

The Lemmens-Seidel conjecture and forbidden subgraphs

Meng-Yue Cao^a, Jack H. Koolen^{*b,c}, Yen-Chi Roger Lin^d, and Wei-Hsuan Yu^e

^aSchool of Mathematical Sciences, Beijing Normal University, 19 Xijiekouwai Street, Beijing, 100875, PR China.

^bSchool of Mathematical Sciences, University of Science and Technology of China, 96 Jinzhai Road, Hefei, 230026, Anhui, PR China.

^cWen-Tsun Wu Key Laboratory of CAS, 96 Jinzhai Road, Hefei, 230026, Anhui, PR China

^dNational Taiwan Normal University, Taipei, 11677, Taiwan

^eNational Central University, Taoyuan, 32001, Taiwan

Dedicated to the 100th birthday anniversary of Professor J. J. Seidel.

Abstract

In this paper we show that the conjecture of Lemmens and Seidel of 1973 for systems of equiangular lines with common angle $\arccos(1/5)$ is true. Our main tool is forbidden subgraphs for smallest Seidel eigenvalue -5 .

1 Introduction

A system of lines through the origin in the r -dimensional Euclidean space \mathbb{R}^r is called equiangular if the angle between any pair of lines is the same. The study of equiangular lines has a long history and is related to many things. For instance, the maximum size of equiangular lines is related to energy minimizing configurations [8], line packing problems [10], and tight spherical designs [2]. Several constructions of equiangular lines come from strongly regular graphs [7] and combinatorial designs [23]. De Caen used association schemes to construct $\frac{2}{9}(r+1)^2$ equiangular lines in \mathbb{R}^r when $r = 3 \cdot 2^{2t-1} - 1$ for any positive integer t [9]. We are interested in determining the maximum cardinality $N(r)$ of a system of equiangular lines in \mathbb{R}^r . Gerzon [19] proved that $N(r) \leq \frac{r(r+1)}{2}$ for all r . However, so far the Gerzon bound is only known to be achieved for

*Corresponding author.

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E-mail addresses: cmy1325@163.com (M.-Y. Cao), koolen@ustc.edu.cn (J.H. Koolen), yclinpa@gmail.com (Y.-C. R. Lin), u690604@gmail.com (W.-H. Yu).

$r = 2, 3, 7$, and 23 . If we have equiangular lines attaining the Gerzon bound, then we immediately have tight spherical 5-designs [2]. The classification of tight spherical 5-designs has been open for decades and the main known necessary condition for the existence of tight spherical 5-designs is $r = 2, 3$, or $r = (2k + 1)^2 - 2$, where $k \in \mathbb{N}$. The history of the study of equiangular lines can be traced back to Haantjes [16], who determined $N(3)$ and $N(4)$ in 1948. After more than 70 years of study, the numbers $N(r)$ are now only known for $r \leq 43$ except for $r = 17, 18, 19, 20$, and 42 . This follows from the works of Van Lint and Seidel [24], Lemmens and Seidel [19], Barg and Yu [3], and Greaves et al. [14]. We summarize the results in the following table. For more references on recent progress of equiangular lines, readers may check [1, 12, 13, 15, 17, 18, 21].

Table 1: Maximum cardinalities of equiangular lines for low dimensions

r	2	3–4	5	6	7–14	15	16	17
$M(r)$	3	6	10	16	28	36	40	48–49
r	18	19	20	21	22	23–41	42	43
$M(r)$	56–60	72–74	90–94	126	176	276	276–288	344

Let $N_\alpha(r)$ be the maximum number of a system of equiangular lines in \mathbb{R}^r with common angle $\arccos \alpha$. Neumann (1973) showed that if $N_\alpha(r) > 2r$, then $\frac{1}{\alpha}$ is an odd integer at least 3. Lemmens and Seidel [19] determined $N_{\frac{1}{3}}(r)$ for all $r \geq 2$. In particular, they showed that $N_{\frac{1}{3}}(r) = 2r - 2$ if $r \geq 15$. They also proposed the following conjecture for the case $\frac{1}{\alpha} = 5$.

Conjecture 1.1. *The maximum cardinality of a system of equiangular lines with angle $\arccos \frac{1}{5}$ in \mathbb{R}^r is 276 for $23 \leq r \leq 185$, and $\lfloor \frac{3r-3}{2} \rfloor$ for $r \geq 185$.*

Neumaier [22] showed Conjecture 1.1 for sufficient large r . He also claimed (without proof) that his method would work for $r \geq N_0$ where $2486 \leq N_0 \leq 45374$. In this paper, we completely solve Conjecture 1.1. Balla, Dräxler, Keevash and Sudakov [1] and Bukh [6] conjectured an asymptotic version of Conjecture 1.1 for other angles as follows:

Conjecture 1.2. *The maximum cardinality of a system of equiangular lines with angle $\arccos \alpha$, where $\frac{1}{\alpha} = 2m + 1$ is an odd integer at least 3, is equal to $\frac{(m+1)(r+1)}{m} + O(1)$, for $r \rightarrow \infty$.*

Jiang and Polyanskii [17] gave partial results for Conjecture 1.2, and it was completely solved by Jiang, Tidor, Yao, Zhang, and Zhao in a recent paper [18].

2 Outline of the paper

All graphs in this paper are simple and undirected. For undefined terminologies, we refer to [5, 11].

First, we transform the problem of determining $N_\alpha(r)$ into a linear algebra problem. To do so, we introduce Seidel matrices.

A Seidel matrix S of order n is a symmetric $(0, \pm 1)$ -matrix with 0 on the diagonal and ± 1 otherwise. Seidel matrices and systems of equiangular lines, are related as follows (see for example, [11, Section 11.1]):

Proposition 2.1. *Let $n > r \geq 2$ be integers. There exists a system of n equiangular lines in \mathbb{R}^r with common angle $\arccos \alpha$ if and only if there exists a Seidel matrix S of order n such that S has smallest eigenvalue at least $-\frac{1}{\alpha}$ and $\text{rk}(S + \frac{1}{\alpha}\mathbf{I}) \leq r$.*

In this paper, we focus on the minimum rank of $S + \frac{1}{\alpha}\mathbf{I}$ for a fixed number n rather than the maximum cardinality of a system of equiangular lines in \mathbb{R}^r with common angle $\arccos \alpha$ for fixed dimension r . Our main result is as follows.

Theorem 2.2. *Let S be a Seidel matrix of order n with the smallest eigenvalue -5 . If $n \geq 277$, then $\text{rk}(S + 5\mathbf{I}) \geq \lfloor \frac{2n}{3} \rfloor + 1$.*

This theorem implies that Conjecture 1.1 is true.

Our main tools are minimal forbidden subgraphs. We will first show that Theorem 2.2 is true when the independence number $\alpha([S])$ of the switching class $[S]$ of a Seidel matrix S (for definitions see next section) is at least 49. This uses, in addition to minimal forbidden subgraphs, also a rank argument, which is done in Section 5. Then, in Section 6, we concentrate on the case when $[S]$ contains a triangle-free graph. If the clique number $\omega([S])$ of $[S]$ is at least 5, then Conjecture 1.1 was already shown by Lemmens-Seidel [19] and Lin-Yu [20]. So we only need to show Theorem 2.2 for the cases when $\omega([S])$ is at most 4. Under this condition, we show Theorem 2.2 is true when $\alpha([S]) \geq 29$ in Section 7. Then, in Section 8, we apply the pillar method to the $(4, 1)$ -pillars and the $(4, 2)$ -pillars. Our bounds for the $(4, 1)$ -pillar and on the $(4, 2)$ -pillar are not yet sharp enough to show Theorem 2.2. So in Section 9 we introduce the gallery with respect to an edge which combines a $(4, 2)$ -pillar with $(4, 1)$ -pillars and finish the proof of Theorem 2.2.

3 Preliminaries

3.1 Matrices

We denote the eigenvalues of a real symmetric matrix M of order n by $\eta_1(M) \geq \eta_2(M) \geq \dots \geq \eta_n(M)$. The largest (resp. smallest) eigenvalue of M is also denoted by $\rho(M)$ (resp. $\eta_{\min}(M)$). The largest eigenvalue of M is also called the spectral radius of M . The rank of M is denoted by $\text{rk}(M)$.

For a real symmetric $n \times n$ matrix B and a real symmetric $m \times m$ matrix C with $n > m$, we say that the eigenvalues of C interlace the eigenvalues of B , if $\eta_{m-m+i}(B) \leq \eta_i(C) \leq \eta_i(B)$ for each $i = 1, \dots, m$. The following result is a special case of interlacing.

Theorem 3.1. (Cf. [11, Theorem 9.1.1]) *Let B be a real symmetric $n \times n$ matrix and C be a principal submatrix of B of order m , where $m < n$. Then the eigenvalues of C interlace the eigenvalues of B .*

3.2 Graphs

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G) \subseteq \binom{V(G)}{2}$. The set $V(G)$ (resp. $E(G)$) is called the vertex set (resp. edge set) of G , and the cardinality of $V(G)$ (resp. $E(G)$) is called the order (resp. size) of G and is denoted by n_G (resp. ε_G). The adjacency matrix of G , denoted by $A(G)$, is a symmetric $(0, 1)$ -matrix indexed by $V(G)$, such that $(A(G))_{xy} = 1$ if and only if xy is an edge in G . The eigenvalues of G are the eigenvalues of $A(G)$, and the spectral radius of G is denoted by $\rho(G)$. The cardinality of a maximum independent set (resp. clique) in G is called the independence number (resp. clique number) of G , denoted by $\alpha(G)$ (resp. $\omega(G)$).

The disjoint union of the graphs G_1 and G_2 is denoted by $G_1 \dot{\cup} G_2$. For a graph G and a subset $U \subseteq V(G)$, we denote by G_U the subgraph of G induced on U , i.e. $V(G_U) = U$ and $E(G_U) = E(G) \cap \binom{U}{2}$. For H an induced subgraph of G , we denote by $N_G(H)$ the subgraph of G induced on the vertices that have a neighbour in H but are not in H , and we denote by $R_G(H)$ the subgraph induced on the vertices of G that are neither in H nor have a neighbour in H . If the graph G is clear from the context, we will simply use $N(H)$ and $R(H)$.

Let G be a graph. We say G is k -regular if the valency of every vertex in G is a non-negative constant integer k . A graph G of order n is said to be strongly regular with parameters (n, k, λ, μ) , if it is k -regular, every pair of adjacent vertices has λ common neighbours, and every pair of distinct nonadjacent vertices has μ common neighbours. The following lemma is well-known (cf. [11, Section 10.1 and 10.2]).

Lemma 3.2. *Let G be an (n, k, λ, μ) strongly regular graph with $k > \mu$. Then G has exactly three distinct eigenvalues $k > \theta > \tau$ satisfying*

$$\theta = \frac{(\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

$$\tau = \frac{(\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$

Moreover, the multiplicity m_θ of θ is given by $m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau}$.

