Cross-ownership and Portfolio Choice

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Abstract

Cross-ownership smooths firms' idiosyncratic shocks but affects their portfolio choice and, therefore, their risk-taking position. The classical intuition on the role of pooling risk in raising welfare is valid when ownership is evenly dispersed. However, when the ownership of some firms is concentrated in the hands of a few others, deeper integration leads to excessive risk-taking and volatility and, consequently, it results in lower aggregate welfare.

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1 Introduction

Back in the 1930s, the influential work of Berle and Means (1932) reported that ownership of capital of large corporations in USA was dispersed among small shareholders and that 44% of the largest 200 corporations were under effective management control. This picture was scrutinised within the corporate finance literature over the years.[1] After the Second World War, the increase in stock market participation of households, buying directly companies' shares, made the dispersed ownership structure even more salient. But from 1980 households' participation in the stock market started to be channeled via the acquisition of mutual funds actively managed by an institution. Davis (2008) investigated the implication of this trend to ownership structure. He reported a widespread increase in cross-ownership accompanied by an increase in ownership concentration. Harford et al. (2011) provide similar insights. Fichtner et al. (2020) revised and confirmed these earlier empirical results, taking into account the growth of passive funds. In a recent study, He and Huang (2020) show that the fraction of U.S. public firms that are cross-held has increased from below 10% in 1980 to about 60% in 2014.

In view of the increase in ownership concentration, empirical research has investigated whether the picture of "separation between ownership and control" depicted by Berle and Means (1932) is still relevant for the modern economy. Many influential papers have confirmed a *de-facto* separation between ownership and control, even when the network of ownership is highly concentrated. This may occur because even sizeable cross-holdings are still too small to create real control (see Harford et al. (2011)). Additionally, legal restrictions and conflict of interests make it costly for companies to interfere on management decisions (Davis (2008)). On the other hand, there is some indirect evidence of collusion across companies with a large share of cross-holding, e.g., He and Huang (2020). We recognise that establishing a causal link between ownership and control is difficult as there could be many mechanisms at work. But it seems uncontroversial that, even if not complete, there is a fair amount of division between ownership and managerial control.

Motivated by these empirical results, we develop a model to study the implications of cross-ownership for firms' portfolio choice and welfare. A collection of firms is located in a network of cross-holding. We focus on cross-holding in the form of shares, but other instruments that channel the performances of one firm on other firms can be included. The network of

[1] The introduction of LaPorta et al. (1999) and the introduction of Davis (2008) provides a concise review of this literature.

cross-holding reflects the claims that each firm has on the value of other firms. There is full separation of who makes these claims (the shareholders) and the decision maker of a firm (the manager who has full control). The manager of a given firm is assumed to be risk averse and has the choice to invest in projects of different risks. For example, a manager can invest on outgoing projects to expand the current capacity or can finance new projects that are riskier. The investments by firms' managers and the network of cross-holding together define the distribution of returns. To simplify the analysis we assume that firm's decision makers have mean-variance preferences and that every firm can invest its endowment in a risk-free asset or in a distinct risky project.

We begin by deriving a summary measure that aggregates all direct and indirect claims induced by the cross-holding network: we refer to this as (the matrix of) ownership. Ownership keeps track of indirect claims of cross-holding and determines the set of final bilateral transfers. In Proposition 1, we characterise decentralised firms' risk-taking behaviour. In Proposition 2, we find the social optimum for risk-taking of firms, when constrained by a given network of cross-holding. These two results clarify who are the firms that take too much or too little risk relative to socially optimal investment and how this depends on their network location. We explain this next.

Portfolio choice in a cross-ownership network leads to two competing forces–one involving diversification and the other risk-shifting. Cross-ownership allows firms to diversify and invest in high return projects despite their risk. But cross-ownership also skews the incentives of how much risk firms take because those that make portfolio decisions do not bear all of the risk they take on. Indeed, a firm will absorb some of the risk taken by the firms it has shares in. How these two effects shape the firm's portfolio choice depends on the local structure of its network of ownership. Low self-ownership incentivises firms to take too much risk. Firms whose ownership is concentrated in the hand of a few others also take too much risk as the risk taken is shifted to a few neighbours. In contrast, firms with high self-ownership and whose ownership is dispersed take too little risk.

At the aggregate level, the cross-ownership network that maximizes aggregate utility is one where self-ownership is minimized and each firm's ownership is uniformly distributed across all other firms (Proposition 3). This is achieved in a complete and symmetric cross-holding network and, in this setting, it is analogous to a perfect insurance scheme. Yet, empirical research has documented highly concentrated cross-ownership networks in the modern economy. Complementary to the body of research discussed above, network scientists, working in

the intersection between computer science, economics and finance, investigated the structure of cross-ownership networks within and across different countries. Glattfelder and Battison (2009) mapped ownership networks focusing on the stock markets of 48 countries. In a subsequent work, Vitali et al. (2011) studied transnational corporations, including both listed and non-listed companies around the world. This analysis revealed that cross-ownership networks have a bow-tie structure, see Figure 1 for a stylised example.

The bow-tie structure provides a taxonomy to group firms in three categories based on their local network of ownership. It also clarifies the nature of asymmetries in cross-ownership networks. Firms that have many shares of other firms in their portfolio but that do not raise equity by issuing their own shares belong to the in-section. Firms located in the core of the bow tie have shares of other core firms and of out-section firms, but they are also cross-held by in-section firms. The out-section firms do not cross-hold other firms, but raise capital by issuing shares that are mainly acquired by the core firms. Glattfelder and Battiston (2009) and Vitali et al. (2011) pointed out that the core section is tightly connected and firms in the out-section are highly exposed to the performance of firms in the core section. The bow-tie describes cross-ownership across transnational firms. However, it also resembles many properties of the ownership structure of traditional national corporations. National corporations are often organized in an almost tree-like structure with monotonous flow of value. For example, see La Porta et al. (1999), who mapped the ownership structures of the 20 largest publicly traded firms in each member of a list of 27 wealthy economies.

Based on this evidence, we study the portfolio choice in a bow-tie network. We parameterize the bow-tie structure and derive the corresponding matrix of ownership. We show that both core and out-section firms can over or under invest relative to the social optimum and we provide conditions for both cases (Proposition 4). We also show that deeper integration among core-section firms increases the welfare of core-firms, because it allows more diversification among them. However, it has a non-monotonic effect on the welfare of in-section firms. In fact, deeper integration can also trigger a reduction in aggregate welfare (Proposition 5). The negative effect of deeper integration in a bow-tie network occurs when there are only a few firms in the in-sector, and each holds major shares of core firms. In such a case, the integration of core firms is analogous to increasing diversifications across core firms, so the core firms take on a lot of risk. But this risk is shifted mainly to the few in-section firms, and this lowers their welfare substantially.

We also illustrate the role of diversification and concentration with the analysis of the Allianz AG cluster (reported in La Porta et al. (1999)). We start from the empirical cross-

holding network and, based on that, derive the ownership structure. We observe that the centre of the cluster, Allianz AG, takes more risk than peripheral firms. We then perform a thought experiment in which the shares of Allianz AG are concentrated in a single hand, and show that this reduces social welfare.

