem to DSM. For two Graphs G, H consider the formal context $\mathbb{G} = (V_G \cup \{\perp\}, E_G \cup \{\{v\} \mid v \in V_G\} \cup \{\emptyset\}, \in)$ and analogously constructed formal context \mathbb{H} . The set of extents of \mathbb{G} is equal to $\{\{v\} \mid v \in V_G\} \cup E_G \cup \{\emptyset, V_G \cup \{\perp\}\}$. This reduction is polynomial in the size of G, H .
- \Rightarrow Let σ be a surjective scale-measure of \mathbb{G} into \mathbb{H} . Then $\sigma^{-1}(\text{Ext}(\mathbb{H})) \subseteq \sigma^{-1}(\text{Ext}(\mathbb{G}))$. In particular for every $e \in E_H$ we have $\sigma^{-1}(e) \in \text{Ext}(\mathbb{G})$. Since σ is surjective, we can infer that $2 \leq |\sigma^{-1}(e)| < |V_G|$. The only extents with a cardinality in that interval are the edge extents of \mathbb{G} . Thus $\sigma^{-1}(e) \in E_G$ and all nodes of e have a unique pre-image. Since $E_H \subseteq \text{Ext}(\mathbb{H})$, all nodes with at least one edge have a unique pre-image. WLOG we assume that the pre-image of all $v \in V_H$ have a unique pre-image, otherwise change the map σ for all but one node to \perp . Hence the map $\sigma^{-1} : V_H \rightarrow V_G$ is edge preserving and an isomorphism of (H, E_H) into a subgraph of G , i.e., into the subgraph given by $(\text{co-dom}(\sigma^{-1}), \{e \in E_G \mid \exists l \in E_H : \sigma^{-1}(l) = e\})$.
- \Leftarrow Let σ be an isomorphism of \mathbb{H} into a subgraph of \mathbb{G} , i.e., an edge preserving map from H into G . Based on this consider the map $\theta : V_G \cup \{\perp\} \rightarrow V_H \cup \{\perp\}$ where $\theta(v) = \sigma^{-1}(v)$ and \perp otherwise. The map θ is surjective by definition. For the node extents, the empty extent and the top extent $V_H \cup \{\perp\}$ of \mathbb{H} we have that their pre-images are in $\text{Ext}(\mathbb{G})$. For an extent e in E_H we have that $\theta^{-1}(e) = \sigma(e) \in E_G$, since σ is edge preserving. Thus θ is a surjective scale-measure from \mathbb{G} into \mathbb{H} .

completeness: An algorithm for identifying if there is a surjective scale-measure for two context \mathbb{O}, \mathbb{K} can be constructed by guessing non-deterministically a mapping σ . The check for a surjective scale-measure can be done deterministically in polynomial time in the size of both contexts.

- b) **hardness:** To show NP-hardness of the DfSM problem, we reduce the induced subgraph isomorphism (ISI) problem to the DSM problem. For two Graphs G, H consider the contexts $\mathbb{G} = (V_G, E_G \cup \{\{v\} \mid v \in V_G\} \cup \{\emptyset\}, \epsilon)$ and \mathbb{H} analogously. The set of extents of \mathbb{G} is equal to $\{\{v\} \mid v \in V_G\} \cup E_G \cup \{\emptyset, V_G\}$. This reduction is polynomial in the size of G, H .

\Rightarrow Let σ be a full scale-measure of \mathbb{H} into \mathbb{G} . Then for every $v \in V_H$ the extent $\{v\} \in \text{Ext}(\mathbb{H})$ is the pre-image of an extent A of $\text{Ext}(\mathbb{G})$. Since $v \in \sigma^{-1}(A)$ we have $\sigma(v) \in A$ and from $\{v\} = \sigma^{-1}(A)$ we can infer that there exists no other $w \in V_H$ with $w \neq v$ and $\sigma(w) = \sigma(v)$. Thus σ is injective.

