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"Information Structures with Unawareness"

by

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Information Structures with Unawareness[†]

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Abstract

I construct a state space model with unawareness following Aumann (1976). Dekel, Lipman and Rustichini (1998a) show that standard state space models are incapable of representing unawareness. The model circumvents the impossibility result by endowing the agent with a subjective state space that differs from the full state space when he has the unawareness problem. Information is modeled as a pair, consisting of both factual information and *awareness information*. The model preserves the central properties of the standard information partition model.

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> "There are things we know that we know. There are known unknowns - that is to say, there are things that we now know we don't know. But there are also unknown unknowns. There are things we do not know we don't know."
>
> Donald Rumsfeld, former U.S. Secretary of Defense

1 Introduction

A person is unaware of an event if he does not know it, and he does not know that he does not know it, and so on *ad infinitum*. In real life, *formulating* a decision problem, including recognizing all relevant uncertainties and available options, is at least as important as finding the solution to the formulated problem. Being unaware of some aspects of the situation is a common problem. For example, prior to the 9/11 attacks, most of us did not know that terrorists might use civilian aircraft as a weapon, and more importantly, we did not know that we did not know this. We were simply unaware of this possibility.[1]

Formalizing the concept of unawareness, however, turns out to be a difficult task. The prevailing model of uncertainty in economics is the information partition model: uncertainties are represented by a state space; at each state, the agent is informed of the corresponding partition element. But then the agent cannot be unaware of anything: having an information partition implies whenever the agent doesn't know an event, he knows he doesn't know it. Dekel, Lipman and Rustichini (1998)(henceforth DLR) further show that the problem is fundamental: *any* model based on the standard state space specification, regardless of information being partitional or not, necessarily imposes either full awareness or full unawareness.

In this paper, I provide a model that generalizes the standard information partition model to allow agents to have nontrivial unawareness as informally defined above. The main idea is as follows. Fix the set of payoff-relevant uncertainties. One can think of them as a set of relevant questions. If the agent is unaware of a question, then a message reminding the agent of the question itself must be informative. Such information is fundamentally different from the kind of *factual information* in the standard models. Modeling unawareness is equivalent to modeling such *awareness information*. Call the state space specifying all the payoff-relevant uncertainties *the full state space*. Since one can only reason about things of which one is aware, if the agent is unaware of an uncertainty, then his reasoning must be contained in a state space that lacks any specification of this uncertainty. Therefore, I model awareness information by the agent's *subjective state spaces* that are less detailed than the full state space when the agent has the unawareness problem.

[1] Understanding such unawareness may also help to develop future models where agents are aware of *the possibility* that they could be unaware of *something*, which several authors have conjectured to be an explanation for issues such as incomplete contracting that are difficult to account for otherwise. See Dekel, Lipman and Rustichini (1998b) for a survey of works along this line.

As an example, consider the following episode: Sherlock Holmes and Watson are investigating a crime. A horse has been stolen and the keeper was killed. From the narration of the local police, Holmes notices the dog in the stable did not bark that night and hence concludes that there was no intruder in the stable. Watson, on the other hand, although he also knows the dog did not bark – he himself mentioned this fact to Holmes – somehow does not come up with the inference that there was no intruder.

The feature I would like to capture in this story is the following. Watson is unaware of the possibility that there was no intruder and hence fails to recognize the factual information "there was no intruder" contained in the message "the dog did not bark." Had someone asked Watson, "Could there have been an intruder in the stable that night?," he would have recognized his negligence and replied, "Of course not, the dog did not bark!"

The relevant question in this example is whether there was an intruder in the stable that night. Call it question t and let 1_t and 0_t denote "yes there was an intruder" and "no there was no intruder," respectively. Write $a = (1_t, \Delta)$, $b = (0_t, \Delta)$, where Δ stands for "*cogito ergo sum*." The full state space is $\{a, b\}$. The dog barked in a and did not bark in b, yielding the full information partition $\{\{a\}, \{b\}\}$. However, in b, Watson is unaware of question t, and hence from his perspective, no state contains any answer to question t. I model this by letting Watson have a subjective state space $\{\Delta\}$ at b, where the answer to question t is dropped from *each* state. Since the question of an intruder never occurs to Watson, the factual information "the dog did not bark," or $\{b\}$, does not "ring a bell" in his mind. To Watson, the information he has is simply $\{\Delta\}$, the projection of $\{b\}$ on his subjective state space. As a consequence, Watson does not know $\{a, b\}$, and does not know that he does not know it.[2]

The question "Could there have been an intruder in the stable that night?" amounts to providing the awareness information $\{t\}$ to Watson, causing Watson to add to his subjective state space specifications regarding whether there was an intruder. Consequently, Watson updates $\{\Delta\}$ to $\{a, b\}$, recognizes the information partition $\{\{a\}, \{b\}\}$, and obtains the knowledge "there was no intruder" as a result of simply being asked a question.

The model is a natural generalization of Aumann (1976). In Aumann's model, a state specifies both resolutions of external uncertainties and the agent's knowledge. In this model, a full state specifies what the agent is or is not aware of, in addition to the resolution of external uncertainties and the agent's knowledge. Information takes the form of a pair, consisting of awareness information, represented by the set of questions of which the agent is aware, and factual information, represented by an event in the full state space. The agent is said to know an event E if and only if his awareness information allows him to recognize E in his subjective state space, and his factual information implies E is

[2]Ely (1998) proposes a similar framework in the context of the Watson example, where the information structure is represented by state-contingent partitions of the state space. Li (2008a) develops a general model along this line, and shows it is equivalent to the current model.

true. Under a "rational" information structure,[3] the model delivers the desired concept of unawareness: the agent is unaware of an event E if and only if he does not know E, and does not know he does not know E, and so on. Moreover, knowledge preserves the central properties of knowledge in the standard information partition model.

There has been fruitful research in modeling unawareness using syntax. For example, Fagin and Halpern (1988) add an additional modal operator of awareness to the standard Kripke structure and postulate that awareness is a prerequisite for any explicit knowledge. Modica and Rustichini (1999), on the other hand, expand the Kripke structure to a lattice of state spaces, including both the objective worlds and the agent's subjective perceptions, which allows them to define knowledge based on the agent's perception, and further define unawareness from knowledge. Halpern (2001) shows the latter can be viewed as a special case of the former.

