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Jack Ochs

In-Uck Park

Center for Economic Institutions

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Institute of Economic Research

Hitotsubashi University

2-1 Naka, Kunitachi, Tokyo, 186-8603 JAPAN

Tel: +81-42-580-8405

Fax: +81-42-580-8333

e-mail: cei-info@ier.hit-u.ac.jp

Overcoming the Coordination Problem: Dynamic Formation of Networks

by

Jack Ochs and **In-Uck Park***

Department of Economics
University of Pittsburgh
4S01 WW Posvar Hall
Pittsburgh, PA 15260, USA

(jochs@pitt.edu and ipark@pitt.edu)

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Abstract

We analyze an entry game with multiple periods. In each period privately informed agents who have not yet joined decide whether to subscribe to a network. Subscribers derive benefits in future periods depending on the network size. We study the case where agents are sufficiently patient and show that there exists a unique symmetric equilibrium if the number of existing subscribers is common knowledge in each period. This resolves the coordination problem which is prevalent in markets with network externalities. (*JEL* Classification Codes: D82, D85)

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* Corresponding author

1. Introduction

Adoption/network externalities arise when complementarities exist across agents in the consumption of certain goods or services. Examples include commodities designed for joint consumption or sharing (telephony and data networks), those with indirect scale economies for complementary goods (hardware-software and durable-good servicing), and adoption of innovations and standards where compatibility is valuable.

Due to complementarity, there typically exist multiple, Pareto ranked equilibria in such markets. The worst is a null equilibrium in which no one adopts because no one is ever anticipated to adopt, while at the other end is a "maximum" equilibrium. The maximum equilibrium refers to a "maximal set of agents" who would indeed adopt when that is what everyone expects to occur. There may be other equilibria intermediate between these two. With no outside force present, the particular equilibrium to be realized is indeterminate. This is a well-known coordination problem. One strand of research has studied inducement schemes as a device to overcome the likelihood of coordination failure in the static, simultaneous move entry game. These schemes provide insurance against low adoption or entry rates. Such insurance warrants a sufficient rate of adoption by those who have a low cost of entry, which, in turn, will induce others with higher entry costs to also enter. Dybvig and Spatt (1983) and Park (2003) devise insurance schemes that will induce certain target equilibria as the unique (symmetric) equilibrium at the minimal expected cost of insurance subsidy. Bagnoli and Lipman (1989) study a refund mechanism to induce private contribution to a public project where a sufficient number of people must contribute before the project produces any benefit.

If agents' types are randomly determined and privately known, but there is common knowledge both that the types are correlated and of the nature of the correlation, then the theory of global games developed by Carlsson and van Damme (1993) would apply. Morris and Shin (2003) show that even when there is only a small amount of heterogeneity in types in such games there will often be a unique equilibrium. The common knowledge of the way in which beliefs are correlated allows individuals through a process of backward induction to condition their beliefs as to how others will act on the knowledge of their own individual types.

In this paper we analyze the effect of a dynamic adoption process on resolving the coordination problem in the market entry game when agent types are privately and independently drawn from a commonly known distribution. The independent nature of types renders the logic of global games inapplicable. A dynamic adoption process, however, introduces a strategic consideration that is absent in the static game. Individuals who chose to enter early may influence the entry decisions of others who have not yet entered. This creates the possibility that early entrants may launch a domino chain reaction of widespread adoption. However, agents considering early entry will be so motivated only if they expect such a domino chain. Such a domino chain itself relies on a nested sequence of optimistic beliefs of future adopters. At first sight, therefore, it appears that the basic intuition of coordination failure due to multiplicity of self-confirming expectation would continue to prevail in dynamic adoption process. Rather surprisingly, we establish that this is not the case. Specifically, we show that there exists a unique symmetric, perfect Bayesian equilibrium if agents decide when as well as whether to adopt and they are sufficiently patient. In this equilibrium entry occurs with positive probability.

In our model, agent's types are ordered by the utility level the agent derives from being a member of the network. Since each member's utility increases as the network gets larger, the higher is the utility an agent derives from the network the lower is the threshold network size for this agent to join profitably. Hence, we describe an agent who derives a *higher* utility level from the network as having a *lower* type. In equilibrium, all agents choose a cutoff strategy in which an agent enters in any period k precisely when his type is no higher than a cutoff level for that period. In this equilibrium some entry always occurs with a positive probability. Intuitively, higher types who would need large numbers of other agents also to be in the network in order to find their own entry profitable, will enter later than lower types who would need smaller numbers of other agents in the network to find it profitable. Therefore, an agent of a particular type who has not yet entered can use the common knowledge of the state of the game in period k to determine the expected number of additional entrants conditional on his own entry in period k and the number of prior entrants. The ability to form these expectations of future entrants generates a backward induction process that uniquely pins down the equilibrium adoption process from the point where all but one agent have entered already all the way back to the point at which no one has entered.

When agents do not discount, equilibrium cutoff level in each period is determined entirely by the number of agents who have entered by then. However, when agents do discount, the equilibrium cutoff levels depend on further details of history such as how many entered in which period. Nevertheless, the basic logic of our argument applies and we show that when agents are sufficiently patient, there exists a unique, symmetric equilibrium. This equilibrium converges to the no-discount equilibrium as the discount factor tends to one.

Since the agents are ex ante identical we find it natural that they follow symmetric strategies in equilibrium. Hence, we primarily focus on symmetric equilibrium in this paper. Asymmetric equilibria, however, may exist in some environments. Nevertheless, such asymmetric equilibria converge to the unique symmetric equilibrium as the number of agents increases without bound, because "the expected behavior of all other agents" to which each agent best-responds differs only by the behavior of one agent from the perspectives of any two agents. Therefore, their respective expectations become arbitrarily close to one another as the number of agents increases. In this sense, the existence of asymmetric equilibria does not undermine our main message that dynamic adoption process resolves the coordination failure.

The coordination problem arising from strategic complementarities has been studied in some other dynamic settings. Rohlfs (1974) considers introductory pricing in his classic paper on telecommunication markets. As mentioned earlier Bagnoli and Lipman (1989) study a refund mechanism in a dynamic setting. Andreoni (1998) examines large "leadership gifts" in charitable fund-raising.[1] These papers analyze environments with (almost) complete information, i.e., either the agents' preferences or their distribution are/is known. Dixit (2003) also obtains unique equilibrium in a dynamic game similar to ours, but his model is one of complete information. Our paper differs from these studies in that we examine an incomplete information environment. Farrell and Saloner (1985) deal with incomplete information and contain an analysis close to ours when there are two agents. Our analysis generalizes theirs to any finite number of agents and to the cases that they derive utility each period so that the dynamics, as well as the final network formed, are of

[1] Organizers of charitable fund drives typically announce at various stages of the drive how much money has been pledged, and possibly the number of individuals who have made donations. According to the logic of our analysis, the fact that these announcement will be made should have an effect on how much will be given in the early stages of a campaign because those who go early have reason to believe that their gifts may encourage others to give at later stages. This is because those who choose not to give early, upon seeing how much has been given, will have a greater degree of confidence that the benefits that they will derive from the completion of the campaign will more than cover the cost to them of their own donation. Of course, fundraisers often select a group of 'leaders' whom they solicit first, prior to announcing a general campaign. This phenomenon may be more closely related to costly transmission of information regarding the 'quality' of the charitable endeavor and informational cascades, a logic quite separate from that underlying our own model. Vesterlund (2003) and Andreoni (2004) discuss how leadership grants may transmit information and how the possibility of this transmittal may affect both the amount raised in the 'leadership' or 'quiet' phase of a fundraising drive and the total amount raised. Marx and Matthews (2000) argue that dynamic contribution tends to enhance efficiency when the cumulative total contribution is publicly known, in a setting of voluntary contribution to a public project.

importance.

