Smallness of a commodity and partial equilibrium analysis

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Abstract

Partial equilibrium analysis has a conceptual dilemma that its object should be negligibly small in order to be free from income effect but then the consumer does not care for it and the notion of willingness to pay for it does not make sense. In the setting of a continuum of commodities, we propose a limiting procedure which transforms the general many-commodity framework into a partial single-commodity framework. In the limit, willingness to pay for a commodity is established as a density notion and it is shown to be free from income effect. This pins down an exact relationship between general equilibrium analysis and partial equilibrium analysis.

1 Introduction

Partial equilibrium analysis isolates the market of a particular commodity from the rest of the economy and looks at changes there by assuming that "other things remain equal (Marshall, *Principle of Economics* [6], p.207)." This presumes that there is no income effect on the commodity under consideration, because otherwise change of consumption of it in general changes expense on the other commodities and this in turn changes the consumer's willingness to pay for it, meaning that the isolation fails and policy recommendation based on such analysis is misleading.

The absence of income effect is usually justified by saying that the commodity is negligibly small compared to the entire set of commodities. Then, however, the consumer does

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not care for it apparently and the notion of willingness to pay for it does not make sense. This is the dilemma we tackle in the current paper.

Vives [10] is the first paper which provides a formal treatment of the above-noted tension. He considers an increasing sequence of sets of commodities, and under certain assumptions shows that income effect on each single commodity vanishes as the number of commodity and income tend to infinity at the same rate.

This approach, however, does not allow us to handle eventually (countably) infinitely many commodities, since (i) it assumes roughly that all commodities have the uniform degree of utility weight, which cannot be true when there are indeed countably infinitely many commodities because if so the entire utility function cannot take a finite value; and (ii) it assumes that income increases at the same rate as the number of commodities, which we cannot think of literally in the limit. Thus, there is a discontinuity between the case of large but finite numbers of commodities and the case of indeed infinite number of commodities, and the asymptotic property of a sequence of preferences over finitely many commodities does not extend to be a property of preference that should make sense *in the limit*. To our knowledge, there has been no study which pins down an *exact* relationship in the limit, between preference in the general equilibrium setting and the notion of willingness to pay in the partial equilibrium setting.

We provide a limiting procedure which resolves the above-noted dilemma and converts the many-commodity general equilibrium framework into the partial equilibrium framework in an exact and operational manner.

We take a reverse direction. We present the whole set of commodities in the outset, which is a continuum, and subdivide it into many pieces so that each piece tends to be arbitrarily small. The continuum assumption might look odd, but it applies not only to the case of finely differentiated commodities, but also to resource allocation under uncertainty with a continuum of states and intertemporal resource allocation with continuous time. Also, more importantly, it is a reasonable framework for precisely describing what we mean by 'negligible.'

Let us illustrate the difference between the Vives approach and the present one. Vives [10] considers a sequence of prices and incomes $\{(p^n, w^n)\}$, where p^n is an n-dimensional price vector, which is uniformly bounded from above and below, and income w^n increases at the same rate as n. Then, for each n consumption vector $z^n \in \mathbb{R}^n_{++}$ must satisfy budget constraint

$$\sum_{k=1}^{n} p_k^n z_k^n = w^n.$$

Then, for a differential income change Δw we have

$$\sum_{k=1}^{n} p_k^n \left(z_k^n + \Delta z_k^n\right) = w^n + \Delta w$$

which implies so-called the Engel aggregation condition

$$\sum_{k=1}^{n} p_k^n \frac{\partial z_k^n}{\partial w} = 1.$$

Under suitable assumptions, the income derivative of demand $\frac{\partial z_k^n}{\partial w}$ uniformly converges to zero at rate $1/n$ (see also Hayashi [4]). This is because given the price sequence $\{p^n\}$ to be uniformly bounded from above and below the sum of income derivatives $\sum_{k=1}^{n} \frac{\partial z_k^n}{\partial w}$ is uniformly bounded from above and below as well and because every $\frac{\partial z_k^n}{\partial w}$ is shown to have the same degree of magnitude.[1] Notice here that income w^n goes to infinity at the same rate as n, and asymptotically the consumer has a large pool of income so that income change Δw is small compared to w^n.

The present approach considers that the set of commodity characteristics is given as a continuum, say the unit interval $[0, 1]$. Let μ be the Lebesgue measure over $[0, 1]$. Then a price system is given as a *density* function $p : [0, 1] \to \mathbb{R}_{++}$ with suitable mathematical properties, and income $w > 0$ is given as a fixed number.

We formulate the process of subdivision in the form an sequence of finite partitions $\{\mathcal{J}^n\}$ of the set of commodity characteristics, which tends to be finer and finer for larger n and converges to the finest partition of singletons. At each step of subdivision, we consider that the consumer is given a finite number of commodities, namely, $|\mathcal{J}^n|$ commodities. That is, given n, each $K \in \mathcal{J}^n$ is taken to be one commodity so that the consumption amount is constant over K, denoted by z_K^n let's say, and the price of subdivision K is given by $p_K^n = \int_K p(t) d\mu(t)$.

[1] This relies on an extra assumption that all commodities are normal. Without this assumption, however, Vives [10] shows that the Euclidian norm of vector $\left(\frac{\partial z_1^n}{\partial w}, \cdots, \frac{\partial z_n^n}{\partial w}\right)$ vanishes at rate $1/\sqrt{n}$.

Then the budget constraint takes the form

$$\sum_{K \in \mathcal{J}^n} p_K^n z_K^n = w$$

Let Δw be the income change then it satisfies

$$\sum_{K \in \mathcal{J}^n} p_K^n \left(z_K^n + \Delta z_K^n \right) = w + \Delta w,$$

which implies the corresponding version of Engel aggregation condition

$$\sum_{K \in \mathcal{J}^n} p_K^n \frac{\partial z_K^n}{\partial w} = 1.$$

The budget constraint and the Engel aggregation condition are rewritten into

$$\sum_{K \in \mathcal{J}^n} \frac{p_K^n}{\mu(K)} z_K^n \mu(K) = w$$

and

$$\sum_{K \in \mathcal{J}^n} \frac{p_K^n}{\mu(K)} \cdot \frac{\partial z_K^n}{\partial w} \mu(K) = 1$$

respectively, where $\frac{p_K^n}{\mu(K)}$ is the *average* of prices of the elements of K.

Because differentiation of integration a function is the original function itself, as we make the subdivisions finer and finer the vector of average prices $\left(\frac{p_K}{\mu(K)} \right)_{K \in \mathcal{J}^n}$ converges to the density p, and the budget constraint converges to the integral form

$$\int_0^1 p(t) f(t) d\mu(t) = w$$

where consumption bundle is given as a function $f : [0, 1] \to \mathbb{R}_{++}$, and the Engel aggregation condition converges to its continuum version

$$\int_0^1 p(t) \frac{\partial f(t)}{\partial w} d\mu(t) = 1.$$

Notice that income effect on each subdivision $\frac{\partial z_K^n}{\partial w}$ converges to the *density* on its element t, a positive term $\frac{\partial f(t)}{\partial w}$, and *does not vanish*. However, the income effect *adjusted* to its smallness $\frac{\partial z_K^n}{\partial w} \mu(K)$ is *negligibly small* as its limit is $\frac{\partial f(t)}{\partial w} d\mu(t)$. This is because the income change associated with having one extra unit of subdivision K is $\Delta w \cdot \mu(K)$ instead of Δw, as its limit at commodity element t is $\Delta w \cdot d\mu(t)$ instead of Δw.

We invoke this point in order to found the notion of willingness to pay. The traditional partial equilibrium analysis after Hicks [5] assumes quasi-linear preference over pairs of consumption and income transfer. Let x denote the quantity of the commodity being focused and a denote the associated income transfer which is either positive (when received) or negative (when paid). The implicit idea behind is that the consumer already has a "large" pool of income and cares only about its relative change described by a.

The analysis assumes that the consumer has so-called quasi-linear preference which is represented in the form

$$u(x, a) = v(x) + a$$

or its monotone transformation. In typical application in which price of the commodity is denoted by p, the income transfer is taken to be the payment, meaning $a = -px$, and the whole expression is taken to be $v(x) - px$. Quasi-linearity implies that the marginal rate of substitution of income transfer for the commodity, marginal willingness to pay for it in other words, takes the form

$$S(x, a) = v'(x),$$

which is independent of a. This means there is no income effect on the commodity under consideration. However, the quasi-linear preference form itself says nothing about *why* there is no income effect.

We think that quasi-linearity should be *derived as a limit property* of preference in some many-commodity general equilibrium setting. Following the Vives approach, Hayashi [4] showed the following asymptotic quasi-linearity result. Given n commodities, consider that one commodity, say j, is to be the object of partial equilibrium analysis and the remaining $n-1$ commodities are to be aggregated, and let $\succsim^{n,j}$ be the preference induced over pairs of consumption of the commodity under analysis and income transfer to be allocated to the other commodities. For $(x, a), (y, b) \in \mathbb{R}_{++} \times (-w^n, \infty)$, define

$$(x, a) \succsim^{n,j} (y, b)$$

by

$$U^n\left(x, z^{n,j}(x, a)\right) \geq U^n\left(y, z^{n,j}(y, b)\right),$$

where $z^{n,j}(x,a) \in \mathbb{R}^{n-1}_{++}$ is the solution to

$$\max_{z_{-j} \in \mathbb{R}^{n-1}_{++}} U^n\left(x, z_{-j}\right)$$
$$\text{subject to} \quad \sum_{k \neq j} p^n_k z_k = w^n + a$$

and similarly for $z^{n,j}(y,b)$.

Hayashi shows that $\succsim^{n,j}$ is asymptotically quasi-linear in the sense that the derivative of marginal rate of substitution with respect to a vanishes as n tends to infinity. Notice again that w^n tends to infinity, which means that income transfer a tends to be *relatively very small* compared to it.

Now we illustrate the present approach. Let U be a representation of preference over the entire consumption space with the continuum of commodity characteristics. Given n, let U^n be the restriction of U on to the finite dimensional subspace $\mathbb{R}^{|\mathcal{J}^n|}$, defined by

$$U^n(z) = U\left(\sum_{K \in \mathcal{J}^n} z_K \mathbf{1}_K\right)$$

for $z = (z_K)_{K \in \mathcal{J}^n} \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$. Also, given n and $J \in \mathcal{J}^n$ to be fixed, $(x, z_{-J}) \in \mathbb{R}^{|\mathcal{J}^n|}$ denotes the vector such that x is its J-component $z_{-J} \in \mathbb{R}^{|\mathcal{J}^n|-1}$ refers to the rest of the entries.