While this research has greatly improved our understanding of unawareness, the tools developed along this line are unfamiliar to many economists. Unlike the above models, the current model uses the state space specification and set operators only, without explicit syntax, to model unawareness and knowledge. In terms of modeling, on the one hand, like Fagin and Halpern (1988), I introduce a separate unawareness operator; on the other hand, like Modica and Rustichini (1999), I define knowledge through the agents' subjective models. The key difference between the current paper and the above two papers is the modeling of the agent's subjective models, which are absent in the former and taken as the primitive in the latter. As a consequence, while a full state in the current model is the analogue of Aumann's state, the same cannot be said about the states in the above models. In an independently conceived work, Heifetz, Meier and Schipper (2006) propose a model using a state-space specification that can be viewed as a set-theoretic version of Modica and Rustichini (1999). They use a lattice of state spaces, ordered by "expressive power," and allow the possibility set at a state to reside in a different state space. Like Modica and Rustichini (1999), Hiefetz, Meier and Schipper derive unawareness from the knowledge hierarchy.

The rest of the paper is organized as follows: Section 2 presents the model of unawareness, which I dub "the product model" for the use of the product structure of the state space. Section 3 characterizes the knowledge hierarchy with nontrivial unawareness. Section **??** discusses extending the product model to the multi-agent environment. Section 4 concludes. Proofs are collected in the Appendix.

2 The Product Model

2.1 A Review of the Standard Model

In the standard model, the primitive is a pair (Ω, P), where Ω is a set of states and P maps Ω to all non-empty subsets of Ω. The interpretation is, at ω, the agent considers

[3] See page 6 for the precise definition of a rational information structure.

the true state to be within the set $P(\omega)$. P is called a *possibility correspondence*. If the image of P partitions Ω and $\omega \in P(\omega)$ for all $\omega \in \Omega$, then the agent is said to have an *information partition* and the model is *partitional*. Otherwise it is *non-partitional*.

For any event $E \subseteq \Omega$, the event "the agent knows E," denoted by $K(E)$, is:

$$K(E) = \{\omega : P(\omega) \subseteq E\}. \tag{2.1}$$

Higher-order knowledge is obtained by iterating K. Consider the following properties:[4]

K1: $K(\Omega) = \Omega$;

K2: $E \subseteq F \Rightarrow K(E) \subseteq K(F)$;

K3: $K(E) \cap K(F) = K(E \cap F)$;

K4: $K(E) \subseteq E$;

K5: $K(E) \subseteq KK(E)$;

K6: $\neg K(E) \subseteq K\neg K(E)$.

K1-3 are basic properties of knowledge, while K4-6 arguably reflect the agent's *rationality* in processing information. It is well understood that knowledge defined in (2.1) always satisfies K1-3, while it satisfies K4-6 only if the model is partitional.[5]

In particular, K6 implies whenever the agent does not know an event, he knows he does not know it, which rules out unawareness as defined at the beginning of the article. DLR further point out the tension actually originates from the state space specification rather than the partitional information structure. They introduce an unawareness operator U, which, for any event $E \subseteq \Omega$, yields the event "the agent is unaware of E." They consider three basic properties of unawareness: for any $E \subseteq \Omega$,

U1: $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$;

U2: $U(E) \subseteq UU(E)$;

U3: $KU(E) = \emptyset$.

U1 says unawareness implies at least lack of knowledge up to the second-order; U2 says unawareness of an event implies unawareness of unawareness of this event; U3 says under no circumstances can one know exactly what one is unaware of. DLR show U1-3 imply one critical property of unawareness: whenever the agent is unaware of

[4]In places where there is no risk of confusion, I omit the parentheses when applying the operators.

[5]Bacharach (1985) shows the reverse is also true: knowledge satisfies K1-6 if and only if the agent has an information partition. For more on the standard information partition model, see Fagin, Halpern, Moses and Vardi(1995).

something, he must not know the state space, i.e., $U(E) \subseteq \neg K(\Omega)$ for any $E \neq \emptyset$. But then adding either K1 (the agent knows tautologies) or K2 (the agent can do logical deductions) eliminates nontrivial unawareness. To circumvent these impossibility results, the following model allows the agent to have subjective state spaces that are less complete than the full state space when he has nontrivial unawareness.

2.2 The Primitives

The model consists of a state space and an information structure defined on the state space, from which the agent's subjective models are derived. In order to differentiate the full model from the derived subjective models, I add a "$*$" to elements in the full model whenever possible.

I explore a product structure on the state space. Without loss of generality, one can think of the set of payoff-relevant uncertainties as a set of questions with binary answers.[6] Let Q^* denote the set of questions. The full state space Ω^* can be written as:

$$\Omega^* = \prod_{q \in Q^*} \{1_q, 0_q\} \times \{\Delta\}. \tag{2.2}$$

Recall that Δ is the symbol for "*cogito ergo sum*." Throughout the paper, I refer to a question q together with its answers $\{1_q, 0_q\}$ as an uncertainty.

I model information by a pair (W^*, P^*). The first component W^* is the awareness function, associating each state in Ω^* with a subset of Q^*, while the second component P^* is a possibility correspondence, associating each state with a nonempty subset of Ω^*. The interpretation is, at each state, the agent receives information that contains two types of contents: the awareness information, indicating which uncertainties prevail, represented by W^*; and the factual information, indicating which answer obtains for *each* question, represented by P^*.[7] I call the pair (W^*, P^*) generalized information structure.

I focus attention on a "partitional" generalized information structure, which seems natural given that (W^*, P^*) represents the information the agent receives.

Definition 1 *The generalized information structure* (W^*, P^*) *is* **rational** *if it satisfies* $\omega^* \in P^*(\omega^*)$ *for all* $\omega^* \in \Omega^*$ *and that* $\omega_1^* \in P^*(\omega_2^*) \Rightarrow (W^*(\omega_1^*), P^*(\omega_1^*)) = (W^*(\omega_2^*), P^*(\omega_2^*))$.

At each state, the agent has a *subjective model* of the world, given his awareness. At ω^*, the agent's subjective state space only contains those uncertainties of which he is

[6] Li (2008a) shows how to construct an equivalent model in arbitrary state spaces.