The remainder of the paper has the following structure. The next section describes the model. Section 3 presents the main analysis that characterizes the unique symmetric equilibrium. Section 4 contains some concluding remarks.

2. Model

There are $N+1$ ex ante identical agents, indexed by $i \in \mathcal{I} = \{1, \cdots, N+1\}$, who are privately informed of their own *types* $t \geq 0$ which are independent draws from a common distribution function $F : \Re_+ \to [0, 1]$. Let $f : \Re_+ \to \Re_+$ denote the corresponding density function. We assume that F is continuous and $F(0) = 0$ (i.e., t is atomless) and f is bounded. For expositional convenience only, we assume $F(t) < 1$ and $f(t) > 0$ for all $t \geq 0$.

There are infinite periods indexed by $k = 1, \cdots$. At the beginning of each period k the number n_{k-1} of agents who adopted/subscribed up until period $k-1$ is common knowledge; Based on the public history $h_k := (n_1, \cdots, n_{k-1})$ the agents who have not adopted already simultaneously choose either to adopt the network product or not. Once adopted, agents cannot reverse their choices in future periods.

An agent who adopts in period k' derives a stage utility from the network product in every period $k \geq k'$, determined by his type t and the network size in period k measured by the number $\nu_k := n_k - 1$ of other adopters (i.e., not counting himself): A t-type agent derives a utility of $u_t(\nu_k) \in \Re$ in period k.[2] The stage utility to a non-adopter is normalized to $u_\phi = 0$. Each agent's objective is to maximize the expected δ-discounted average of utility stream with a discount factor $\delta \leq 1$: That is, each agent maximizes the expected value of

$$(1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} \mathrm{u}_k \tag{1}$$

if $\delta < 1$, where u_k is the utility in period k, which is 0 if the agent has not adopted yet and is $u_t(\nu_k)$ if the agent of type t has adopted; and maximizes the limit of the expected value of (1) as $\delta \to 1$ if the discount factor is $\delta = 1$.

An agent's type, t, measures how reluctant he is to join the network, so a higher type means a more conservative agent who needs a larger network to benefit by joining.

[2] After development of this paper, we became aware of Xue (2003). Xue studies a dynamic version of the stag hunt game. His model has special features that are not present in our own model, namely, the benefit from network does not realize unless everybody adopts and the type enters in the utility function linearly. His result and analysis for no discounting case are similar to ours, however there are some steps (e.g., Lemmas 3 and 4) that we found necessary to prove our result but are not used in his proof. This may reflect nontrivial differences between the two models.

Hence, we assume that $u_t(\nu)$ is strictly increasing in $\nu = 0, \cdots, N$, strictly decreasing and continuous in t, and that

$$u_0(0) = 0 \quad \text{and} \quad \exists\, \bar{t} \;\; s.t. \;\; u_{\bar{t}}(N) = 0. \tag{2}$$

The first equality says that the "best" type is indifferent between being a sole member of the network and being a non-member.[3] Clearly, $\bar{t} > 0$ defined above is unique because $u_t(N)$ strictly decreases in t and $u_0(N) > u_0(0) = 0$. We denote this game by Γ.

An agent i's *period-k strategy* when he has not adopted yet, given a history $h_k = (n_1, \cdots, n_{k-1})$, is an integrable function that maps types to adoption probabilities, i.e.,

$$a^i(\cdot|h_k) \, : \, \Re_+ \to [0, 1]$$

where $a^i(t|h_k)$ is the probability that the agent i adopts (the network product) when his type is t, if he has not adopted up to the previous period. A function $a^i(\cdot|h_k)$ is a *cutoff strategy at (a cutoff level)* $\hat{t} \geq 0$, if $a^i(t|h_k) = 1$ for all $t < \hat{t}$ and $a^i(t|h_k) = 0$ for all $t > \hat{t}$. An agent i's *strategy* is a collection $\{a^i(\cdot|h_k)\}$ for all possible h_k, which we denote by a^i as shorthand. A strategy a^i is a *cutoff strategy* if $a^i(\cdot|h_k)$ is a cutoff strategy for every possible h_k.

A strategy profile $(a^i)_{i \in \mathcal{I}}$ is a *(perfect Bayesian) equilibrium* of Γ if each agent i's period-k strategy after each possible h_k is a best response to $(a^j)_{j \neq i}$ conditional on h_k.

3. Unique Symmetric Equilibrium

In this section we focus our attention on symmetric equilibrium and show that there exists a unique symmetric equilibrium of Γ when δ is sufficiently large. First, we construct it for $\delta = 1$ and show that it is a cutoff equilibrium and the cutoff level in each period depends only on the total number of agents who already adopted. Then, we show (details in the Appendix) that there exists a threshold $\delta^* < 1$ such that the same argument can be extended to all $\delta > \delta^*$ to establish that there is a unique symmetric equilibrium and it is a cutoff equilibrium, however, the cutoff level in each period depends on the full adoption history up to then.

From now until Theorem 1, we analyze the case that $\delta = 1$. Since there are finite agents the adoption process stops within finite periods, so that any utility stream $\{\mathrm{u}_k\}$ that an agent might expect has a constant utility level after a finite number of periods. Hence, the limit of (1) as $\delta \to 1$ for any utility stream is this constant utility level. Consequently,

[3] All the main results of this paper hold when $u_0(0)$ is negative and sufficiently small and $u_0(1) > 0$.

the agent's objective amounts to maximizing the "terminal" stage utility level that will prevail after the adoption process has stopped.

The observation that agents only care about the final network size of any adoption process simplifies the analysis for the case $\delta = 1$ because the details of the adoption process leading to the final network can be ignored. However, it allows an inessential indifference of an agent between adopting now and adopting later so long as the final network will be the same. For example, if all but one agent already adopted the remaining agent is indifferent between adopting now and adopting in any later period. To circumvent this problem in this paper we adopt a stopping rule that if no one adopted the network in some period k, then no further adoption is allowed and only those agents who adopted by then benefit from the network in future periods.[4] Below we characterize the symmetric equilibrium for the case $\delta = 1$. Note that $a^i = a^j$ in symmetric equilibrium.