We then develop further the analysis of social welfare. We derive an expression for the welfare of networks in which linkages are small (thin networks). In thin networks, welfare is well approximated by only the first two layers of investments. We show that being thin is a sufficient condition, independently of the structure of cross-holding, for integration to increase welfare (Proposition 6). Yet, also in thin networks the variance across investments reduces the welfare benefits that more integration creates.

Finally, we relax the assumption that the returns of projects are uncorrelated. We show existence of an equilibrium and provide sufficient conditions for uniqueness. In the case of weak positive correlation we find that correlation typically mitigates risk taking. More precisely, we show that the larger is the partial insurance that i provides to other firms, the larger is the moderating effect of positive weak correlation on firm i's risk-taking.

We build on two important strands of research. The first line of work is the research on cross-holdings and linkages.[2] From this literature we borrow the formalization of cross-holding networks. The second strand is the literature on portfolio choice. In a complete market setting, any uncertainty on returns is washed out and only expected returns matter. However, when markets are incomplete, maybe because access is restricted, risk matters.[3] This motivates a richer model of firm risk-taking choices. We build on a prominent strand of the literature that has used the portfolio model of Pyle (1971) and Hart and Jaffee (1974). Within this framework, firms are assumed to behave as competitive portfolio managers, taking prices and yields as given and choosing their portfolio (composition of their balance sheets and liabilities) in order to maximize the expected utility of the firms's financial net worth.[4]

An important assumption of our model is that we take the cross-ownership structure, and therefore the exposure to the risk-taking behavior of other firms, as given. Our analysis and result should be interpreted as the study of how exogenous regulations and rigidities in

[2]See, for example, Azar et al. (2018), Brioschi et al. (1989), Eisenberg, and Noe (2001), Fedenia et al. (1994) and the recent work of Elliott et al. (2014)

[3]As stressed by Rochet (1992), "it is hard to believe that a deep understanding of the banking sector can be obtained within the set-up of complete contingent markets, essentially because of the Modigliani-Miller indeterminacy principle".

[4]The portfolio choice model has been successfully used to evaluate the effect of capital regulations on risk-taking, see e.g., Koehn and Santomero (1980), Kim and Santomero (1988), Keeley and Furlong (1990), Zhou (2010) and Gersbach and Rochet (2012).

cross-holding affect the level of investment of a firm in a risky project that is not directly accessible to other firms. This is in line with a large literature that has focussed on situations in which not all the elements of a firm's balance sheet can be chosen. In particular, Rochet (1992) reevaluates the work of Koehn and Santomero (1980) and Kim and Santomero (1988) in a model in which the firm equity capital is fixed, in the short run over which the model spans; this reflects the real distinction in the way equity capital can be altered in the short run relative to other securities.

In the recent work on contagion in financial networks, attention has focused on the role of the distribution of shocks and the architecture of networks in the case of bankruptcy.[5] There are two distinguishing features of our work. First, the origin of the shocks – the investments in risky projects – is itself an object of individual decision making.[6] Second, the economics system is not "damaged", i.e., there is no risk of bankruptcy. Thus, the focus of our work is, first, on how the network of linkages shapes the level of risk-taking by agents and, second, on how it spreads the rewards of the risky choices across different parts of the system. Our work on the effects of integration and on optimal network design should be seen as complementary to the existing body of work.[7]

Section 2 introduces the model. Section 3 presents our main results. Section 4 extends the main result to correlation across firms. Section 5 concludes. The proofs of the results are presented in the Appendix.

2 Model

There are $\mathcal{N} = \{1, ..., n\}$, $n \geq 2$ firms. Firm i has an endowment $w \in \mathbb{R}$ and chooses to allocate it between a safe asset, with return normalized to zero, and a (personal) risky project i, with return z_i. We assume that z_i is normally distributed with mean $\mu > 0$ and variance σ^2;

[5] See, for example, Allen and Gale (2000), Babus (2016), Farboodi (2014), Cabrales et al. (2017), Elliott et al. (2014), Elliott and Hazell (2015), Greenwood, Landier, and Thesmar (2015) and Gai and Kapadia (2010). For a survey, see Cabrales et al. (2015).

[6] Jackson and Pernoud (2019) consider a model in which linkages affect incentives but focus on contagion due to bankruptcy. See also Shu (2019). A recent paper of Vohra et al (2020) studies how agency-conflict between a firm interplays with ownership networks and its effect on shocks propagation.

[7] In a recent paper, Belhaj and Deroian (2012) study risk-taking by agents located within a network. The main modeling difference is that they assume bilateral output sharing with no spillovers. So, with independent assets, there are no network effects in their model; also in their model, strategic effects in risk-taking derives from the assumption of positive correlation in returns to risky assets. Our focus is on how the structure of the ownership network shapes risk-taking, and the effects of integration and diversification and the design of optimal networks (with weights on systemic risk). These issues are not addressed in their paper.

we impose that $w > \mu/\sigma^2$. We assume that the n risky projects are uncorrelated. Investments by firm i in the risky asset and the safe asset are denoted by $\beta_i \in [0, w]$ and $w - \beta_i$, respectively. Let $\beta = \{\beta_1, ..., \beta_n\}$ denote the profile of investments.

Firms are embedded in a network of cross-holdings; we represent the network as a $n \times n$ matrix S, with $s_{ii} = 0$, $s_{ij} \geq 0$ and $\sum_{j \in \mathcal{N}} s_{ji} < 1$ for all $i \in \mathcal{N}$. The link s_{ij} represents the fractional claim that firm i has on firm j's economic value V_j. The economic value V_j is in turn determined by the profile of investments β, the realization of projects' returns, $z = \{z_1, ..., z_n\}$, and the cross-holding network S. This is at the essence of the formulation of cross-holdings that we adopt and we present it next; see e.g., Brioschi, Buzzacchi, and Colombo (1989), Elliott, Golub and Jackson (2014), Eisenberg and Noe (2001), and Fedenia, Hodder, and Triantis (1994).

Let $W_i = \beta_i z_i$ be firm i's returns; firm i's inflated value is then

$$\overline{V}_i = W_i + \sum_{j=1} s_{ij} \overline{V}_j,$$

and the corresponding vector equation $\overline{V} = W + S\overline{V}$ delivers a fixed point $\overline{V} = (I - S)^{-1} W$. Firm i's value V_i is then the fraction of $\overline{V}_i$ that firm i owns, i.e.,

$$V_i = (1 - \sum_j s_{ji}) \overline{V}_i.$$

Let D be a $n \times n$ diagonal matrix, in which the i-th diagonal element is the self ownership of firm i, $1 - \sum_{j \in \mathcal{N}} s_{ji}$, without taking account of indirect linkages, and define $\Gamma = D[I - S]^{-1}$. It follows that the vector of economic values of different firms is defined by:

$$V = \Gamma W \tag{1}$$

The element (i, j) of Γ, denoted by γ_{ij}, is firm i's shares of each firm j's "primitive" returns, W_j. Since, for each $i \in \mathcal{N}$, $\sum_{j \in \mathcal{N}} s_{ji} < 1$, we can express

$$\Gamma = D \sum_{k=0}^{\infty} S^k,$$

and therefore γ_{ij} is the sum of all walks in S that starts at i and ends at j, where each walk

is discounted based on its length, i.e., for every $i \neq j$,

$$\gamma_{ij} = [1 - \sum_{j \in \mathcal{N}} s_{ji}] \left[0 + s_{ij} + \sum_{k} s_{ik} s_{kj} + .. \right].$$

Since Γ is column-stochastic, $\gamma_{ii} = 1 - \sum_{j \neq i} \gamma_{ji}$.