Given n, pick $J \in \mathcal{J}^n$ to be the object of partial equilibrium analysis in the approximate sense. In the limit, the set J is supposed to shrink to a point. Let x be the consumption amount which is constant over J. Let a be the amount of income transfer which is accompanied with the consumption of each element of J. Since the mass of the piece is $\mu(J)$, the total income transfer which is accompanied with the consumption of the commodity piece is $a\mu(J)$. Because income is fixed to be finite here, this adjustment corresponds to the nature that income transfer is relatively very small compared to the pool of income.

Thus, given n, $J \in \mathcal{J}^n$, the preference relation induced over pairs of consumption and income transfer, denoted $\succsim^{n,J}$, is defined by:

$$(x,a) \succsim^{n,J} (y,b)$$

hold for $(x,a),(y,b) \in \mathbb{R}_{++} \times \left(-\frac{w}{\mu(J)}, \infty\right)$ if and only if

$$U^n\left(x, z^{n,J}(x,a)\right) \geq U^n\left(y, z^{n,J}(y,b)\right),$$

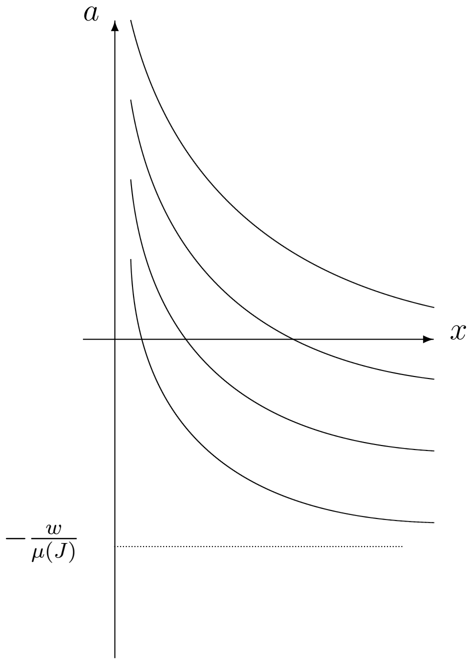

Figure 1: Induced Preference

where $z^{n,J}(x,a) = (z_K^{n,J}(x,a))_{K \in \mathcal{J}^n \setminus \{J\}} \in \mathbb{R}_{++}^{|\mathcal{J}^n|-1}$ is the solution to

$$\max_{z_{-J} \in \mathbb{R}_{++}^{|\mathcal{J}^n|-1}} U^n(x, z_{-J})$$
$$\text{subject to} \quad \sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K z_K = w + a\mu(J)$$

and similarly for $z^{n,J}(y,b)$.

See Figure 1 for how the induced preference typically looks like.

The paper shows that the sequence of induced 2-good preferences $\{\succsim^{n,J}\}$ converges and the limit preference is quasi-linear, as is illustrated in Figure 2. This is a stronger result in the sense that the previous one shows only that the corresponding sequence exhibits asymptotic constancy of MRS with respect to income transfers but there may not be a limit preference.

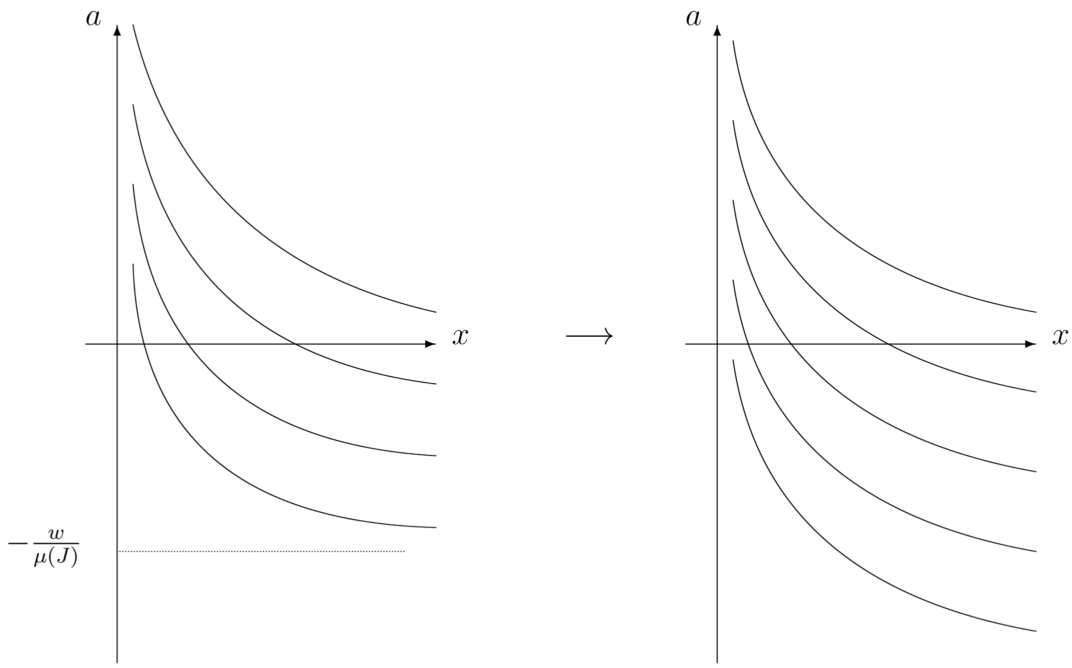

Figure 2: Limit Preference

2 The quasi-linearity limit theorem: separable preferences

For the presentation purpose, we limit our argument to the case of separable preferences. More general arguments on non-separable preferences are provided in the appendix.

2.1 Consumption space

Let $T = [\underline{t}, \overline{t}]$ be a finite interval, Σ be the family of Lebesgue measurable sets, and μ be the Lebesgue measure.

Let $L^\infty(T)$ be the space of essentially bounded measurable functions from T to $\mathbb{R}$, which is endowed with the sup norm. Denote the norm dual of $L^\infty(T)$ by $L^\infty(T)^*$. It is known that the norm dual of $L^\infty(T)$ is the set of finitely additive signed measured over T endowed with the total variation norm denoted by $ba(T)$, that is, $L^\infty(T)^* = ba(T)$.

Let $L^1(T)$ be the space of Lebesgue integrable functions from T to $\mathbb{R}$, which is endowed with the integral norm. It is known that $L^1(T)$ can be viewed as a subset of $L^\infty(T)^* = ba(T)$. There the dual operation takes the form $\langle p, f\rangle = \int_T p(t)f(t)d\mu(t)$, where $f \in L^\infty(T)$

and $p \in L^1(T)$. Given an integrable function $p : T \to \mathbb{R}$ and a measurable set $K \in \Sigma$, let $p_K = \int_K p(t)d\mu(t)$.

It is known that $L^1(T)^* = L^\infty(T)$, with the dual operation given by $\langle f, p\rangle = \int_T f(t)p(t)d\mu(t)$ where $f \in L^\infty(T)$ and $p \in L^1(T)$.

Let $L^\infty_+(T)$ be the set of essentially bounded measurable functions which are non-negative almost everywhere. Define $L^1_+(T)$ similarly. Also, let

$$L^\infty_{++}(T) = \{f \in L^\infty_+(T) : a.e.\ t \in T,\ f(t) > 0\}$$

and

$$L^\infty_{+++}(T) = \{f \in L^\infty_+(T) : \exists l > 0,\ a.e.\ t \in T,\ f(t) \geq l\}.$$

and define $L^1_{++}(T)$ and $L^1_{+++}(T)$ similarly.

We take $L^\infty_{+++}(T)$ to be the consumption space.

For simplicity, the process of subdivisions is described by a sequence of increasing partitions, denoted $\{\mathcal{J}^n\}$, which is generated by the binary expansion as

$$\begin{aligned}\mathcal{J}^n \quad = \quad & \left\{ \left[\underline{t}, \underline{t} + \frac{\bar{t}-\underline{t}}{2^n}\right), \left[\underline{t} + \frac{\bar{t}-\underline{t}}{2^n}, \underline{t} + \frac{2(\bar{t}-\underline{t})}{2^n}\right), \right.\\ & \left. \cdots, \left[\underline{t} + \frac{(2^n-2)(\bar{t}-\underline{t})}{2^n}, \underline{t} + \frac{(2^n-1)(\bar{t}-\underline{t})}{2^n}\right), \left[\underline{t} + \frac{(2^n-1)(\bar{t}-\underline{t})}{2^n}, \bar{t}\right] \right\}\end{aligned}$$

for each n. Notice that for every $t \in T$, there exists a unique sequence $\{J^n(t)\}$ such that $J^n(t) \in \mathcal{J}^n$ and $t \in J^n(t)$ for all n and $\liminf J^n(t) = \limsup J^n(t) = t$.

2.2 Assumptions

Separable Preference: Preference over $L^\infty_{+++}(T)$ is represented by a function U which has the form

$$U(f) = \int_T v(f(t), t)d\mu(t),$$

for $f \in L^\infty_{+++}(T)$, with the following properties:

1. For almost all $t \in T$, $v(\cdot, t) : \mathbb{R}_{++} \to \mathbb{R}$ is a C^2 function, and for all $z \in \mathbb{R}_{++}$, $\frac{\partial v(z,\cdot)}{\partial z} : T \to \mathbb{R}$ and $\frac{\partial^2 v(z,\cdot)}{\partial z^2} : T \to \mathbb{R}$ are Lebesgue measurable.

2. There exist non-increasing functions $\underline{\phi}, \overline{\phi}$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$ such that
   (i) $\underline{\phi}(y) \leq \overline{\phi}(y)$ for all $y \in \mathbb{R}_{++}$;
   (ii) $\underline{\phi}(y) \to \infty$ as $y \to 0$ and $\overline{\phi}(y) \to 0$ as $y \to \infty$; and
   (iii) for all $z \in \mathbb{R}_{++}$ and almost every $t \in T$,
$$\underline{\phi}(z) \leq \frac{\partial v(z,t)}{\partial z} \leq \overline{\phi}(z).$$
3. For any fixed $\underline{z}, \overline{z} > 0$, there exist $\underline{\beta} < \overline{\beta} < 0$ such that
$$\underline{\beta} \leqq \frac{\partial^2 v(z,t)}{\partial z^2} \leqq \overline{\beta}$$
   for all $z \in [\underline{z}, \overline{z}]$ and almost all $t \in T$.

Wealth and Prices: (i) $w > 0$;
(ii) $p \in L^1_{+++}(T)$ and there exist $\underline{p}, \overline{p}$ with $0 < \underline{p} < \overline{p}$ such that $p(t) \in [\underline{p}, \overline{p}]$ for almost all $t \in T$.