[7] Although mathematically identical to the possibility correspondence P in the standard model, P^* in this model has a different interpretation. In the standard model (Ω, P), $P(\omega)$ is interpreted as the set of states *the agent considers possible at* ω; in the product model, $P^*(\omega^*)$ is interpreted as the factual content of the agent's information at ω^*. In this sense P is subjective – it describes possible worlds from the agent's perspective – while P^* is objective. The counterpart of P in the product model is the subjective factual information $P_{\omega^*}(s(\omega^*))$ – see below.

aware. Let $\Omega(\omega^*)$ denote the agent's subjective state space at ω^*, then,

$$\Omega(\omega^*) = \prod_{q \in W^*(\omega^*)} \{1_q, 0_q\} \times \{\Delta\} . \tag{2.3}$$

Every subjective state $\omega \in \Omega(\omega^*)$ leaves some questions unanswered and thus gives a "blurry" picture of the environment. Different subjective state spaces blur the full state space in different ways. For simplicity, for any $Q \subseteq Q^*$, I write $\Omega(Q)$ as a shorthand for the subjective state space defined by Q:

$$\Omega(Q) \equiv \prod_{q \in Q} \{1_q, 0_q\} \times \{\Delta\} .$$

The collection of all possible subjective state spaces is:

$$\mathcal{S} = \{\Omega(Q) : Q \subseteq Q^*\} .$$

There is a natural order over $\mathcal{S}$. I say $\Omega(Q_1)$ is weakly coarser than $\Omega(Q_2)$, or equivalently, $\Omega(Q_2)$ is weakly finer than $\Omega(Q_1)$, if $Q_1 \subseteq Q_2$.[8] Finally, for any $\Omega \in \mathcal{S}$, let the operator $\mathbb{P}^{\Omega}$ map elements or subsets of any state space weakly finer than Ω to their projections on Ω.

Intuitively, the agent can only recognize those factual contents of his information regarding uncertainties of which he is aware. Thus for any $\omega^* \in \Omega^*$, I define the subjective possibility correspondence at ω^*, denoted by P_{ω^*}, as follows: for any $\omega \in \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*))$,

$$P_{\omega^*}(\omega) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)). \tag{2.4}$$

The map P_{ω^*} represents the agent's subjective factual information structure at ω^*. Thus, the pair $(\Omega(\omega^*), P_{\omega^*})$ represent the agent's subjective model at ω^*. Let $s(\omega^*) = \mathbb{P}^{\Omega(\omega^*)}(\omega^*)$ denote the projection of the true state on the agent's subjective state space at that state. At ω^*, the agent considers subjective states in $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*))$ possible, which equals to $P^*(\omega^*)$ only if the agent is fully aware.

Example 1. Suppose Charlie has an episodic hearing problem that causes him to hear nothing when he experiences it, which prevents him from telling whether it rains outside. Suppose Charlie is never aware of the hearing problem.

This can be modeled as follows. Let r, p denote the questions "whether it rains" and "whether Charlie experiences the hearing problem," respectively.

$Q^* = \{r, p\}$;
$\Omega^* = \prod_{q \in Q^*} \{1_q, 0_q\} \times \{\Delta\} = \{(1_r, 1_p, \Delta), (1_r, 0_p, \Delta), (0_r, 1_p, \Delta), (0_r, 0_p, \Delta)\}$;
$W^*(\omega^*) = \{r\}$ for all $\omega^* \in \Omega^*$;

[8] Note that sets in $\mathcal{S}$ form a lattice under this order.

P^* induces the full information partition $\{\{(1_r, 1_p, \Delta), (0_r, 1_p, \Delta)\}, \{(1_r, 0_p, \Delta)\}, \{(0_r, 0_p, \Delta)\}\}$.

At $(1_r, 1_p, \Delta)$, Charlie receives the factual information $\{(1_r, 1_p, \Delta), (0_r, 1_p, \Delta)\}$. However, being unaware of the hearing problem, Charlie realizes only that he does not hear whether it rains. This is reflected in his subjective model at $(1_r, 1_p, \Delta)$:

$\Omega((1_r, 1_p, \Delta)) = \prod_{q \in W^*((1_r,1_p,\Delta))} \{1_q, 0_q\} \times \{\Delta\} = \{(1_r, \Delta), (0_r, \Delta)\}$,
$P_{(1_r,1_p,\Delta)}(s(1_r, 1_p, \Delta)) = P_{(1_r,1_p,\Delta)}((1_r, \Delta)) = \mathbb{P}^{\{(1_r,\Delta),(0_r,\Delta)\}}(\{(1_r, 1_p, \Delta), (0_r, 1_p, \Delta)\}) = \{(1_r, \Delta), (0_r, \Delta)\}$.

2.3 The Events

In the standard model, two events are different if and only if they contain different facts. In the product model, two events can differ in their descriptions which involve different awareness, as well as in their factual contents. For example, in the hearing problem example, the events "it rains, and there is a possibility that Charlie has a hearing problem" and "it rains," although expressing essentially the same facts, are different events because their descriptions involve different levels of awareness.

By construction, subjective states from different subjective state spaces embed different levels of awareness. Take any $\Omega \in \mathcal{S}$ and $\omega \in \Omega$, the set function

$$q(\omega) = \{q \in Q^* : 1_q \text{ or } 0_q \text{ is an element in the tuple } \omega\}$$

is well-defined and one-to-one. It is easy to see that $\omega_1, \omega_2 \in \Omega$ implies $q(\omega_1) = q(\omega_2)$, which allows one to extend the domain of the function q to the set of all nonempty subsets of any subjective state space: for any $\Omega \in \mathcal{S}$ and $E \subseteq \Omega, E \neq \emptyset$, $q(E) = q(\omega)$ for any $\omega \in E$.

I further introduce a collection of special empty sets to complete the construction. For any $\Omega \in \mathcal{S}$, I add to its power set 2^Ω the object $\emptyset_\Omega$, which represents the empty set tagged with the awareness information $q(\Omega)$. This object behaves in the same way as the usual empty set, except that it is confined to its state space. That is, $\emptyset_\Omega$ is a subset of E if and only if $E \subseteq \Omega, E \neq \emptyset$. To fully incorporate $\emptyset_\Omega$ to 2^Ω, I let the usual empty set be a subset of $\emptyset_\Omega$ and that the intersection of all disjoint $E, F \in 2^\Omega$ be $\emptyset_\Omega$.[9,10]

[9]Strictly speaking, this requires one to redefine the set operations. For notational ease, I use the conventional symbols for them: for any sets $E, F \in 2^\Omega$, the set inclusion, intersection, union and complement notions are defined in the usual way, except that for disjoint E and F, $E \cap F = \emptyset_\Omega$ instead of $\emptyset$; for any $E \subseteq \Omega, E \neq \emptyset$, one has $\emptyset_\Omega \subseteq E, \emptyset_\Omega \cup E = E$, $\emptyset_\Omega \cap E = \emptyset_\Omega$, $E \setminus \emptyset_\Omega = E$.