Lemma 1: *If $\delta = 1$, in any symmetric equilibrium every agent adopts with a positive probability in period 1.*

Proof: Consider a symmetric strategy profile $(a^i)_{i \in \mathcal{I}}$ such that $\int a^i(\cdot|h_1)dF = 0$. Let t^1 be the unique type such that $u_{t^1}(1) = 0$. Consider an ϵ-type agent in period 1 where $\epsilon < t^1$ so that $u_\epsilon(1) > 0$. If this agent deviates by adopting in period 1, then in the next period other agents would adopt with a positive probability, say $p > 0$, because adopting is beneficial when their types are lower than t^1. The expected utility from such a deviation, therefore, exceeds $pu_\epsilon(1) + (1-p)u_\epsilon(0)$ which tends to $pu_0(1) > 0$ as $\epsilon \to 0$, so that such deviation is beneficial for sufficiently small ϵ. Hence, the considered strategy profile cannot be an equilibrium. *Q.E.D.*

By Lemma 1 in every symmetric equilibrium there is a positive probability that the game reaches a period with any number of existing adopters, i.e., with a history $h_k = (n_1, \cdots, n_{k-1})$ for any $n_{k-1} = 0, \cdots, N$. As will become clear in the analysis, what matters in the strategic decisions in the remaining part of the game is the total number n_{k-1} of adopters by then (equivalently, the number of agents who have not adopted), not how it evolved. So, we define the *state (variable)* s for a period k with a history

[4] The effect of this stopping rule is to shorten the completion time of the network to be formed by speeding up the adoption process. This is entirely immaterial when $\delta = 1$. For $\delta < 1$, an equilibrium under this stopping rule remains an equilibrium under a more general, χ-stopping rule for any natural number χ: the process stops if no one adopted for χ consecutive periods. There may exist additional equilibria under χ-stopping rule if $\chi > 1$, but the final network to emerge in these equilibria converge, as $\delta \to 1$, to that in the unique equilibrium under the 1-stopping rule. (A proof is available from authors.) In this sense, the main results of this paper are robust to the stopping rule we adopt.

$h_k = (n_1, \cdots, n_{k-1})$ as $s = N + 1 - n_{k-1}$, i.e., the number of non-adopters after h_k, who we refer to as the "remaining" agents. With a slight abuse of notation, we write $a^i(\cdot|s)$ if a strategy a^i has the property that $a^i(\cdot|h_k) = a^i(\cdot|h'_{k'})$ whenever both h_k and $h'_{k'}$ have the same state s. We now proceed with an induction argument that characterizes symmetric equilibrium $(a^i)_{i\in\mathcal{I}}$ when $\delta = 1$.

STEP 1: As shown above, the game reaches any possible state $s = 0, \cdots, N + 1$, with a strictly positive probability. The game ends if it reaches $s = 0$. Suppose that the game reached a state $s = 1$, i.e., only one agent remains in some period k. It is trivial that this last agent will adopt precisely when his type does not exceed $\bar{t}$ defined in (2). That is, the equilibrium strategy of the remaining agent when he is the only remaining agent (i.e., in state $s = 1$) is a cutoff strategy at $\tau_1 \equiv \bar{t}$:

$$a^i(t|1) \;=\; \begin{cases} 1 & \text{if} \quad t < \tau_1 \\ 0 & \text{if} \quad t > \tau_1 \end{cases} \tag{3}$$

STEP 2: Suppose that the game reached a state $s = 2$ in period k with a history h_k. Consider one remaining agent, say i, of type $t_i \leq \tau_1$. If the other remaining agent, say j, were to adopt in this period, agent i would get a utility of $u_{t_i}(N)$ by adopting in this period; if agent i waited in this period he would adopt in the next period (because $t_i \leq \tau_1$), hence again get a utility of $u_{t_i}(N)$ eventually. Therefore, agent i's optimal decision in this period depends on what would happen in the contingency that agent j were to not adopt in this period. In this contingency, agent i would get a utility $u_\phi = 0$ by not adopting in this period because no further adoption would ensue due to the postulated stopping rule; if agent i adopted in this period, he would get $u_{t_i}(N)$ eventually in case agent j joins next period and $u_{t_i}(N-1)$ otherwise. Since agent j's response in the next period is independent of t_i, the expected utility of agent i from adopting decreases in t_i, whereas that from waiting is 0. Consequently, agent i (and j by symmetry) should employ a cutoff strategy at a level, say $\hat{t}$. Note that agent i strictly prefers waiting in this period if his type is sufficiently close to τ_1, hence $\hat{t} < \tau_1$.

Let $g(\cdot|h_k)$ denote the posterior density function updated from f by h_k on the type of the remaining agent. Then, the condition that characterizes $\hat{t}$ is the following: agent i of $\hat{t}$-type is indifferent between adopting and waiting in this period given that agent j follows a cutoff strategy at $\hat{t}$ in this period and a cutoff strategy at τ_1 in state $s = 1$, i.e.,

$$u_{\hat{t}}(N) \int_{\hat{t}}^{\tau_1} g(t|h_k)dt + u_{\hat{t}}(N-1) \int_{\tau_1}^{\infty} g(t|h_k)dt \;=\; 0. \tag{4}$$

The left hand side, LHS, of (4) is the expected utility of a $\hat{t}$-type agent when he adopts in the current period conditional on the other remaining agent waits, while the RHS is that when he waits in the current period. Note that the LHS of (4) is strictly decreasing in $\hat{t}$, clearly from a positive value when $\hat{t} = 0$ to a negative value when $\hat{t} = \tau_1$. Hence, there exists a unique value of $\hat{t}$ that solves (4), which is the equilibrium cutoff level in the period following the history h_k, denoted by $\tau_2(h_k)$. Summarizing,

Lemma 2: *If $\delta = 1$, the equilibrium strategy in state $s = 2$ with a history h_k is a cutoff strategy at $\tau_2(h_k)$, the unique level of $\hat{t}$ that solves (4).*

STEP 3: Fix a state $\tilde{s}$ and any possible history $\tilde{h}$ whose state is $\tilde{s}$. (A history $h'_{k'}$ is an *extension* of a history h_k if $k' \geq k$ and the first k components of $h'_{k'}$ coincide with h_k.) Consider the following property in an equilibrium:

[A] The strategy after any extension h of $\tilde{h}$ whose state is $s < \tilde{s}$, is a cutoff strategy at a level that only depends on the state, denoted by $\tau_s(\tilde{h})$, and decreases in s (conditional of $\tilde{h}$).

Note that this property holds along an equilibrium when $\tilde{s} = 3$ by Lemma 2, and trivially if $\tilde{s} < 3$. (For $\tilde{s} = 3$, note that, given the equilibrium strategy after $\tilde{h}$, the posterior $g(\cdot|h')$ is uniquely determined for h' that extends $\tilde{h}$ and has a state 2.) We now make an induction hypothesis that the property [A] holds for all $\tilde{s} \leq r$ where $r = 3, \cdots, N$, along an equilibrium. Below we establish that under this hypothesis the property [A] holds for $\tilde{s} = r + 1$ as well. In short, we try to show that any extension of $\tilde{h}$ entails a cutoff strategy that only depends on the state, with the cutoff level strictly decreasing in the state.