3.3 Seidel matrices

Recall that a Seidel matrix S of order n is a symmetric $(0, \pm 1)$ -matrix with 0 on the diagonal and ± 1 otherwise. The graph $G = G(S)$ corresponding to a Seidel matrix S is the graph on $\{1, \dots, n\}$ such that two distinct vertices i and j are adjacent if and only if $S_{ij} = -1$. It follows immediately that $A(G) = \frac{1}{2}(\mathbf{J} - \mathbf{I} - S)$, where \mathbf{J} is the all-ones matrix and \mathbf{I} is the identity matrix. Conversely, the Seidel matrix $S = S(G)$ corresponding to a graph G can be obtained by $S = \mathbf{J} - \mathbf{I} - 2A(G)$.

Let $U \subseteq \{1, \dots, n\}$. Define the diagonal matrix D_U by $(D_U)_{ii} = 1$ if $i \in U$ and $(D_U)_{ii} = -1$ if $i \notin U$. For a Seidel matrix S we define the Seidel matrix $S_{sw}(U)$ by $S_{sw}(U) := D_U S D_U$. For a graph G with Seidel matrix $S(G)$ we denote by $G_{sw}(U)$ the graph $G((S(G))_{sw}(U))$. In other words, the graph $G_{sw}(U)$ is obtained from G by switching with respect to U . If G and H are switching equivalent, then $S(G)$ and $S(H)$ are similar and hence have the same spectrum. The collection of graphs that can be obtained from G by switching is called the switching class of G , denoted by $[G]$. For a Seidel matrix S , we define $[S]$ as $[S] := [G]$, where G is the corresponding graph of S . We call $[S]$ the switching class of S .

Let S be a Seidel matrix of order $n \geq 2$. Let $S' := \begin{pmatrix} S & -S+\mathbf{I} \\ -S+\mathbf{I} & S \end{pmatrix}$. The graph $\mathcal{S}w(S) := G(S')$ is called the switching graph of S . Note that $\mathcal{S}w(S)$ only depends on $[S]$, that is, $\mathcal{S}w(S_1) \cong \mathcal{S}w(S_2)$ if and only if S_1 and S_2 are switching equivalent. We define the independence number (resp. clique number) of $[S]$ as $\alpha([S]) := \alpha(\mathcal{S}w(S))$ (resp. $\omega([S]) := \omega(\mathcal{S}w(S))$). Note that $\alpha(\mathcal{S}w(S)) \geq 2$ and $\alpha(\mathcal{S}w(S)) = 2$ if and only if $[S] = [\mathbf{I} - \mathbf{J}]$. Similarly, $\omega(\mathcal{S}w(S)) \geq 2$ and $\omega(\mathcal{S}w(S)) = 2$ if and only if $[S] = [\mathbf{J} - \mathbf{I}]$.

3.4 Some bounds on the smallest eigenvalue

Let G be a graph. From now on, we will use λ_i to denote the eigenvalues of the Seidel matrix $S(G)$, and by θ_i to denote the eigenvalues of the adjacency matrix $A(G)$.

Lemma 3.3. *Let S be a Seidel matrix, and G be its corresponding graph. Then,*

(i) $\lambda_{\min}(S) \geq -2\rho(G) - 1;$

(ii) *For any induced subgraph H of G , we have $\lambda_{\min}(S(H)) \geq \lambda_{\min}(S)$.*

Proof. The first item follows immediately from the fact that $S = \mathbf{J} - \mathbf{I} - 2A(G)$. The second item is an easy consequence of Theorem 3.1. \square

Note that the Perron-Frobenius Theorem (cf. [4, Theorem 3.1.1]) implies that the spectral radius $\rho(G)$ of a connected graph G is simple, and we can take an eigenvector for $\rho(G)$ with

positive entries only. This means that, for any graph G , there exists an eigenvector \mathbf{v} for the eigenvalue $\rho(G)$ with non-negative entries only.

Lemma 3.4. *Let S be a Seidel matrix with the smallest eigenvalue λ_{\min} . Let G be its corresponding graph of S with adjacency matrix A and spectral radius $\rho (= \frac{-\lambda_{\min} - 1}{2})$. Assume that \mathbf{v} is an eigenvector of A with eigenvalue ρ , that is, $A\mathbf{v} = \rho\mathbf{v}$, and that \mathbf{v} is not perpendicular to the all-ones vector \mathbf{j} . If there exists another eigenvector \mathbf{w} of A not perpendicular to the all-ones vector \mathbf{j} , say with eigenvalue $\theta \neq \rho$, then $\theta < \frac{-\lambda_{\min} - 1}{2}$.*

Proof. We denote by \mathbf{U} the 2-dimensional space spanned by \mathbf{v} and \mathbf{w} . Then there exists a non-zero vector $\mathbf{u} \in \mathbf{U} - \{\mathbf{0}\}$ such that $\mathbf{u} \perp \mathbf{j}$. We find $S\mathbf{u} \leq (-1 - 2\theta)\mathbf{u}$, but equality would imply that $\rho = \theta$, as both \mathbf{v} and \mathbf{w} are not perpendicular to \mathbf{j} . Hence, $\lambda_{\min} < -1 - 2\theta$ and the conclusion holds. \square

Lemma 3.4 immediately implies the following proposition.

Proposition 3.5. *If $\rho(G) > \frac{-\lambda_{\min} - 1}{2}$, then any eigenvector for the eigenvalue $\frac{-\lambda_{\min} - 1}{2}$ is perpendicular to \mathbf{j} .*

The next proposition says that there exists at most one connected component of a graph G whose spectral radius is larger than $\frac{-\lambda_{\min}(S(G)) - 1}{2}$.

Proposition 3.6. *Let S be a Seidel matrix with the smallest eigenvalue λ_{\min} . Let G be its corresponding graph of S . Let H be an induced subgraph of G . Let $R(H)$ be the subgraph of G induced by the vertices which are neither in H nor are adjacent to any vertex in H , that is, $V(R(H)) = \{x \in V(G) \mid x \notin V(H), x \approx y, \forall y \in V(H)\}$. If $\rho(H) > \frac{-\lambda_{\min} - 1}{2}$, then $\rho(R(H)) < \frac{-\lambda_{\min} - 1}{2}$.*

Proof. Let H' be the disjoint union of H and $R(H)$. Then the adjacency matrix of H' is a diagonal block matrix A' with two blocks, namely, the adjacency matrix $A(H)$ and $A(R(H))$ of H and $R(H)$, respectively. Let \mathbf{u} (resp. \mathbf{v}) be an non-negative eigenvector for $\rho(H)$ (resp. $\rho(R(H))$), that is, $A(H)\mathbf{u} = \rho(H)\mathbf{u}$ and $A(R(H))\mathbf{v} = \rho(R(H))\mathbf{v}$.

Define \mathbf{w} by

$$\mathbf{w}_x = \begin{cases} \mathbf{u}_x, & \text{if } x \in V(H), \\ 0, & \text{if } x \in V(R(H)), \end{cases}$$

and, in similar fashion, define \mathbf{w}' from \mathbf{v} . Note that $A'\mathbf{w} = \rho(H)\mathbf{w}$, $A'\mathbf{w}' = \rho(R(H))\mathbf{w}'$, $\mathbf{w} \not\perp \mathbf{j}$, $\mathbf{w}' \not\perp \mathbf{j}$ and $\mathbf{w} \perp \mathbf{w}'$.

Let S' be the Seidel matrix of H' . By Theorem 3.1, we have $\lambda_{\min}(S') \geq \lambda_{\min}$. So, if $\rho(H) > \frac{-\lambda_{\min} - 1}{2}$, then $\rho(R(H)) < \frac{-\lambda_{\min} - 1}{2}$ by Lemma 3.4. This shows the proposition. \square

Let M be a symmetric $n \times n$ matrix and $\pi := \{V_1, \dots, V_r\}$ be a partition of $\{1, \dots, n\}$. Let M_{ij} be the submatrix of M whose rows are indexed by V_i and whose columns are indexed by V_j . We say π is an equitable partition with respect to M if M_{ij} has constant row sum for all $1 \leq i, j \leq r$. For an equitable partition π with respect to M , let q_{ij} be the row sum of M_{ij} , for $1 \leq i, j \leq r$. The quotient matrix Q of M with respect to π is defined as $Q = (q_{ij})_{1 \leq i, j \leq r}$.

Lemma 3.7. *Let M be a symmetric $n \times n$ matrix. If π is an equitable partition of M and Q is the quotient matrix with respect to π of M , then every eigenvalue of Q is an eigenvalue of M .*

Proof. Let $\pi := \{V_1, \dots, V_r\}$ be an equitable partition of M . Let λ be an eigenvalue of Q and \mathbf{v} be an eigenvector of Q with λ . Let \mathbf{w} be the vector in \mathbb{R}^n such that $\mathbf{w}_k = \mathbf{v}_i$ for $k \in V_i, i \in \{1, \dots, r\}$.

Let $k \in V_i$. Then $\sum_{l \in V_j} M_{kl} = q_{ij}$. It follows that $(M\mathbf{w})_k = \sum_{l=1}^n M_{kl}\mathbf{w}_l = \sum_{j=1}^r q_{ij}\mathbf{v}_j = \lambda\mathbf{v}_i = \lambda\mathbf{w}_k$. This shows the lemma. \square

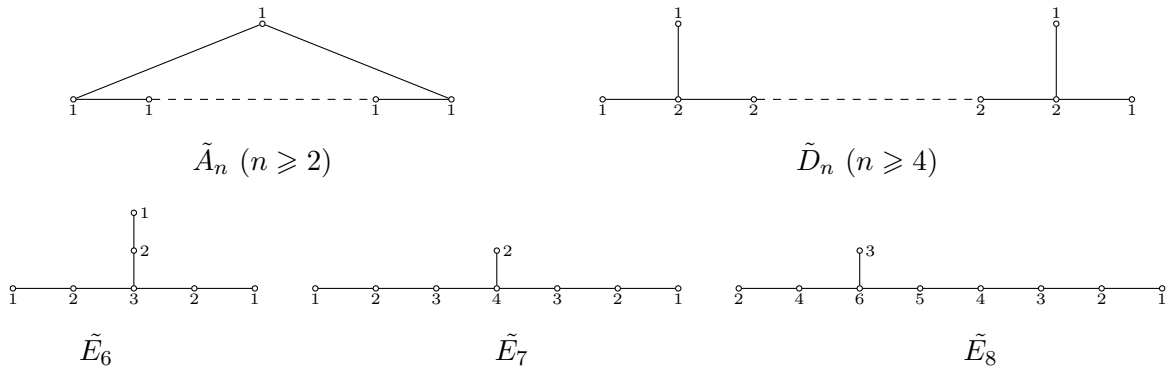
If M is the adjacency matrix of a graph G , and π is an equitable partition of $\{1, \dots, n\}$ with respect to M , then we say that π is an equitable partition of G . Note that in this case π is also an equitable partition with respect to the Seidel matrix of G .