[10]An alternative way to think about the multiple empty sets is that an event E in the model is actually a pair (Ω, E) where $E \subseteq \Omega$. The first object in the pair, $\Omega \in \mathcal{S}$, represents the awareness embedded in E, while the second object represents the involved facts. In particular, $\emptyset_\Omega$ should be interpreted as the pair $(\Omega, \emptyset)$. Then the usual set inclusion can be extended to this space by letting $(\Omega, E) \subseteq (\Omega', F)$ if and only if $\Omega = \Omega'$ and $E \subseteq F$, and similarly for other set operations.

I extend the domain of the function q to include the collection $\{\emptyset_\Omega : \Omega \in \mathcal{S}\}$ by defining: $q(\emptyset_\Omega) = q(\Omega)$.

Finally, using the redefined set operations, I let the collection of events in this model be:

$$\mathcal{E}^p = \{E \subseteq \Omega : \Omega \in \mathcal{S}, E \neq \emptyset\}.$$

Notice $\{\emptyset_\Omega : \Omega \in \mathcal{S}\} \subseteq \mathcal{E}^p$ and the function $q : \mathcal{E}^p \to 2^{Q^*}$ is well-defined.

Definition 2 *For any $E \in \mathcal{E}^p$, let $\Omega \in \mathcal{S}$ be such that $q(E) \subset q(\Omega)$, the event*

$$\left\{\omega \in \Omega : \mathbb{P}^{\Omega(q(E))}(\omega) \in E\right\}$$

is called the **elaboration of E in Ω**, *denoted by E_Ω.*

Elaborations of an event are events that contain the same factual content as E but incorporate more awareness. For example, the event $\{(1_r, 1_p, \Delta), (1_r, 0_p, \Delta)\}$ is the elaboration of $\{(1_r, \Delta)\}$ in the full state space, i.e., $\{(1_r, 1_p, \Delta), (1_r, 0_p, \Delta)\} = \{(1_r, \Delta)\}_{\Omega^*}$. Both events represent the fact "it rains," but the former also contains awareness information regarding Charlie's hearing problem, while the latter does not.

Since the logical relations between events, such as logical consequences, conjunctions and disjunctions, concern only facts, they are preserved by elaborations. Thus one can investigate logical relations between two arbitrary events E and F using their minimal elaborations that live in the same space, i.e., the space $\Omega(q(E) \cup q(F))$. This observation suggests the following definitions of extended set relations and operations on elements of $\mathcal{E}^p$ and their connections to logical relations:

Definition 3 Extended set relations and operations[11]

1. **Extended set inclusion** *(logical consequence): E is an extended (weak) subset of F, denoted by $E \subseteq_* F$, if $E_{\Omega^*} \subseteq F_{\Omega^*}$;*

2. **Extended set intersection** *(conjunction): $E \cap_* F \equiv E_{\Omega(q(E)\cup q(F))} \cap F_{\Omega(q(E)\cup q(F))}$;*

3. **Extended set union** *(disjunction): $E \cup_* F \equiv E_{\Omega(q(E)\cup q(F))} \cup F_{\Omega(q(E)\cup q(F))}$.*

Finally, the **negation** of a subjective event involves the same amount of awareness, and hence is identified with the set complement operation *with respect to the corresponding subjective state space*:

$$\neg E = \Omega(q(E)) \setminus E.$$

[11] Notice these relations and operations reduce to the usual ones for events from the same space.

2.4 The Knowledge and Unawareness Operators

As in the standard model, knowledge is defined as "truth in all possible states." Given any event E, I say the agent knows E if there is some version of E in his subjective state space – either E itself or some elaboration of it – and in all *subjective* states he considers possible, this event is true. For example, does Charlie know it rains in the full state $(1_r, 0_p, \Delta)$? At $(1_r, 0_p, \Delta)$, Charlie's subjective state space is $\{(1_r, \Delta), (0_r, \Delta)\}$, and his subjective factual information is $(1_r, \Delta)$. The event "it rains" is represented by the singleton set $\{(1_r, \Delta)\}$ in this subjective state space, and indeed it is true at $(1_r, \Delta)$. Thus Charlie knows it rains in his subjective model at $(1_r, 0_p, \Delta)$, and hence *from the modeler's perspective*, Charlie knows it rains at $(1_r, 0_p, \Delta)$. Similarly for higher-order knowledge: I say the agent knows that he knows $\cdots$ that he knows E at ω^* if he knows that he knows $\cdots$ that he knows the corresponding version of E in his subjective model at $s(\omega^*)$.

Formally, I define knowledge in two steps. First, for any $\omega^* \in \Omega^*$, let $\mathcal{A}(\omega^*) \equiv \{E \subseteq \Omega^* : E \neq \emptyset\}$ denote the set of events of which the agent is aware at ω^*.[12] I define a *subjective knowledge operator* $\tilde{K}_{\omega^*} : \mathcal{A}(\omega^*) \to \mathcal{A}(\omega^*)$, representing knowledge in the subjective model at ω^*. Intuitively, this is knowledge *from the agent's perspective* at ω^*. Notice the subjective model $(\Omega(\omega^*), P_{\omega^*})$ is just a standard information partition model when restricted to subjective states contained in $P_{\omega^*}(s(\omega^*))$. For notational ease, for any event E, let $\emptyset_E$ be a shorthand for $\emptyset_{\Omega(q(E))}$, the empty set that embeds the same amount of awareness information as in E. For all $E \in \mathcal{E}^p$,

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \{\omega \in \Omega(\omega^*) : P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)}\} & \text{if } q(E) \subseteq W^*(\omega^*); \\ \emptyset_E & \text{if } q(E) \nsubseteq W^*(\omega^*). \end{cases} \tag{2.5}$$

Higher-order subjective knowledge is obtained by iterating $\tilde{K}_{\omega^*}$. Let $\tilde{K}^1_{\omega^*}(E) = \tilde{K}_{\omega^*}(E)$, for $n = 2, 3, \cdots$,

$$\tilde{K}^n_{\omega^*}(E) = \tilde{K}_{\omega^*}(\tilde{K}^{n-1}_{\omega^*}(E)). \tag{2.6}$$

The *objective* description of the knowledge hierarchy, viewed from the modeler's perspective, is then obtained by tracking the subjective knowledge hierarchies state by state. Formally, the *n-th order objective knowledge* is defined as:

$$K^n(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in \tilde{K}^n_{\omega^*}(E)\right\}. \tag{2.7}$$

To be consistent with the literature, I use the customary notation K^n to represent n-th order knowledge.[13] Similarly, let $(\neg K)^n$ denote the objective n-th order lack of

[12] Notice this set includes the special empty set $\emptyset_{\Omega(\omega^*)}$.