For an edge $e \in E_G$ where $e \subseteq \text{co-dom}(\sigma)$ we have $\sigma^{-1}(e) \in \text{Ext}(\mathbb{H})$ and since σ is injective we can infer $|\sigma^{-1}(e)| = 2$ and thus $\sigma^{-1}(e) \in E_H$. For an edge $e \in E_H$ there must be an $A \in \text{Ext}(\mathbb{G})$ with $\sigma^{-1}(A) = e$. Thus $\sigma(e) \subseteq A$. Since the only extents of \mathbb{G} for which this applies are V_G extents of cardinality two, i.e., the edges of G . Thus $\sigma(e) \in \text{Ext}(\mathbb{G})$ and further $\sigma(e) \in E_G$. Concluding, σ is an isomorphism between H and $\sigma(H)$.

\Leftarrow Let σ be an isomorphism between H and an induced subgraph of G . Then for every $v \in V_G$ is $\sigma^{-1}(\{v\})$ either in V_H or empty since σ is injective. For edges $e \in E_G$ where $e \subseteq \text{co-dom}(\sigma)$ we have that $\sigma^{-1}(e) \in E_H \subseteq \text{Ext}(\mathbb{H})$ since σ is an isomorphism restricted to $\text{co-dom}(\sigma)$. In case $e \not\subseteq \text{co-dom}(\sigma)$ does not hold, the pre-image is equal to a node or the emptyset. Thus $\sigma^{-1}(E_G) \subseteq \text{Ext}(\mathbb{H})$. Furthermore, $\sigma^{-1}(\emptyset) = \emptyset \in \text{Ext}(\mathbb{H})$ and $\sigma^{-1}(V_G) = V_H \in \text{Ext}(\mathbb{H})$. Thus σ is a scale-measure of \mathbb{H} into \mathbb{G} . For an edge $e \in E_H$ we have that $\sigma^{-1}(\{\sigma(v) \mid v \in e\}) = e$ and $\{\sigma(v) \mid v \in e\} \in E_G \subseteq \text{Ext}(\mathbb{G})$ since σ is isomorphism restricted to co-dom . Thus σ is a full scale-measure.

completeness: An algorithm for identifying if there is a full scale-measure for two context \mathbb{O}, \mathbb{K} can be constructed by guessing non-deterministically a mapping σ . The check for a full scale-measure can be done deterministically in polynomial time in the size of both contexts.

□

Unfortunately all these problems are NP-complete which makes the task computational costly. Further studying the computational complexity of the local variations or preserving maps will not help either, since these problems are only slight variations from the ones studied here. For example a reduction for local scale-measures can be done analogously by removing the \perp node that was introduced to capture all nodes that were not in the co-domain of the isomorphism, or consider a map σ from \mathbb{G} to \mathbb{H} of the local variant of the full scale-measures reduction.

Now that we understand the computational complexities for both problems, we want to present an interesting property of scales that may actually help to reduce the computational efforts.

3 Heredity of Ordinal Motifs

For the use of ordinal motifs for the analysis of a data set, it is meaningful to consider a set of ordinal motifs \mathcal{O} . Moreover, there are particularly meaningful families of scale contexts, such as the standardized (or elementary) scales of ordinal type [8]. These have a special property, called *heredity*, i.e., every subscale of a scale belonging to the same family is equivalent to a scale of the same family.

In this section we will demonstrate how the notion for heredity of scales impacts scale-measures.

Lemma 3.1 (Heredity of Scale-Measures). *Let \mathbb{K} be a formal context and \mathbb{S} a scale with $\sigma : G_{\mathbb{K}} \rightarrow G_{\mathbb{S}}$ a surjective (full) scale-measure. For any $H \subseteq G_{\mathbb{K}}$ is the map $\sigma|_H$ a surjective (full) scale-measure from $\mathbb{K}[H, M_{\mathbb{K}}]$ into $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$.*