Lemma 3: *Suppose $\delta = 1$. Pick an arbitrary remaining agent i after the game reached a state $\tilde{s}$ in period k with a history $\tilde{h}$, such that [A] holds. Consider the contingency that $m > 0$ of the other $\tilde{s} - 1$ remaining agents were to adopt in period k according to the equilibrium strategy $a^i(\cdot|\tilde{h})$. Then, the final network size that would realize when the agent i adopts in this period is the same as that that would realize when he adopts in the next period.*

Proof: Consider the case that the agent i adopted in period k, so that the state in period $k + 1$ is $s_1 = \tilde{s} - m - 1 < \tilde{s}$, hence all remaining agents of types lower than the equilibrium cutoff level τ_{s_1} would adopt in period $k + 1$ by [A]. (We use τ_s as shorthand for $\tau_s(\tilde{h})$ for $s < \tilde{s}$ in this proof.) Let s_2 be the state of period $k + 2$ that arises as a result. If $s_2 = s_1$ then no further adoption comes forth by [A], in which case s_2 is called the terminal state; otherwise, i.e., if $s_2 < s_1$ then all remaining agents of types lower than the equilibrium cutoff level τ_{s_2} would adopt in period $k+2$, resulting in a state s_3 of period $k+3$. If $s_3 = s_2$ then s_3 is the terminal state; otherwise, the state keeps being updated analogously for subsequent periods. The updating should stop because there are finite states. Denoting the terminal state by s_x, we have a sequence of states $s_1 > s_2 > \cdots > s_x$ and associated cutoff levels $\tau_{s_1} < \tau_{s_2} < \cdots < \tau_{s_x}$ for periods $k+1, \cdots, k+x$, respectively. Note $s_{x-1} = s_x$ by construction.

Consider the alternative case that the agent i did not adopt in period k, so that the state in period $k+1$ is $s'_1 = r - m = s_1 + 1$, hence all remaining agents of types lower than the equilibrium cutoff level $\tau_{s'_1}$ would adopt in period $k+1$. Let s'_2 be the state of period $k+2$ that arises as a result. If $s'_2 = s'_1$ then no further adoption comes forth by [A], hence s'_2 is the terminal state; otherwise, i.e., if $s'_2 < s'_1$ then all remaining agents of types lower than the equilibrium cutoff level $\tau_{s'_2}$ would adopt in period $k+2$, resulting in a state s'_3 of period $k+3$. If $s'_3 = s'_2$ then s'_3 is the terminal state; otherwise, the state keeps being updated analogously. Denoting the terminal state by s'_y, we have a sequence of states $s'_1 > s'_2 > \cdots > s'_y$ and associated cutoff levels $\tau_{s'_1} < \tau_{s'_2} < \cdots < \tau_{s'_y}$ for periods $k+1, \cdots, k+y$, respectively. Again, $s'_{y-1} = s'_y$ by construction.

The claim of the Lemma is proved if $s_x = s'_y$. In fact, it is easy to see that

[B] $s_x = s'_y$ ensues if $s_j = s'_\ell$ for some $1 \le j \le x$ and $1 \le \ell \le y$, because then $s_{j+1} = s'_{\ell+1}$ and the subsequent updating of the state is the same between the two sequences.

Note $s'_1 > s'_2$ because the agent i adopts in period $k+1$. Since $s_1 = s'_1 - 1$ by construction as noted earlier, $s_1 \ge s'_2$. If $s_1 = s'_2$ then the claim is proved by [B].

Suppose otherwise, i.e., $s_1 > s'_2$. By construction, $s'_2 = s'_1 - 1 - \#(0, \tau_{s'_1}] = s_1 - \#(0, \tau_{s'_1}]$ where $\#(0, \tau]$ is the number of agents other than i who remain after period $k+1$ and have types in $(0, \tau]$. Similarly, $s_2 = s_1 - \#(0, \tau_{s_1}]$ by construction. Since $s'_1 > s_1$ implies $\tau_{s'_1} < \tau_{s_1}$, it follows that $\#(0, \tau_{s'_1}] \le \#(0, \tau_{s_1}]$, hence $s'_2 \ge s_2$.

The claim follows by [B] if $s'_2 = s_2$, hence suppose $s'_2 > s_2$ in the sequel. By construction, $s'_3 = s'_1 - 1 - \#(0, \tau_{s'_2}] = s_1 - \#(0, \tau_{s'_2}]$. Since $s_1 > s'_2$ it follows that $s_2 \ge s'_3$. Since the claim follows if $s_2 = s'_3$, suppose $s_2 > s'_3$ in the sequel.

Proceeding analogously, we deduce that $s_x = s'_y$ unless $s'_j > s_j > s'_{j+1} > s_{j+1}$ for all $j = 1, 2, \cdots$. However, these inequalities contradict $s_{x-1} = s_x$, an equality that must hold by construction, hence we conclude that $s_x = s'_y$, i.e., the final network sizes are the same. *Q.E.D.*

Lemma 4: *Suppose $\delta = 1$ and that along the equilibrium path a state $\tilde{s}$ is reached in period k with a history $\tilde{h}$ such that [A] holds. The equilibrium strategy in period k is a cutoff strategy whose cutoff level is uniquely determined by $\tilde{h}$ and is lower than the cutoff level for the state $\tilde{s} - 1$.*

Proof: Let $g(\cdot|\tilde{h})$ and $G(\cdot|\tilde{h})$ denote the posterior density and cdf functions, respectively, updated by $\tilde{h}$ on the type of each remaining agent. In light of [A], let $\tau_s(\tilde{h})$ denote the equilibrium cutoff level after the history $\tilde{h}^s = (\tilde{h}, s)$ for $s < \tilde{s}$.

Consider an arbitrary remaining agent i in period k. Suppose his type is $t_i \leq \tau_{\tilde{s}-1}(\tilde{h})$. If he waited while $m > 0$ other agents adopted in this period, by adopting in the next period he can induce the same final network size as when he adopted in this period, according to Lemma 3—In fact, he will indeed adopt in the next period because $t_i \leq \tau_{\tilde{s}-1}(\tilde{h}) < \tau_{\tilde{s}-m}(\tilde{h})$. Hence, adopting and waiting is equivalent in this contingency and, therefore, the optimal decision of remaining agent in this period is determined by what would happen in the contingency that no agent other than i would adopt in this period. In this latter contingency, if the agent i adopted, then his expected utility is

$$\sum_{j=1}^{\tilde{s}} u_{t_i}(N - \tilde{s} + j)\mathrm{Prob}(j|\tilde{h})$$

where $\mathrm{Prob}(j|\tilde{h})$ is the probability conditional on $\tilde{h}$ that no other agent adopts in period k and j more other agents adopt eventually. If the agent i did not adopt, the adoption process would end and he would get $u_\phi = 0$. Again, since other remaining agents' behavior does not depend on t_i, the sum above strictly decreases in t_i. Hence, the equilibrium strategy in this period is a cutoff strategy at, say $\hat{t}$. The equilibrium level of $\hat{t}$ is characterized by

$$\sum_{j=1}^{\tilde{s}} u_{\hat{t}}(N - \tilde{s} + j)\mathrm{Prob}(j|\tilde{h}) = 0 \tag{5}$$