[13] Iterations of K are technically well-defined, but they do not have meaningful interpretations. By definition, $K(E)$ is beyond the agent's understanding.

knowledge, i.e., the agent does not know that he does not know $\cdots$ that he does not know. Just like K^n, this operator is defined through its subjective counterpart:

$$(\neg K)^n(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in (\neg\tilde{K}_{\omega^*})^n(E)\right\},$$

where $(\neg\tilde{K}_{\omega^*})^n(E) = \neg\tilde{K}_{\omega^*}((\neg\tilde{K}_{\omega^*})^{n-1}(E))$ is the subjective knowledge "I don't know that I don't know that $\cdots$ that I don't know E" from the agent's perspective at ω^*.[14] Other knowledge such as "the agent knows that he does not know" or "the agent does not know that he knows" are defined and denoted analogously.

The first-order knowledge can be directly computed from (Ω^*, W^*, P^*): When $n = 1$, equation (2.7) reduces to:

$$K(E) \quad = \quad \{\omega^* \in \Omega^* : q(E) \subseteq W^*(\omega^*),\ P^*(\omega^*) \subseteq E_{\Omega^*}\}. \tag{2.8}$$

For any E, the event "the agent is unaware of E" by definition must be an event from the modeler's perspective, i.e., an objective event. The *unawareness operator* U is defined as follows:

$$U(E) = \{\omega^* \in \Omega^* : q(E) \nsubseteq W^*(\omega^*)\}. \tag{2.9}$$

To see the connection with the standard model, notice that if the agent is fully aware in all full states, then the product model simply reduces to the standard model: the agent has the same subjective model – the full model – in all full states; consequently the subjective knowledge operator $\tilde{K}$ reduces to the usual K in the standard model, all subjective knowledge hierarchies become identical, and the objective knowledge hierarchy coincides with the subjective one.

3 The Knowledge Hierarchy with Unawareness

Recall that in the standard information partition model (Ω, P), the agent's knowledge hierarchy is completely characterized at the first level: for any $E \subseteq \Omega$,

1. $K(E) = KK(E)$;
2. $\neg K(E) = K\neg K(E)$.

In words, given any event, the agent either knows it or does not know it, and he always knows whether he knows it. It turns out that natural generalization of the above characterization obtains in the product model.

[14] Note that the negation of the K^n operator is also meaningful: $\neg K^n(E) = \Omega \setminus K^n(E)$ is the objective knowledge "the agent does not know that he knows that $\cdots$ that he knows E." However, it does not make sense to iterate on $\neg K$ directly: just like $K(E)$, $\neg K(E)$ is the objective knowledge "the agent does not know E" and this knowledge is beyond the agent.

Theorem 1 *In the product model* (Ω^*, W^*, P^*)*, let* (W^*, P^*) *be rational. Then the agent's knowledge hierarchy satisfies: for any* $E \in \mathcal{E}^p$*,*

1. $U(E) = \neg K(E) \cap \neg K \neg K(E)$;
2. $K(E) = KK(E)$;
3. $\neg K(E) \cap \neg U(E) = K \neg K(E)$.

Theorem 1 says, given any event, the agent is either unaware of it, in which case he does not know it and does not know he does not know it; or he is aware of it, in which case he either knows it or does not know it, and always knows whether he knows it. Analogous to the standard model, the knowledge hierarchy is completely pinned down at the first level. In particular, note the entire knowledge hierarchy can be derived from the pair (W^*, P^*) directly.

The product model satisfies the unawareness axioms proposed in the literature, including the DLR axioms. In addition, it parallels the standard information partition model in that it satisfies appropriate analogues of K1-6 under parallel structural assumptions. The results are organized into two lemmas containing properties under arbitrary information structures and those under rational information structures, respectively.

The first lemma deals with the basic properties of knowledge and unawareness without imposing rationality of (W^*, P^*). For all $E,\ F \in \mathcal{E}^p$,

U0* *Symmetry*: $U(E) = U(\neg E)$

U1$'$ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K)^n (E)$

U2* *AU introspection*:[15] $U(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K)^n U(E)$

U3$'$ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

K1* *Subjective necessitation*: $\omega^* \in K(\Omega(\omega^*))$ for all $\omega^* \in \Omega^*$

K2* *Generalized monotonicity*: $E \subseteq_* F,\ q(E) \supseteq q(F) \Rightarrow K(E) \subseteq K(F)$

K3* *Conjunction*:[16] $K(E) \cap K(F) = K(E \cap_* F)$

[15] The operator $K^n U$, mapping each event E to the event "I know that I know that $\cdots$ that I am unaware of E," is defined similarly to K^{n+1}, by tracking the agent's subjective knowledge of the subjective event "I am unaware of E." Note that to define the latter event, one needs to specify in the subjective model the agent's perception of his *awareness* information. See Section **??** for a brief discussion and Li (**?**) for details. For the current purpose, it suffices to note that the subjective event "I am unaware of E" in the agent's subjective model must contain at least as much awareness information as E does and hence is empty whenever the agent is indeed unaware of E.

[16] Parallel to the standard model, conjunction implies generalized monotonicity.

Symmetry is proposed by Modica and Rustichini (1999). It says that one is unaware of an event if and only if one is unaware of the negation of it. The other three unawareness properties correspond to the three axioms proposed by DLR. Strong plausibility strengthens DLR's plausibility axiom. Plausibility requires that whenever one is unaware of something, one does not know it and does not know that one does not know it. I require such a lack of knowledge to be extended to an arbitrarily high order. While KU introspection requires the agent never know exactly what he is unaware of, the weak KU introspection weakens it by allowing the agent to have false knowledge of his being unaware of a particular event.[17]

$K1^* - 3^*$ are natural analogues of K1-3 in the context of nontrivial unawareness. Recall that necessitation says the agent knows all tautological statements: $K(\Omega^*) = \Omega^*$. However, while all theorems are tautologies, arguably Newton does not know the theory of general relativity because he is unaware of it. This is reflected in subjective necessitation, which says the agent knows all tautological statements *of which he is aware*.