Corollary 3.8. *If π is an equitable partition of a graph G and Q is the quotient matrix with respect to the Seidel matrix S of G , then every eigenvalue of Q is an eigenvalue of S .*

3.5 Smith's Theorem

Now we present Smith's Theorem in the year of 1970, in which Smith determined all graphs with spectral radius 2. Note that the corresponding Seidel matrices of these graphs have their smallest eigenvalues at least -5 .

Theorem 3.9. (Cf. [4, Section 3.2]) *The only connected graphs having spectral radius 2 are the following graphs (the number of vertices is one more than the index given).*



For each graph, the corresponding eigenvector is indicated by the integers at the vertices. Moreover, each connected graph with spectral radius less than 2 is a subgraph of the above graphs, and each connected graph with spectral radius greater than 2 contains one of these graphs.

Remark 3.10. This theorem shows that each graph with spectral radius less than 2 is a forest.

As an easy consequence of Theorem 3.9, we determine the minimal graphs with spectral radius larger than 2, that is, the graphs with spectral radius larger than 2 such that any proper induced subgraph has spectral radius at most 2.

Corollary 3.11. *The minimal graphs with spectral radius greater than 2 are the 18 graphs listed in Figure 1.*

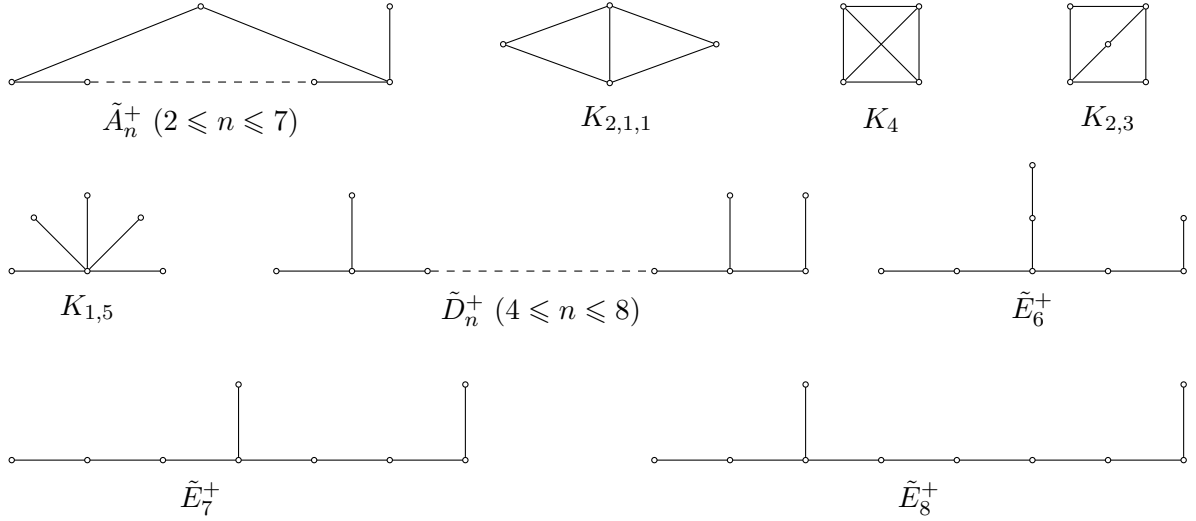


Figure 1: The 18 minimal graphs with spectral radius greater than 2

4 Forbidden subgraphs

For $\lambda < 0$, let \mathcal{F}_λ denote the set of minimal forbidden graphs for the smallest Seidel eigenvalue λ , that is,

$$\mathcal{F}_\lambda := \{G \mid \lambda_{\min}(S(G)) < \lambda \text{ and any induced proper subgraph } H \text{ of } G \text{ satisfies } \lambda_{\min}(S(H)) \geq \lambda\}.$$

Jiang and Polyanskii [17, Theorem 1] showed that the set \mathcal{F}_{-5} is finite. Since we are talking about Seidel eigenvalues, only the switching classes of such graphs are needed. Now we determine some graphs inside \mathcal{F}_{-5} . In order to do so, we define the following. For a graph G , let $G(s, t)$ be the disjoint union of G , s isolated vertices, and t copies of K_2 , where s, t are non-negative integers. In particular, we write $G(s)$ for $G(s, 0)$.

Using the graphs of Corollary 3.11, we obtain the following lemma.

Lemma 4.1. *Table 2 gives 18 graphs that belong to \mathcal{F}_{-5} .*

Proof. For each of the graphs G of Corollary 3.11, we determine the smallest integer s that satisfies $\lambda_{\min}(S(G(s))) < -5$. That the obtained graphs $G(s)$ belong to \mathcal{F}_{-5} , follows from the fact

G	\tilde{A}_2^+	\tilde{A}_3^+	\tilde{A}_4^+	\tilde{A}_5^+	\tilde{A}_6^+	\tilde{A}_7^+	$K_{2,1,1}$	K_4	$K_{2,3}$
s such that $G(s) \in \mathcal{F}_{-5}$	41	67	97	130	165	201	11	5	19
G	$K_{1,5}$	\tilde{D}_4^+	\tilde{D}_5^+	\tilde{D}_6^+	\tilde{D}_7^+	\tilde{D}_8^+	\tilde{E}_6^+	\tilde{E}_7^+	\tilde{E}_8^+
s such that $G(s) \in \mathcal{F}_{-5}$	41	137	225	327	439	557	465	966	2477

Table 2: 18 minimal forbidden graphs for $\lambda = -5$.

that any such G is a minimal graph with spectral radius larger than 2 (by Corollary 3.11) and Lemma 3.3. \square

Remark 4.2. This lemma shows that there exists a graph of order 2487 inside \mathcal{F}_{-5} . We do not know whether this graph has the largest order inside \mathcal{F}_{-5} . Nevertheless this suggests that it may be difficult to find all graphs in \mathcal{F}_{-5} . This explains the lower bound of Neumaier's claim in the introduction.

The following lemma is of crucial importance for this paper.

Lemma 4.3. *Let $r \geq 2, s \geq 0, t \geq 0$ be integers such that $s + t \geq 1$. Then, the following hold.*

- (i) *The smallest eigenvalue of $S(K_{1,r}(s, t))$ satisfies $\lambda_{\min}(S(K_{1,r}(s, t))) \geq -5$ if and only if $(r - 4)(s + 4t - 4) \leq 36$;*
- (ii) *Given a graph G with r vertices, let $C(G)$ be the cone of G , that is, adding a new vertex to G and joining it with all vertices of G . Then $\lambda_{\min}(S(C(G)(s, t))) \leq \lambda_{\min}(S(K_{1,r}(s, t)))$.*

Proof. (i) First, we consider the case when $r \geq 2, s \geq 1$ and $t \geq 1$. Let v be the vertex of valency r in $K_{1,r}$, $V_1 = V(K_{1,r} - v)$, $V_2 = V(\overline{K}_s)$ and $V_3 = V(tK_2)$. Consider a partition $\pi = \{\{v\}, V_1, V_2, V_3\}$ of $K_{1,r}(s, t)$. The partition π is equitable with quotient matrix Q with respect to $S(K_{1,r}(s, t))$:

$$Q = \begin{pmatrix} 0 & -r & s & 2t \\ -1 & r-1 & s & 2t \\ 1 & r & s-1 & 2t \\ 1 & r & s & 2t-3 \end{pmatrix}.$$

Note that $\det(Q + 3\mathbf{I}) = -16t(r - 1)$. As $r \geq 2$ and $t \geq 1$, we see that $\lambda_{\min}(Q) < -3$. By Theorem 3.1, we observe that $S(K_{1,r}(s, t))$ has at most one eigenvalue at most -3 , as $\lambda_{\min}(S(\overline{K}_{r+s} \cup tK_2)) = -3$. This implies that $\lambda_{\min}(Q) = \lambda_{\min}(S(K_{1,r}(s, t)))$, by Lemma 3.7. Next, we find that

$$\det(Q + 5\mathbf{I}) = -8((r - 4)(s + 4t - 4) - 36).$$

This shows that (i) is correct, if $r \geq 2, s \geq 1$ and $t \geq 1$.

If $s = 0$ or $t = 0$, then with a similar argument we see that (i) is true.

$$(ii) \text{ Fix } a, b, c \in \mathbb{R} \text{ such that } Q \begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix} = \lambda_{\min}(Q) \begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}.$$

We find

$$\lambda_{\min}(Q) = -ra + sb + 2tc$$

$$a\lambda_{\min}(Q) = -1 + (r-1)a + sb + 2tc.$$

If $a \leq 0$, then

$$a\lambda_{\min}(Q) + 1 \leq sb + 2tc \leq \lambda_{\min}(Q).$$

This gives a contradiction, as $\lambda_{\min}(Q) < 0$ and $a\lambda_{\min}(Q) + 1 > 0$. It follows that $a > 0$.

Let $\{w\} = V(C(G)) - V(G)$, $W_1 = V(G)$, $W_2 = V(\overline{K}_s)$ and $W_3 = V(tK_2)$. Let \mathbf{w} be a vector in $\mathbb{R}^{V(C(G)(s,t))}$ such that

$$\mathbf{w}_x = \begin{cases} 1, & \text{if } x = w, \\ a, & \text{if } x \in W_1, \\ b, & \text{if } x \in W_2, \\ c, & \text{if } x \in W_3. \end{cases}$$

For any vertex x in $V(C(G)(s,t))$, note that $(S(C(G)(s,t))\mathbf{w})_x \leq (S(K_{1,r}(s,t))\mathbf{w})_x = (\lambda_{\min}(Q)\mathbf{w})_x$.

This implies that $\lambda_{\min}(S(C(G)(s,t))) \leq \lambda_{\min}(Q)$. It shows (ii). \square

Analogous computations show that the following graphs also belong to \mathcal{F}_{-5} . We omit the details here.

Lemma 4.4. *The graphs $B_1(14)$ and $B_2(9)$ belong to \mathcal{F}_{-5} , where the graphs B_1 and B_2 are listed in Figure 2.*



Figure 2: Two more forbidden graphs in \mathcal{F}_{-5}

5 The independence number is at least 49

Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. In this section, we show that, if the switching class of S has independence number at least 49, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$. This

shows that in this case Theorem 2.2 is true.

We start with the small spectral radius.

Proposition 5.1. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order n . If the switching class of S contains a graph G with spectral radius $\rho(G) \leq 2$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. If $\rho(G) < 2$, then $S(H') + 5\mathbf{I}$ has full rank, by Lemma 3.3. Next we may assume $\rho(G) = 2$. Clearly, $\lambda_{\min}(S) \geq -2\rho(G) - 1 = -5$, by Lemma 3.3. The multiplicity of -5 of $S(G)$ is one less than the number of connected components of G with spectral radius 2. As each connected component with spectral radius 2 has at least 3 vertices, it follows that $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$. This shows the proposition. \square

The next lemma gives a lower bound for the rank of $S + 5\mathbf{I}$, where S is a Seidel matrix.