We assume that firms seek to maximize a mean-variance utility function[8]:

$$U_i(\beta_i, \beta_{-i}) = E[V_i(\beta)] - \frac{\alpha}{2} Var[V_i(\beta)].$$

Using expression (1), it follows that

$$E[V_i(\beta)] = \sum_{j \in \mathcal{N}} \gamma_{ij} E[W_j] \quad \text{and} \quad Var[V_i(\beta)] = \sigma^2 \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \beta_j^2. \tag{2}$$

Therefore, firm i's utility is

$$U_i(\beta_i, \beta_{-i}) = \mu \sum_{j \in \mathcal{N}} \gamma_{ij} \beta_j - \frac{\sigma^2}{2} \sum_{j \in \mathcal{N}} \gamma_{ij}^2 \beta_j^2. \tag{3}$$

The assumption that the n risky projects are independent implies that investment choices are strategic independent, i.e., the cross partial derivative of expression (3) between firms' investment choices is zero. We extend the model to allow for correlation in Section 4. Let β^* denote the vector of optimal choices. Our aim is to develop a systematic understanding of the relationship between the network of cross-holdings, S, portfolio choice β^*, and its consequences for welfare.

3 Risk-taking in cross-ownership networks

The optimal investment by firm i can be written as:

$$\beta_i^* = \arg \max_{\beta_i \in [0,w]} \quad \mu \gamma_{ii} \beta_i - \frac{\sigma^2}{2} \gamma_{ii}^2 \beta_i^2.$$

[8]For a discussion of the foundations of mean-variance utility, see Gollier (2001).

If firm i has no cross-holding –i.e., $s_{ij} = s_{ji} = 0$ for all $i \neq j \in \mathcal{N}$– then $\gamma_{ii} = 1$, and, therefore, the optimal investment is $\hat{\beta} = \mu/\sigma^2$.[9] We shall refer to $\hat{\beta}$ as firm i's *autarchy* investment. With this definition in place, we state our characterization result on the individualy optimal risk-taking. Note that as actions are strategic independent, the result delivers the equilibrium investment.

Proposition 1 *The equilibrium investment of firm i is:*

$$\beta_i^* = \min\left\{w, \frac{\hat{\beta}}{\gamma_{ii}}\right\}. \tag{4}$$

Hereafter we assume that w is large and so optimal investments are interior. We note that relative to autarchy, cross-holding raises firms' propensity to take risk: firm i's risk-taking investment is negatively related to his self-ownership, as captured by γ_{ii}. Thus, firm i invests more than firm j in the risky project if, and only if, $\gamma_{ii} < \gamma_{jj}$.

The intuition is the following. Cross-ownership provides implicit insurance to firms. A firm with a low γ_{ii} is a firm with lot of direct and/or indirect shareholders. In this case, when firm i's returns are high the shareholders extract large benefits from the firm, but when i's returns are low, the shareholders extract much less. This implicit insurance encourages firm i's risk-taking.

We now compare risk-taking behavior with socially optimal choice. We assume that the planner seeks to maximize aggregate utilities. The objective of the planner is

$$\max_{\beta} W(\beta, S) = \sum_{i\in\mathcal{N}}\left[E[V_i] - \frac{1}{2}Var[V_i]\right].$$

We obtain:

Proposition 2 *The optimal investment of the social planner in risky projects is*

$$\beta_i^P = \min\left[w, \frac{\hat{\beta}}{\sum_{j\in\mathcal{N}}\gamma_{ji}^2}\right]$$

Hence, firm i over-invests in the risky project as compared to the social planner, $\beta_i^* > \beta_i^P$ if

[9]Note that $\hat{\beta} < w$ because we have assumed that $w > \mu/\sigma^2$.

and only if

$$\gamma_{ii} < \sum_{j \in \mathcal{N}} \gamma_{ji}^2$$

In order to understand the externalities created by the network of cross-holding, we compare the marginal utility of increasing β_i for firm i, with the marginal utility of the utilitarian planner. We have respectively:

$$\frac{\partial W(\beta, S)}{\partial \beta_i} - \frac{\partial U_i(\beta_i, S)}{\partial \beta_i} \quad = \quad \mu(1 - \gamma_{ii}) - \sigma^2 \beta_i \sum_{j \in \mathcal{N} \setminus \{i\}} \gamma_{ji}^2$$

The first term reflects that firm i does not internalize the positive impact that investing on its risky asset has on aggregate returns of the economy. This lack of internalization leads to under investment and it is driven by the fact that the cross-holding network does not fully insure firm i ($\gamma_{ii} > 0$). At the same time, firm i does not internalize that its own risky investment creates costs to its shareholders. This effect, summarised by the second term, leads to over investment relative to socially optimal choice, and it is large when the shares of i are concentrated in the hands of a few others.

This analysis points out to two conflicting forces. Cross-ownership allows firms to be diversified and, so, take higher risks. But cross-ownership also implies that the risk taken by a firm is shifted, in part, to its neighbours. When the ownership of a firm is concentrated in the hands of a few others, taking high risks is not desirable. In contrast, when the ownership of a firm is dispersed, then risk taking does not lead to large risk exposures. In this case firms tend to take too little risk relative to the socially optimal level. The implications of these two forces on welfare are investigated next.

3.1 Utilities and welfare across cross-holding networks

We now investigate the utilities and welfare consequences of different topology of cross-holding networks. Under private optimal investments, firm i's utility is

$$U_i(\beta_i^*, S) = \hat{\beta} \mu \sum_j \left[\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2} \frac{\gamma_{ij}^2}{\gamma_{jj}^2} \right], \tag{5}$$

and aggregate welfare is:

$$W(\beta^*, S) \;=\; \frac{\mu^2}{\sigma^2}\sum_i\sum_j\left[\frac{\gamma_{ij}}{\gamma_{jj}} - \frac{1}{2}\frac{\gamma_{ij}^2}{\gamma_{jj}^2}\right].$$

Define $\rho_{ij} = \gamma_{ij}/\gamma_{jj}$ and let $\rho_i = \{\rho_{i1}, ..., \rho_{in}\}$. Note that ρ_{ij} determines the impact of firm j's investment on i's expected value. This should be interpreted as the ratio between the partial insurance that j gets from firm i, γ_{ij}, and the fraction of j's risk-taking that is not insured through cross-holding, γ_{jj}. We can rewrite expression 5 and observe that firm i's utility depends on the "average" of ρ_{ij} across j and the concentration of ρ_{ij} across j. Define

$$\bar{\rho}_i = \frac{1}{n}\sum_j \rho_{ij} \quad and \quad \epsilon_i^2 = \frac{1}{n}\sum_j(\bar{\rho}_i - \rho_{ij})^2$$

and note that $\bar{\rho}_i \in [\frac{1}{n}, 1]$. We obtain:

$$U_i(\beta^*, S) \propto \underbrace{\bar{\rho}_i}_{\text{average } i\text{'s partial insurance to others}} \qquad -\frac{1}{2} \qquad \underbrace{[\bar{\rho}_i^2 + \epsilon_i^2]}_{\text{diversification } i\text{'s partial insurance to others}}$$

Since $\bar{\rho}_i \leq 1$, firms whose location in the cross-holding network determines a large $\bar{\rho}_i$ are better off, ceteris paribus. A higher $\bar{\rho}_i$ means that i plays a large role in partially insuring other firms through direct and indirect cross-ownership. This encourages risk-taking and it increases expected returns. Of course, this risk is shifted in part to connected firms and this creates a cost because returns are more volatile. This cost is increasing in ϵ_i^2, which is large in cross-holding networks for which the entries of vector ρ_{ij} are very dispersed. Intuitively, a high ϵ_i^2 captures a network in which firm i's ownership is in the hand of a few firms, and so those firms have large exposure to the risk taken by firm i.