Proof. First we show that $\sigma|_H$ is a scale-measure from $\mathbb{K}[H, M_{\mathbb{K}}]$ into $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$. Since $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$ is an induced subcontext of \mathbb{S} with equal attribute set, we can write every extent $A \in \text{Ext}(\mathbb{S}[\sigma(H), M_{\mathbb{S}}])$ as the intersection $\check{A} \cap \sigma(H)$ for some $\check{A} \in \text{Ext}(\mathbb{S})$. The pre-image $(\sigma|_H)^{-1}(\check{A} \cap \sigma(H))$ is equal to $(\sigma|_H)^{-1}(\check{A}) \cap (\sigma|_H)^{-1}(\sigma(H))$. Since \check{A} and $\sigma(H)$ are entailed in the image of σ on H we can follow that $(\sigma|_H)^{-1}(\check{A}) = \sigma^{-1}(\check{A})$ and $(\sigma|_H)^{-1}(\sigma(H)) = H$. Moreover, since σ is a scale-measure we can follow that $\sigma^{-1}(\check{A})$ is an extent of \mathbb{K} . Summarizing, the preimage $(\sigma|_H)^{-1}(A)$ is equal to the intersection of an extent of \mathbb{K} and H . Hence, $(\sigma|_H)^{-1}(A)$ is an extent of $\mathbb{K}[H, M_{\mathbb{K}}]$ and $\sigma|_H$ a scale-measure of $\mathbb{K}[H, M_{\mathbb{K}}]$ into $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$.

In case σ is a full scale-measure it remains to be shown that for every $D \in \text{Ext}(\mathbb{K}[H, M])$ there exists a $C \in \mathbb{S}[\sigma(H), M_{\mathbb{S}}]$ with $(\sigma|_H)^{-1}(C) = D$. We can write the extent D as the intersection $\check{D} \cap H$ where $\check{D} \in \text{Ext}(\mathbb{K})$. Since σ is a full scale-measure we can follow for \check{D} that there is a $\check{C} \in \text{Ext}(\mathbb{S})$ with $\sigma^{-1}(\check{C}) = \check{D}$. Since $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$ is an induced subcontext of \mathbb{S} with equal attribute set we find that $\check{C} \cap \sigma(H)$ is an extent of $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$. Thus, for $C := \check{C} \cap \sigma(H)$ we find that $(\sigma|_H)^{-1}(\check{C} \cap \sigma(H)) = (\sigma|_H)^{-1}(\check{C}) \cap (\sigma|_H)^{-1}(\sigma(H))$ and furthermore that $(\sigma|_H)^{-1}(\check{C}) \cap (\sigma|_H)^{-1}(\sigma(H)) = \check{D} \cap H = D$. Hence, $\sigma|_H$ is a full scale-measure.

The map $\sigma|_H$ is surjective, since the object set of $\mathbb{S}[\sigma(H), M_{\mathbb{S}}]$ is equal to the co-domain of $\sigma|_H$. \square

Proposition 3.1 (Heredity of Surjective Scale-Measures). *Let \mathbb{K} be a formal context and \mathbb{S} a heredity scale with $\sigma : G_{\mathbb{K}} \rightarrow G_{\mathbb{S}}$ a surjective (full) scale-measure. Then for any $H \subseteq G_{\mathbb{K}}$ is the map $\sigma|_H$ a surjective (full) scale-measure from $\mathbb{K}[H, M_{\mathbb{K}}]$ into an ordinal motif of the same family as \mathbb{S} .*

Proof. This proposition follows directly from Lemma 3.1 and the definition of heredity scales. \square

This proposition is essential when applying ordinal motifs for the analysis of ordinal data set using heredity scales. When computing all candidates for (full) scale-measures this statement allows to discard a large proportion. Many families of scales, such as the nominal scales, ordinal scales, inter-ordinal scales, contra-nominal scales, etc, have the heredity property [8, Proposition 123]. Unfortunately,

the crown scales do not have this property. The problem for deciding surjective scale-measures for crown scales is related to the hamiltonian path problem and therefore we do not expect there to be an easy solution to this problem. However, as far as our preliminary investigations on real-world data suggest, large crown scales are rare. Nonetheless, this claim has to be studied more thoroughly.

4 Applying Ordinal Motifs to Data Sets

We demonstrate the applicability of ordinal motifs on real-world data using a medium sized formal context: the *spices planner* data set [10]. This context contains 56 meals (objects) and 37 spices and food categories (attributes). The incidence encodes that a spice is recommended to be used to cook a meal or a meal belongs to a food category. The context has 531 formal concepts. We conduct our experiment on the dual context, i.e., $\mathbb{K}^d := (M, G, I^d)$, to derive ordinal motifs within the spices and food categories. For our application we employ the standard scales [7], as they are the most commonly used.