2.3 Induced 2-good preference

Given n, pick $J \in \mathcal{J}^n$ to be the object of partial equilibrium analysis in the approximate sense. Now consider the preference induced over pairs of the quantity of commodity piece J and associated income transfer, which is determined by aggregating the other commodity pieces in the Hicksian manner.[2]

[2] One might think of establishing the preference induced over the two commodity groups, J and $T \setminus J$, by working directly on a continuum rather than by going through finite subdivisions as above. For example, given any interval piece $J \subset T$, one may consider Hicksian aggregation over the complement $T \setminus J$ by solving the problem
$$\begin{aligned} &\max_{f_{-J} \in L^\infty_{+++}(T \setminus J)} U\left(x\mathbf{1}_I, f_{-J}\right) \\ &\text{subject to } \; \langle p_{-J}, f_{-J} \rangle = w + a\mu(J), \end{aligned}$$
and then making J arbitrarily small. However, on infinite-dimensional vector spaces like $L^\infty(T)$ or $L^\infty(T \setminus J)$, it is unclear if the demand analysis with differential comparative statics as we do here can work, where we also need to vary J. Therefore we start with taking finite subdivisions of the set of commodity characteristics, and make it arbitrarily finer in order to obtain the limit preference. This is somewhat parallel to the strategy taken by Bewley [1].

Given n, the restriction of U onto the finite-dimensional subspace $\mathbb{R}^{|\mathcal{J}^n|}$, denoted $U^n : \mathbb{R}^{|\mathcal{J}^n|}_{++} \to \mathbb{R}$, is defined by

$$U^n(z) = U\left(\sum_{K\in\mathcal{J}^n} z_K \mathbf{1}_K\right)$$

for $z = (z_K)_{K\in\mathcal{J}^n} \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$. Also, given n and $J \in \mathcal{J}^n$ to be fixed, $(x, z_{-J}) \in \mathbb{R}^{|\mathcal{J}^n|}$ denotes the vector such that x is its J-component and $z_{-J} \in \mathbb{R}^{|\mathcal{J}^n|-1}$ refers to the rest.

Definition 1 Given n, $J \in \mathcal{J}^n$ and $(x, a), (y, b) \in \mathbb{R}_{++} \times \left(-\frac{w}{\mu(J)}, \infty\right)$, the relation

$$(x, a) \succsim^{n,J} (y, b)$$

holds if

$$U^n\left(x, z^{n,J}(x, a)\right) \geq U^n\left(y, z^{n,J}(y, b)\right),$$

where $z^{n,J}(x, a) = (z^{n,J}_K(x, a))_{K\in\mathcal{J}^n\setminus\{J\}} \in \mathbb{R}^{|\mathcal{J}^n|-1}_{++}$ and $z^{n,J}(y, b) = (z^{n,J}_K(y, b))_{K\in\mathcal{J}^n\setminus\{J\}} \in \mathbb{R}^{|\mathcal{J}^n|-1}_{++}$ are solutions to

$$\begin{aligned} &\max_{z_{-J}\in\mathbb{R}^{|\mathcal{J}^n|-1}_{++}} U^n(x, z_{-J}) \\ &\text{subject to} \sum_{K\in\mathcal{J}^n\setminus\{J\}} p_K z_K = w + a\mu(J) \end{aligned}$$

and

$$\begin{aligned} &\max_{z_{-J}\in\mathbb{R}^{|\mathcal{J}^n|-1}_{++}} U^n(y, z_{-J}) \\ &\text{subject to} \sum_{K\in\mathcal{J}^n\setminus\{J\}} p_K z_K = w + b\mu(J) \end{aligned}$$

respectively.

Under separable preference, the Hicksian aggregation problem reduces to

$$\max_{z_{-J}\in\mathbb{R}^{|\mathcal{J}^n|-1}_{++}} \left\{\int_J v(x, t)d\mu(t) + \sum_{K\in\mathcal{K}^n\setminus\{J\}} \int_K v(z_K, t)\mu(t)\right\}$$

$$\text{subject to} \sum_{K\in\mathcal{J}^n\setminus\{J\}} p_K z_K = w + a\mu(J),$$

and the conditional demand is independent of x. Thus denote the solution by $z^{n,J}(a) = (z^{n,J}_K(a))_{K \in \mathcal{J}^n \setminus \{J\}}$, and let us call it *conditional demand.*

Under the current assumption the finite dimensional problem above has a unique interior solution (see Debreu [3]) and with the Lagrange multiplier $\lambda^{n,J}(a) > 0$ we have the first order condition

$$\frac{\partial}{\partial z_K} \int_K v(z_K, t) d\mu(t) = \lambda^{n,J}(a) p_K$$

for each $K \in \mathcal{J}^n \setminus \{J\}$.

By differentiating the budget equation by a, we obtain the Engel aggregation condition

$$\sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K \frac{\partial z^{n,J}_K(a)}{\partial a} = \mu(J).$$

Now let

$$V^{n,J}(x, a) = \int_J v(x, t) d\mu(t) + \sum_{K \in \mathcal{J}^n \setminus \{J\}} \int_K v(z^{n,J}_K(a), t) \mu(t)$$

be the indirect utility function given by the conditionally optimal consumption. Then we have

$$\begin{aligned} \frac{\partial V^{n,J}(x,a)}{\partial x} &= \frac{\partial}{\partial x} \int_J v(x,t) d\mu(t) \\ &= \int_J \frac{\partial}{\partial x} v(x,t) d\mu(t) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial V^{n,J}(x,a)}{\partial a} &= \sum_{K \in \mathcal{J}^n \setminus \{J\}} \frac{\partial}{\partial z_K} \int_K v(z_K, t) d\mu(t) \frac{\partial z^{n,J}_K(a)}{\partial a} \\ &= \lambda^{n,J}(a) \sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K \frac{\partial z^{n,J}_K(a)}{\partial a} \\ &= \lambda^{n,J}(a) \mu(J) \end{aligned}$$

Therefore the marginal rate of substitution of income transfer for the commodity piece J at $(x, a) \in \mathbb{R}_{++} \times \left(-\frac{w}{\mu(J)}, \infty\right)$ is given by

$$\begin{aligned} S^{n,J}(x,a) &= \frac{\partial V^{n,J}(x,a)}{\partial x} \Bigg/ \frac{\partial V^{n,J}(x,a)}{\partial a} \\ &= \frac{1}{\lambda^{n,J}(a)} \cdot \frac{\int_J \frac{\partial}{\partial x} v(x,t) d\mu(t)}{\mu(J)} \end{aligned}$$

2.4 The limit theorem

Let $\tau \in T$ be the object of the partial equilibrium analysis. Recall that there exists a unique sequence $\{J^n(\tau)\}$ such that $J^n(\tau) \in \mathcal{J}^n$ and $t \in J^n(\tau)$ for all n and $\liminf J^n(\tau) = \limsup J^n(\tau) = \tau$.

Here is the limit theorem.

Theorem 1 Given almost every $\tau \in T$ and any compact set $C \subset \mathbb{R}_{++} \times \mathbb{R}$, there exists a subsequence of $\{n\}$, denoted $\{k(n)\}$, such that

$$\sup_{(x,a)\in C} \left| S^{k(n),J^{k(n)}(\tau)}(x,a) - S^{\tau}(x) \right| \to 0$$

where

$$S^{\tau}(x) = \frac{1}{\lambda} \cdot \frac{\partial}{\partial x} v(x,\tau)$$

and λ is the Lagrange multiplier to the problem

$$\max_{f \in L^{\infty}_{+++}(T)} U(f)$$

$$\text{subject to} \quad \int_T p(t) f(t) d\mu(t) = w.$$

Proof. Fix any $\tau \in T$ as the object of partial equilibrium analysis in the limit. Take the sequence of intervals which contains τ, and denote it by $\{J^n(\tau)\}$.

As $n \to \infty$, $J^n(\tau)$ converges to $\{\tau\}$. Then, from the fundamental theorem of calculus we have

$$\frac{\int_{J^n(\tau)} \frac{\partial}{\partial x} v(x,t) d\mu(t)}{\mu(J^n(\tau))} \longrightarrow \frac{\partial}{\partial x} v(x,\tau),$$

which is a uniform convergence on compacta under the current assumption.

Under the current assumptions we can apply Lemma 8 in the appendix, which delivers that the sequence $\{\lambda^{n,J^n(\tau)}(a)\}$ has a subsequence which converges uniformly to a constant λ, which is the Lagrange multiplier for the problem

$$\max_f \int_T v(f(t),t)\mu(t)$$

$$\text{subject to} \quad \int_T p(t) f(t) d\mu(t) = w.$$

■

Thus, the limit of marginal rate of substitution for the sequence of induced 2-good preferences, $S^\tau(x)$, is understood to be the marginal willingness to pay for commodity τ when the consumer is given x units of it.

Let $\succsim^\tau$ be the preference relation over $\mathbb{R}_{++} \times \mathbb{R}$ which corresponds to the limit. Then by integrating the above marginal rate of substitution formula in the limit we can represent it by

$$(x, a) \succsim^\tau (y, b) \quad \Longleftrightarrow \quad \frac{v(x,\tau)}{\lambda} + a \geq \frac{v(y,\tau)}{\lambda} + b.$$

Here is an example of closed-form solution.

Example 1 Consider that a consumer has CES preference, which is represented by

$$\int_T \gamma(t) \frac{f(t)^{1-\rho}}{1-\rho} d\mu(t),$$

where $f(t)$ denotes the consumption of commodity $t \in T$, $\gamma(t)$ denotes the weight on it, and $\rho > 0$ denotes the coefficient of elasticity of substitution.

Then the marginal willingness to pay for commodity $\tau \in T$ when she is given x units of it has the form

$$S^\tau(x) = \frac{\gamma(\tau)}{\left(\int_T \gamma(t)^{\frac{1}{\rho}} p(t)^{1-\frac{1}{\rho}} d\mu(t)\right)^\rho} \cdot \frac{w^\rho}{x^\rho},$$

and by integrating this back we can represent $\succsim^\tau$ in the form

$$\frac{\gamma(\tau)}{\left(\int_T \gamma(t)^{\frac{1}{\rho}} p(t)^{1-\frac{1}{\rho}} d\mu(t)\right)^\rho} w^\rho x^{1-\rho} + a.$$

2.5 General equilibrium and partial equilibrium

The current approach allows us to characterize an *exact* relationship between general equilibrium and partial equilibrium. To emphasize that willingness to pay depends on the price system and income at the *general equilibrium level*, denote consumer i's marginal willingness to pay for an extra one unit of commodity $\tau \in T$ when she is consuming x units of it by $S^{i,\tau}(x; p, w^i)$, which is given by

$$S^{i,\tau}(x; p, w^i) = \frac{1}{\lambda^i(p, w^i)} \cdot \frac{\partial}{\partial x} v^i(x, \tau),$$

where $\lambda^i(p, w^i)$ is the Lagrange multiplier to the problem

$$\max_f \int_T v^i(f(t), t) d\mu(t)$$
$$\text{subject to} \quad \int_T p(t) f(t) d\mu(t) = w^i.$$

To illustrate, consider a pure exchange economy in which consumers' initial endowments are given by $\omega^i \in L^\infty_{+++}(T)$, $i = 1, \cdots, n$. Then the following proposition is immediate from the first-order condition.