At a more aggregate level, a similar representation can be extended to describe aggregate welfare generated by cross-ownership network S. Indeed, let

$$\bar{\rho} = \frac{1}{n}\sum_i \bar{\rho}_i \quad and \quad \epsilon^2 = \frac{1}{n}\sum_i(\bar{\rho} - \bar{\rho}_i)^2 + \epsilon_i^2,$$

where again $\bar{\rho} \in [\frac{1}{n}, 1]$. Then

$$W(\beta^*, S) = n^2\hat{\beta}\mu\left[\bar{\rho} - \frac{1}{2}\bar{\rho}^2 - \frac{1}{2}\epsilon^2\right]$$

Cross-ownership networks with a large $\bar{\rho}$ creates lot of partial insurance. This stimulates risk-taking thereby increasing average economic returns. Networks with higher ϵ^2 are networks with great asymmetries in ownership structure across firms. This, in turn, increases concentration of risk exposure thereby creating costs for the economy.

Based on this interpretation, the cross-ownership network that maximize aggregate utility will be the one that maximize $\bar{\rho}$ and, at the same, time minimizes ϵ^2. This is the cross-ownership network that generates the most effective insurance across firms.

Proposition 3 *The cross-ownership network S^* that solves $\arg\max_S W(\beta^*(S), S)$ is the complete network with symmetric cross-ownership, i.e., $s^*_{ij} = 1/(n-1)$ for all $i \neq j$. This generates*

$$W(\beta^*, S^*) = \frac{1}{2} n^2 \hat{\beta} \mu$$

Note that network S^* leads to $\gamma_{ij} = 1/n$ for all ij. Hence, $\bar{\rho}_i = \bar{\rho} = 1$ for all i and $\epsilon_i^2 = \epsilon^2 = 0$. The network S^* is the cross-ownership network that generates complete insurance.

3.2 Bow-tie cross-holding networks

In the last part of the paper we will highlight properties of the cross-ownership network S that leads to more or less imperfect insurance, and we will look at the consequences for welfare. We start by focusing on the empirically relevant bow-tie structure (see discussion in the introduction). A bow-tie structure has three sections: the in-section, core-section and out-section. In the in-section there are firms investing in firms in the core-section, whereas the out-section firms are those without ownership stakes in other firms. Firms belonging to the core-section have cross-holding with each other and with out-section firms. Formally:

Definition 1 *In a bow-tie cross-holding network S the set of firms is partitioned in three sets $\{\mathcal{N}_I, \mathcal{N}_C, \mathcal{N}_O\}$ and*

- *For all $i \in \mathcal{N}_I$, s_{ij} is equal to s_I for all $j \in \mathcal{N}_C$ and 0 otherwise;*
- *For all $i \in \mathcal{N}_C$, s_{ij} is equal to s_C for all $j \in \mathcal{N}_C$, s_F for all $j \in \mathcal{N}_O$ and 0 otherwise;*
- *For all $i \in \mathcal{N}_O$, s_{ij} is equal to 0 for all $j \in \mathcal{N}$.*

Finally, $n_I s_I + (n_C - 1)s_C < 1$ and $n_C s_F < 1$, where $n_I = |\mathcal{N}_{IN}|$, $n_C = |\mathcal{N}_C|$ and $n_F = |\mathcal{N}_O|$.

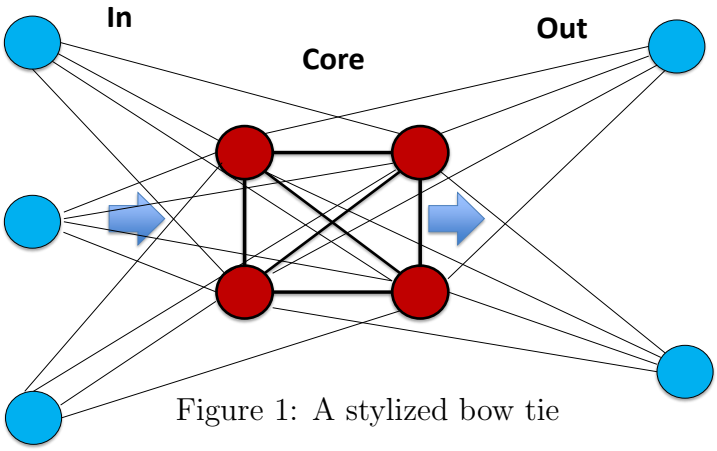

Figure 1: A stylized bow tie

The bow tie structure can be seen as hierarchical, where cross-holding is top-down, with the core firms also cross-holding shares laterally.

The following proposition describes optimal investment of each type of firm in a bow-tie network and also determines over and under investment relative to socially optimal risk-taking.

Proposition 4 *Suppose the cross-holding network has a bow-tie structure. Firms in the in-section invest* $\beta_I^* = \hat{\beta}$*, firms in the out-section invests* $\beta_F^* = \hat{\beta}/(1 - n_C s_F)$*, and firms in the core-section invest*

$$\beta_C^* = \frac{(s_C + 1)[1 - (n_C - 1)s_C]}{[1 - (n_C - 1)s_C - n_I s_I][1 - (n_C - 2)s_C]}\hat{\beta}$$

Relative to the social planner investment: a.) firms in the in-section invest efficiently, b.) core-firms over-invest in the risky projects if and only if

$$\frac{[1 - (n_C - 1)s_C - n_I s_I][1 - (n_C - 2)s_C]}{(s_C + 1)[1 - (n_C - 1)s_C]} < \frac{1}{2} \tag{6}$$

and, c.) for any given n_I, n_F, s_I, s_C*, there exists* $b_F, 0 < b_F < 1$*, such that out-section firms over-invest in the risky projects if and only if* $n_C s_F > b_F$*.*

Condition 6 tells us that core-firms will over-invest when most of their shares are held by the upstream firms, i.e., the in-section firms. Note that for small cross-holdings across core firms (small s_C) the LHS of inequality 6 is roughly $(1 - n_I s_I)$ and so the condition for core firms to over invest is roughly that $n_I s_I > 1/2$. Intuitively, when most of the shares of core firms are held by the upstream firms, self-ownership of core-firms is low and so core firms invest

substantially in the risky project. At the same time, the risk they take is shifted to the in-section firms, who are disproportionally exposed to core firms' actions. Similar interpretations apply for the case of over-investment of out-section firms.

The deviation from optimal investment affects welfare via changes in expected return and volatility. To gain insight we consider the simplest possible hierarchical structure that entails cross-holding between core firms, the "fat bow tie" network. In this configuration, there is only one firm in the in-section and the out-section, but there are two core firms. This is a stylised example of the empirical evidence that ownership networks have a bow-tie structure with a dense core, and few in-section firms exposed to it. What we want to understand is the effect of an increase in cross-holding integration.