The essence of monotonicity is the intuitive notion that knowledge should be monotonic with respect to the information content of events. In the standard model, an event is more informative than another if and only if it conveys more facts. In the product model, an event is more informative than another if and only if it contains both more facts and *more awareness*. Alternatively, note that monotonicity means the agent knows the logical consequences of his knowledge, while generalized monotonicity, which explicitly takes into account that the agent may not be fully aware, says the agent knows those logical consequences of his knowledge *of which he is aware*.[18]

Lemma 2 *The product model* $\{\Omega^*, W^*, P^*\}$ *satisfies* $U0^*$, $U1'$, $U2^*$, $U3'$ *and* $K1^* - 3^*$.

The product model circumvents DLR's impossibility results by introducing subjective state spaces that may be different from the full state space. Recall that the key property implied by DLR's unawareness axioms is that, *fixing a state space*, whenever the agent is unaware of an event in this state space, he must not know the entire state space. In the product model, this property can be written as follows:

$$U(E) \subseteq \neg K(\Omega(q(E))),$$

which is consistent with the agent's having nontrivial knowledge about events in spaces weakly coarser than his own subjective state space. Consequently, although knowledge of tautologies does imply awareness of all relevant events, i.e. $\omega^* \in K(\Omega)$ for some $\Omega \in \mathcal{S}$ implies $\omega^* \in \neg U(E)$ for all $E \subseteq \Omega$ (DLR's first impossibility result), the agent can still be unaware of events in spaces that are not weakly coarser than Ω; although unawareness of an event does imply ignorance of any event from the same state space under monotonicity

[17] KU introspection requires a version of the truth axiom. In this sense it has some rationality flavor. Indeed, I show shortly that KU introspection is satisfied when the information structures are rational.

[18] Notice that subjective necessitation and generalized monotonicity mean that the agent is no longer logically omniscient when he has unawareness.

of knowledge, i.e., $U(E) = U(G) \subseteq \neg K(G)$ for all E, G such that $q(E) = q(G)$ (DLR's second impossibility result), the agent can still have nontrivial knowledge of events in some space that is strictly coarser than $\Omega(q(E))$.[19]

Remark 1. DLR consider the weaker property plausibility, which is sufficient for the negative result in which they are interested. However, when it comes to providing *positive* results in a model that deals with unawareness, strong plausibility, or an even stronger property that equates unawareness with the lack of knowledge of all orders which I discuss shortly, seems more interesting. It is also worth noting that subjective necessitation is equivalent to the "weak necessitation" property DLR discussed in the context of propositional models.

Lemma 2 does not require (W^*, P^*) to be rational. This is because analogous to the standard model, rationality in information processing mainly has implications for higher-order knowledge, while essentially none of the above properties involves nontrivial higher-order reasoning.[20]

For all $E \in \mathcal{E}^p$,

U1* *UUU (Unawareness = unknown unknowns)*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n(E)$

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

K4.a^* *The axiom of knowledge I*: $K(E) \subseteq_* E$

*K*4.b^* *The axiom of knowledge II*: $K^n(E) \subseteq K^{n-1}(E)$

K5* *The axiom of transparency*: $K(E) \subseteq KK(E)$

K6* *The axiom of limited wisdom*: $\neg K(E) \cap \neg U(E) \subseteq K \neg K(E)$

Lemma 3 *The product model* (Ω^*, W^*, P^*) *satisfies* $U1^*$*,* $U3^*$*,* $K4.a^*$*,* $K4.b^*$*,* $K5^*$ *and* $K6^*$ *if* (W^*, P^*) *is rational.*

The axiom of limited wisdom extends the axiom of wisdom to an environment with unawareness by requiring the agent to know that he does not know only when he *is aware of the involved event.* UUU says the agent is unaware of an event *if and only if* he does not know it, he does not know that he does not know it, and so on. The extra strength added to strong plausibility is due to the axiom of limited wisdom: if the

[19]Generalized monotonicity and monotonicity differ in that the former does not require knowledge to be monotonic when $q(E) \nsubseteq q(F)$, while in the standard model, one necessarily has $q(E) = q(F)$.

[20]Although U1$'$ and U2* do involve higher-order knowledge, they both simply follow from the observation that subjective knowledge is an empty set for any event of which the agent is unaware, which immediately implies all higher-order ignorance. There is no nontrivial knowledge calculation involved in these two properties.

agent always knows that he does not know if he is aware of the event, then the only circumstance where he does not know that he does not know must be that he is unaware of it. Last, the axiom of knowledge says the agent can never have false knowledge, both with respect to the set of "natural" events $\mathcal{E}^p$ (K4.a^*), and in introspection of his own knowledge hierarchy (K4.b^*). Such a "nondelusion" property, combined with weak KU introspection, yields KU introspection.

Notice theorem 1 follows from Lemma 3: K5* combined with K4.b^* implies $K(E) = KK(E)$; K6*, K4.b^* and $U1^*$ imply $\neg K(E) \cap \neg U(E) = K\neg K(E)$, which in turn implies $K\neg K\neg K(E) = \emptyset$, reducing UUU to $U(E) = \neg K(E) \cap \neg K\neg K(E)$.

Remark 2. Information has more dramatic effects on the agent's knowledge hierarchy when the agent has nontrivial unawareness than when he does not. Upon receipt of new information, the agent updates his subjective state space as well as his subjective factual information. Formally, given ω^*, let the agent's initial information be $(W_0^*(\omega^*), P_0^*(\omega^*))$. The agent has subjective factual information:

$$\mathbb{P}^{\Omega(W_0^*(\omega^*))}[P_0^*(\omega^*)]$$

Upon receipt of new information $(W_1^*(\omega^*), P_1^*(\omega^*))$, the agent updates his subjective factual information to

$$\mathbb{P}^{\Omega(W_0^*(\omega^*)\cup W_1^*(\omega^*))}[P_0^*(\omega^*) \cap P_1^*(\omega^*)]$$

As long as $W_1^*(\omega^*) \setminus W_0^*(\omega^*) \neq \emptyset$, the agent gains new knowledge.