Proposition 1 Maintain the previous assumptions. Then, an interior allocation $(f^i)_{i=1,\cdots,m}$ constitutes competitive general equilibrium under price system p and if and only if for almost all $\tau \in T$, $(f^i(\tau))_{i=1,\cdots,m}$ satisfies

$$S^{i,\tau}(f^i(\tau); p, \langle p, \omega^i \rangle) = p(\tau)$$

for every $i = 1, \cdots, m$.

This also implies that competitive equilibrium allocation is viewed as the solution to an *unconstrained* maximization problem for the sum of the *integrals* of consumers' surplus across commodities.

Proposition 2 Maintain the previous assumptions. Then, an interior allocation $(f^i)_{i=1,\cdots,m}$ constitutes competitive general equilibrium under price system p if and only if it is a solution to

$$\max_{g^1,\cdots,g^m} \sum_{i=1}^m \int_T \left\{ \int_0^{g^i(t)} S^{i,t}(x; p, \langle p, \omega^i \rangle) dx - p(t) g^i(t) \right\} d\mu(t).$$

Note that the above maximization problem is equivalent to

$$\max_{g^1,\cdots,g^m} \sum_{i=1}^m \frac{1}{\lambda^i(p, \langle p, \omega^i \rangle)} \int_T v^i(g^i(t), t) d\mu(t) - \sum_{i=1}^m \int_T p(t) g^i(t) d\mu(t),$$

where the first term is so-called Negishi's social welfare function in which welfare weights are endogenously determined as the inverse of marginal utilities of income (Negishi [8]). These two are in general different for non-separable preferences, however, because one's willingness to pay for a particular commodity depends implicitly on her entire consumption profile in general equilibrium as well as the price system and her income (see appendix for details).

3 Concluding remarks

We conclude by presenting possible future directions of the research. First thing is whether the convergence result holds with some uniformity across commodities. Second is how fast or slow the convergence is. Thirdly, we have assumed in the current paper that the market is complete, in particular that all the commodities are marketable, and we wonder what is the right way to obtain a notion of willingness to pay when the overall market is incomplete.

A Appendix: General non-separable case

A.1 Mathematical preliminaries

Let $T = [\underline{t}, \overline{t}]$ be a finite interval, Σ be the family of Lebesgue measurable sets, and μ be the Lebesgue measure.

Let $L^\infty(T)$ be the space of essentially bounded measurable functions from T to $\mathbb{R}$, which is endowed with the sup norm. Denote the norm dual of $L^\infty(T)$ by $L^\infty(T)^*$. It is known that the norm dual of $L^\infty(T)$ is the set of finitely additive signed measured over T endowed with the total variation norm, which is denoted by $ba(T)$. Thus, $L^\infty(T)^* = ba(T)$.

Let $L^1(T)$ be the space of Lebesgue integrable functions from T to $\mathbb{R}$, which is endowed with the integral norm. It is known that $L^1(T)$ can be viewed as a subset of $L^\infty(T)^* = ba(T)$. There the dual operation takes the form $\langle p, f\rangle = \int_T p(t)f(t)d\mu(t)$, where $f \in L^\infty(T)$ and $p \in L^1(T)$. Given an integrable function $p : T \to \mathbb{R}$ and a measurable set $K \in \Sigma$, let $p_K = \int_K p(t)d\mu(t)$.

It is known that $L^1(T)^* = L^\infty(T)$, with the dual operation given by $\langle f, p\rangle = \int_T f(t)p(t)d\mu(t)$ where $f \in L^\infty(T)$ and $p \in L^1(T)$. Hence one can consider weak convergence in $L^1(T)$ and weak-$*$ convergence in $L^\infty(T)$. Say that a sequence in $L^1(T)$, denoted $\{p^\nu\}$, weakly converges to p if

$$\langle f, p^\nu\rangle \quad \to \quad \langle f, p\rangle$$

for all $f \in L^\infty(T)$. Say that a sequence in $L^\infty(T)$, denoted $\{f^\nu\}$, weak-$*$ converges to f if

$$\langle f^\nu, p\rangle \quad \to \quad \langle f, p\rangle$$

for all $p \in L^1(T)$.

Let $C \subset \mathbb{R}^m$ be a compact set and consider a sequence of functions from C to $L^1(T)$, denoted by $\{p^\nu\}$. Say that $\{p^\nu\}$ weakly converges to p, a function from C to $L^1(T)$, uniformly on C if

$$\sup_{s\in C} |\langle f, p^\nu(s) - p(s)\rangle| \to 0$$

for all $f \in L^\infty(T)$. Also, consider a sequence of functions from C to $L^\infty(T)$, denoted by $\{f^\nu\}$. Then say that $\{f^\nu\}$ weak-$*$ converges to f, a function from C to $L^\infty(T)$, uniformly on C if

$$\sup_{s\in C} |\langle f^\nu(s) - f(s), p\rangle| \to 0$$

for all $p \in L^1(T)$.

We have three mathematical claims. First follows from the sequential Banach-Alaoglu theorem, since $L^1(T)$ is separable.

Lemma 1 $[\underline{z}\mathbf{1}, \overline{z}\mathbf{1}]$ is weak-$*$ sequentially compact.

Second is about denseness of $\mathcal{J}^n$-measurable simple functions.

Lemma 2 The subspace of $\mathcal{J}^n$-measurable simple functions is weak-$*$ dense in $L^\infty_{+++}(T)$.

Third is a generalization of the Ascoli-Arzela theorem.

Lemma 3 Let C be a compact metric space. Let $\{f^\nu\}$ be a sequence of functions from C to $L^\infty(T)$. Suppose
(i) there exists a weak-$*$ sequentially compact subset $G \subset L^\infty(T)$ such that $f^\nu(s) \in G$ for all ν and $s \in C$;
(ii) for all $q \in L^1(T)$, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all ν and $s, s' \in C$,

$$d(s, s') < \delta \implies |\langle f^\nu(s), q\rangle - \langle f^\nu(s'), q\rangle| < \varepsilon.$$

Then there exists a subsequence $\{f^{k(\nu)}\}$ and a f function from C to $L^\infty(T)$ such that for all $q \in L^1(T)$,

$$\sup_{s \in C} |\langle f^{k(\nu)}(s), q\rangle - \langle f(s), q\rangle| \to 0 \text{ as } \nu \to \infty,$$

where f is continuous in the sense that $f(s^l)$ weak-$*$ converges to $f(s)$ as $s^l \to s$.

Proof. Given $s \in C$ and a natural number m, let $B_m(s) = \{s' : d(s, s') < 1/m\}$. Since $C \subset \bigcup_{s \in S} B_m(s)$ and C is compact, the family of open balls $\{B_m(s)\}_{s \in C}$ has a finite subfamily which covers C. Let C_m be the set of center of the open balls which form the finite subcovers. Now let $C^d = \bigcup_{m=1}^\infty C_m$, which is a countable dense subset of C. Let C^d be presented in the form $C^d = \{s_1, s_2, \cdots\}$.

For s_1, $\{f^\nu(s_1)\}$ is a sequence in $[\underline{z}\mathbf{1}, \overline{z}\mathbf{1}]$, hence has a weak-$*$ convergent subsequence. Denote the convergent subsequence by $\{f^{1,\nu}(s_1)\}$. Denote the corresponding subsequence of $\{f^\nu\}$ by $\{f^{1,\nu}\}$.
Next, consider $\{f^{1,\nu}(s_2)\}$. It is a sequence in $[\underline{z}\mathbf{1}, \overline{z}\mathbf{1}]$, hence has a weak-$*$ convergent subsequence. Denote the convergent subsequence by $\{f^{2,\nu}(s_2)\}$. Denote the corresponding

subsequence of $\{f^{1,\nu}\}$ by $\{f^{2,\nu}\}$. Note that both $f^{2,\nu}(s_1)$ and $f^{2,\nu}(s_2)$ are convergent in the weak-$*$ sense.

Now follow the diagonal argument, and let $\{f^{\nu,\nu}\}$ be the diagonal sequence that weak-$*$ converges at each point of C^d. Rewrite it by $\{g^\nu\}$.

Pick any $q \in L^1(T)$ and $\varepsilon > 0$. By the equicontinuity condition, there is $\delta > 0$ such that for all $d(s, s') < \delta$ implies $|\langle g^\nu(s), q\rangle - \langle g^\nu(s'), q\rangle| < \varepsilon/3$ for all $s, s' \in C$ and for all ν.

Let $M > 1/\delta$ and C_M be the finite subset of C^d as constructed in the previous step where $m = M$, which is δ-dense in C. Since $\{g^n\}$ converges at each point of C_M in the weak-$*$ sense, there is N such that

$$\nu, \nu' > N \implies |\langle g^\nu(s), q\rangle - \langle g^{\nu'}(s), q\rangle| < \varepsilon/3, \quad \forall s \in C_M$$

Pick any $s \in C$. Then there exists $s' \in C_M$ such that $d(s, s') < \delta$, hence for all $\nu, \nu' > \max\{N, M\}$,

$$\begin{aligned} |\langle g^\nu(s), q\rangle - \langle g^{\nu'}(s), q\rangle| &\leqq |\langle g^\nu(s), q\rangle - \langle g^\nu(s'), q\rangle| \\ &\quad + |\langle g^\nu(s'), q\rangle - \langle g^{\nu'}(s'), q\rangle| \\ &\quad + |\langle g^{\nu'}(s'), q\rangle - \langle g^{\nu'}(s), q\rangle| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

Since q is arbitrary, $\{g^\nu(s)\}$ is a weak-$*$ Cauchy sequence when $s \in C$ is fixed. Hence $\{g^\nu(s)\}$ weak-$*$ converges. Denote the limit by $f(s)$. Since $[\underline{z}\mathbf{1}, \overline{z}\mathbf{1}]$ is weak-$*$ compact, we have $f(s) \in [\underline{z}\mathbf{1}, \overline{z}\mathbf{1}]$.