where $\mathrm{Prob}(j|\tilde{h})$ is calculated using the fact that the posterior density on the type of remaining agent after this period is $g(\cdot|\tilde{h})|_{t \geq \hat{t}}$, the truncated density of $g(\cdot|\tilde{h})$ above $\hat{t}$. As $\hat{t}$ increases, $g(\cdot|\tilde{h})|_{t \geq \hat{t}}$ deteriorates in the sense of first-order stochastic dominance, hence so does the distribution of total number of future adopters. Together with the fact that utility decreases in type, we deduce that the LHS of (5) is strictly decreasing in $\hat{t}$, hence there is a unique value of $\hat{t}$ that solves (5), denoted by $\tau_{\tilde{s}}(\tilde{h})$. Clearly, $\tau_{\tilde{s}}(\tilde{h}) > 0$ because the LHS of (5) is positive when $\hat{t} = 0$. Consider a $\tau_{\tilde{s}-1}(\tilde{h})$-type agent: his expected utility would be 0 if he already adopted and $\tilde{s}-2$ agents remain whose type is distributed according to $g(\cdot|\tilde{h})$ truncated at $\tau_{\tilde{s}-1}(\tilde{h})$. So, his expected utility would be negative if $\tilde{s}-1$ agents remain with the same type distribution. This means that the LHS of (5) is negative at $\hat{t}=\tau_{\tilde{s}-1}(\tilde{h})$ and, therefore, $0<\tau_{\tilde{s}}(\tilde{h})<\tau_{\tilde{s}-1}(\tilde{h})$. *Q.E.D.*

Recall the induction hypothesis that the property [A] holds for all $\tilde{s}\le r$ where $r=3,\cdots,N$, along an equilibrium. We now establish that the property [A] holds for $\tilde{s}=r+1$ as well. Suppose a state $\tilde{s}=r+1$ is reached in some period k after a history $\tilde{h}$. By induction hypothesis and Lemma 4, the equilibrium strategy after $\tilde{h}^r=(\tilde{h},r)$ is a cutoff strategy at a level $\tau_r(\tilde{h}^r)$ and $\tau_r(\tilde{h}^r)<\tau_{r-1}(\tilde{h}^r)<\cdots<\tau_1(\tilde{h}^r)=\bar{t}$.

We now show that the history $(\tilde{h}^r,s)$ and any other extension $\tilde{h}^s$ of $\tilde{h}$ with state s entail the same cutoff strategy for $s<r$. By induction hypothesis we only need to show this for $\tilde{h}^s$ that is not an extension of $\tilde{h}^r$, which we assume below. Let $g(\cdot|\tilde{h})$, $g(\cdot|\tilde{h}^s)$ and $g(\cdot|\tilde{h}^r,s)$ be the posterior densities after $\tilde{h}$, $\tilde{h}^s$ and $(\tilde{h}^r,s)$, respectively. If $s=1$, the cutoff levels are clearly $\tau_s(\tilde{h}^s)=\tau_s(\tilde{h}^r)=\bar{t}$. Now suppose $\tau_s(\tilde{h}^s)=\tau_s(\tilde{h}^r)$ for all $s=1,\cdots,z-1$, and consider $s=z(<r)$. For the cutoff level $\tau_z=\tau_z(\tilde{h}^r)$, the expected utility of a τ_z-agent when he finds himself to be the sole adopter in the current period, say k_z, is 0:

$$\sum_{j=0}^{z} u_{\tau_z}(N-z+j)\text{Prob}(j|\tilde{h}^r,z-1)=0 \tag{6}$$

where $\text{Prob}(j|\tilde{h}^r,z-1)$ is the probability that j more agents adopt eventually when $z-1$ agents who remain after period k_z follow cutoff levels $\tau_s(\tilde{h}^r)$, $s=1,\cdots,z-1$, in future periods. The posterior density of the $z-1$ remaining agents is $g(\cdot|\tilde{h}^r)|_{t>\tau_z}$ because cutoff strategies would have been followed in period $k+1$ and later.

Next, consider $\tau_z'=\tau_z(\tilde{h}^z)$. Similarly as above, the expected utility of a τ_z'-agent when he finds himself to be the sole adopter in the current period, say k_z', is 0:

$$\sum_{j=0}^{z} u_{\tau_z'}(N-z+j)\text{Prob}(j|\tilde{h},z-1)=0 \tag{7}$$

where $\text{Prob}(j|\tilde{h},z-1)$ is the probability that j more agents adopt eventually when $z-1$ agents who remain after period k_z' follow cutoff levels $\tau_s(\tilde{h}^z)$, $s=1,\cdots,z-1$, in future periods. Note $\tau_s(\tilde{h}^z)=\tau_s(\tilde{h}^r)$ for $s=1,\cdots,z-1$ by supposition and that the posterior density of the $z-1$ remaining agents in this case is $g(\cdot|\tilde{h}^s)|_{t>\tau_z'}$ because, again, cutoff strategies would have been followed in period $k+1$ and later. Assume $\tau_z\ge\tau_z'$ so that $g(\cdot|\tilde{h}^s)|_{t>\tau'_z} = c \cdot g(\cdot|\tilde{h}^s)|_{t>\tau_z}$ for some constant $c > 0$ for all $t > \tau_z$ and, therefore, $\text{Prob}(j|\tilde{h}, z-1) = c^{z-1} \cdot \text{Prob}(j|\tilde{h}^r, z-1)$ for all $t > \tau_z$. From (6), therefore, it follows that (7) holds precisely when $\tau'_z = \tau_z$. A symmetric argument works when $\tau_z < \tau'_z$. Hence, we have proved that $\tau_s(\tilde{h}^s) = \tau_s(\tilde{h}^r)$ for all $s < r$, as desired. This completes the induction argument that [A] holds for $\tilde{s} = r + 1$ as well.

Finally, applying Lemma 4 to histories with state $\tilde{s} = r+1$, we prove that the equilibrium strategy in state $r + 1$ and onwards is a cutoff strategy that is uniquely determined by the posterior on the remaining agents' type shaped by the history up to then. Applying the same logic inductively all the way back to the the state $N + 1$, i.e., to the null history, we find a unique symmetric equilibrium. We state this as:

Theorem 1: *If $\delta = 1$, there exists a unique symmetric equilibrium of Γ. In this equilibrium, the remaining agents' strategy after any history is a cutoff strategy at a level that depends only on the state s (i.e., the number of remaining agents), denoted by τ_s, and $0 < \tau_{N+1} < \tau_N < \cdots < \tau_1 = \bar{t}$.*

Part of the analysis up to now relies on the fact $\delta = 1$, hence is not readily applicable to the case $\delta < 1$. If $\delta < 1$, an agent would prefer adopting earlier rather than later if adopting later delays the adoption process although it leads to the same network eventually. In evaluating the benefit of adopting now as opposed to waiting, therefore, the time paths following adoption by some other agents come into the equation even if the final network will be the same regardless of whether the agent in question adopts now or in the next period, because the differences along the two paths now matters. Due to such additional considerations the future cutoff levels depends not only on how many agents have adopted by then but also on when they (including the agent in question) adopted. This implies that the final network can be different depending on when the agent in question adopts. So, the Lemma 3 does not hold. Nevertheless, the effects of these complications become negligible as δ approaches 1 because then the discrepancy in argument from the case $\delta = 1$ either happens with negligible probability or has a negligible effect because it applies only to a finite number of periods before the terminal network is reached. Therefore, the basic intuition of Theorem 1 extends to large values of δ: We establish that there is a unique symmetric equilibrium, and that it is a cutoff equilibrium and converges to the one described in Theorem 1 as δ tends to 1. This result is formally stated in the next theorem and is proved in the Appendix. Note that it is no longer the case that the equilibrium cutoff level depends only on the number of total adopters by then, but it depends on the full adoption history up to then.