Lemma 5.2. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order n . Assume the switching class of S contains a graph G with $\rho(G) > 2$. Let d_{\max} be the maximum valency of G . Let H be an induced subgraph of G with $\rho(H) > 2$. Let n_H (resp. ε_H) be the order (resp. size) of H . Let $\alpha(H)$ be the independence number of H . Then $\text{rk}(S + 5\mathbf{I}) \geq n - n_H(1 + d_{\max}) + 2\varepsilon_H + \alpha(H)$.*

Proof. For a vertex x in G , denote by $d_x(G)$ the valency of x in G . Let $d_{\max}(V(H))$ be the maximum valency among all vertices in $V(H)$ in G , that is, $d_{\max}(V(H)) := \max\{d_x \mid x \in V(H)\}$. Let $R(H)$ be the subgraph of G induced on the vertices that are neither vertices of H nor have a neighbour in H . Then $R(H)$ has at least $n - (n_H(1 + d_{\max}(V(H))) - 2\varepsilon_H) = n - n_H(1 + d_{\max}(V(H))) + 2\varepsilon_H$ vertices. As $\rho(R(H)) < 2$, by Lemma 3.3, it follows that $S(R(H)(\alpha(H))) + 5\mathbf{I}$ has full rank. This shows that

$$\begin{aligned} \text{rk}(S + 5\mathbf{I}) &\geq n - n_H(1 + d_{\max}(V(H))) + 2\varepsilon_H + \alpha(H) \\ &\geq n - n_H(1 + d_{\max}) + 2\varepsilon_H + \alpha(H). \end{aligned}$$

\square

As a consequence of Lemma 5.2, we have the following theorem.

Theorem 5.3. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. If the switching class of S contains a graph G with maximum valency $d_{\max} \leq 16$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. Let G be a graph in the switching class of S with $d_{\max} \leq 16$. By Proposition 5.1, we may assume $\rho(G) > 2$. For any vertex x in G , we denote the valency of x in G by d_x . Let H be a minimal induced subgraph of G with $\rho(H) > 2$. Let $d_{\max}(V(H))$ be the maximum valency of all vertices in $V(H)$ in G , that is, $d_{\max}(V(H)) := \max\{d_x \mid x \in V(H)\}$. Clearly, $d_{\max}(V(H)) \geq 3$.

If $d_{\max}(V(H)) = 3$, then, by Corollary 3.11, we have $n_H \leq 10$, $\varepsilon_H \geq n_H - 1$ and $\alpha(H) \geq 1$. By Lemma 5.2, we have

$$\begin{aligned} \text{rk}(S + 5\mathbf{I}) &\geq n - n_H(1 + d_{\max}(V(H))) + 2\varepsilon_H + \alpha(H) \\ &\geq n - n_H(1 + d_{\max}(V(H))) + 2(n_H - 1) + 1 \\ &\geq n - 21 \\ &> \frac{2n}{3} + 1, \end{aligned}$$

as $n \geq 277$. On the other hand, if $d_{\max}(V(H)) \geq 4$, then $n_H \leq 6$, $\varepsilon_H \geq n_H - 1$ and $\alpha(H) \geq 1$, by Corollary 3.11. By Lemma 5.2, we have

$$\begin{aligned} \text{rk}(S + 5\mathbf{I}) &\geq n - n_H(1 + d_{\max}) + 2\varepsilon_H + \alpha(H) \\ &\geq n - n_H(1 + d_{\max}) + 2(n_H - 1) + 1 \\ &\geq n - 91 \\ &> \frac{2n}{3} + 1, \end{aligned}$$

as $n \geq 277$. This shows the theorem. \square

Now, we show the main result of this section.

Theorem 5.4. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. If the independence number $\alpha([S])$ of $[S]$ satisfies $\alpha([S]) \geq 49$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. Let $\alpha := \alpha([S])$. Take a graph G in the switching class of S with independence number $\alpha(G) = \alpha$. Let C be an independent set of G of order α . We may assume that all vertices, that are not in $V(C)$, have at most $\lfloor \frac{\alpha}{2} \rfloor$ neighbours in C .

Let x be a vertex outside $V(C)$ and assume that x has r neighbours in C . The subgraph of G induced on $V(C) \cup \{x\}$ is isomorphic to $K_{1,r}(\alpha - r)$ with $\alpha - r \geq \lfloor \frac{\alpha}{2} \rfloor \geq 25$, as $\alpha \geq 49$. As $\lambda_{\min}(S(K_{1,r}(s))) \geq -5$ if and only if $(r - 4)(s - 4) \leq 36$, by Lemma 4.3 (i), we see $r \leq 4$ when $r + s = \alpha \geq 49$. That is, every vertex x outside C has at most 4 neighbours in C .

Now we show the following claim.

Claim 5.5. *The maximum valency d_{\max} of G is at most 6.*

Proof of Claim 5.5: Assume $d_{\max} \geq 7$. Let x be a vertex with valency at least 7, and let y_1, \dots, y_7 be 7 of its neighbours. Let H be the subgraph of G induced by $\{x, y_1, \dots, y_7\}$. The number of vertices in C that are in H or have at least one neighbour in H is at most $4 \times 8 = 32$. Note that $\lambda_{\min}(S(H(17))) \leq \lambda_{\min}(S(K_{1,7}(17))) < -5$, by Lemma 4.3 (i) and (ii). This shows the claim. \square

Therefore, $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$, by Theorem 5.3. \square

Note that, by Theorem 5.4 and the Ramsey theory, it follows that Conjecture 1.1 is true when n is sufficiently large, a result also obtained by Neumaier [22].

6 The switching class contains a triangle-free graph

Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. In this section, we will show that Theorem 2.2 is true when the switching class contains a triangle-free graph. Our main result of this section is as follows.

Theorem 6.1. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. Assume that the switching class of S contains a triangle-free graph G . Then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. Let G be a triangle-free graph in $[S]$. By Proposition 5.1 and Theorem 5.3, we may assume that $\rho(G) > 2$ and $d_{\max} \geq 17$. Note that $\alpha([S]) \geq 1 + d_{\max}$, as $K_{1,t}$ is switching equivalent to an independent set of order $t + 1$. Hence, if $d_{\max} \geq 48$, then, by Theorem 5.4, $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$. So we only need to consider $17 \leq d_{\max} \leq 47$. For a subgraph H of G , we denote by n_H the order of H , and by $R(H)$ the subgraph of G induced on the vertices of G that are neither in H nor have a neighbour in H .

Claim 6.2. *The graph G contains an induced subgraph isomorphic to $K_{2,3}$.*

Proof of Claim 6.2: Let x be a vertex of G with valency $d_x = d_{\max} \geq 17$. We partition the neighbours of x into 3 sets, say S_1, S_2 and S_3 , such that $|S_1| = |S_2| = 5$. Let H_i be the subgraph induced on $S_i \cup \{x\}$, for $i = 1, 2, 3$. Then $R(H_i)$ is triangle-free and satisfies $\rho(R(H_i)) < 2$, by Proposition 3.6. It follows that $n_{R(H_i)} \leq 2\alpha(R(H_i))$, by Theorem 3.9.

As $K_{1,5}(41)$ and $K_{1,7}(17)$ are in \mathcal{F}_{-5} , by Lemma 4.3 (i), we find

$$n_{R(H_1)} \leq 2\alpha(R(H_1)) \leq 2 \times 40 = 80,$$

$$n_{R(H_2)} \leq 2\alpha(R(H_2)) \leq 2 \times 40 = 80,$$

$$n_{R(H_3)} \leq 2\alpha(R(H_3)) \leq 2 \times 16 = 32.$$

Then $1 + |S_1| + |S_2| + |S_3| + n_{R(H_1)} + n_{R(H_2)} + n_{R(H_3)} \leq 1 + 47 + 80 + 80 + 32 = 240$, as $|S_1| + |S_2| + |S_3| = d_{\max} \leq 47$. This implies that there exists a vertex y satisfies $y \not\sim x$ and y has at least one neighbour in S_i , for $i = 1, 2, 3$. This shows the claim. \square

Let K be an induced subgraph of G isomorphic to $K_{2,3}$. As $K_{2,3}(19) \in \mathcal{F}_{-5}$, by Lemma 4.1, we find that $\alpha(R(K)) \leq 18$. Since $\rho(R(K)) < 2$, we have $n_{R(K)} \leq 2\alpha(R(K)) \leq 2 \times 18 = 36$. This

means that

$$\begin{aligned}
n &\leq n_K(d_{\max} + 1) - 2\varepsilon_K + n_{R(K)} \\
&\leq 5(47 + 1) - 12 + 36 \\
&= 264,
\end{aligned}$$

a contradiction. This finishes the proof of this theorem. \square

7 A new bound for the independence number

We start with the following result.

Theorem 7.1. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ of order $n \geq 277$. If the clique number $\omega([S])$ of $[S]$ satisfies $\omega([S]) \in \{2, 3, 5, 6\}$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Indeed, the case $\omega([S]) = 6$ was shown by Lemmens and Seidel [19]; the case $\omega([S]) = 2$ is trivial, as shown in Section 5; and the cases $\omega([S]) \in \{3, 5\}$ were shown by Lin and Yu [20]. Note that the cases $\omega([S]) \leq 3$ also follow from Theorem 6.1, since one can always isolate a vertex and the rest of the graph is triangle-free.

In this section we will show a new bound for the independence number for the case $\omega([S]) = 4$.

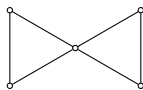
Theorem 7.2. *Let S be a Seidel matrix of order $n \geq 277$ with $\lambda_{\min}(S) = -5$ and $\omega([S]) = 4$. If the independence number $\alpha([S])$ of $[S]$ satisfies $\alpha([S]) \geq 39$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. Let $\alpha := \alpha([S])$. By Theorem 5.4, we may assume $\alpha \leq 48$. Let G be a graph in the switching class of S with $\alpha(G) = \alpha$. Let C be an independent set of G of order α . We may assume that every vertex outside C has at most $\lfloor \frac{\alpha}{2} \rfloor$ neighbours in C . Note that G contains a triangle say with vertices x, y and z , by Theorem 6.1. The set $V := \{u \in V(G) \setminus \{x, y, z\} \mid u \not\sim x, u \not\sim y, u \not\sim z\}$ is an independent set, and $|V| \leq \alpha - 1 \leq 47$, as $\omega([S]) = 4$ and $\alpha \leq 48$. Without loss of generality, we may assume that the valency of x , d_x , is at least

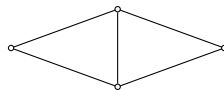
$$\lceil \frac{n - |V| - 3}{3} + 2 \rceil \geq \frac{277 - 47 - 3}{3} + 2 = 78,$$

as $n \geq 277$. This implies the following claim.