To show that the convergence is uniform, note that we already have

$$\lim_{\nu,\nu' \to \infty} \sup_{s \in C} |\langle g^\nu(s), q\rangle - \langle g^{\nu'}(s), q\rangle| = 0.$$

Now let $\nu' \to \infty$ while ν is fixed, then we have

$$\lim_{\nu \to \infty} \sup_{s \in C} |\langle g^\nu(s), q\rangle - \langle f(s), q\rangle| = 0.$$

Since q is arbitrary, we have the uniform weak-$*$ convergence.

To show that f is continuous in the above-noted sense, pick any $q \in L^1(T)$. Then the result follows from

$$\begin{aligned} |\langle f(s^l), q\rangle - \langle f(s), q\rangle| &\leqq |\langle f(s^l), q\rangle - \langle f^\nu(s^l), q\rangle| \\ &\quad + |\langle f^\nu(s^l), q\rangle - \langle f^\nu(s), q\rangle| \\ &\quad + |\langle f^\nu(s), q\rangle - \langle f(s), q\rangle|. \end{aligned}$$

■

A.2 Basic assumptions

Let $U : L^\infty_{+++}(T) \to \mathbb{R}$ be a representation of the preference. Here we list the basic assumptions on the representation U. The assumptions below involve some cardinal information about the representation. One may write them down in terms of marginal rate of substitution as is done in Hayashi [4]. However, for simplicity of the argument we take the current description.

Regular Preference:

(i) $U : L^\infty_{+++}(T) \to \mathbb{R}$ is norm-continuous and Frechet differentiable. Moreover, $DU(f) \in L^1_{+++}(T)$ for all $f \in L^\infty_{+++}(T)$, and the mapping $DU(\cdot) : L^\infty_{+++}(T) \to L^1(T)$ is continuous in the following sense: given any compact set $C \subset \mathbb{R}^m$, for any sequence of functions from C to $L^\infty_{+++}(T)$, denoted by $\{f^\nu\}$, and a function from C to $L^\infty_{+++}(T)$ denoted by f, if $\{f^\nu\}$ weak-$*$ converges to f uniformly on C then $DU(f^\nu)$ weakly converges to $DU(f)$ uniformly on C.

(ii) $U : L^\infty_{+++}(T) \to \mathbb{R}$ is strictly quasi-concave.

Under first-order differentiability and quasi-concavity, the following claim holds.

Lemma 4 For all $f, g \in L^\infty_{+++}(T)$,

$$U(g) \geqq U(f) \quad \Longrightarrow \quad \langle DU(f), g - f \rangle \geqq 0$$

and

$$U(g) > U(f) \quad \Longrightarrow \quad \langle DU(f), g - f \rangle > 0.$$

Proof. To show the first part, suppose $U(g) \geqq U(f)$. Then by quasi-concavity we have

$$\frac{U(f + \alpha(g - f)) - U(f)}{\alpha} \geqq 0$$

for all $\alpha \in (0, 1)$. As $\alpha \to 0$, the first-order differentiability assures that the left-hand-side converges to $\langle DU(f), g - f \rangle$, which is non-negative in the limit.
To show the second part, suppose $U(g) > U(f)$ and $\langle DU(f), g - f \rangle \leqq 0$. Since $DU(f) \neq \mathbf{0}$, by norm continuity of U and the linear operator $DU(f)$ one can take $h \in L^\infty_{+++}(T)$ so that $U(h) > U(f)$ and $\langle DU(f), h - f \rangle < 0$, which is a contradiction to the previous part. ■

We make some assumptions about the preference induced on the finite dimensional subspaces generated by $\{\mathcal{J}^n\}$.

Regular Preference on Finite Dimensions:

(i) For all n, the restriction of U onto the finite-dimensional subspace $\mathbb{R}^{|\mathcal{J}^n|}$, denoted $U^n : \mathbb{R}^{|\mathcal{J}^n|}_{++} \to \mathbb{R}$, which is defined by

$$U^n(z) = U\left(\sum_{K \in \mathcal{J}^n} z_K \mathbf{1}_K\right)$$

for $z = (z_K)_{K \in \mathcal{J}^n} \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$, is twice continuously differentiable.

(ii) Denote the first derivative of U^n by DU^n. Then,

$$DU^n(z) \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$$

for all $z \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$.

(iii) Denote the second derivative of U^n by D^2U^n. Then, for all $z \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$, the $(|\mathcal{J}^n| + 1) \times (|\mathcal{J}^n| + 1)$ matrix

$$H^n(z) = \begin{pmatrix} D^2U^n(z) & DU^n(z)^t \\ DU^n(z) & 0 \end{pmatrix}$$

is invertible.

We assume that the Inada-type condition holds in the uniform manner across n, which is parallel to what Vives [10] assumes for increasing numbers of commodities.

Uniform Inada Property: There exist non-increasing functions $\underline{\phi}, \overline{\phi}$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$ such that

(i) $\underline{\phi}(y) \leq \overline{\phi}(y)$ for all $y \in \mathbb{R}_{++}$;

(ii) $\underline{\phi}(y) \to \infty$ as $y \to 0$ and $\overline{\phi}(y) \to 0$ as $y \to \infty$;

(iii) for all n, $z = (z_K)_{K \in \mathcal{J}n} \in \mathbb{R}^{|\mathcal{J}^n|}_{++}$ and $K \in \mathcal{J}^n$,

$$\underline{\phi}(z_K) \leq \frac{\partial U^n(z)}{\partial z_K} \Big/ \mu(K) \leq \overline{\phi}(z_K).$$

The last basic assumption is about prices and the base income level.

Wealth and Prices: (i) $w > 0$;

(ii) $p \in L^1_{+++}(T)$ and there exist $\underline{p}, \overline{p}$ with $0 < \underline{p} < \overline{p}$ such that $p(t) \in [\underline{p}, \overline{p}]$ for almost all $t \in T$.

A.3 Induced 2-good preference

Fix n and $J \in \mathcal{J}^n$, and consider the maximization problem

$$\max_{z_{-J} \in \mathbb{R}^{|\mathcal{J}^n|-1}_{++}} U^n(x, z_{-J})$$
$$\text{subject to} \quad \sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K z_K = w + a\mu(J)$$

From the assumptions made on the finite-dimensional subspaces, the above maximization problem has a unique solution in the interior (see Debreu [3], Mas-Colell [7]). Hence we can resort to the first order condition: there exists $\lambda^{n,J} > 0$ such that

$$\frac{\partial}{\partial z_K} U^n(x, z_{-J}) = \lambda^{n,J} p_K$$

for all $K \in \mathcal{J}^n \setminus \{J\}$, where $p_K = \int_K p(t) d\mu(t)$. From the second-order differentiability assumption, the solution, denoted $z^{n,J}(x,a) = (z_K^{n,J}(x,a))_{K \in \mathcal{J}^n \setminus \{J\}}$, is differentiable in (x,a). Also, we have the corresponding Lagrange multiplier as a differentiable function of (x,a), hence we denote it by $\lambda^{n,J}(x,a)$.

By differentiating the budget equation by x, we have

$$\sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K \frac{\partial z_K^{n,J}(x,a)}{\partial x} = 0.$$

By differentiating the budget equation by a, we have

$$\sum_{K \in \mathcal{J}^n \setminus \{J\}} p_K \frac{\partial z_K^{n,J}(x,a)}{\partial a} = \mu(J).$$

Now let

$$V^{n,J}(x,a) = U^n\left(x, z^{n,J}(x,a)\right)$$

be the indirect utility function given by the conditional optimal consumption. Then we have

$$\begin{aligned}
\frac{\partial V^{n,J}(x,a)}{\partial x} &= \left.\frac{\partial}{\partial x}U^n\left(x, z_{-J}\right)\right|_{z_{-J}=z^{n,J}(x,a)} \\
&\quad + \sum_{K\in\mathcal{J}^n\setminus\{J\}} \left.\frac{\partial}{\partial z_K}U^n\left(x, z_{-J}\right)\right|_{z_{-J}=z^{n,J}(x,a)} \frac{\partial z_K^{n,J}(x,a)}{\partial x} \\
&= \left.\frac{\partial}{\partial x}U^n\left(x, z_{-J}\right)\right|_{z_{-J}=z^{n,J}(x,a)} + \lambda^{n,J}(x,a) \sum_{K\in\mathcal{J}^n\setminus\{J\}} p_K \frac{\partial z_K^{n,J}(x,a)}{\partial x} \\
&= \left.\frac{\partial}{\partial x}U^n\left(x, z_{-J}\right)\right|_{z_{-J}=z^{n,J}(x,a)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial V^{n,J}(x,a)}{\partial a} &= \sum_{K\in\mathcal{J}^n\setminus\{J\}} \left.\frac{\partial}{\partial z_K}U^n\left(x, z_{-J}\right)\right|_{z_{-J}=z^{n,J}(x,a)} \frac{\partial z_K^{n,J}(x,a)}{\partial a} \\
&= \lambda^{n,J}(x,a) \sum_{K\in\mathcal{J}^n\setminus\{J\}} p_K \frac{\partial z_K^{n,J}(x,a)}{\partial a} \\
&= \lambda^{n,J}(x,a)\mu(J)
\end{aligned}$$

Thus we obtain the characterization of the induced preference.

Proposition 3 Given n, the marginal rate of substitution of income transfer for the neighboring good $J \in \mathcal{J}^n$, at $(x,a) \in \mathbb{R}_{++} \times \left(-\frac{w}{\mu(J)}, \infty\right)$, takes the form

$$\begin{aligned}
S^{n,J}(x,a) &= \frac{\partial V^{n,J}(x,a)}{\partial x} \Bigg/ \frac{\partial V^{n,J}(x,a)}{\partial a} \\
&= \frac{1}{\lambda^{n,J}(x,a)} \cdot \frac{\frac{\partial}{\partial x}U^n\left(x, z_{-J}\right)\big|_{z_{-J}=z^{n,J}(x,a)}}{\mu(J)}.
\end{aligned}$$

A.4 Behavior of the conditional demand

First we show that the conditional demand choice is uniformly bounded from above and away from zero.

Lemma 5 There exist $\underline{z}, \overline{z}$ with $0 < \underline{z} < \overline{z}$ and $\underline{\lambda}, \overline{\lambda}$ with $0 < \underline{\lambda} < \overline{\lambda}$, such that

$$z_K^{n,J}(x,a) \in [\underline{z}, \overline{z}]$$

$$\lambda^{n,J}(x,a) \in [\underline{\lambda}, \overline{\lambda}]$$

for all n, $(x,a) \in C$ and $J \in \mathcal{J}^n$, $K \in \mathcal{J}^n \setminus \{J\}$.