Theorem 2: *Suppose i) $\dot{u}_t(\nu)$, the derivative of $u_t(\nu)$ with respect to t, exists for all $t \in (0, \bar{t})$ and $\nu = 0, \cdots, N$, and ii) there is $\theta > 0$ such that $|\dot{u}_t(\nu)| > \theta$ for all $t \in (0, \bar{t})$ and $\nu = 0, \cdots, N$. Then, there is $\delta^* < 1$ such that if $\delta > \delta^*$ there is a unique symmetric equilibrium of Γ. Furthermore, this equilibrium is a cutoff equilibrium and converges to the equilibrium described in Theorem 1 as $\delta \to 1$.*

Up to now we have focused on symmetric equilibrium and characterized it fully. We note, however, that equilibria in asymmetric cutoff strategies may also exist in some environments. This is because, given a history up to a certain period, individual agents may have different beliefs regarding the distributions of types of all other remaining agents. Such differences in beliefs may support differences in the anticipated number of other people who will enter in the future and, therefore, may result in asymmetric equilibrium cutoff levels.[5] Note, however, that the differences in beliefs of any pair of agents can only exist with respect to beliefs about each other's types since their respective sets of "all other agents" differ only with respect to each other, and this difference becomes insignificant when the number of agents, N, is large.[6] As N grows without bound, therefore, the differences in expectations of any two agents become ever smaller and consequently, all asymmetric equilibria converge to the unique symmetric equilibrium. This is stated below.

Theorem 3: *Suppose i) $\bar{t}^N \to \bar{t}^\infty < \infty$ as $N \to \infty$, where $\bar{t}^N$ is the unique t such that $u_t(N) = 0$, ii) $\dot{u}_t(\nu)$ exists and $|\dot{u}_t(\nu)| > \theta$ for some $\theta > 0$, for all $t \in (0, \bar{t}^\infty)$ and all $\nu = 0, \cdots$. For δ larger than δ^* described in Theorem 2, asymmetric equilibria of Γ may exist, however they converge, if exist, to the unique symmetric equilibrium as the number of agents tends to infinity.*

Proof: See Appendix.

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[5] We provide a two-agent example. Let $u_t(\nu) = \nu - t$ be the utility functions for $\nu = 0, 1$, and consider a cdf function F such that $F(0.2) = 1/6$, $F(0.4) = 3/8$ and $F(1) = 1/2$. Clearly, $\bar{t} = 1$ is the cutoff level when only one agent remains. Let t_1 and t_2 be the cutoff levels of agents 1 and 2, respectively, when $\delta = 1$ and neither of them adopted, i.e., in state $s = 2$. The condition for the agent 1 of t_1 type to be indifferent between adopting and not, is $F(1)u_{t_1}(1) + (1 - F(1))u_{t_1}(0) = F(t_2)u_{t_1}(1)$, or equivalently, $(\frac{1}{2} - F(t_2))(1 - t_1) = (1 - \frac{1}{2})t_1$. An analogous condition for agent 2 of t_2 type is $(\frac{1}{2} - F(t_1))(1 - t_2) = (1 - \frac{1}{2})t_2$. One can easily verify from $F(0.2) = 1/6$ and $F(0.4) = 3/8$ that these two conditions are satisfied when $t_1 = 0.2$ and $t_2 = 0.4$ and when $t_1 = 0.4$ and $t_2 = 0.2$, hence asymmetric cutoff equilibria exist.

[6] This is so even if the adoption process went a long way and only a small number of agents remain, because "all other remaining agents" can be any subset of the initial set of agents with the right cardinality. Here, we implicitly assume anonymity in the sense that each agent cannot tell other agents apart except by their past adoption decisions.

4. Concluding Remarks

New products and services that have network externalities are often adopted by at least some of the potential users of such products and services. A satisfactory model of the adoption process should be able to account for the size of the group that chooses to enter a network. A static model of network formation cannot do this because the existence of complementarities implies that there are a multiplicity of equilibria. However, it is natural to think of the formation of a network as a dynamic process in which each agent can observe at each moment in time how many people have already entered the network and can use this information to update his/her beliefs with respect to the expected number of additional agents who will eventually join the network. Modelling the entry process as a dynamic game of incomplete information is not only more realistic, but, as our analysis of a simple dynamic market entry game shows, may yield a unique equilibrium. Unlike the static network entry game, if individuals can choose when to enter then the decision of one agent can influence the decisions of others. Therefore, the existence of complementarities in the payoff function of agents cannot, in general, support an equilibrium in which no one enters because every agent believes that no one else will enter. Our model also has the testable implication that the expected number of entrants is related to the properties of the distribution of types in the population in a quite natural way. In particular, the model implies that the more enthusiastic is the population about adopting the network (in the sense of first-order stochastic dominance), the greater is the expected number of people who will enter in equilibrium.

Appendix

Proof of Theorem 2: It is straightforward (hence, omitted) to extend the logic of Lemma 1 to δ sufficiently close to 1 and verify that there is a threshold $\underline{\delta} < 1$ such that in any symmetric equilibrium of Γ with $\delta > \underline{\delta}$, every agent adopts with a positive probability in period 1. Throughout this Appendix we consider $\delta \in (\underline{\delta}, 1)$. We now characterize symmetric equilibrium $(a^i)_{i\in\mathcal{I}}$ by an induction argument.

<u>*STEP A1:*</u> As argued above, the game reaches any possible state $s = 0, \cdots, N+1$, with a strictly positive probability. It is trivial that if the game reached a state $s = 1$, i.e., only one agent remains in some period k, then this last agent will adopt precisely when his type does not exceed $\bar{t}$ defined in (2). That is, the equilibrium strategy of the remaining agent when he is the only remaining agent (i.e., in state $s = 1$) is a cutoff strategy at $\tau_1 \equiv \bar{t}$.

<u>*STEP A2:*</u> Suppose that the game reached a state $s = 2$ in period k with a history h_k. Consider one remaining agent, say i, of type $t_i \leq \tau_1$. If the other remaining agent, say j, were to adopt in this period (which happens with probability p_1, say), agent i would get a stage utility of $u_{t_i}(N)$ forever by adopting in this period; if agent i waited in this period he would adopt in the next period (because $t_i \leq \tau_1$), hence again get a stage utility of $u_{t_i}(N)$ forever but from next period onwards.

Next consider the contingency that agent j were to not adopt in this period, which happens with probability p_0 $(= 1 - p_1)$. In this contingency, agent i would get a utility $u_\phi = 0$ by not adopting in this period; if agent i adopted in this period, he would get $u_{t_i}(N-1)$ this period, and from next period on he would get $u_{t_i}(N)$ in case agent j joins next period (which happens with probability q, say, conditional on j does not join this period) and $u_{t_i}(N-1)$ otherwise. Note that the agent j's response in the next period is independent of t_i.