Definition 2 *We say that S is more integrated than S' if $s_{ij} \geq s'_{ij}$ for all $i, j \in \mathcal{N}$ and $s_{ij} > s'_{ij}$ for some $i, j \in \mathcal{N}$.*

The definition of integration reflects the idea that links between firms have become stronger. What is the effect of an increase in integration across core firms in the fat bow tie structure?

Proposition 5 *Suppose a fat bow tie structure, i.e., $n_I = n_F = 1$ and $n_C = 2$. An increase in integration between the two core firms, i.e., an increase in s_C, leads to the following: a) the welfare of each core firm increases, b) the welfare of the out-section firm does not change and c) there exists a $\overline{s}_I > 0$, such that for $s_I > \overline{s}_I$, the welfare of in-sector firms first increases and then decreases (with an increase of s_C).*

As the two core firms become more integrated, their individual self-ownership decreases and so each take higher risk. This higher risk translates in higher and higher variance for the in-section firm and, eventually, this decreases its welfare. This effect can be very large and can lead to a decrease in aggregate welfare. Figure 2, obtained with $s_I = 0.3, s_F = 0.2$, provides an example in which an increase in integration across the two core firms results in lower aggregate welfare.

3.3 Allianz AG

To illustrate the role of diversification and concentration we consider the case of Allianz AG, a public company with six large shareholders reported in La Porta and al. (1999). In this

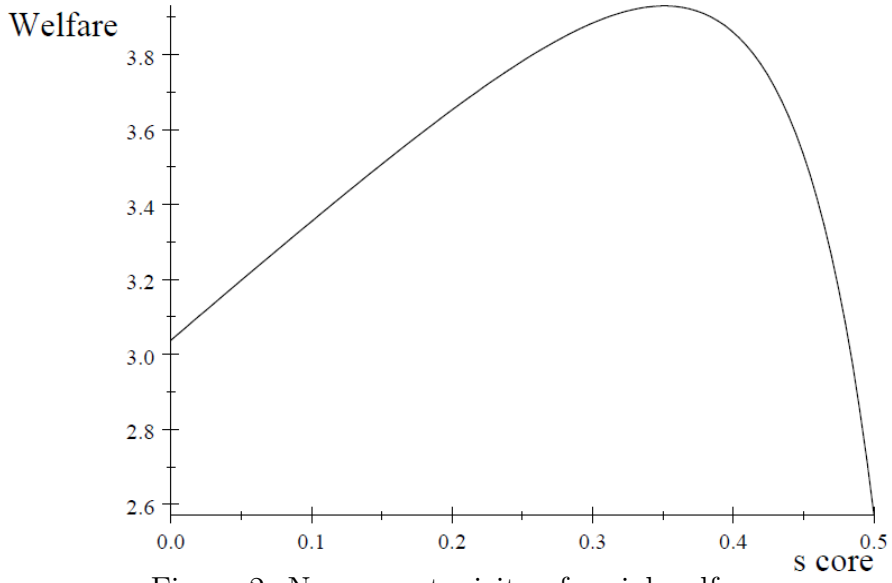

Figure 2: Non monotonicity of social welfare

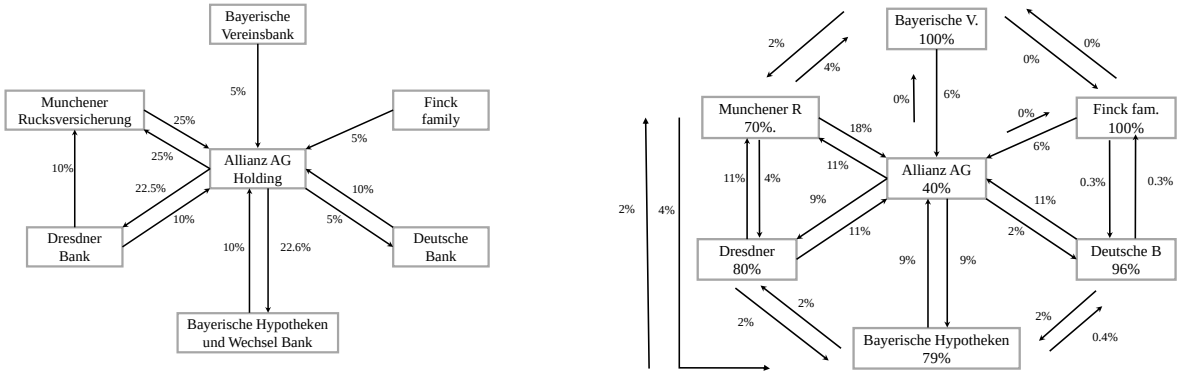

Figure 3: Cross holding network (a) and ownership network (b)

cluster of firms, Allianz AG holds shares of four other core firms and these, plus two in-firms, invest in Allianz AG (see panel A of Figure 3.3).[10]. The resulting ownership network[11] is represented in panel B Figure 3.3.[12].

The centre of the cluster, Allianz AG., has the smallest self-ownership and is the firm taking the maximum risk. On the other hand, the two in-firms, Bayerishe V. and Finck, take the lowest risk.[13].

[10]The numerical analysis uses the analytical formulation obtained in the Appendix and the code can be obtained from the authors.

[11]Note that firms in this cluster also have cross-holding with firms outside the cluster. We allocate these shares to an external investor, which, in line with our model, is represented by the manager

[12]In this picture we ignored, for clarity, three linkages of less than 3%.

[13]The capital available to the firm for private investment in the risky project may vary. However, as we assume mean variance preferences for the manager and sufficiently large capital to ensure interiority, investment

In this network, six firms own Allianz AG shares adding to a total of 65% of Allianz AG. To illustrate the effect of concentration of shareholding we suppose that instead these shares are concentrated in a single hand, Munchener R. (we maintain the assumption that Allianz AG is the manager). The increase in concentration implies that the volatility that Allianz AG investment creates is absorbed by large only by a single firm. This is quantified by the significant rise in the value of $\overline{\rho}_{Munchener}$ from 0.23 to 0.36 while the average of $\overline{\rho}_i$ over all firms falls from 0.194 to 0.188. In turn, this increases the variance of $\overline{\rho}_i$, increasing ϵ^2 (as defined in section 3.1). Finally, this has an effect on welfare: the increases in concentration generates a loss in social welfare, from $0.175n^2\hat{\beta}\mu$ to $0.168n^2\hat{\beta}\mu$.

3.4 The welfare effects in thin networks

In this section we develop further the analysis of social welfare. The difficulty stems from the complexity of the dependence of $\bar{\rho}_i$ with the elements of S. We consider economies where the interlikages are sufficiently small to legitimate the focus on links with directly connected firms and their own neighbours.

We know that γ_{ij} can be obtained as an infinite series. Truncating the series when it involves three interlinkages, i.e., terms like $s_{ik}s_{kh}s_{hj}$, we obtain

$$\overline{\rho}_i = \frac{1}{n}\sum_j \frac{\gamma_{ij}}{\gamma_{jj}} \simeq \frac{1}{n}\left(\eta_i^{out}(1-\eta_i^{in}) + \sum_j s_{ij}\left(\eta_j^{in}+\eta_j^{out}\right)+1\right) \tag{7}$$

where $\eta_i^{in} = \sum_{j\in\mathcal{N}} s_{ji}$ and $\eta_i^{out} = \sum_{j\in\mathcal{N}} s_{ij}$ are the in-degree and out-degree of $i \in \mathcal{N}$, respectively.