Proof. First we show the uniform boundedness from above. Suppose not. Then without loss of generality there is $\{(x^n, a^n), J^n, K^n\}$ such that

$$z_{K^n}^{n,J^n}(x^n, a^n) \to \infty.$$

Then we have

$$\frac{\partial}{\partial z_{K^n}} U^n\left(x^n, z_{-J^n}\right) \Big/ \mu(K^n) \quad \leq \quad \overline{\phi}\left(z_{K^n}^{n,J^n}(x^n, a^n)\right)$$

By the assumed property of $\overline{\phi}$, we have $\overline{\phi}\left(z_{K^n}^{n,J^n}(x^n, a^n)\right) \to 0$.

Since

$$\begin{aligned} \frac{\partial}{\partial z_{K^n}} U^n\left(x^n, z_{-J^n}\right) \Big/ \mu(K^n) &= \lambda^{n,J^n}(x^n, a^n) \frac{p_{K^n}}{\mu(K^n)} \\ &\geq \lambda^{n,J^n}(x^n, a^n) \underline{p}, \end{aligned}$$

from the first-order condition, we have $\lambda^{n,J^n}(x^n, a^n) \to 0$.

On the other hand, also from the first order condition, we have

$$\begin{aligned} \max_{K \in \mathcal{J}^n \setminus \{J^n\}} \frac{\partial}{\partial z_K} U^n\left(x^n, z_{-J^n}\right) \Big/ \mu(K) &= \lambda^{n,J^n}(x^n, a^n) \max_{K \in \mathcal{J}^n \setminus \{J^n\}} \frac{p_K}{\mu(K)} \\ &\leq \lambda^{n,J^n}(x^n, a^n) \overline{p}, \end{aligned}$$

hence the left hand side converges to zero.

From the uniform Inada condition again, we have

$$\begin{aligned} \max_{K \in \mathcal{J}^n \setminus \{J^n\}} \frac{\partial}{\partial z_K} U^n\left(x^n, z_{-J^n}\right) \Big/ \mu(K) &\geq \max_{K \in \mathcal{J}^n \setminus \{J^n\}} \underline{\phi}\left(z_K^{n,J^n}(x^n, a^n)\right) \\ &= \underline{\phi}\left(\min_{K \in \mathcal{J}^n \setminus \{J^n\}} z_K^{n,J^n}(x^n, a^n)\right). \end{aligned}$$

From the assumed property of $\underline{\phi}$, we have $\min_{K \in \mathcal{J}^n \setminus \{J^n\}} z_K^{J^n}(x^n, a^n) \to \infty$.

However, since

$$w + a^n \mu(J^n) = \sum_{K \in \mathcal{J}^n \setminus \{J^n\}} p_K z_K^{n,J^n}(x^n, a^n) \geq \underline{p}\mu(T) \min_{K \in \mathcal{J}^n \setminus \{J^n\}} z_K^{n,J^n}(x^n, a^n),$$

we have

$$\min_{K\in\mathcal{J}^n\setminus\{J^n\}} z_K^{n,J^n}(x^n,a^n) \leq \frac{w+a^n\mu(J^n)}{\underline{p}\mu(T)} \leq \frac{w+\max_{(x,a)\in C}|a|\mu(T)}{\underline{p}\mu(T)},$$

which is a contradiction to the previous claim.

Uniform boundedness away from zero can be shown similarly. Since consumption is uniformly bounded from above and below, the corresponding Lagrangean multipliers are also bounded from above and away from zero. ■

Next we derive comparative statics properties of the conditional demand. From the second-order argument, we have

$$\begin{pmatrix} D_{-J}^2U^n(x,z_{-J}) & D_{-J}U^n(x,z_{-J})^t \\ D_{-J}U^n(x,z_{-J}) & 0 \end{pmatrix}\begin{pmatrix} dz_{-J}^t \\ d\lambda \end{pmatrix} = \begin{pmatrix} D_JD_{-J}U^n(x,z_{-J})^t & \mathbf{0}^t \\ 0 & \mu(J) \end{pmatrix}\begin{pmatrix} dx \\ da \end{pmatrix},$$

where D_J refers to the derivative with regard to z_J and D_{-J} refers to the derivative with regard to z_{-J}.

Given n and $J\in\mathcal{J}^n$, let $H^{n,J}(z)$ be the $|\mathcal{J}^n|\times|\mathcal{J}^n|$ matrix obtained by deleting the J-row and the J-column of $H^n(z)$. That is,

$$H^{n,J}(z) = \begin{pmatrix} D_{-J}^2U^n(z) & D_{-J}U^n(z)^t \\ D_{-J}U^n(z) & 0 \end{pmatrix}.$$

For each $K\in\mathcal{J}^n\setminus\{J\}$, let $H_K^{n,J}(z)$ be the matrix obtained by replacing the K-column of $H^{n,J}(z)$ by $(D_JD_{-J}U^n(z),0)^t$. Also, for each $K\in\mathcal{J}^n\setminus\{J\}$, let $\widetilde{H}_K^{n,J}(z)$ be the matrix obtained by replacing the K-column of $H^{n,J}(z)$ by $(\mathbf{0},\mu(J))^t$.

Then, by Cramer's rule we have

$$\frac{dz_K}{dx} = \frac{\left|H_K^{n,J}(x,z_{-J})\right|}{|H^{n,J}(x,z_{-J})|}.$$

and

$$\frac{dz_K}{da} = \frac{\left|\widetilde{H}_K^{n,J}(x,z_{-J})\right|}{|H^{n,J}(x,z_{-J})|}$$

for each n, $J\in\mathcal{J}^n$ and $K\in\mathcal{J}^n\setminus\{J\}$. Thus we can characterize the sensitivity of conditional demand by means of the differential properties of the preference.

Here we assume that the sensitivity terms given above are uniformly bounded as the consumption vectors are uniformly bounded.

Uniform Boundedness of Sensitivity: For any fixed $\underline{z}, \overline{z} > 0$, there exist $\underline{\alpha}, \overline{\alpha}$ and $\underline{\beta}, \overline{\beta}$ such that

$$\underline{\alpha} \leqq \frac{\left|H_K^{n,J}(z)\right|}{|H^{n,J}(z)|} \leqq \overline{\alpha}$$

and

$$\underline{\beta} \leqq \frac{\left|\widetilde{H}_K^{n,J}(z)\right|}{|H^{n,J}(z)|} \leqq \overline{\beta}$$

for all n, $z \in [\underline{z}, \overline{z}]^{\mathcal{J}^n}$ and $J, K \in \mathcal{I}^n$.

Remark 1 When the preference is additively separable we have $\frac{\left|H_K^{n,J}(z)\right|}{|H^{n,J}(z)|} = 0$ for all n, $z \in [\underline{z}, \overline{z}]^{\mathcal{J}^n}$ and $J, K \in \mathcal{I}^n$, hence the first assertion of the assumption is met in a straightforward manner.

Here we are assuming the condition which just makes income effects uniformly bounded, and not assuming that the income effect on each commodity piece vanishes, though it turns out to be true eventually.

Also note that the above conditions are stated directly as a property of the preference, the primitive, not as a property of the derived conditional demand function.

A.5 The limit theorem

Now consider making the subdivision finer and finer. We show that the induced 2-good preference converges to a quasi-linear preference (see Figure 2).

Hereafter, fix $\tau \in T$ arbitrarily and let $J = J^n(\tau)$ for each n, and fix a compact set $C \subset \mathbb{R}_{++} \times \mathbb{R}$. Also, for all sufficiently large n's, let $\{f^{n,J^n(\tau)}\}$ be the sequence of functions from C to $[\underline{z}\mathbf{1}, \overline{z}\mathbf{1}] \subset L^\infty_{+++}(T)$ given by

$$f^{n,J^n(\tau)}(x,a) = x\mathbf{1}_{J^n(\tau)} + \sum_{K \in \mathcal{J}^n \setminus \{J^n(\tau)\}} z_K^{n,J^n(\tau)}(x,a)\mathbf{1}_K$$

for each n and $(x,a) \in C$.

Lemma 6 For all $q \in L^1(T)$, for all $\varepsilon > 0$, there is $\delta > 0$ such that for all n and $(x,a),(y,b) \in C$,

$$d((x,a),(y,b)) < \delta \implies |\langle f^{n,J^n(\tau)}(x,a), q\rangle - \langle f^{n,J^n(\tau)}(y,b), q\rangle| < \varepsilon.$$

Proof. Note that for any $(x,a),(y,b)\in C$,

$$\langle f^{n,J^n(\tau)}(x,a),q\rangle-\langle f^{n,J^n(\tau)}(y,b),q\rangle=(x-y)q_{J^n(\tau)}+\sum_{K\in\mathcal{J}^n\setminus\{J^n(\tau)\}}\left(z_K^{n,J^n(\tau)}(x,a)-z_K^{n,J^n(\tau)}(y,b)\right)q_K.$$

Hence the proof is done if it is shown that for all $\varepsilon>0$ there is $\delta>0$ such that $d((x,a),(y,b))<\delta$ implies $\left|z_K^{n,J^n(\tau)}(x,a)-z_K^{n,J^n(\tau)}(y,b)\right|<\varepsilon$ for all n and $K\in\mathcal{J}^n\setminus\{J^n(\tau)\}$.

By the mean value theorem,

$$z_K^{n,J^n(\tau)}(x,a)-z_K^{n,J^n(\tau)}(y,b)=\left(\frac{z_K^{n,J^n(\tau)}(v,e)}{\partial v},\frac{z_K^{n,J^n(\tau)}(v,e)}{\partial e}\right)\begin{pmatrix}x-y\\a-b\end{pmatrix}$$

for some (v,e) between (x,a) and (y,b). By the Uniform Boundedness of Sensitivity assumption, $\frac{z_K^{n,J^n(\tau)}(v,e)}{\partial v}$ and $\frac{z_K^{n,J^n(\tau)}(v,e)}{\partial e}$ are uniformly bounded. This delivers the equi-continuity property. ■

Lemma 7 The sequence $\{f^{n,J^n(\tau)}\}$ has a convergent subsequence $\{f^{k(n),J^{k(n)}(\tau)}\}$ with the limit $f\in[\underline{z}\mathbf{1},\overline{z}\mathbf{1}]$ which is constant over (x,a), in the sense that

$$\sup_{(x,a)\in C}|\langle f^{k(n),J^{k(n)}(\tau)}(x,a),q\rangle-\langle f,q\rangle|\to 0$$

for all $q\in L^1(T)$.
Moreover, f is the unique solution to the problem (we call it *unconditional problem*)

$$\begin{aligned}&\max_{f\in L^\infty_{+++}(T)} U(f)\\&\text{subject to }\int_T p(t)f(t)d\mu(t)=w.\end{aligned}$$

Proof. From the equi-continuity property, $\{f^{n,J^n(\tau)}\}$ has a subsequence $\{f^{k(n),J^{k(n)}(\tau)}\}$ which weak-$*$ converges uniformly on C. Denote its limit by f^τ, then for all $(x,a)\in C$, we have $f^\tau(x,a)\in[\underline{z}\mathbf{1},\overline{z}\mathbf{1}]\subset L^\infty_{+++}(T)$.