Combining the two contingencies, the benefit of adopting this period as opposed to waiting is

$$p_1(1-\delta)u_{t_i}(N) + p_0[u_{t_i}(N-1) + q\delta(u_{t_i}(N) - u_{t_i}(N-1))]$$

which is strictly decreasing in t_i regardless of p_1 and q, with a negative value at $t_i = \tau_1$. Therefore, agent i (and j by symmetry) should employ a cutoff strategy at a level, say $\hat{t} < \tau_1$.

Let $g(\cdot)$ denote the posterior density function updated from f by h_k on the type of the remaining agent. Since $p_1 = \int_0^{\hat{t}} g(t)dt$ and $q = \int_{\hat{t}}^{\tau_1} g(t)dt / \int_{\hat{t}}^{\infty} g(t)dt$, the cutoff level $\hat{t}$

satisfies

$$
\begin{aligned}
&(1-\delta)u_{\hat{t}}(N)\int_0^{\hat{t}} g(t)dt \\
&\qquad + \Big(1-\int_0^{\hat{t}} g(t)dt\Big)\Big[u_{\hat{t}}(N-1) + \frac{\delta\int_{\hat{t}}^{\tau_1} g(t)dt}{\int_{\hat{t}}^{\infty} g(t)dt}(u_{\hat{t}}(N)-u_{\hat{t}}(N-1))\Big] \;=\; 0 \\
\iff\quad & (1-\delta)u_{\hat{t}}(N)\int_0^{\hat{t}} g(t)dt\int_{\hat{t}}^{\infty} g(t)dt + \Big(1-\int_0^{\hat{t}} g(t)dt\Big)\times \\
&\qquad \Big[u_{\hat{t}}(N-1)\Big(\int_{\hat{t}}^{\infty} g(t)dt - \delta\int_{\hat{t}}^{\tau_1} g(t)dt\Big) + \delta u_{\hat{t}}(N)\int_{\hat{t}}^{\tau_1} g(t)dt\Big] \;=\; 0.
\end{aligned}
\tag{8}
$$

Note that as $\delta \to 1$, i) the first term of the LHS of (8) becomes negligible, and ii) the second term is strictly decreasing in $\hat{t}$ (with the derivative bounded away from 0), clearly from a positive value when $\hat{t} = 0$ to a negative value when $\hat{t} = \tau_1$. Hence, for δ sufficiently close to 1 there exists a unique value of $\hat{t}$ that solves (8), which is the equilibrium cutoff level in the period following the history h_k, or equivalently, in the period with $s = 2$ and density g, denoted by $\tau_2(g|\delta)$. Furthermore, since the first derivative of the LHS w.r.t. $\hat{t}$ when $\delta = 1$ is bounded away from 0 by a number independent of g (because this derivative is bounded above by $\max_t \dot{u}_t(N-1) \le -\theta < 0$), for any $\epsilon > 0$ there exists $\delta_\epsilon < 1$ (independent of g) such that if $\delta > \delta_\epsilon$ then $\tau_2(g|\delta)$ uniquely exists and $|\tau_2(g|\delta) - \tau_2(g|1)| < \epsilon$. Summarizing,

Lemma A2: *For any $\epsilon > 0$, there is $\delta_\epsilon(2) < 1$ such that if $\delta > \delta_\epsilon(2)$ then the equilibrium strategy in state $s = 2$ with any density g is a cutoff strategy at $\tau_2(g|\delta)$, the unique level of $\hat{t}$ that solves (8), and $|\tau_2(g|\delta) - \tau_2(g|1)| < \epsilon$.*

<u>*STEP A3:*</u> Fix a state $\tilde{s}$ and consider the following property in a symmetric equilibrium:

[A'] For any $\epsilon > 0$, there is $\delta_\epsilon(\tilde{s}) < 1$ such that if $\delta > \delta_\epsilon(\tilde{s})$ then the equilibrium strategy in any state $s < \tilde{s}$ with any density g is a unique cutoff strategy at $\tau_s(g|\delta)$ and $|\tau_s(g|\delta) - \tau_s(g|1)| < \epsilon$.

Note that this property holds along an equilibrium when $\tilde{s} = 3$ by Lemma A2, and trivially if $\tilde{s} < 3$. We now make an induction hypothesis that the property [A'] holds for all $\tilde{s} \le r$ where $r = 3, \cdots, N$, along an equilibrium. Then, we establish that under this hypothesis the property [A'] holds for $\tilde{s} = r + 1$ as well. For this it suffices to show Lemma A3 below.

Lemma A3: *Suppose [A'] holds for some $\tilde{s}$. Then, for any $\epsilon > 0$, there is $\delta'_\epsilon(\tilde{s}) < 1$ such that if $\delta > \delta'_\epsilon(\tilde{s})$ then the equilibrium strategy in state $\tilde{s}$ with any density g is a unique cutoff strategy at $\tau_{\tilde{s}}(g|\delta)$ and $|\tau_{\tilde{s}}(g|\delta) - \tau_{\tilde{s}}(g|1)| < \epsilon$.*

Proof: Fix $\epsilon > 0$. Consider an equilibrium of the subgame, $\Gamma(\tilde{s}, g)$, starting with a state $\tilde{s}$ and a density g. Let g' denote the equilibrium density after the first period of this subgame. Note from Section 3 that when $\delta = 1$ the equilibrium cutoff levels after the first period of this subgame only depends on the state, which we denote by $\tau_s(g'|1)$ for $s < \tilde{s}$. Also note that $\tau_s(g'|1) < \tau_r(g'|1)$ if $r < s < \tilde{s}$ and therefore, by supposition of Lemma A3, the cutoff level decreases in the state for δ sufficiently close to 1.

Consider a remaining agent i of type $t_i < \tau_{\tilde{s}-1}(g'|\delta)$ in the first period of this subgame. First, consider the contingency that at least one other agent adopts in this period. If $\delta = 1$ and the agent i did not adopt in this period, by adopting in the next period he can ensure the same final network size as the one that would have resulted if he adopted in the first period, by the same argument as the proof of Lemma 3. For δ sufficiently close to 1 so that $\tau_s(g''|\delta)$ is arbitrarily close to $\tau_s(g''|1) = \tau_s(g'|1)$[7] for all $s < \tilde{s}$ and g'' that may arise in future periods, the following holds: If the agent i did not adopt in this period, by adopting in the next period he can ensure with arbitrarily large probability the same final network size as the one that would have resulted if he adopted in the first period; and in this case agent i's utility differential when adopt now and when adopt in the next period (which he will surely do because $t_i < \tau_{\tilde{s}-1}(g'|\delta)$) vanishes as $\delta \to 1$. The utility differential for the case that the final network is not the same also vanishes as $\delta \to 1$ because the probability vanishes that such a case gets realized.