From this expression we learn that the extent to which firm i provides insurance to other firms is increasing in its own out-degree and is decreasing in its in-degree. In addition, $\overline{\rho}_i$ also rises when firm i invests in firms that have both high in-degree and high out-degree.

Next, in the appendix we show that social welfare can be approximated by

$$\frac{\sigma}{\mu}W(\beta^*,S) \simeq \frac{n}{2} + \sum_i \eta_i^{out} + \frac{1}{2}\left(\sum_i \left(\eta_i^{in}\right)^2 - \sum_{ij}\left(s_{ij}-\overline{s}_j\right)^2\right)$$

A rise in integration, linked to an increase in the economy wide in-degree and out-degree,

is independent of the size of the firm.

increases welfare. However, concentration of investment in any given firm j into a few hands generates a rise in the variance $\sum_i (s_{ij} - \overline{s}_j)^2$. In the aggregate, any rise of variances leads to an increase in the sum of variances across all firms which reduces social welfare[14].

These insights are formalised in the next proposition.

Proposition 6 *Let $V(S) = \sum_{i,j} (s_{ij} - \overline{s}_j)^2$. There exist $\overline{w} > 0$ and $\bar{s} > 0$ so that if $w > \overline{w}$ and $||S||_{\max} < \bar{s}$ and $||S'||_{\max} < \bar{s}$, it holds that*

1. *If S is more integrated than S', then $W(\beta^*, S) > W(\beta^*, S')$.*
2. *If $\sum_i \eta_i^{out} = \sum_i \eta_i'^{out}$, $\sum_i (\eta_i^{in})^2 = \sum_i (\eta_i'^{in})^2$ and $V(S') > V(S)$ then $W(\beta^*, S) > W(\beta^*, S')$.*

When the strength of each link in S is sufficiently small, i.e., the network of cross-holdings S is thin, the effect of integration on welfare is dominated by the first order term $\sum_i \eta_i^{out}$. Regardless of the topological structure of the cross-holding, integration increases the extent that firms partially insure each other and welfare rises. In other words, risk-shifting is always second order compared to gains from insurance. However, as integration grows welfare still grows but the variance in investment $\sum_i (s_{ij} - \overline{s}_j)^2$, which is second order in s, gains importance and reduce the benefits of integration. Concentration of investments in few investors is detrimental to welfare. Finally, as integration continues to grow further, second and third order effects may dominate, and integration may reduce welfare in asymmetric networks, as in the bow-tie example.

4 Correlated returns

We now relax the assumption that the returns of projects are uncorrelated. We show that there exists an equilibrium and provide sufficient conditions for uniqueness and for interiority of the equilibrium. We then show how our main results extend to the case of weak correlation.

Recall that each project z_i is normally distributed with mean μ and variance σ^2 and therefore $z = \{z_1, ..., z_n\}$ is a multivariate normal distribution. Let Ω be the covariance

[14]Equivalently this amount to maximise $\sum_{i,j,p\neq j} s_{ij} s_{pj}$ while keeping the terms $\sum_{ij} s_{ij}^2$ minimal, that is possibly with $s_{ij} s_{pj}$ with $p \neq j$.

matrix. Under the assumption that z is a non-degenerate multivariate normal distribution, it follows that Ω is positive definite. Note that.

$$U_i(\beta_i, \beta_{-i}) = \mu \sum_{j \in \mathcal{N}} \gamma_{ij}\beta_j - \frac{\alpha}{2} \sum_{j \in \mathcal{N}} \sum_{j' \in \mathcal{N}} \gamma_{ij}\beta_j \gamma_{ij'}\beta_{j'}\sigma^2_{jj'}, \tag{8}$$

and the sign of $\partial^2 U_i/(\partial\beta_i \partial\beta_j)$ is the same as the sign of $-\sigma_{ij}$; that is, investments in risky project i and j are strategic substitutes (strategic complement) whenever the returns from the two projects are positively correlated (negatively correlated).

Let $\circ$ be the Hadamard product. Let $\mathbf{b}$ be an n dimensional vector where the i-th element is μ.

Proposition 7 *There always exists an equilibrium and the equilibrium is unique if* $\sum_j s_{ij} < 1/2$ *for all* i. *Furthermore*

1. *There exists a* $\bar{w} > 0$ *and a* $\bar{s} > 0$ *such that if* $w > \bar{w}$ *and* $||S||_{max} < \bar{s}$ *the unique equilibrium is interior and takes the following form* $\beta = \{\beta_1, ..., \beta_n\}$*:*

$$\beta = \frac{1}{\alpha}[\Gamma \circ \Omega]^{-1}\mathbf{b}.$$

2. *Let* $\sigma^2_{ii} = \sigma^2$ *for all* i *and* $\sigma^2_{ij} = \delta\sigma^2$ *for all* $i \neq j$ *where* $\delta \in [-1, 1]$. *Then, in an interior equilibrium,*

$$\beta_i = \frac{\mu}{\sigma^2\alpha} \sum_j \{[I + \delta\hat{\Gamma}]^{-1}\}_{ij} \frac{1}{\gamma_{jj}}$$

where $\hat{\Gamma}$ *is the matrix of entries* $\hat{\gamma}_{ij} = \gamma_{ij}/\gamma_{ii}$. *For weak correlation, that is when* $\delta \to 0$, *we obtain*

$$\beta_i \simeq_{\delta \to 0} \beta_i^* \left(1 - N\delta\overline{\rho}_i\right)$$

Proposition 7 has an interesting feature. Positive weak correlation reduces investment of a firm i in the risky project by a factor $\delta \sum_j \frac{\gamma_{ij}}{\gamma_{ii}} = N\delta\overline{\rho}_i$. Note that $\bar{\rho}_i$ captures the role that i plays in partially insuring, through direct and indirect cross-ownership, other firms. Therefore, the proposition implies that the larger the partial insurance that i provides to other firms, the larger the moderating effect of positive weak correlation on firm i's risk-taking. The intuition is the following: when firm i provides low partial insurance to others, the other firm will

take more risk, but then, as risky projects are positively correlated, this decreases more the incentives of firm i to take risks.

5 Conclusion

Financial linkages have the potential to smoothen the shocks and uncertainties faced by individual components of the system. However, they also channel the financial shocks and so create costs for risk averse agents. In the case of cross-holding, each firm benefits from owning a specific optimal portfolio of shares, and deviations from this may annihilate the gains from diversification of risk.

In this paper we explore situations in which the portfolio of shares owned by a firm is exogenously fixed, and each firm can decide on the level of investment of a single private project. The analysis also assumes separation between ownership and control. We find that integration, in the sense of deeper cross-holdings, increases risk taking and investment. As the unconstrained social optimum is symmetric and associated to significant risk taking, integration may result in an increase in social welfare. This intuition is always correct when cross-holdings are small, but as these become larger, integration may reduce social welfare. The origin is a decrease in welfare of firms that are shareholders of firms that are central and take high risks. Cross-holding imposes a risk shift to connected firms. When the shares of a firm are concentrated in a few hands, risk shifting creates large exposures to a few firms. Finally, we find that positive correlation typically mitigates risk taking. More precisely, the larger the partial insurance provided to other firms, the larger the moderating effect of positive weak correlation on risk-taking.