In particular, if $P_0^*(\omega^*)$ is not an elaboration of $\mathbb{P}^{\Omega(W_0^*(\omega^*))}[P_0^*(\omega^*)]$, that is, if it contains factual information concerning questions beyond $W_0^*(\omega^*)$, then the agent could learn new facts from introspection of the first-period factual information alone. For example, in the Watson story, at b, Watson's initial information is the pair $(W_0^*(b), P_0^*(b)) = (\emptyset, \{b\})$. His subjective factual information is $\mathbb{P}^{\{\Delta\}}\{b\} = \{\Delta\}$. The question "could there have been an intruder?" is represented by the pair $(W_1^*(b), P_1^*(b)) = (\{t\}, \{a, b\})$. Now Watson updates his subjective state space and recognizes the factual information he has had all along but neglected:

$$\mathbb{P}^{\Omega(\emptyset\cup\{t\})}(\{b\} \cap \{a, b\}) = \mathbb{P}^{\{a,b\}}(\{b\}) = \{b\}.$$

Remark 3. An important question is how unawareness differs from assigning probability zero. Although a thorough answer to this question is well beyond the scope of this paper, the product model suggests some interesting directions.

In the hearing problem example, Charlie is unaware of the events "it rains and I have a hearing problem" and "it rains and I do not have a hearing problem," while he may assign positive probability to the event "it rains." In contrast, assigning zero probability to the former two events dictates that Charlie also assigns zero probability to the event "it rains."

Such difference seems especially stark in dynamic environments. Suppose Charlie is informed that "it may rain, and you could have a hearing problem." This event is represented by the pair $(\{r,p\},\Omega^*)$. If Charlie was unaware of question p, this information would cause him to update his subjective state space, and in principle, he may assign *any* probability to events such as $\{(1_r,1_p,\Delta)\}$, which (or whose factually equivalent event) was not present in his probability space before due to his unawareness. On the other hand, if Charlie was aware of both questions r and p, but only assigned probability zero to the event $\{(1_r,1_p,\Delta),(0_r,1_p,\Delta)\}$, i.e., "I have the hearing problem," then Charlie must have assigned probability 1 to the universal event Ω^*, and hence upon receipt of which Charlie makes no updating of probabilities. In particular, he must still assign probability zero to the event $\{(1_r,1_p,\Delta),(0_r,1_p,\Delta)\}$ and hence $\{(1_r,1_p,\Delta)\}$.[21]

4 Concluding Remarks

In a companion paper, I extend the product model to the multi-agent environment (Li 2008c). In contrast to the single-agent case, adding unawareness to interactive reasoning gives rise to knowledge hierarchies that significantly differ from those in the standard model. For example, while i's knowledge about a "natural" event or his own knowledge is never false, it could be false when it comes to j's knowledge, where $j \neq i$. The presence of unawareness can also reduce higher-order informational uncertainties. For example, when having unawareness, i could "know" that j knows E, but once i's unawareness is lifted, he may become uncertain about whether j knows E. As a consequence, Aumann's classic characterization of common knowledge does not immediately apply in this environment, even if there is "common awareness" of the event involved. Li (2008c) discusses and characterizes common knowledge in the presence of unawareness.

There are many additional important topics of unawareness one may further pursue using the product model. One obvious direction is to incorporate a probabilistic language in the current framework, which is necessary for a rigorous and thorough analysis of the difference between being unaware of an event and assigning probability zero to it. Li (2008b) has an extensive discussion on this issue using a decision-theoretic approach. Another direction is to enrich the current model to accommodate "self-awareness of unawareness," i.e., the agent's ability to reason about events such as "there *exists some* event of which I am unaware." Intuitively, unawareness can have non-trivial effects on individual decision-making only through the agent's self-awareness of possible unawareness.

[21] In the zero probability case, suppose Charlie construes the universal event "it may rain, and you could have a hearing problem" as indicating the zero-probability event "I have a hearing problem" has occurred, then he may actually update his probabilities. In other words, although unawareness and zero probability seem different in naive comparison, it is not clear if there exists *some* standard probabilistic model, perhaps involving an expanded state space and updating on zero probability events, that delivers the same implications a model of unawareness would deliver. See Li (2008b) for an extensive discussion on this issue.

5 Appendix.

5.1 Proof of Lemma 2.

U0* *Symmetry*: $U(E) = U(\neg E)$

Follows from $q(E) = q(\neg E)$.

U1$'$ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty}(\neg K)^n(E)$

Let $\omega^* \in U(E)$. Then $q(E) \nsubseteq W(\omega^*)$. By 2.5, $\tilde{K}_{\omega^*}(E) = \emptyset_E$. By 2.7, $\omega^* \in \neg K(E)$. Note that $q(\neg\tilde{K}_{\omega^*}(E)) = q(\tilde{K}_{\omega^*}(E)) = q(\emptyset_E) = q(E) \nsubseteq W(\omega^*)$, now it follows $\tilde{K}_{\omega^*}\neg\tilde{K}_{\omega^*}(E) = \emptyset_E$ and $\omega^* \in \bar{K}_2(E)$. It is easy to see that $q(\neg\tilde{K}_{\omega^*}^{n-1}(E)) = q(E)$ for all n, which implies $\tilde{K}_{\omega^*}(\neg\tilde{K}_{\omega^*}^{n-1}(E)) = \emptyset_E$ for all n, and hence, $\omega^* \in (\neg K)^n(E)$ for all n.

U2**AU introspection*: $U(E) \subseteq (\neg K)^n U(E)$

Let $\omega^* \in U(E)$. Then $q(E) \nsubseteq W^*(\omega^*)$. Notice:

$$(\neg K)^n U(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in \neg\tilde{K}_{\omega^*}[(\neg\tilde{K})^{n-1}\tilde{U}]_{\omega^*}(E)\right\}.$$

But $q(E) \nsubseteq W^*(\omega^*)$ implies all subjective awareness information does not contain $q(E)$, and hence all higher-order knowledge and unawareness is $\emptyset_E$, i.e., $\tilde{K}_{\omega^*}[(\neg\tilde{K})^{n-1}\tilde{U}]_{\omega^*}(E) = \emptyset_E$ for all n. Therefore we have $s(\omega^*) \in \neg\tilde{K}_{\omega^*}[(\neg\tilde{K})^{n-1}\tilde{U}]_{\omega^*}(E)$, and hence $\omega^* \in (\neg K)^n U(E)$.