We recall their definitions, where $[n] := \{1, \dots, n\}$ and $\mathbb{K}_1 \mid \mathbb{K}_2$ is the context apposition operator. For crown scales we further require that $n \geq 3$.

$$\begin{aligned}
 \mathbb{N}_n &:= ([n], [n], =) && \text{(Nominal Scale)} \\
 \mathbb{O}_n &:= ([n], [n], \leq) && \text{(Ordinal Scale)} \\
 \mathbb{I}_n &:= ([n], [n], \leq) \mid ([n], [n], \geq) && \text{(Interordinal Scale)} \\
 \mathbb{B}_n &:= ([n], [n], \neq) && \text{(Contranominal Scale)} \\
 \mathbb{C}_n &:= ([n], [n], J), \text{ where } (a, b) \in J \iff a = b \text{ or } (a, b) = (n, 1) \text{ or } b = a + 1 && \text{(Crown Scale)}
 \end{aligned}$$

We depict all these scales, more precisely their contextual representations, in Figure 1. Additionally, we show their corresponding concept lattices. Hence, our goal is to identify these ordinal motifs (or ordinal patterns) in the spices data set.

The number of identified local full scale-measure of the spices data set per standard scale can be found in Table 1. In this table we distinguish between local and maximal local (with respect to the heredity). We observe that the spices data set entails a large number of ordinal motifs. The interordinal scale motifs are the most frequent in both cases, i.e., local and maximal local. For crown scales both values are equally 2145, since crown scales do not have the heredity property. All found ordinal scale motifs are trivial, i.e., all 37 found motifs are of size 1. In the last row of Table 1 we printed the size of the largest ordinal motif of the respective kind. Thus, the biggest motif is nominal and of size nine. The largest crown is of size six. We depicted all largest motifs in the Appendix Figures 2 to 5.

Basic Meanings The discovered ordinal motifs allow us to interpret parts of the spices data set in terms of their *basic meaning* of standard scales [8]. In the following we provide basic meanings of the largest local full scale-measure with respect to the found motifs.

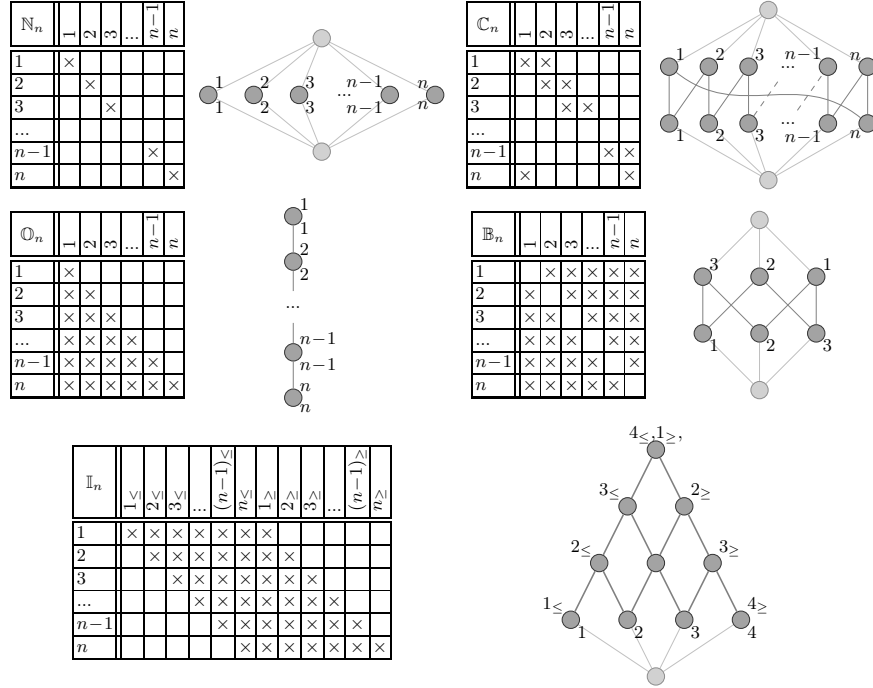


Figure 1. In this figure we depict the formal contexts and concept lattices of standard scales. From top left to bottom right we depicted the *nominal scale* \mathbb{N}_n , the *crown scale* \mathbb{C}_n , the *ordinal scale* \mathbb{O}_n , the *contranominal scale* \mathbb{B}_n and the *interordinal scale* \mathbb{I}_n .