Claim 7.3. *The graph G contains one of the following graphs as an induced subgraph.*



B_1



$K_{2,1,1}$



K_4

Proof of Claim 7.3: Let N_x be the set of all neighbours of x in G . Let N denote the subgraph of G induced on N_x . Since $N_x \geq 78 > 48 \geq \alpha$, N contains an edge uv . If the edge uv is isolated, then the subgraph induced by $N_x \setminus \{u, v\}$ must contain another edge, and N contains a $2K_2$. If the edge uv is not isolated, then N contains a $K_{1,2}$ or a K_3 . Putting back the vertex x implies the claim. \square

Claim 7.4. *Let H be the induced subgraph as in Claim 7.3.*

(i) *If H is isomorphic to B_1 , then $\alpha \leq 38$;*

(ii) *If H is isomorphic to $K_{2,1,1}$, then $\alpha \leq 30$;*

(iii) *If H is isomorphic to K_4 , then $\alpha \leq 28$.*

Proof of Claim 7.4: We first show (i). Assume that G contains an induced subgraph H isomorphic to B_1 and $\alpha \geq 39$. Note that every vertex outside C has at most 5 neighbours in C , as $K_{1,t}(39-t) \in \mathcal{F}_{-5}$ unless $t \leq 5$, by Lemma 4.3 (i). Then, the number of vertices in C that are neither vertices of H nor have a neighbour in H is at least $\alpha - 5n_H \geq 39 - 25 = 14$. By Lemma 4.4, we have $B_1(14) \in \mathcal{F}_{-5}$ and this gives a contraction. This shows case (i).

(ii) and (iii) follow in similar manner as (i). In (ii), we use that $K_{1,8}(14)$, $K_{1,7}(17)$, $K_{1,6}(23)$ and $K_{2,1,1}(11)$ are all in \mathcal{F}_{-5} , by Lemma 4.3 (i) and Lemma 4.1. In (iii), we use that $K_{1,8}(14)$, $K_{1,7}(17)$, $K_{1,6}(23)$ and $K_4(5)$ are all in \mathcal{F}_{-5} , by Lemma 4.3 (i) and Lemma 4.1. This shows the claim. \square

This implies that, if $\alpha \leq 48$, then we have $\alpha \leq 38$ or $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$. By Theorem 5.4, the proof of Theorem 7.2 is now finished. \square

Once the bound $\alpha([S]) \leq 39$ is shown, we may use it again to further slash this bound, as the following theorem shows.

Theorem 7.5. *Let S be a Seidel matrix of order $n \geq 277$ with $\lambda_{\min}(S) = -5$ and $\omega([S]) = 4$. If the independence number $\alpha([S])$ of $[S]$ satisfies $\alpha([S]) \geq 29$, then $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$.*

Proof. Let $\alpha := \alpha([S])$. By Theorem 7.2, we may assume $\alpha \leq 38$. Take a graph G in the switching class of S with $\alpha(G) = \alpha$. Let C be an independent set of G of order α . We may assume that every vertex outside C has at most $\lfloor \frac{\alpha}{2} \rfloor$ neighbours in C . By Theorem 6.1, it follows that G contains a triangle, say with vertices x, y and z . The set $V := \{u \in V(G) \setminus \{x, y, z\} \mid u \not\sim x, u \not\sim y, u \not\sim z\}$ is an independent set, and $|V| \leq \alpha - 1 \leq 37$, as $\omega([S]) = 4$ and $\alpha \leq 38$. Without loss of generality, we may assume the valency d_x of x is at least

$$\lceil \frac{n - |V| - 3}{3} + 2 \rceil \geq \lceil \frac{277 - 37 - 3}{3} + 2 \rceil = 81 > 2\alpha,$$

as $n \geq 277$. Therefore the subgraph N_x induced by the neighbors of x contains an edge which is not isolated, and this implies that N_x (and G) contains $K_{2,1,1}$ or K_4 as an induced subgraph.

Claim 7.6. *Assume that G contains an induced subgraph H isomorphic to $K_{2,1,1}$ or K_4 . Then the following hold.*

(i) *If H is isomorphic to $K_{2,1,1}$, then $\alpha \leq 28$;*

(ii) *If H is isomorphic to K_4 , then $\alpha \leq 24$.*

Proof of Claim 7.6: (i) Assume that G contains an induced subgraph H isomorphic to $K_{2,1,1}$ and $\alpha \geq 29$. Then every vertex outside C has at most 5 neighbours in C , as $K_{1,8}(14)$, $K_{1,7}(17)$ and $K_{1,6}(23)$ are all in \mathcal{F}_{-5} , by Lemma 4.3 (i). Note that the number of vertices in C , that are neither in H nor have a neighbour in H , is at most 10, as $K_{2,1,1}$. Then all vertices of H have at least $\alpha - 10 \geq 29 - 10 = 19$ and at most $5n_H = 20$ neighbours in C ; in particular, none of the vertices in H belongs to C . It follows that there exists a vertex u of H with valency 3 that has two neighbours, say v and w , in C that are not adjacent to any of the other three vertices of H . This means that the subgraph H' induced on $V(H) \cup \{v, w\}$ is isomorphic to B_2 of Lemma 4.4. As $B_2(9) \in \mathcal{F}_{-5}$, by Lemma 4.4, this gives a contraction. This shows case (i).

(ii) Assume that G contains an induced subgraph H isomorphic to K_4 and $\alpha \geq 25$. Note that every vertex outside C has at most 6 neighbours in C , as $K_{1,9}(12)$, $K_{1,8}(14)$ and $K_{1,7}(17)$ are in \mathcal{F}_{-5} , by Lemma 4.3 (i). So, as $\alpha - 6n_H \geq 25 - 6 \times 4 = 1$, there is at least 1 vertex in C that is neither in H nor has a neighbour in C . This implies that G contains $K_4(1)$ as an induced subgraph, which gives a contradiction, as $\omega([S]) = 4$. This finishes the proof of case (ii). \square

This implies that, if $\alpha \leq 38$, then we have $\alpha \leq 28$ or $\text{rk}(S + 5\mathbf{I}) \geq \frac{2n}{3} + 1$. Now the theorem immediately follows from Theorem 7.2. \square

8 Pillar

Let S be a Seidel matrix, and $\omega := \omega([S])$ be the clique number of $[S]$. Take a graph G in the switching class of S with $\omega(G) = \omega$.

Let $B := \{x_1, \dots, x_\omega\}$ be the vertex set of an ω -clique inside G . We call B a base of order ω . For $U \subseteq B$, the pillar \mathcal{P}_U with respect to B is the set $\{y \notin B \mid y \sim u, \text{ if } u \in U, y \not\sim u, \text{ if } u \in B - U\}$ of vertices. Without loss of generality, we may assume $|U| \leq \lfloor \frac{\omega}{2} \rfloor$ and if $|U| = \frac{\omega}{2}$, then $x_1 \in U$. Note that $\mathcal{P}_\emptyset = \emptyset$, as otherwise, $\omega([S]) \geq \omega + 1$. Let p_U denote the cardinality of \mathcal{P}_U . Let $p_{\omega,t}$ denote the maximum cardinality of \mathcal{P}_U , where $|U| = t \leq \lfloor \frac{\omega}{2} \rfloor$. We call that \mathcal{P}_U is a (ω, t) pillar when $|U| = t$.

Let S be a Seidel matrix with $\lambda_{\min}(S) \geq -5$. In the rest of the article we will show that for the case $\omega([S]) = 4$, the Theorem 2.2 is true.

Let S be a Seidel matrix of order $n \geq 277$ with $\lambda_{\min}(S) = -5$ and $\omega([S]) = 4$. Take a graph G in $[S]$ with vertex set $V(G)$ and $\omega(G) = 4$. Let $\{x_1, \dots, x_4\}$ be a 4-base. Let $S' := S(G)$. Note that $S' + 5\mathbf{I}$ is positive semidefinite, so there exists a map $\tau : \{1, \dots, n\} \mapsto \mathbb{R}^t : x \mapsto \hat{x}$ for some positive integer t such that $\langle \hat{x}, \hat{y} \rangle = (S' + 5\mathbf{I})_{xy}$ for $x, y \in V(G)$. Let W be the vector space spanned by $\{\hat{x}_1, \dots, \hat{x}_4\}$.

8.1 A new bound for $p_{4,1}$

Let $y \in \mathcal{P}_{\{x_1\}}$. Decompose \hat{y} into $\hat{y} = h_1 + c_y$ such that $h_1 \in W$ and $c_y \in W^\perp$. Then, as $\langle \hat{y}, \hat{x}_1 \rangle = -1$ and $\langle \hat{y}, \hat{x}_i \rangle = 1$ for $i = 2, 3, 4$, it follows that $h_1 = \frac{1}{3}(\hat{x}_2 + \hat{x}_3 + \hat{x}_4)$ and $\langle c_y, c_y \rangle = 4$. Let $y' \in \mathcal{P}_{\{x_1\}} - \{y\}$. Then $\langle c_y, c_{y'} \rangle = 0$. Likewise, for $u \in \mathcal{P}_{\{x_2\}}$, \hat{u} can be decomposed into $\hat{u} = h_2 + c_u$ where $h_2 = \frac{1}{3}(\hat{x}_1 + \hat{x}_3 + \hat{x}_4) \in W$ and $c_u \in W^\perp$. Now $u \sim y$ implies that $\langle c_u, c_y \rangle = -\frac{4}{3}$ and $u \not\sim y$ implies that $\langle c_u, c_y \rangle = \frac{2}{3}$.

Lin and Yu [20] showed that if $p_{4,1} \geq 25$, then there exists exactly one $(4, 1)$ pillar having more than one vertex. The next theorem gives a similar result.

Theorem 8.1. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$ and $\omega([S]) = 4$. If there exists an edge between two different $(4, 1)$ pillars, then each of the other two $(4, 1)$ pillars contains at most 19 vertices.*

Proof. Without loss of generality, we may assume that $u \in \mathcal{P}_{\{x_2\}}, v \in \mathcal{P}_{\{x_3\}}$ and u and v are adjacent. Let $\mathcal{P}_{\{x_1\}} = \{y_1, \dots, y_p\}$, where p is the number of vertices in $\mathcal{P}_{\{x_1\}}$. Write

$$\begin{aligned}\hat{y}_i &= h_1 + c_{y_i}, \text{ where } h_1 = \frac{1}{3}(\hat{x}_2 + \hat{x}_3 + \hat{x}_4) \text{ for } i = 1, \dots, p; \\ \hat{u} &= h_2 + c_u, \text{ where } h_2 = \frac{1}{3}(\hat{x}_1 + \hat{x}_3 + \hat{x}_4); \\ \hat{v} &= h_3 + c_v, \text{ where } h_3 = \frac{1}{3}(\hat{x}_1 + \hat{x}_2 + \hat{x}_4).\end{aligned}$$

The Gram matrix M of $c_{y_1}, \dots, c_{y_p}, c_u, c_v$ is

$$M = \begin{pmatrix} 4\mathbf{I}_p & B \\ B^T & 4 & -\frac{4}{3} \\ & -\frac{4}{3} & 4 \end{pmatrix},$$

where B is a $(p \times 2)$ -matrix with each row is one of $(\frac{2}{3}, \frac{2}{3})$, $(\frac{2}{3}, -\frac{4}{3})$, $(-\frac{4}{3}, \frac{2}{3})$, and $(-\frac{4}{3}, -\frac{4}{3})$, and we denote the number of occurrences of these rows by α_{00} , α_{01} , α_{10} , and α_{11} , respectively.