U3$'$ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

Since $\neg KU(E) = \Omega^* \setminus KU(E)$, the result follows from AU introspection.

K1**Subjective necessitation*: for all $\omega^* \in \Omega^*$, $\omega^* \in K(\Omega(\omega^*))$

For any $\omega^* \in \Omega^*$, $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \subseteq \Omega(\omega^*)$, which implies $s(\omega^*) \in \tilde{K}_{\omega^*}(\Omega(\omega^*))$, and hence $\omega^* \in K(\Omega(\omega^*))$.

K2**Generalized monotonicity*: $E \subseteq_* F$, $q(F) \subseteq q(E) \Rightarrow K(E) \subseteq K(F)$

This is implied by conjunction. Take E and F such that $E \subseteq_* F$, $q(F) \subseteq q(E)$. By conjunction,

$$\begin{aligned} K(E) \cap K(F) &= K(E \cap_* F) \\ &= K(E \cap F_{\Omega(q(E))}) \\ &= K(E) \end{aligned}$$

It follows $K(E) \subseteq K(F)$.

K3* *Conjunction*: $K(E) \cap K(F) = K(E \cap_* F)$

Let $\omega^* \in K(E) \cap K(F)$. Then $q(E) \subseteq W^*(\omega^*)$, $q(F) \subseteq W^*(\omega^*)$ and $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$. Note that:

1. $q(E) \subseteq W^*(\omega^*)$, $q(F) \subseteq W^*(\omega^*)$ if and only if $q(E) \cup q(F) \subseteq W^*(\omega^*)$, which implies $q(E \cap_* F) \subseteq W^*(\omega^*)$. Thus, the event $E \cap_* F$ has an elaboration in the space $\Omega(\omega^*)$.

2. $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$ if and only if $P^*(\omega^*) \subseteq (E_{\Omega^*} \cap F_{\Omega^*}) = (E \cap_* F)_{\Omega^*}$.

Using (1), $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \subseteq \mathbb{P}^{\Omega(\omega^*)}((E \cap_* F)_{\Omega^*}) = (E \cap_* F)_{\Omega(\omega^*)}$, hence $s(\omega^*) \in \tilde{K}_{\omega^*}((E \cap_* F)_{\Omega(\omega^*)})$, and hence $\omega^* \in K(E \cap_* F)$. □

5.2 Proof of Lemma 3.

U1* *UUU*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n(E)$

Strong plausibility gives ⇒; Applying De Morgan's law on the axiom of limited wisdom gives the other direction.

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

Suppose not. Let $\omega^* \in KU(E)$. This implies: (See definitions in section 4.2.)
(1): $q(\tilde{U}_{\omega^*}(E)) \subseteq W^*(\omega^*)$; and
(2) $P_{\omega^*}(s(\omega^*)) \subseteq \tilde{U}_{\omega^*}(E)$.
By (1), $q(E) \subseteq W^*(\omega^*)$. By (**??**), $W_{\omega^*}(s(\omega^*)) = W^*(\omega^*)$, hence $q(E) \subseteq W_{\omega^*}(s(\omega^*))$, thus $s(\omega^*) \notin \tilde{U}_{\omega^*}(E)$, which obviously contradicts (2), because $s(\omega^*) \in P_{\omega^*}(s(\omega^*))$ by (2.4) and that $\omega^* \in P^*(\omega^*)$ (rational information).

For the next four properties, observe that when (W^*, P^*) is rational, the subjective model is a standard information partition model at the set of possible subjective states, and hence the standard results apply.

K4.a^*, K4.b^* *The axiom of knowledge*: $K(E) \subseteq_* E$, $K^n(E) \subseteq K^{n-1}(E)$

Let $\omega^* \in K(E)$. By (2.8), $P^*(\omega^*) \subseteq E_{\Omega^*}$, but then since P^* induces an information partition over Ω^*, $\omega^* \in P^*(\omega^*)$; it follows that $\omega^* \in E_{\Omega^*}$, and hence $K(E) \subseteq E_{\Omega^*}$;

To see K4.b^*, let $\omega^* \in K^n(E)$. By (2.7), $s(\omega^*) \in \tilde{K}^n_{\omega^*}(E)$; by (2.6), $P_{\omega^*}(s(\omega^*)) \subseteq K^{n-1}_{\omega^*}(E)$. But then again $\omega^* \in P^*(\omega^*)$ implies $s(\omega^*) \in P_{\omega^*}(s(\omega^*)) \Rightarrow s(\omega^*) \in K^{n-1}_{\omega^*}(E)$, and hence $\omega^* \in K^{n-1}(E)$.

K5* *The axiom of transparency*: $K(E) \subseteq KK(E)$

Let $\omega^* \in K(E)$. It suffices to show $s(\omega^*) \in \tilde{K}_{\omega^*}\tilde{K}_{\omega^*}(E)$.
Since (W^*, P^*) is rational, $P_{\omega^*}(\omega) = P_{\omega^*}(s(\omega^*))$ for all $\omega \in P_{\omega^*}(s(\omega^*))$. Now

$$\begin{aligned}
& s(\omega^*) \in \tilde{K}_{\omega^*}(E) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & \omega \in \tilde{K}_{\omega^*}(E) \text{ for all } \omega \in P_{\omega^*}(s(\omega^*)) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq \tilde{K}_{\omega^*}(E) \\
\Rightarrow\ & s(\omega^*) \in \tilde{K}^2_{\omega^*}(E)
\end{aligned}$$

K6* *The axiom of limited wisdom*: $\neg U(E) \cap \neg K(E) \subseteq K\neg K(E)$

Let $\omega^* \in \neg U(E) \cap \neg K(E)$. Then $q(E) \subseteq W^*(\omega^*)$, so we only need to show $P_{\omega^*}(s(\omega^*)) \subseteq \neg\tilde{K}_{\omega^*}(E)$. Let $\omega \in P_{\omega^*}(s(\omega^*))$. By (2.4), $P_{\omega^*}(\omega) = P_{\omega^*}(s(\omega^*))$, but $\omega^* \in \neg K(E) \Rightarrow P_{\omega^*}(s(\omega^*)) \nsubseteq E_{\Omega(\omega^*)}$, it follows $P_{\omega^*}(\omega) \nsubseteq E_{\Omega(\omega^*)} \Rightarrow \omega \in \neg\tilde{K}_{\omega^*}(E)$. □

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