Nominal: The food categories *miscellaneous (group)*, *fish (group)*, *potato (group)*, *vegetables (group)*, *meat (group)*, *sauce (group)*, *poultry (group)*, *rice (group)* and *pastries (group)* **form a partition.**

Ordinal: There are no non trivial local full ordinal scale-measures. If this motif would exist in the spices data set, it would **form a rank order.**

Interordinal: The spices and food categories *ginger*, *mugwort*, *meat (group)*, *black pepper* and *juniper berries* **form a linear betweenness relation.**

Table 1. Results for ordinal motifs of the spices planner context. Every column represents ordinal motifs of a particular standard scale family. Maximal lf-sm is the number of local full scale-measures for which there is no lf-sm with a larger domain. Largest lf-sm refers to the largest domain size that occurs in the set of local full scale-measures.

	nominal	ordinal	interordinal	contranominal	crown
local full sm	2342	37	4643	2910	2145
maximal lf-sm	527	37	2550	1498	2145
largest lf-sm	9	1	5	5	6

Contranominal: The spices *Thyme, Sweet Paprika, Oregano, Caraway* and *Black Pepper* form a partition and are independent.

Crown: The literature, precisely Ganter and Wille [8], does not provide a basic meaning for crowns.

The ordinal motifs obviously allow a far more complex and meaningful explanation of the substructures found. To develop this is the task of future investigations.

5 Discussion and Conclusion

With our work we have shown a new approach to the analysis and interpretation of ordinal data. By using scale-measures, we have found an expressive representation for ordinal motifs that also allows us to calculate and measure them. The necessary and useful extension of the notion of scale-measures to include a local variant is a result that will find applications in Formal Concept Analysis and beyond, independent of ordinal motifs.

While our approach is capable to extract preset frequent recurring ordinal patterns in order structures, there is room for improvement. First, apart from our theoretical considerations of computational complexity, we did not address the development of specific algorithms. On the one hand, it is certainly possible to find better algorithms in general than the naive implementations we used in our experiments. On the other hand, there are special classes of interesting ordinal motifs, such as the standard scales, which certainly allow easier computations or even simpler computation classes.

Second, in our example application, we have resorted to a very simple interpretation of the ordinal motifs found. Here we can imagine that with the help of researchers from the field of human computer interaction, general as well as area-specific explanatory methods can be derived. A third line of research would be an extension of the notion of ordinal motifs towards other context-based pattern languages, such as *clones* [5], *p-clones* [4] or *complements* [18]. Fourth, the new ability to identify standard scales may help a common conceptual data reduction method which is based on nested representations of concept lattices [17]. Last, among the identified ordinal motifs are artifacts of the underlying conceptual scaling [7]. Those include a lot of trivial scales such as *small < medium < large* which one may want to remove.

We disclosed many lines of research on how to extend and improve our methods. Improvements can be made both algorithmically for optimized identification of specific ordinal motifs and in terms of the textual explanations, by providing more understandable or domain specific textual templates. Finally, studying the occurrence of ordinal motifs quantitatively on a large number of data sets is the next task at hand.