Let S_1, S_2, S_3 , and S_4 be the sets of rows of M which have an occurrence of $(\frac{2}{3}, \frac{2}{3})$, $(\frac{2}{3}, -\frac{4}{3})$, $(-\frac{4}{3}, \frac{2}{3})$, and $(-\frac{4}{3}, -\frac{4}{3})$, respectively. Let $\pi = \{S_1, \dots, S_4, \{u\}, \{v\}\}$ be a partition of $\mathcal{P}_{\{x_1\}} \cup \{u\} \cup$

$\{v\}$. Observe that π is an equitable partition of M . The quotient matrix of M with respect to π is given by

$$Q = \frac{1}{3} \begin{pmatrix} 12 & 0 & 0 & 0 & 2 & 2 \\ 0 & 12 & 0 & 0 & 2 & -4 \\ 0 & 0 & 12 & 0 & -4 & 2 \\ 0 & 0 & 0 & 12 & -4 & -4 \\ 2\alpha_{00} & 2\alpha_{01} & -4\alpha_{10} & -4\alpha_{11} & 12 & -4 \\ 2\alpha_{00} & -4\alpha_{01} & 2\alpha_{10} & -4\alpha_{11} & -4 & 12 \end{pmatrix}.$$

As M is a positive semidefinite, all the eigenvalues of Q must be non-negative, by Lemma 3.7.

Let Q_i be the matrix obtained from Q by removing the i^{th} row and column of Q for $i = 5, 6$. Then eigenvalue of Q are all non-negative if and only if $\det(Q_5) \geq 0$, $\det(Q_6) \geq 0$, and $\det(Q) \geq 0$, as $4\mathbf{I}$ is positive definite.

We find that $\det(Q_5) \geq 0$ if and only if

$$\alpha_{00} + 4\alpha_{01} + \alpha_{10} + 4\alpha_{11} \leq 36, \quad (1)$$

and $\det(Q_6) \geq 0$ if and only if

$$\alpha_{00} + \alpha_{01} + 4\alpha_{10} + 4\alpha_{11} \leq 36. \quad (2)$$

Formulae (1) and (2) imply

$$\alpha_{00} + 4\alpha_{11} \leq 36 - \frac{5}{2}(\alpha_{01} + \alpha_{10}). \quad (3)$$

Furthermore, $\det(Q) \geq 0$ if and only if

$$3\alpha_{01}\alpha_{10} + (3(\alpha_{00} + 4\alpha_{11}) - 44)(\alpha_{01} + \alpha_{10}) - 32(\alpha_{00} + 4\alpha_{11} - 12) \geq 0. \quad (4)$$

Define $\beta_1 := \alpha_{00} + 4\alpha_{11}$ and $\beta_2 := \alpha_{01} + \alpha_{10}$. As $\alpha_{01}\alpha_{10} \leq (\frac{\beta_2}{2})^2$, equation (4) implies

$$\frac{1}{4}(3\beta_2 - 32)(4\beta_1 + \beta_2 - 48) \geq 0. \quad (5)$$

Equation (3) give

$$\beta_1 \leq 36 - \frac{5}{2}\beta_2. \quad (6)$$

We need to consider two cases $\beta_2 \geq 11$ and $\beta_2 \leq 10$. If $\beta_2 \geq 11$, then (5) combined with (6) gives

$$\frac{3}{4}(3\beta_2 - 32)(32 - 3\beta_2) \geq 0.$$

This is a contradiction.

So $\beta_2 \leq 10$. Now (5) implies $4\beta_1 + \beta_2 \leq 48$. Hence, $p = \alpha_{00} + \alpha_{01} + \alpha_{10} + \alpha_{11} \leq \beta_1 + \beta_2 = \frac{(4\beta_1 + \beta_2) + 3\beta_2}{4} \leq \lfloor \frac{48 + 3 \times 10}{4} \rfloor = 19$. This concludes the proof of this theorem. \square

Remark 8.2. With some extra calculations, it can be shown that $p = 19$ implies that $\alpha_{00} = 9$, $\alpha_{01} = \alpha_{10} = 5$ and $\alpha_{11} = 0$. This result can also be obtained by semidefinite integer programming if we follow the similar approach in Lin-Yu [20].

8.2 A bound for $p_{4,2}$

In this subsection, we are going to bound the order of a $(4, 2)$ -pillar.

Let S be a Seidel matrix of order n and $\lambda_{\min}(S) = -5$. Assume $\omega([S]) = 4$. Let $B := \{x_1, \dots, x_4\}$ be a 4-base and take a graph G in $[S]$ such that $\{x_1, \dots, x_4\}$ is a clique.

Consider the $\{x_1, x_2\}$ -pillar $\mathcal{P}_{\{x_1, x_2\}}$ with respect to B , say with order p . Let $G_{\mathcal{P}}$ be the subgraph of G induced on $\mathcal{P} = \mathcal{P}_{\{x_1, x_2\}}$. Now switch G with respect to $\{x_1, x_2\}$ to obtain $G_{sw}(\{x_1, x_2\})$. So the subgraph induced $\{x_1, \dots, x_4\} \cup \mathcal{P}$ is $G_{\mathcal{P}} \dot{\cup} 2K_2$. Note that $G_{\mathcal{P}}$ is triangle-free and $\alpha(G_{\mathcal{P}}) \leq \alpha([S]) - 2$.

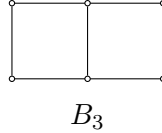
We will show the following result, and, as a corollary, we obtain a bound of the order of a $(4, 2)$ -pillar.

Theorem 8.3. *Let G be a triangle-free graph with order n_G such that $\lambda_{\min}(S(G \dot{\cup} 2K_2)) \geq -5$. Assume further that $\alpha(G) \leq 26$. For a vertex x of G , let a_x be the valency of x in G . Then the following hold:*

(i) $n_G \leq 68$;

(ii) If $n_G \geq 66$, then there exists an edge xy in G such that $a_x + a_y \leq 20$.

Before we give the proof of this theorem, we start with a few lemmas that can be verified by straightforward computations. For B_3 , see the picture below.



Lemma 8.4. *The following graphs are in \mathcal{F}_{-5} , where P_n is a path with length $n - 1$.*

(i) $K_{1,14} \dot{\cup} 2K_2$;

(ii) $H \dot{\cup} P_4 \dot{\cup} K_2$, where $H \in \{K_{2,3}, K_{1,6}, B_3\}$;

(iii) $H \dot{\cup} K_{1,3} \dot{\cup} K_2$, where $H \in \{K_{2,3}, K_{1,6}, B_3\}$.

Lemma 8.5. *For a graph G , let $G(t_1, t_2, t_3)$ be the disjoint union of G , t_1 isolated vertices, t_2 copies of K_2 and t_3 copies of P_3 , where t_1, t_2 and t_3 are non-negative integers. Then, the following hold.*

(i) *The smallest eigenvalue of $K_{2,3}(t_1, t_2 + 1, t_3)$ satisfies $\lambda_{\min}(S(K_{2,3}(t_1, t_2 + 1, t_3))) \geq -5$ if and only if $t_1 + 4t_2 + 10t_3 \leq 14$;*

(ii) The smallest eigenvalue of $K_{1,6}(t_1, t_2 + 2, t_3)$ satisfies $\lambda_{\min}(S(K_{1,6}(t_1, t_2 + 2, t_3))) \geq -5$ if and only if $t_1 + 4t_2 + 10t_3 \leq 14$;

(iii) The smallest eigenvalue of $B_3(t_1, t_2 + 2, t_3)$ satisfies $\lambda_{\min}(S(B_3(t_1, t_2 + 2, t_3))) \geq -5$ if and only if $t_1 + 4t_2 + 10t_3 \leq 16$.

Lemma 8.6. *Let G be a triangle-free graph with order n_G such that $\lambda_{\min}(G \dot{\cup} 2K_2) \geq -5$. Let a_{\max} be the maximum valency of G . If $n_G \geq 66$, then*

(i) $a_{\max} \leq 12$;

(ii) G does not contain $K_{2,3}$ as an induced subgraph.

Proof. As $K_{1,14} \dot{\cup} 2K_2 \in \mathcal{F}_{-5}$, by Lemma 8.4 (i), we have $a_{\max} \leq 13$. Suppose that $a_{\max} = 13$, and let x be a vertex with valency 13 in G . Consider a vertex $y \neq x$ which is not adjacent to x . Assume that x and y have t common neighbours. Let K be the subgraph of G induced on $\{x, y\} \cup \{z \in V(G) \mid z \sim x\}$. We observe that $\det(S(K \dot{\cup} 2K_2) + 5\mathbf{I}) \geq 0$ if and only if $(t - 3)^2 \leq 0$. It follows that every vertex, that is not adjacent to x , has 3 common neighbours with x in G . As $a_{\max} \leq 13$, we have

$$\begin{aligned} n_G &\leq 1 + a_{\max} + \frac{a_{\max}(a_{\max} - 1)}{3} \\ &\leq 1 + 13 + \frac{13(13 - 1)}{3} \\ &= 66. \end{aligned}$$

By the assumption that $n_G \geq 66$, we see that the equality must hold, and every vertex in G has valency 13. Hence, in this case, G is a strongly regular graph with parameters $(66, 13, 0, 3)$. Such a strongly regular graph does not exist as the multiplicities of the eigenvalues are non-integral, by Lemma 3.2. Hence $a_{\max} \leq 12$.

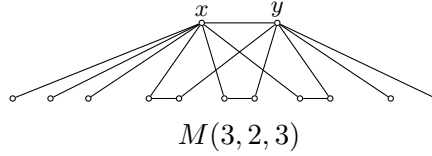
Suppose that G contains $K_{2,3}$ as an induced subgraph, say H . There are at most $n_H(1 + a_{\max}) - 2\varepsilon_H$ vertices in G that are either in H or have a neighbour in H . Let $R(H)$ be the subgraph induced on the vertices of G that are neither in H nor have a neighbour in H . Now $R(H)$ has neither P_4 nor $K_{1,3}$ as an induced subgraph by Lemma 8.4. It follows by Lemma 8.5 (i) that $R(H)$ has at most 10 vertices, as there are 2 non-adjacent K_2 outside $R(H)$ in G . So

$$\begin{aligned} n_G &\leq n_H(1 + a_{\max}) - 2\varepsilon_H + n_{R(H)} \\ &\leq 5 + 5a_{\max} - 12 + 10 \\ &\leq 63 < 66, \end{aligned}$$

a contradiction. Therefore G may not contain a $K_{2,3}$ as an induced subgraph. This finishes the proof. \square

The next lemma is needed when G does not contain a $K_{2,3}$ as an induced subgraph and it can be proved by straightforward computations.