We have emphasised in the introduction the mixed empirical evidence of the extent to which ownership translates into managerial control. Although there is a recent literature that attacks this problem in some specific circumstances (for example, Azar et al. (2018)), we leave for further research the task of integrating cross-ownership, endogenous risk taking and control. Allowing managers to optimize the whole portfolio rather than the sole private project is the other main avenue for further research. In a frictionless world, the optimal portfolio would generate a symmetric cross-holding network. But realistically there are frictions, and these might be the source of asymmetries.

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7 Appendix

Proof of Proposition 1 Suppose that the solution is interior. As the objective function is concave, the first-order condition is sufficient. Taking derivatives in (3) with respect to β_i and setting it equal to 0, immediately yields the required expression for optimal investments. Substituting the optimal investments in the expressions for the expected value and variance yields the expressions in the statement of the result.

Proof of Proposition 2 Rewriting the objective function of the planner we obtain that

$$W(S) = \sum_{i \in \mathcal{N}} \beta_i \mu - \frac{\alpha}{2} \sum_{i \in \mathcal{N}} \beta_i \sigma^2 A_i, \tag{9}$$

where $A_i \equiv \sum_{j \in \mathcal{N}} \gamma_{ji}^2$. Suppose the optimum is interior. Then, under the assumption that projects are independent, we obtain that for every $i \in \mathcal{N}$, the first order condition is

$$\mu - \sigma^2 \beta_i \alpha A_i = 0. \tag{10}$$

We obtain that the optimal level of investment of the social planner is, for every i,

$$\beta_i^P = \min \left[w_i, \hat{\beta} \frac{1}{A_i} \right]. \tag{11}$$

Since for individual i we have $\beta_i^* = \min \left\{ w, \frac{\hat{\beta}}{\gamma_{ii}} \right\}$ there is over-investment iff $\sum_{j \in \mathcal{N}} \gamma_{ji}^2 > \gamma_{ii}$.

Proof of Proposition 3 First we prove the expression for social welfare. From the definition of ϵ_i^2 we have

$$\epsilon_i^2 = \frac{1}{n} \sum_j (\bar{\rho}_i^2 + \rho_{ij}^2 - 2\bar{\rho}_i \rho_{ij}) \Rightarrow \sum_j \rho_{ij}^2 = n \left(\epsilon_i^2 + \bar{\rho}_i^2 \right)$$

Therefore

$$W(\beta^*, S)\frac{n\mu^2}{\sigma^2}\sum_i\left[\overline{\rho}_i - \frac{1}{2}\left(\epsilon_i^2 + \bar{\rho}_i^2\right)\right]$$

Let ϵ_ρ^2 be defined by

$$\epsilon_\rho^2 = \frac{1}{n}\sum_i\ (\bar{\rho} - \bar{\rho}_i\ )^2 = \frac{1}{n}\left(\left(\sum_i\ \bar{\rho}_i^2\right)\ - n\bar{\rho}^2\right) \Rightarrow \sum_i\ \bar{\rho}_i^2 = n\left(\ \epsilon_\rho^2 + \ \bar{\rho}^2\right)$$

Finally

$$W(\beta^*\ , S) = \frac{n\mu^2}{\sigma^2}\left(\ n\bar{\rho}\ - \frac{1}{2}\left(\sum_i \epsilon_i^2 + \sum_i \bar{\rho}_i^2\right)\right) = \frac{n\mu^2}{\sigma^2}\left(\ n\bar{\rho}\ - \frac{1}{2}\left(\sum_i \epsilon_i^2 + n\left(\ \epsilon_\rho^2 + \ \bar{\rho}^2\right)\right)\right)$$

introducing ϵ^2 we obtain the stated expression.

We now look at the first-best design problem. Substituting in expression (9) the centralised solution $\beta^P = \{\beta_1^P, ..., \beta_n^P\}$, we obtain that

$$W(S, \beta^P) = \frac{1}{2}\sum_{i\in\mathcal{N}}\hat{\beta}\mu\frac{1}{A_i}$$

Recall that $A_i = \sum_{j\in\mathcal{N}}\gamma_{ji}^2$ and therefore A_i only depends on $\{\gamma_{1i}, ..., \gamma_{ni}\}$. Moreover, if we fix i, the expression

$$\hat{\beta}\mu\frac{1}{A_i}$$

is declining in A_i. Next note that if, for some i, $\gamma_{li} > \gamma_{ki}$ for some $l \neq i$ and $k \neq i$, then, we can always find a small enough $\epsilon > 0$ so that, by making the local change $\gamma'_{li} = \gamma_{li} - \epsilon$ and $\gamma'_{ki} = \gamma_{ki} + \epsilon$, we strictly decrease A_i, without altering A_j for all $j \neq i$. Hence, such a local change strictly increases welfare. This implies that at the optimum $\gamma_{li} = \gamma_{ki}$ for all $l, k \neq i$. Set $\gamma_{li} = \gamma_{ki} = \gamma$; hence,$\gamma_{ii} = 1 - (n-1)\gamma$. Then, W is maximized when A_i is minimized, or, equivalently, γ minimizes

$$(n-1)\gamma^2 + [1 - \gamma(n-1)]^2$$

which implies that $\gamma = 1/n$. Note that Γ such that $\gamma_{ij} = 1/n$ for all i and for all j is obtained when S is complete and $s_{ij} = 1/(n-1)$ for all i and for all $j, j \neq i$.

Now we consider the second-best design problem. Note that by setting $s_{ij} = 1/(n-1)$ for all $i \neq j$, we obtain that $\gamma_{ij} = 1/n$ for all i, j and that, as a consequence β^* coincides with the socially optimal choice. Hence, the planner can replicate the first best outcome just by setting $s_{ij} = 1/(n-1)$ for all $i \neq j$.

Proof of Proposition 4. Investment in the bow tie

Let there be n_I investor firms with share s_I in each of the n_C core firms. These have share s_C in each of the other $n_C - 1$ core firms (fully connected core). In addition they have s_F shares in each of the n_F final firms.

The S matrix reads

$$S = \begin{bmatrix} 0 & s_I \Uparrow & 0 \\ 0 & s_C \Uparrow - s_C I & s_F \Uparrow \\ 0 & 0 & 0 \end{bmatrix}_{(n_I+n_C+n_F)\times(n_I+n_C+n_F)}$$

where $\Uparrow$ is a matrix of ones. Let

$$G = I - S = \begin{bmatrix} I & A & 0 \\ 0 & B & C \\ 0 & 0 & I \end{bmatrix}$$

with

$$\begin{aligned} A &= -s_I \Uparrow_{n_I \times n_C} \\ B &= -s_C \Uparrow_{n_C \times n_C} + (1+s_C) I_{n_C \times n_C} \\ C &= -s_F \Uparrow_{n_C \times n_F} \end{aligned}$$

There are three important preliminary results. Note that the condition

$$\sum_j s_{ji} < 1.$$

introduces an upper bound on the elements of S, which is here

$$\begin{aligned} n_C s_I + (n_C - 1)s &< 1 \\ n_C s_F &< 1 \end{aligned}$$

Lemma 1 *If $[\Uparrow]_{n\times p}$ is the $n \times p$ matrix of ones and $M_{p\times q}$ a $p \times q$ matrix then*

$$[\Uparrow]_{n\times p} M_{p\times q} [\Uparrow]_{q\times r} = \left(\sum M\right) [\Uparrow]_{n\times r}$$

where $(\sum M)$ is the sum of the elements of M. In addition

$$[\Uparrow]_{n\times p} [\Uparrow]_{p\times q} = p\, [\Uparrow]_{n\times q}$$

Proof: Right multiplying a matrix M by a matrix of ones delivers a matrix with rows made up of the sums of all elements in the row of the original matrix M. Left multiplying a matrix M by a matrix of ones, delivers a matrix with columns made up of the sums of all elements in the column of the original matrix M.