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A Appendix

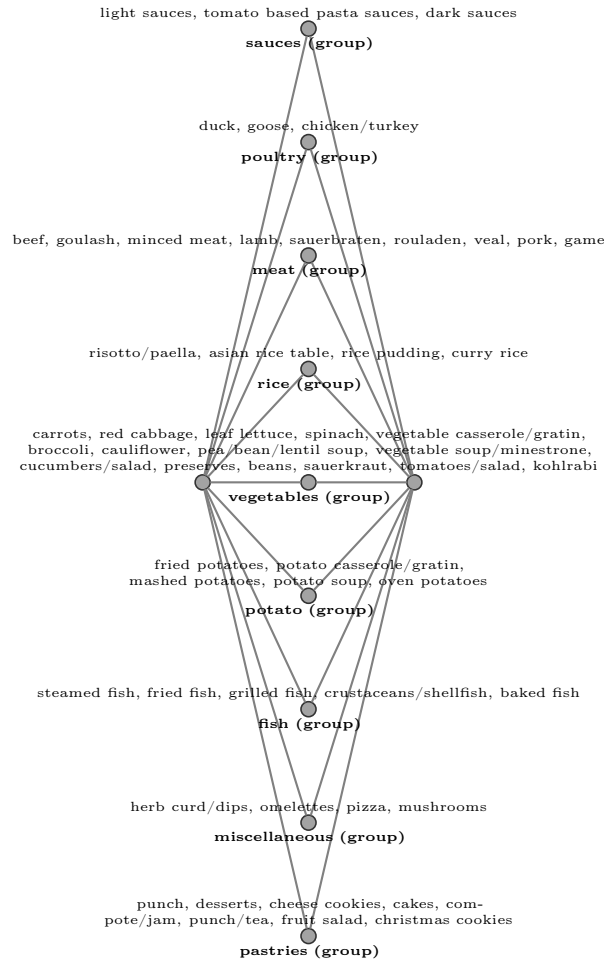


Figure 2. The largest local full nominal scale-measure of the spices data sets. We employed the dual context to get conceptual explanations of the attributes (spices). The attributes that induce the local full scale-measure are highlighted with bold font. The diagram was rotated by 90 degrees counter clockwise to improve readability, i.e., the top concept is on the left.

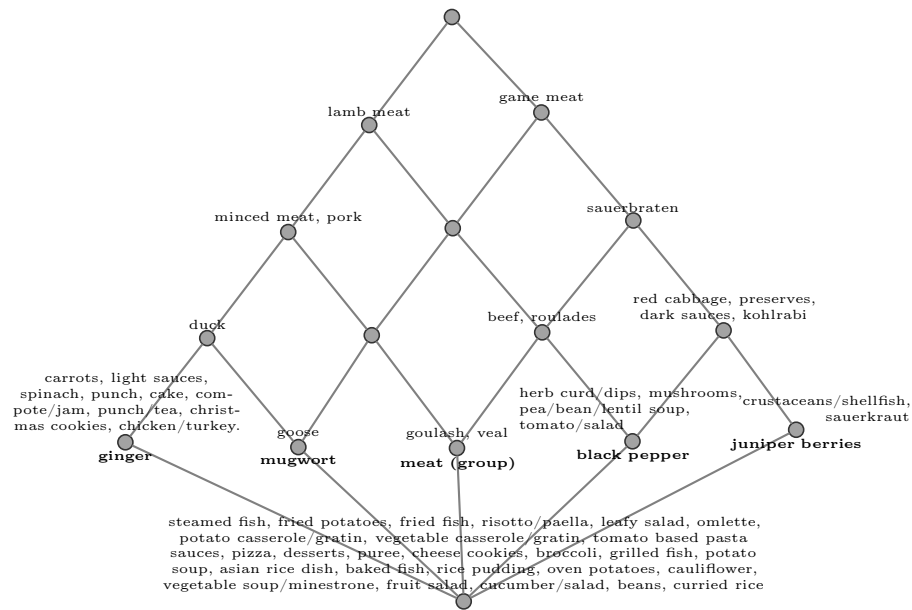


Figure 3. The largest local full interordinal scale-measure of the spices data set. We employed the dual context to get conceptual explanations of the attributes (spices). The attributes that induce the local full scale-measure are highlighted with bold font.