Lemma 8.7. *Let H be the disjoint union of tK_2 and $s_1 + s_2$ isolated vertices, where t , s_1 and s_2 are non-negative integers. Let $M(s_1, s_2, t)$ be the graph obtained by adding an edge xy to H such that x is adjacent to t isolated vertices in tK_2 and s_1 isolated vertices, and y is adjacent to all vertices in H that are not adjacent to x . For example, see $M(3, 2, 3)$ as below. Then, the smallest eigenvalue of $M(s_1, s_2, t) \dot{\cup} 2K_2$ satisfies $\lambda_{\min}(S(M(s_1, s_2, t) \dot{\cup} 2K_2)) \geq -5$ if and only if $3s_1 + 3s_2 + 4t \leq 36$.*



Remark 8.8. $M(0, 0, 2)$ is the graph B_3 . Therefore if a graph does not contain B_3 , it does not contain $M(0, 0, 2)$ either.

Lemma 8.9. *Let G be a triangle-free graph with order $n_G \geq 66$ such that $\lambda_{\min}(G \dot{\cup} 2K_2) \geq -5$. For a vertex x in G , let a_x be the valency of x in G . If the independence number $\alpha(G)$ of G satisfies $\alpha(G) \leq 26$, then*

(i) $a_x + a_y \leq 20$ for all $xy \in E(G)$;

(ii) $n_G \leq 68$.

Proof. (i) Let xy be an edge of G . Let $N(xy)$ be the subgraph of G induced on the vertices outside $\{x, y\}$ and have a neighbour in $\{x, y\}$. As G is triangle-free and does not contain $K_{2,3}$ as an induced subgraph, we observe that $N(xy)$ is isomorphic to $M(s_1, s_2, t)$ of Lemma 8.7, for some non-negative integers s_1 , s_2 and t . By Lemma 8.7, $3s_1 + 3s_2 + 4t \leq 36$. Note that

$$\begin{aligned} a_x + a_y &= 2 + s_1 + s_2 + 2t \\ &= 2 + \frac{2s_1 + 2s_2 + 4t}{2} \\ &\leq 2 + \frac{36 - s_1 - s_2}{2} \\ &\leq 20. \end{aligned}$$

This shows the case (i).

(ii) We denote by $\rho(G)$ the spectral radius of G . If $\rho(G) \leq 2$, then $n_G \leq \frac{5}{2}\alpha(G) \leq \frac{5 \times 26}{2} = 65$, which is a contradiction. This implies that $a_{\max} \geq 3$. By Lemma 8.6, we have $3 \leq a_{\max} \leq 12$. Next, we will consider two cases $6 \leq a_{\max} \leq 12$ and $3 \leq a_{\max} \leq 5$.

In the following proof, we will denote by $N(H)$ (resp. $R(H)$) subgraph of G induced on the vertices outside $V(H)$ that have a neighbour (resp. no neighbours) in H , for an induced subgraph, H , of G .

Case 1. $6 \leq a_{\max} \leq 12$. First we assume that G does not contain B_3 of Lemma 8.4 as an induced subgraph. Note that the graph $M(s_1, s_2, t)$ in (i) satisfies that $t \leq 1$ since $M(0, 0, 2)$ is the graph B_3 . It follows that

$$\begin{aligned} a_x + a_y &= 2 + s_1 + s_2 + 2t \\ &= 2 + \frac{3s_1 + 3s_2 + 6t}{3} \\ &\leq \lfloor 2 + \frac{36 + 2t}{3} \rfloor \\ &= 14. \end{aligned}$$

Let v be a vertex in G with valency $a_v = a_{\max}$ and v_1, \dots, v_6 be 6 neighbours of v . Then $a_v + a_{v_i} \leq 14$ for $i = 1, \dots, 6$. Let H_1 be the subgraph of G induced on $\{v, v_1, \dots, v_6\}$. Then

$$\begin{aligned} n_{N(H_1)} &\leq a_v + \sum_{i=1}^6 a_{v_i} - 2 \times 6 \\ &= \sum_{i=1}^6 (a_v + a_{v_i}) - 5a_v - 12 \\ &\leq 6 \times 14 - 5 \times 6 - 12 \\ &= 42. \end{aligned}$$

Note that $R(H_1)$ has neither P_4 nor $K_{1,3}$ as an induced subgraph by Lemma 8.4. It follows by Lemma 8.5 (ii) that $R(H_1)$ has at most 14 vertices, as $\rho(R(H_1)) < 2$. It follows that $n_G = n_{H_1} + n_{N(H_1)} + n_{R(H_1)} \leq 7 + 42 + 14 = 63$, which is a contradiction.

So G_φ contains B_3 as an induced subgraph. Let $H_2 \cong B_3$ be an induced subgraph of G , and $V(H_2) = \{w_1, \dots, w_6\}$ and $E(H_2) = \{w_1w_2, w_3w_4, w_5w_6, w_1w_3, w_2w_4, w_3w_5, w_4w_6\}$. Note that $a_{w_i} + a_{w_{i+1}} \leq 20$ for $i = 1, 3, 5$, by (i). Then

$$\begin{aligned} n_{N(H_2)} &\leq \sum_{i=1}^6 a_{w_i} - 2 \times 7 \\ &\leq 3 \times 20 - 2 \times 7 \\ &= 46. \end{aligned}$$

Note that $R(H_2)$ has neither P_4 nor $K_{1,3}$ as an induced subgraph by Lemma 8.4. It follows by Lemma 8.5 (iii) that $R(H_2)$ has at most 16 vertices, as $\rho(R(H_2)) < 2$. So $n_G = n_{H_2} + n_{N(H_2)} + n_{R(H_2)} \leq 6 + 46 + 16 = 68$. This shows the case $6 \leq a_{\max} \leq 12$.

Case 2. $3 \leq a_{\max} \leq 5$. Let K be a minimal subgraph of G with $\rho(K) > 2$. Then, $\rho(R(K)) < 2$, by Proposition 3.6. It follows that $n_{R(K)} \leq 2\alpha(R(K)) \leq 2(\alpha(G) - \alpha(K)) \leq 52 - 2\alpha(K)$, as $\alpha(G) \leq 26$. So, $n_G \leq n_K + n_K a_{\max} - 2\varepsilon_K + 2\alpha(R(K)) \leq n_K(1 + a_{\max}) - 2\varepsilon_K + 52 - 2\alpha(K)$.

First we assume that $3 \leq a_{\max} \leq 4$. If G contains \tilde{D}_4^+ as an induced subgraph, say H_3 . Then $n_{R(H_3)} \leq 52 - 2\alpha(H_3) \leq 52 - 2 \times 4 = 44$. Note that $n_{N(H_3)} \leq 6a_{\max} - 2 \times 5 \leq 6 \times 4 - 10 = 14$. So, $n_G = n_{H_3} + n_{N(H_3)} + n_{R(H_3)} \leq 6 + 14 + 44 = 64$, this is a contradiction. Assume that G does not contain \tilde{D}_4^+ as an induced subgraph. Let H_4 be a minimal subgraph of G with $\rho(H_4) > 2$. We observe that every vertex in H_4 has valency at most 3 in G . In this case, we have $n_{H_4} \leq 10$, $\varepsilon_{H_4} \geq n_{H_4} - 1$ and $\alpha(H_4) \geq \lceil \frac{n_{H_4}}{2} \rceil$, by Corollary 3.11. This shows that

$$\begin{aligned} n_G &= n_{H_4} + n_{N(H_4)} + n_{R(H_4)} \\ &\leq n_{H_4}(1 + a_{\max}) - 2\varepsilon_{H_4} + 52 - 2\alpha(H_4) \\ &\leq 4n_{H_4} - 2(n_{H_4} - 1) + 52 - n_{H_4} \\ &\leq n_{H_4} + 54 \\ &\leq 64, \end{aligned}$$

this is a contradiction.

This follows that $a_{\max} = 5$. Let $H_5 \cong K_{1,5}$ be an induced subgraph of G . We obtain that

$$\begin{aligned} n_G &= n_{H_5} + n_{N(H_5)} + n_{R(H_5)} \\ &\leq n_{H_5}(1 + a_{\max}) - 2\varepsilon_{H_5} + 52 - 2\alpha(H_5) \\ &\leq 6 + 6 \times 5 - 2 \times 5 + 52 - 2 \times 5 \\ &= 68. \end{aligned}$$

This finishes the proof of (ii). □

Theorem 8.3 follows immediately from Lemma 8.6 and 8.9. As an easy consequence of Theorem 8.3, we obtain a bound of the order of (4, 2) pillar.

Corollary 8.10. *Let S be a Seidel matrix with $\lambda_{\min}(S) = -5$. Assume that $\omega([S]) = 4$ and $\alpha([S]) \leq 28$. Let G be a graph in the switching class of S such that $\omega(G) = 4$. Let \mathcal{P} be a (4, 2) pillar. Let $G_{\mathcal{P}}$ be the subgraph of G induced on \mathcal{P} . Let a_x be the valency of $x \in V(G_{\mathcal{P}})$ in $G_{\mathcal{P}}$. Let p denote the number of vertices in \mathcal{P} . If $p \geq 66$, then the following hold.*

(i) $a_x + a_y \leq 20$ for all $xy \in E(G_{\mathcal{P}})$.

(ii) $p \leq 68$.

9 Gallery

Let S be a Seidel matrix of order n and $\omega([S]) = 4$. Let H be a graph in $[S]$ such that there exist two adjacent vertices x_1 and x_2 satisfy that x_1 and x_2 have no common neighbours. Define the gallery with respect to $\{x_1, x_2\}$, $\mathcal{G}a(x_1, x_2)$, as the subgraph of H induced on $U(x_1, x_2) := \{y \in V(H) \mid y \text{ is not adjacent to } x_1 \text{ nor } x_2\}$.

Lemma 9.1. *Let $u, v \in U(x_1, x_2)$ such that $u \sim v$, where $U(x_1, x_2)$ is defined as above. Then the following hold:*

- (i) *The set $W_{\{u\}} := \{y \in U(x_1, x_2) \mid y \sim u, y \not\sim v\} \setminus \{v\}$ is the $(4, 1)$ pillar $\mathcal{P}_{\{v\}}$ with respect to the 4-base $\{u, v, x_1, x_2\}$.*
- (ii) *The set $W_{\emptyset} := \{y \in U(x_1, x_2) \mid y \not\sim u, y \not\sim v\} \setminus \{u, v\}$ is the $(4, 2)$ pillar $\mathcal{P}_{\{u, v\}}$ with respect to the 4-base $\{u, v, x_1, x_2\}$.*
- (iii) *$U(x_1, x_2)$ is the disjoint union of $\{u, v\}$, $W_{\{u\}}$, $W_{\{v\}}$, and W_{\emptyset} .*

Proof. This follows straightforward from the definition. In particular, (iii) follows from the assumption that $\omega([S]) = 4$. □

Now we come to our main result in this section.