Then it is easy to see that the corresponding subsequence of $\left\{\sum_{K\in\mathcal{J}^n\setminus\{J^n(\tau)\}}z_K^{n,J^n(\tau)}(x,a)\mathbf{1}_K\right\}$ also weak-$*$ converges to $f^\tau(x,a)$ uniformly on K.

Since

$$\begin{aligned}\left\langle\sum_{K\in\mathcal{J}^n\setminus\{J^n(\tau)\}}z_K^{n,J^n(\tau)}(x,a)\mathbf{1}_K,p\right\rangle&=\sum_{K\in\mathcal{J}^n\setminus\{J^n(\tau)\}}p_Jz_K^{n,J^n(\tau)}(x,a)\mathbf{1}_K\\&=w+a\mu(J^n(\tau))\end{aligned}$$

for all n, the uniform weak-$*$ convergence of the corresponding subsequence implies $\langle f^\tau(x,a), p\rangle = w$ for all $(x,a) \in C$.

Fix any $(x,a) \in C$. We show that $f^\tau(x,a)$ is a solution to the unconditional problem. Suppose not. Then there exists $g \in L^\infty_{+++}(T)$ with $\langle p, g\rangle = w$ such that $U(g) > U(f^\tau(x,a))$. Since the uniform weak-$*$ convergence implies pointwise weak-$*$ convergence, one can find $\widetilde{U}$ with $U(g) > \widetilde{U} > U(f)$ such that $\widetilde{U} > U\left(f^{n,J^n(\tau)}(x,a)\right)$ for all sufficiently large n.

Since the subspace of $\mathcal{J}^n$-measurable simple functions is weak-$*$ dense, one can find sufficiently large n and $x\mathbf{1}_{J^n(\tau)} + \sum_{K\in\mathcal{J}^n\setminus\{J^n(\tau)\}} z_K \mathbf{1}_K$ so that it satisfies the corresponding budget constraint and its value is larger than $\widetilde{U}$. However, it contradicts to the optimality given n.

From strict quasi-concavity, the unconditional problem has at most one solution. Therefore, $f^\tau(x,a)$ is constant over (x,a) and τ, hence rewrite it by f. ■

Lemma 8 The corresponding subsequence of $\{\lambda^{n,J^n(\tau)}(x,a)\}$ converges to $\lambda > 0$ uniformly on C, which is the Lagrange multiplier associated with the solution f given above.

Proof. Pick any $K \in \mathcal{J}^r \setminus \{J^r(\tau)\}$ for some fixed r. From the first-order condition we have

$$\lambda^{n,J^n(\tau)}(x,a) = \frac{\int_T DU(f^{n,J^n(\tau)}(x,a))(t)\mathbf{1}_K d\mu(t)}{\int_K p(t)d\mu(t)}$$

for all $n \geqq r$.

As the subsequence $\{f^{k(n),J^{k(n)}(\tau)}\}$ uniformly weak-$*$ converges to f, from the Regular Preference assumption the sequence $\{DU(f^{k(n),J^{k(n)}(\tau)}(x,a))\}$ uniformly weakly converges to $DU(f)$. Therefore the right-hand-side uniformly converges to $\frac{\int_T DU(f)(t)\mathbf{1}_K d\mu(t)}{\int_K p(t)d\mu(t)}$. Since the limit of the right-hand-side is independent of (x,a) and τ, so is the limit of the left-hand-side. Thus, let λ^τ be the uniform limit of $\lambda^{n,J^{k(n)}(\tau)}$, which is constant over (x,a). Summing up, we have

$$\lambda^\tau = \frac{\int_T DU(f^*)(t)\mathbf{1}_K d\mu(t)}{\int_K p(t)d\mu(t)}$$

Since this is true for arbitrary $K \in \mathcal{J}^r \setminus \{J^r(\tau)\}$ and r, by picking almost any $\sigma \neq \tau$ and letting $J = J^{k(n)}(\sigma)$, from the Lebesgue differentiation theorem we have

$$\lambda^\tau = \lim_{n\to\infty} \frac{\int_{J^{k(n)}(\sigma)} DU(f)(t)d\mu(t)/\mu(J^{k(n)}(\sigma))}{\int_{J^{k(n)}(\sigma)} p(t)d\mu(t)/\mu(J^{k(n)}(\sigma))} = \frac{DU(f)(\sigma)}{p(\sigma)}$$

Thus, λ^τ is the Lagrange multiplier corresponding to f. Since f is independent of τ, so is λ^τ and we rewrite it by λ. ■

We make the following assumption with regard to the limit of shrinking neighborhoods.

Continuous Marginal Utility Density: For almost every $\tau \in T$, for any $D \subset \mathbb{R}_{++}$ and $f \in L^{\infty}_{+++}(T)$, there exists $\Delta U(x, \tau; f)$ such that

$$\sup_{x \in D} \left| \frac{\partial}{\partial x} U\left(x\mathbf{1}_J + f\mathbf{1}_{T \setminus J}\right) - \Delta U(x, \tau; f)\mu(J) \right| = o(\mu(J)),$$

where J is any interval containing τ with $\mu(J) > 0$.

Moreover, $\Delta U(x, \tau; f)$ is continuous in f in the following sense: Given any compact set $C \subset \mathbb{R}^m$, if a sequence of functions from C to $L_{+++}(T)$, denoted $\{f^{\nu}\}$, weak-$*$ converges to f uniformly on C, then

$$\sup_{s \in C} \sup_{x \in D} |\Delta U(x, \tau; f^{\nu}(s)) - \Delta U(x, \tau; f(s))| \to 0.$$

Remark 2 In the additive separable case, this is nothing but the result of the Lebesgue differentiation theorem which is applied to the function $\frac{\partial v(x, \cdot)}{\partial x} : T \to \mathbb{R}$.

Here we state the main result.

Theorem 2 Given almost every $\tau \in T$ and any compact set $C \subset \mathbb{R}_{++} \times \mathbb{R}$, there exists a subsequence of $\{n\}$, denoted $\{k(n)\}$, such that

$$\sup_{(x,a) \in C} |S^{n, J^{k(n)}(\tau)}(x, a) - S^{\tau}(x)| \to 0$$

where

$$S^{\tau}(x) \equiv \frac{1}{\lambda} \cdot \Delta U(x, \tau; f),$$

f is the unique solution to the problem

$$\max_{f \in L^{\infty}_{+++}(T)} U(f)$$

$$\text{subject to} \quad \int_T p(t) f(t) d\mu(t) = w$$

and λ is the corresponding Lagrange multiplier.

Proof. From the assumption of Continuous Marginal Utility Density, we have

$$\sup_{(x,a)\in C}\left|\frac{\frac{\partial}{\partial x}U\left(x\mathbf{1}_{J^{k(n)}(\tau)}+f\mathbf{1}_{T\setminus J^{k(n)}(\tau)}\right)}{\mu(J^{k(n)}(\tau))}\right.$$

$$\left.-\frac{\frac{\partial}{\partial x}U\left(x\mathbf{1}_{J^{k(n)}(\tau)}+\sum_{K\in\mathcal{J}^{k(n)}\setminus\{J^{k(n)}(\tau)\}}z_K\mathbf{1}_K\right)\Big|_{z=z^{n,J^{k(n)}(\tau)}(x,a)}}{\mu(J^{k(n)}(\tau))}\right|$$

$$\leqq \sup_{(x,a)\in C}\frac{\left|\frac{\partial}{\partial x}U\left(x\mathbf{1}_{J^{k(n)}(\tau)}+f\mathbf{1}_{T\setminus J^{k(n)}(\tau)}\right)-\Delta U(x;\tau,f)\mu(J^{k(n)}(\tau))\right|}{\mu(J^{k(n)}(\tau))}+$$

$$\sup_{(x,a)\in C}\frac{\left|\Delta U(x;\tau,f)\mu(J^{k(n)}(\tau))-\Delta U(x;\tau,f^{k(n),J^{k(n)}(\tau)}(x,a))\mu(J^{k(n)}(\tau))\right|}{\mu(J^{k(n)}(\tau))}+$$

$$\sup_{(x,a)\in C}\left|\frac{\frac{\partial}{\partial x}U\left(x\mathbf{1}_{J^{k(n)}(\tau)}+\sum_{K\in\mathcal{J}^{k(n)}\setminus\{J^{k(n)}(\tau)\}}z_K\mathbf{1}_K\right)\Big|_{z=z^{n,J^{k(n)}(\tau)}(x,a)}}{\mu(J^{k(n)}(\tau))}\right.$$

$$\left.-\frac{\Delta U(x;\tau,f^{k(n),J^{k(n)}(\tau)}(x,a))\mu(J^{k(n)}(\tau))}{\mu(J^{k(n)}(\tau))}\right|$$

$$=\sup_{(x,a)\in C}\left|\Delta U(x;\tau,f)-\Delta U(x;\tau,f^{k(n),J^{k(n)}(\tau)}(x,a))\right|+\frac{o(\mu(J^{k(n)}(\tau)))}{\mu(J^{k(n)}(\tau)))}$$

where $f^{k(n),J^{k(n)}(\tau)}(x,a)=x\mathbf{1}_{J^{k(n)}(\tau)}+\sum_{K\in\mathcal{J}^{k(n)}\setminus\{J^{k(n)}(\tau)\}}z_K^{n,J^{k(n)}(\tau)}(x,a)\mathbf{1}_K$.
Since $f^{k(n),J^{k(n)}(\tau)}(x,a)$ uniformly weak-$*$ converges to f, the right-hand-side uniformly converges to zero, from the assumption of Continuous Marginal Utility Density.