Next consider the contingency that no other agent adopts in the first period. In this contingency, agent i would get a utility $u_\phi = 0$ by not adopting in this period. If agent i adopted in this period, other agents would adopt in future periods according to the equilibrium cutoff levels. As $\delta \to 1$, agent i's utility in this case is arbitrarily approximated by the expected utility level of $u_{t_i}(\nu)$ calculated using the probabilities that ν is the number of other agents who eventually adopt. (Note these probabilities is independent of t_i.) This expected utility level is strictly decreasing in t_i to a negative value at $\tau_i = \tau_{\tilde{s}-1}(g'|\delta)$, and the rate at which it decreases is bounded away from 0 independently of g' (because the rate each $u_{t_i}(\nu)$ decreases is bounded away from 0). Therefore, there is $\delta'' < 1$ such that the expected benefit of agent i of adopting in this period as opposed to waiting is strictly decreasing in t_i if $\delta > \delta''$ in any equilibrium of the subgame $\Gamma(\tilde{s}, g)$ for any g, hence the equilibrium strategy in this period is a cutoff strategy at a level $\tau_{\tilde{s}}(g|\delta) < \tau_{\tilde{s}-1}(g'|\delta)$.

Finally, let $Eu_{\hat{t}}(g|\delta)$ denote the expected benefit of agent i of type $\hat{t} < \tau_{\tilde{s}-1}(g'|\delta)$ of

[7] This equality follows because from next period on the cutoff level depends only on the state when $\delta = 1$, as shown in Section 3.

adopting in this period as opposed to waiting when g is the density and $\hat{t}$ is the cutoff level in this period and $\tau_s(g''|\delta)$ is the cutoff level of relevant future periods when the state is $s < \tilde{s}$ and g'' is the density. ($\tau_s(g''|\delta)$ is well-defined by [A'].) As shown in Section 3 (in the proof of Lemma 4), $Eu_{\hat{t}}(g|1)$ is strictly decreasing in $\hat{t}$ due to 2 factors: i) $u_{\hat{t}}(\nu)$ strictly decreases in $\hat{t}$ for each ν, and ii) the distribution of the final number of future adopters in case only agent i adopts in this period, deteriorates as $\hat{t}$ increases in the sense of first-order stochastic dominance. Since the factor ii) only reinforces the decrease, the rate at which $Eu_{\hat{t}}(g|1)$ decreases is bounded above by $-\theta$, hence bounded away from 0 independently of g. Since $Eu_{\hat{t}}(g|\delta)$ is arbitrarily closely approximated by $Eu_{\hat{t}}(g|1)$ as $\delta \to 1$, there is $\delta''' < 1$ such that if $\delta < \delta'''$ then the solution value of $\hat{t}$ to $Eu_{\hat{t}}(g|\delta) = 0$ is arbitrarily close to the solution value of $\hat{t}$ to $Eu_{\hat{t}}(g|1) = 0$. Setting $\delta_\epsilon(\tilde{s}) = \min\{\delta'', \delta'''\}$ proves Lemma A3. *Q.E.D.*

Recall that Lemma A3 establishes the induction argument that if the property [A'] holds for all $\tilde{s} \le r$ where $r = 3, \cdots, N$, then [A'] holds for $\tilde{s} = r + 1$ as well. Applying this result repeatedly, we conclude that [A'] holds for $\tilde{s} = N + 1$, i.e., at the beginning of period 1, thereby establishing that there is a threshold $\delta^* < 1$ such that if $\delta > \delta^*$ then there is a unique symmetric equilibrium and this equilibrium converges to the unique equilibrium characterized in Theorem 1 as $\delta \to 1$. This completes the proof of Theorem 2.

Proof of Theorem 3: First we consider the case $\delta = 1$. The arguments that prove Lemmas 3 and 4 do not rely on the equilibrium being symmetric. Therefore, it is straightforward (hence, details omitted) to extend these arguments to show that all equilibria are cutoff equilibria and that the claim of Lemma 3 holds for asymmetric equilibrium, too.

Consider any sequence of equilibria, $(a^{iN})_{i=1}^{N+1}$, $N = 1, 2, \cdots$, where the superscript N denotes the number of agents minus 1. (If asymmetric equilibrium does not exist for some N, take the symmetric equilibrium.) Represent each equilibrium by the cutoff levels $\tau^{iN}(h_k)$ for each N, each $i = 1, \cdots, N+1$, and each possible history h_k. For each h_k, let $\underline{\tau}^N(h_k) = \min_i\{\tau^{iN}(h_k)\}$ and $\bar{\tau}^N(h_k) = \max_i\{\tau^{iN}(h_k)\}$. It suffices to show that for each h_k, the two sequences $\underline{\tau}^N(h_k)$ and $\bar{\tau}^N(h_k)$ converge to the same point as $N \to \infty$.

To reach a contradiction, suppose to the contrary that they do not for some h_k. By taking subsequences if necessary, this amounts to supposing that $\underline{\tau}^N(h_k) \to a$ and $\bar{\tau}^N(h_k) \to b$ and $a < b$, owing to supposition i) of Theorem 3. Let ℓ be the last element of h_k, i.e., ℓ agents adopted in the last period of h_k. Clearly, $\bar{t}^\ell \le a$ because a $\bar{t}^\ell$-type agent would certainly join if ℓ other agents already adopted. Consider the agent with the cutoff level $\bar{\tau}^N(h_k)$. Recall that if this agent is the sole adopter in period k and he is of $\bar{\tau}^N(h_k)$-type, his expected payoff is 0. In this contingency (i.e., when he is the sole adopter in period k), if the probability of additional adoption converges to 0 as N tends to infinity, then $\bar{\tau}^N(h_k)$ would have to converge to $\bar{t}^\ell$. Since this would contradict $a < b$, the probability of additional adoption converges to a positive number. Then, the expected payoff of this agent, say agent i, in this contingency is strictly decreasing in his type, and the rate at which it does so is bounded away from 0 due to supposition ii). Consider the expected payoff of any other agent, say j, in the contingency that agent j is the sole adopter in period k. For sufficiently large N, the effect of agent i in agent j's expected payoff is negligible, and so is that of agent j in the corresponding expected payoff of agent i. Therefore, the expected payoff schedule of agent j (as a function of t) is arbitrarily close to that of agent i. Since the slope of the latter is bounded away from 0 as discussed above, it follows that $\tau^{jN}(h_k)$, the type for which agent j's expected payoff is 0, converges to $\bar{\tau}^N(h_k)$, that for agent i. Since agent j was chosen arbitrarily and $\bar{\tau}^N(h_k) \to b$ as $N \to \infty$, we end up with a contradiction to $\underline{\tau}^N(h_k) \to a < b$.

For $\delta \in (\delta^*, 1)$ we provide a proof based on an alternative argument. Note that if one agent adopts in period 1, every other agent would have adopted by period 2 with a probability at least $F(\bar{t}^1) > 0$, hence the expected number of other adopters by period 2 increases with N. Since $u_0(1) > 0$, this implies that every agent adopts in period 1 with a probability that is bounded away from 0, say by $\alpha > 0$, for large enough N. Then, the expected number of adopters in period 1 increases without bound as $N \to \infty$ and, therefore, the first period cutoff level in any asymmetric equilibrium converges to $\bar{t}^\infty$. Since the cutoff levels of the symmetric equilibrium also converge to $\bar{t}^\infty$, it follows that the set of types that would have adopted by any particular period in any asymmetric equilibrium converges to that of the unique symmetric equilibrium as $N \to \infty$. This completes the proof of Theorem 3.

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