Theorem 9.2. *Let S be a Seidel matrix of order $n \geq 277$ with $\lambda_{\min}(S) = -5$, $\alpha([S]) \leq 28$ and $\omega([S]) = 4$. We define $\mathcal{G}a(x_1, x_2)$ and $U(x_1, x_2)$ as above. For $y \in \mathcal{G}a(x_1, x_2)$, let b_y be the valency of y in $\mathcal{G}a(x_1, x_2)$. Let b_{\max} be the maximum valency of $\mathcal{G}a(x_1, x_2)$. Let κ be the order of $\mathcal{G}a(x_1, x_2)$. If $b_{\max} \leq 20$, then the following hold:*

- (i) $\kappa \leq 105$;
- (ii) *If $\kappa = 105$, then there exists an edge uv inside $\mathcal{G}a(x_1, x_2)$ such that $b_u = b_v = 20$.*

Proof. Assume that $\kappa \geq 105$. Let uv be an edge in $\mathcal{G}a(x_1, x_2)$. As $b_{\max} \leq 20$, note that

$$p_{\{u, v\}} = \kappa - (b_u + b_v) \geq 105 - 2 \times 20 = 65.$$

Moreover, if equality holds, then we have $p_{\{u\}} = p_{\{v\}} = 20$, $p_{\{u, v\}} = 65$ and $\kappa = 105$. Now we assume that $p_{\{u, v\}} \geq 66$. Since $p_{\{u, v\}} > 28 \geq \alpha([S])$, there exist two adjacent vertices in $\mathcal{P}_{\{u, v\}}$, say x and y . By Lemma 9.1 and Corollary 8.10, we have $p_{\{u, v\}} \leq 68$. It follows that

$$b_x + b_y = \kappa - p_{\{u, v\}} \geq 105 - 68 = 37.$$

Let a_x and a_y be the valencies of x and y in the subgraph of $\mathcal{G}a(x_1, x_2)$ induced on $\mathcal{P}_{\{u, v\}}$, respectively. By Corollary 8.10, we see that $a_x + a_y \leq 20$. This means that x and y have

$(b_x + b_y) - (a_x + a_y) \geq 37 - 20 = 17$ neighbours outside $\mathcal{P}_{\{u,v\}}$ in $\mathcal{G}a(x_1, x_2)$. Without loss of generality, we may assume that u and x have at least $\lceil \frac{17}{4} \rceil = 5$ common neighbours in $\mathcal{G}a(x_1, x_2)$.

Let w_1, \dots, w_5 are 5 common neighbours of u and x .

Let K be the subgraph of $\mathcal{G}a(x_1, x_2)$ induced on $\{u, x, w_1, w_2, w_3\}$. Then there are at most $5 + b_u + b_x + b_{w_1} + b_{w_2} + b_{w_3} - 2 \times 6 - 2 = b_u + b_x + b_{w_1} + b_{w_2} + b_{w_3} - 9$ vertices in $\mathcal{G}a(x_1, x_2)$ that are either in K or have a neighbour in K . Let $R(K)$ be the subgraph induced on the vertices of $\mathcal{G}a(x_1, x_2)$ that are neither in K nor have a neighbour in K . Now $R(K)$ has neither P_4 nor $K_{1,3}$ as an induced subgraph by Lemma 8.4. By Lemma 8.5 (i), we see that $R(K)$ has at most 14 vertices, as $\rho(R(K)) < 2$. It follows that

$$\begin{aligned} \kappa &\leq b_u + b_x + b_{w_1} + b_{w_2} + b_{w_3} - 9 + 14 \\ &\leq 5b_{\max} + 5 \\ &\leq 5 \times 20 + 5 \\ &= 105, \end{aligned}$$

and if $\kappa = 105$ holds, then we have $b_u = b_x = b_{w_1} = b_{w_2} = b_{w_3} = 20$. This shows the theorem. \square

We will show the following theorem in the remaining of this section. This finishes the proof of Theorem 2.2.

Theorem 9.3. *Let S be a Seidel matrix of order n and $\lambda(S) = -5$. If $\omega([S]) = 4$ and $\alpha([S]) \leq 28$, then $n \leq 276$.*

Proof. Let $\mathcal{G}a(x, y)$ be the gallery with respect to xy with order q_{xy} . For a 4-base $\{x_1, \dots, x_4\}$ denote for $U \subseteq \{x_1, \dots, x_4\}$ the U -pillar with respect to $\{x_1, \dots, x_4\}$ by \mathcal{P}_U with order p_U .

Let $\{x_1, \dots, x_4\}$ be a 4-base. If $p_{\{x_1\}} \geq 20$, then $p_{\{x_2\}} + p_{\{x_3\}} + p_{\{x_4\}} \leq \alpha([S])$, by Theorem 8.1. It follows that

$$\begin{aligned} n &= 4 + p_{\{x_1\}} + p_{\{x_2\}} + p_{\{x_3\}} + p_{\{x_4\}} + p_{\{x_1, x_2\}} + p_{\{x_1, x_3\}} + p_{\{x_1, x_4\}} \\ &\leq 4 + 2\alpha([S]) + 3p_{4,2} \\ &\leq 4 + 2 \times 28 + 3 \times 68 \\ &= 264, \end{aligned}$$

as $\alpha([S]) \leq 28$ and $p_{4,2} \leq 68$.

Now we may assume that $p_{\{x_i\}} \leq 19$ for $i = 1, \dots, 4$. As $q_{x_1x_2} = 2 + p_{\{x_3\}} + p_{\{x_4\}} + p_{\{x_1, x_2\}}$ and $q_{x_3x_4} = 2 + p_{\{x_1\}} + p_{\{x_2\}} + p_{\{x_1, x_2\}}$, we observe that $q_{x_1x_2} + q_{x_3x_4} = 4 + p_{\{x_1\}} + p_{\{x_2\}} + p_{\{x_3\}} + p_{\{x_4\}} + 2p_{\{x_1, x_2\}}$. By Theorem 9.2, we have $q_{x_1x_2} \leq 105$ and, if $q_{x_1x_2} = 105$, then we may assume

that $p_{\{x_3\}} = p_{\{x_4\}} = 19$ and hence $p_{\{x_1, x_2\}} = 65$. It follows that

$$\begin{aligned} n &= 4 + p_{\{x_1\}} + p_{\{x_2\}} + p_{\{x_3\}} + p_{\{x_4\}} + p_{\{x_1, x_2\}} + p_{\{x_1, x_3\}} + p_{\{x_1, x_4\}} \\ &= \frac{1}{2}(q_{x_1x_3} + q_{x_2x_4} + q_{x_1x_4} + q_{x_2x_3}) + p_{\{x_1, x_2\}} \\ &\leq \frac{4 \times 105}{2} + 65 \\ &= 275. \end{aligned}$$

So we may assume that $q_{x_ix_j} \leq 104$ for all $1 \leq i < j \leq 4$. We find

$$\begin{aligned} n &= 4 + p_{\{x_1\}} + p_{\{x_2\}} + p_{\{x_3\}} + p_{\{x_4\}} + p_{\{x_1, x_2\}} + p_{\{x_1, x_3\}} + p_{\{x_1, x_4\}} \\ &= \frac{1}{2}(q_{x_1x_3} + q_{x_2x_4} + q_{x_1x_4} + q_{x_2x_3}) + p_{\{x_1, x_2\}} \\ &\leq \frac{4 \times 104}{2} + 68 \\ &= 276. \end{aligned}$$

This finishes the proof. □

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References

- [1] I. Balla, F. Dräxler, P. Keevash, and B. Sudakov. Equiangular lines and spherical codes in euclidean space. *Inventiones Mathematicae*, 211(1):179–212, 2018.
- [2] E. Bannai and E. Bannai. A survey on spherical designs and algebraic combinatorics on spheres. *European Journal of Combinatorics*, 30(6):1392–1425, 2009.

- [3] A. Barg and W.-H. Yu. New bounds for equiangular lines. *Contemporary Mathematics*, 625:111–121, 2014.
- [4] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs*. Springer Berlin Heidelberg, 1989.
- [5] A.E. Brouwer and W.H. Haemers. *Spectra of graphs*. Springer New York, 2012.
- [6] B. Bukh. Bounds on equiangular lines and on related spherical codes. *SIAM Journal on Discrete Mathematics*, 30(1):549–554, 2016.
- [7] P.J. Cameron. Strongly regular graphs. *Topics in Algebraic Graph Theory*, 102:203–221, 2004.
- [8] H. Cohn and A. Kumar. Universally optimal distribution of points on spheres. *Journal of the American Mathematical Society*, 20(1):99–148, 2007.
- [9] D. de Caen. Large equiangular sets of lines in euclidean space. *The Electronic Journal of Combinatorics*, 7(1):55, 2000.
- [10] M. Fickus, D.G. Mixon, and J. Jasper. Equiangular tight frames from hyperovals. *IEEE Transactions on Information Theory*, 62(9):5225–5236, 2016.
- [11] C. Godsil and G. Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.
- [12] G. Greaves. Equiangular line systems and switching classes containing regular graphs. *Linear Algebra and its Applications*, 536:31–51, 2018.
- [13] G. Greaves, J.H. Koolen, A. Munemasa, and F. Szöllösi. Equiangular lines in Euclidean spaces. *Journal of Combinatorial Theory, Series A*, 138:208–235, 2016.
- [14] G. Greaves, J. Syatriadi, and P. Yatsyna. Equiangular lines in low dimensional euclidean spaces. *arXiv preprint arXiv:2002.08085*, 2020.
- [15] G. Greaves and P. Yatsyna. On equiangular lines in 17 dimensions and the characteristic polynomial of a seidel matrix. *Mathematics of Computation*, 88(320):3041–3061, 2019.
- [16] J. Haantjes. Equilateral point-sets in elliptic two- and three-dimensional spaces. *Nieuw Archief voor Wiskunde*, 22(2):355–362, 1948.
- [17] Z.-L. Jiang and A. Polyanskii. Forbidden subgraphs for graphs of bounded spectral radius, with applications to equiangular lines. *Israel Journal of Mathematics*, 2019.

- [18] Z.-L. Jiang, J. Tidor, Y. Yao, S.-T. Zhang, and Y.-F. Zhao. Equiangular lines with a fixed angle. *arXiv preprint arXiv:1907.12466*, 2019.
- [19] P.W.H. Lemmens and J.J. Seidel. Equiangular lines. *Journal of Algebra*, 24(3):494–512, 1973.
- [20] Y.-C.R. Lin and W.-H. Yu. Equiangular lines and the Lemmens-Seidel conjecture. *Discrete Mathematics*, 2019. doi: 10.1016/j.disc.2019.111667.
- [21] Y.-C.R. Lin and W.-H. Yu. Saturated configuration and new construction of equiangular lines. *Linear Algebra and its Applications*, 588:272–281, 2020.
- [22] A. Neumaier. Graph representations, two-distance sets, and equiangular lines. *Linear Algebra and its Applications*, 114:141–156, 1989.
- [23] D.E. Taylor. *Some topics in the theory of finite groups*. PhD thesis, University of Oxford, 1971.
- [24] J.H. van Lint and J.J. Seidel. Equilateral point sets in elliptic geometry. *Indagationes Mathematicae*, 28(3):335–348, 1966.