Combining this with the fact that $\{\lambda^{n,J^{k(n)}(\tau)}(x,a)\}$ converges to $\lambda>0$ uniformly on C, we obtain the desired result. ■

Let $\succsim^\tau$ be the preference relation over $\mathbb{R}_{++}\times\mathbb{R}$ which corresponds to the limit. Then by integrating the above marginal rate of substitution formula in the limit we can represent it by

$$(x,a)\succsim^\tau(y,b)\quad\Longleftrightarrow\quad\frac{1}{\lambda}\int_0^x\Delta U(z,\tau;f)dz+a\geq\frac{1}{\lambda}\int_0^y\Delta U(z,\tau;f)dz+b.$$

We provide some examples of how to calculate marginal utility density.

Example 2 Consider the weighted expected utility preference (Chew and MacCrimmon [2]) represented in the form

$$U(f)=\frac{\int_T v(f(t))d\nu(t)}{\int_T w(f(t))d\nu(t)},$$

where ν is absolutely continuous with respect to μ.

By direct calculation, we have

$$\frac{\frac{\partial}{\partial x}U\left(x\mathbf{1}_J + f\mathbf{1}_{T\setminus J}\right)}{\mu(J)} =$$

$$\frac{\nu(J)}{\mu(J)} \cdot \frac{v'(x)\left(w(x)\nu(J) + \int_{T\setminus J} w(f(t))d\mu(t)\right) - \left(v(x)\nu(J) + \int_{T\setminus J} v(f(t))d\mu(t)\right)w'(x)}{\left(w(x)\nu(J) + \int_{T\setminus J} w(f(t))d\mu(t)\right)^2}.$$

Hence the marginal utility density of commodity $\tau \in T$ at quantity x is

$$\Delta U(x, \tau; f) = \frac{d\nu(\tau)}{d\mu(\tau)} \cdot \frac{v'(x)\int w(f(t))d\nu(t) - w'(x)\int v(f(t))d\nu(t)}{(\int w(f(t))d\nu(t))^2}$$

Notice that in the expected utility case with w being a constant, say 1, it reduces to $\frac{d\nu(\tau)}{d\mu(\tau)}v'(x)$.

Example 3 Let $T = [0, T]$. Consider Uzawa preference (Uzawa [9]) represented in the form

$$U(f) = \int_0^T u(f(t))e^{-\int_0^t \beta(f(s))ds}dt.$$

By direct calculation, we have

$$\frac{\frac{\partial}{\partial x}U\left(x\mathbf{1}_J + f\mathbf{1}_{T\setminus J}\right)}{\mu(J)} = e^{-\int_0^{\inf J}\beta(f(s))ds}\left(u'(x) - \beta'(x)\int_{\sup J}^T u(f(t))e^{-\int_{\sup J}^t \beta(f(s))ds}dt\right).$$

Hence the marginal utility density of commodity $\tau \in T$ at quantity x is

$$\Delta U(x, \tau; f) = e^{-\int_0^\tau \beta(f(s))ds}\left(u'(x) - \beta'(x)\int_\tau^T u(f(t))e^{-\int_\tau^t \beta(f(s))ds}dt\right).$$

Notice that in the additive case with β being a constant it reduces to $e^{-\beta\tau}u'(x)$.

Here we provide one characterization of the marginal utility density, under an additional assumption. Given a consumption vector f, the marginal utility density of commodity $\tau \in T$ at quantity $f(\tau)$ is equal to the Frechet derivative, an integrable function from T to $\mathbb{R}$ under our assumption, evaluated at τ.

Proposition 4 Assume additionally that for all $f \in L^\infty_{+++}(T)$ and almost all $\tau \in T$, and $J \in \Sigma$ with $\tau \in I$,

$$\langle DU(f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J}), \mathbf{1}_J\rangle = \langle DU(f), \mathbf{1}_J\rangle + o(\mu(J)).$$

Then, for all $f \in L^\infty_{+++}(T)$ and almost all $\tau \in T$,

$$\Delta U(f(\tau), \tau; f) = DU(f)(\tau).$$

Proof. From the derivative formula for a composite function, we have

$$\frac{\partial}{\partial x} U\left(x\mathbf{1}_J + f\mathbf{1}_{T\setminus J}\right)\Big|_{x=f(\tau)} = \langle DU(f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J}), \mathbf{1}_J\rangle$$

From the assumption of Continuous Marginal Utility Density we have

$$\frac{\partial}{\partial x} U\left(x\mathbf{1}_J + f\mathbf{1}_{T\setminus J}\right)\Big|_{x=f(\tau)} = \Delta U(f(\tau), \tau; f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J})\mu(J) + o(\mu(J)).$$

Therefore,

$$\begin{aligned}
\Delta U(f(\tau), \tau; f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J}) &= \frac{\frac{\partial}{\partial x} U\left(x\mathbf{1}_J + f\mathbf{1}_{T\setminus J}\right)\big|_{x=f(\tau)}}{\mu(J)} + \frac{o(\mu(J))}{\mu(J)} \\
&= \frac{\langle DU(f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J}), \mathbf{1}_J\rangle}{\mu(J)} + \frac{o(\mu(J))}{\mu(J)} \\
&= \frac{\langle DU(f), \mathbf{1}_J\rangle + o(\mu(J))}{\mu(J)} + \frac{o(\mu(J))}{\mu(J)},
\end{aligned}$$

where the third line follows from the additional assumption.

As J converges to $\{\tau\}$, $f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J}$ converges to f in the weak-$*$ topology. By the assumption of Continuous Marginal Utility Density, $\Delta U(f(\tau), \tau; f(\tau)\mathbf{1}_J + f\mathbf{1}_{T\setminus J})$ converges to $\Delta U(f(\tau), \tau; f)$.

By the Lebesgue differentiation theorem, the right-hand-side converges to $DU(f)(\tau)$. ■

B Application: general equilibrium and partial equilibrium

As an application of our approach, here we provide an *exact* relationship between general equilibrium and partial equilibrium. To emphasize that willingness to pay depends on the price system and income at the *general equilibrium level*, denote consumer i's marginal willingness to pay for an extra one unit of commodity $\tau \in T$ when she is consuming x units of it by $S^{i,\tau}(x; p, w^i)$, which is given by

$$S^{i,\tau}(x; p, w^i) = \frac{1}{\lambda^i(p, w^i)} \cdot \Delta U(x, \tau; f^i(p, w^i)),$$

where $f^i(p, w^i)$ is the solution to the problem

$$\max_{f \in L^{\infty}_{+++}(T)} U^i(f)$$
$$\text{subject to } \int_T p(t)f(t)d\mu(t) = w^i$$

and $\lambda^i(p, w^i)$ is the corresponding Lagrange multiplier.

To illustrate, consider a pure exchange economy in which consumers' initial endowments are given by $\omega^i \in L^{\infty}_{+++}(T)$, $i = 1, \cdots, n$.

Proposition 5 Maintain the previous assumptions, and also assume that for each $i = 1 \cdots, m$, for almost all $\tau \in T$ and every $f \in L^{\infty}_{+++}(T)$, $\Delta U^i(x; \tau, f)$ is decreasing in x.

Then, an interior allocation $(f^i)_{i=1,\cdots,m}$ constitutes competitive general equilibrium under price system p and if and only if for almost all $\tau \in T$, $(f^i(\tau))_{i=1,\cdots,m}$ satisfies

$$S^{i,\tau}(f^i(\tau); p, \langle p, \omega^i \rangle) = p(\tau)$$

for every $i = 1, \cdots, m$.

Proof. ($\Longrightarrow$ part) The interior equilibrium condition tells that

$$DU^i(f^i)(\tau) = \lambda^i p(\tau)$$

for all $i = 1, \cdots, m$ and almost all $\tau \in T$, where λ^i is the Lagrange multiplier for the problem

$$\max_g U^i(g)$$
$$\text{subject to } g \in L^{\infty}_{+++}(T), \quad \int_T p(t)g(t)d\mu(t) = w^i.$$

Pick almost any $\tau \in T$, then since $\Delta U^i(f^i(\tau), \tau; f^i) = DU^i(f^i)(\tau)$ for each i, we have

$$\frac{\Delta U^i(f^i(\tau), \tau; f^i)}{\lambda^i} = p(\tau)$$

for each i.

($\Longleftarrow$ part) Given p and $(w^i)_{i=1,\cdots,m}$, suppose

$$\frac{\Delta U^i(f^i(\tau), \tau; \widehat{f^i})}{\widehat{\lambda}^i} = p(\tau)$$

for each i and almost all $\tau \in T$, where $\widehat{f^i}$ is the interior solution to the problem

$$\max_g U^i(g)$$

$$\text{subject to} \quad g \in L^\infty_{+++}(T), \quad \int_T p(t)g(t)d\mu(t) = w^i$$

and λ^i is the corresponding Lagrange multiplier.

The interior optimality condition for $\widehat{f^i}$ is that

$$DU^i(\widehat{f^i})(\tau) = \widehat{\lambda}^i p(\tau)$$

for almost all $\tau \in T$, hence we have

$$\Delta U^i(f^i(\tau), \tau; \widehat{f^i}) = DU^i(\widehat{f^i})(\tau)$$

for almost all $\tau \in T$.

From the assumption made above, this implies $f^i(\tau) = \widehat{f}^i(\tau)$ for almost all $\tau \in T$ and f^i is optimizing under the budget constraint. ■

This result also implies that competitive equilibrium allocation is viewed as the solution to an *unconstrained* maximization problem for the *integral* of consumers' surplus across commodities.

Proposition 6 Maintain the previous assumptions. Then, an interior allocation $(f^i)_{i=1,\cdots,m}$ constitutes competitive general equilibrium under price system p if and only if it is a solution to

$$\max_{g^1,\cdots,g^m} \sum_{i=1}^m \int_T \left\{ \int_0^{g^i(t)} S^{i,t}(x; p, \langle p, \omega^i \rangle) dx - p(t)g^i(t) \right\} d\mu(t).$$

The above maximization problem is equivalent to

$$\max_{g^1,\cdots,g^m} \sum_{i=1}^m \frac{1}{\lambda^i(p, \langle p, \omega^i \rangle)} \int_T \int_0^{g^i(t)} \Delta U^i(x, \tau; f^i) dx d\mu(t) - \sum_{i=1}^m \int_T p(t)g^i(t)d\mu(t),$$

where the first term is in general different from Negish's social welfare functions $\sum_{i=1}^m \frac{1}{\lambda^i(p,\langle p,\omega^i \rangle)} U^i(g^i)$, in contrast to the case of separable preferences, because the marginal utility density $\Delta U^i(x, \tau; f^i)$ may depend on the entire consumption profile f^i in general equilibrium.

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