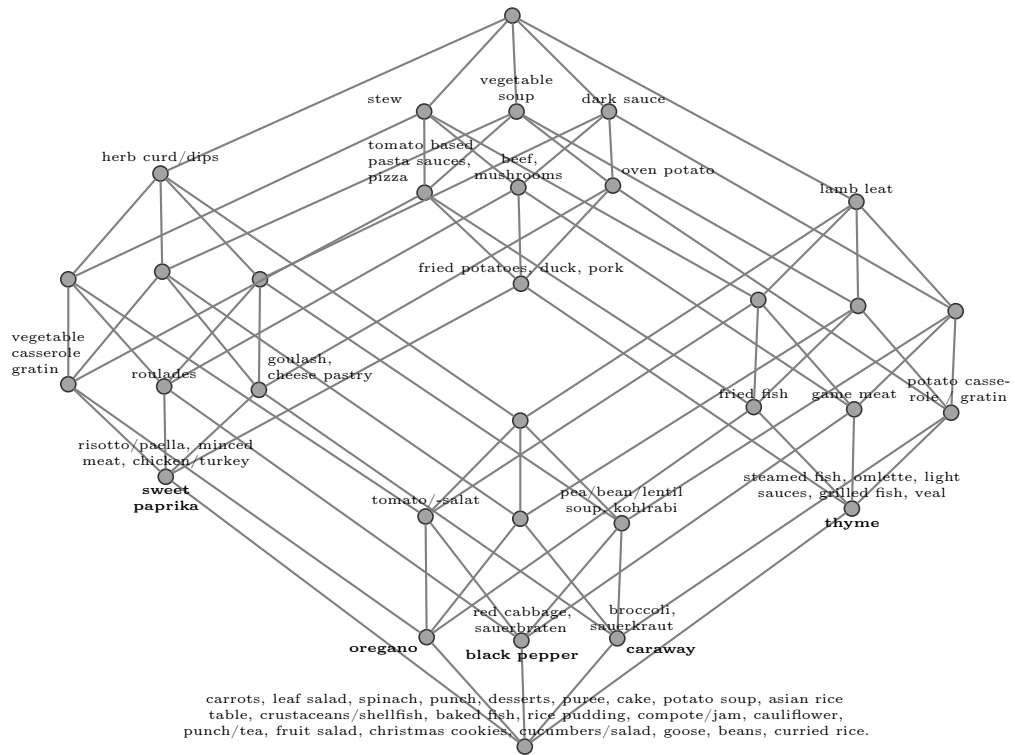


Figure 4. The largest local full contranomial scale-measure of the spices data set. We employed the dual context to get conceptual explanations of the attributes (spices). The attributes that induce the local full scale-measure are highlighted with bold font.

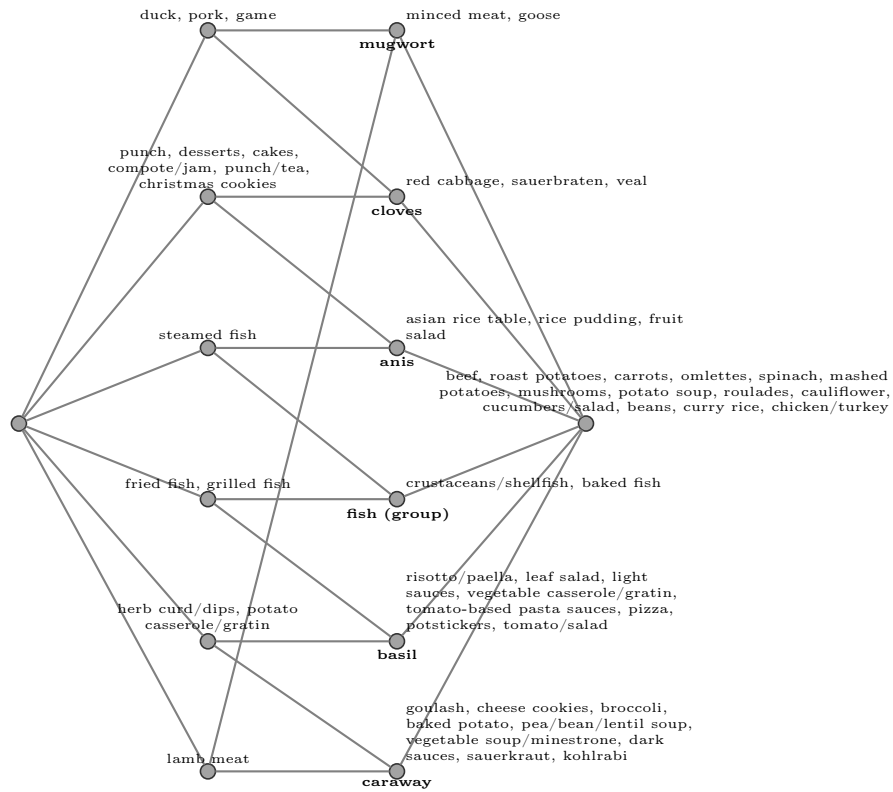


Figure 5. The largest local full crown scale-measure of the spices data set. We employed the dual context to get conceptual explanations of the attributes (spices). The attributes that induce the local full scale-measure are highlighted with bold font. The diagram was rotated by 90 degrees counter clockwise to improve readability, i.e., the top concept is on the left.