End Proof

Lemma 2 *The Leontief inverse of G is*

$$G^{-1} = \begin{bmatrix} I & -AB^{-1} & AB^{-1}C \\ 0 & B^{-1} & -B^{-1}C \\ 0 & 0 & I \end{bmatrix}$$

Proof:

$$\begin{bmatrix} I & A & 0 \\ 0 & B & C \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -AB^{-1} & AB^{-1}C \\ 0 & B^{-1} & -B^{-1}C \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

End Proof.

Lemma 3 *The inverse of B is* $B^{-1} = 1$ *if* $n_C = 1$ *and otherwise*

$$\begin{aligned} B^{-1} &= \frac{1}{(n_C - 1)\, s_C^2 + (n_C - 2)s_C - 1} \left[-s_C \Uparrow_{n_C \times n_C} + s_C I_{n_C \times n_C} + ((n_C - 2)s_C - 1)\, I_{n_C \times n_C}\right] \\ &= \frac{1}{\Delta} \left[-s_C \Uparrow_{n_C \times n_C} + ((n_C - 1)s_C - 1)\, I_{n_C \times n_C}\right] \end{aligned}$$

with $\Delta = (n_C - 1)\, s_C^2 + (n_C - 2)s_C - 1$.

Proof: As $B = -s_C \Uparrow_{n_C \times n_C} + (1 + s_C) I_{n_C \times n_C}$ we need to verify that $BB^{-1} = I$. That is

$$\left(-s_C \Uparrow_{n_C \times n_C} + (1 + s_C) I_{n_C \times n_C}\right) \left(-s_C \Uparrow_{n_C \times n_C} + ((n_C - 1)s_C - 1)\, I_{n_C \times n_C}\right) = \Delta I_{n_C \times n_C}$$

Expanding the product and using the fact that $\left(\Uparrow_{n_C \times n_C}\right)^2 = n_C \Uparrow_{n_C \times n_C}$ we obtain

$$\begin{aligned} &= s_C^2 \left(\Uparrow_{n_C \times n_C}\right)^2 - n_C s_C^2 I_{n_C \times n_C} + \left((n_C - 1)\, s_C^2 + (n_C - 2)s_C - 1\right) I_{n_C \times n_C} \\ &= \Delta I_{n_C \times n_C} \end{aligned}$$

End Proof.

We now use these results to prove the Proposition. First note that

$$\begin{aligned} & [s_I \Uparrow_{n_I \times n_C}] \left[-s_C n_C s_F \Uparrow_{n_C \times n_C} + ((n_C - 1)s_C - 1)\, I_{n_C \times n_C}\right] \\ = \; & \left[-n_C s_C s_I + s_I ((n_C - 1)s_C - 1)\right] \Uparrow_{n_I \times n_C} \\ & [-s_I \Uparrow_{n_I \times n_C}] \left[-s_C \Uparrow_{n_C \times n_C} + ((n_C - 1)s_C - 1)\, I_{n_C \times n_C}\right] [-s_F \Uparrow_{n_C \times n_F}] \\ = \; & -\left[n_C^2 s_C s_I s_F - n_C s_I s_F ((n_C - 1)s_C - 1)\right] \Uparrow_{n_I \times n_F} \\ & -\left[-s_C \Uparrow_{n_C \times n_C} + ((n_C - 1)s_C - 1)\, I_{n_C \times n_C}\right] [-s_F \Uparrow_{n_C \times n_F}] \\ = \; & -\left[n_C s_C s_F - s_F ((n_C - 1)s_C - 1)\right] \Uparrow_{n_C \times n_F} \end{aligned}$$

The normalisation matrix D with diagonal elements $d_{ii} = 1 - \sum_j s_{ji}$ is

$$D = \begin{bmatrix} I & 0 & 0 \\ 0 & (1-(n_C-1)s_C - n_I s_I)\, I & 0 \\ 0 & 0 & (1-n_C s_F)\, I \end{bmatrix}$$

and the matrix Γ is

$$\frac{D}{\Delta}\begin{bmatrix} \Delta I_{n_I \times n_I} & \left[-n_C s_C s_I + s_I\left((n_C-1)s_C - 1\right)\right] \Uparrow_{n_I \times n_C} & -\left[n_C^2 s_C s_I s_F - n_C s_I s_F\left((n_C-1)s_C - 1\right)\right] \Uparrow_{n_I \times n_F} \\ 0 & \left[-s_C \Uparrow_{n_C \times n_C} + \left((n_C-1)s_C - 1\right) I_{n_C \times n_V}\right] & -\left[n_C s_C s_F - s_F\left((n_C-1)s_C - 1\right)\right] \Uparrow_{n_C \times n_F} \\ 0 & 0 & \Delta I_{n_F \times n_F} \end{bmatrix}$$

leading to

$$\Gamma = \frac{1}{\Delta}\begin{bmatrix} \Delta I_{n_I \times n_I} & a \Uparrow_{n_I \times n_C} & b \Uparrow_{n_I \times n_F} \\ 0 & c \Uparrow_{n_C \times n_C} + d I_{n_C \times n_C} & h \Uparrow_{n_C \times n_F} \\ 0 & 0 & (1-n_C s_F)\, \Delta I_{n_F \times n_F} \end{bmatrix}$$

with

$$\begin{aligned} a &= \left(-n_C s_C s_I + s_I\left((n_C-1)s_C - 1\right)\right) \\ b &= -\left(n_C^2 s_C s_I s_F - n_C s_I s_F\left((n_C-1)s_C - 1\right)\right) \\ c &= -s_C\left(1-(n_C-1)s_C - n_I s_I\right) \\ d &= \left(1-(n_C-1)s_C - n_I s_I\right)\left((n_C-1)s_C - 1\right) \\ h &= -\left(1-(n_C-1)s_C - n_I s_I\right)\left(n_C s_C s_F - s_F\left((n_C-1)s_C - 1\right)\right) \end{aligned}$$

The diagonal reads

$$\Gamma_D = \frac{1}{\Delta}\begin{bmatrix} \Delta I_{n_I \times n_I} & 0 & 0 \\ 0 & \left(1-(n_C-1)s_C - n_I s_I\right)\left((n_C-2)s_C - 1\right) I_{n_C \times n_C} & 0 \\ 0 & 0 & (1-n_C s_F)\, \Delta I_{n_F \times n_F} \end{bmatrix}$$

We now show that investment by core firms is increasing in n_I, s_I, n_C,s_C while investment of the final firms is increasing in both n_F and s_F. Clearly investment in the core increases with the number of external firms, n_I investing in it and the size of the investment s_I. We want to explore the dependence on s_C and n_C. To simplify notation, let $m = n_I s_I$, $n_C - 1 = p$ and $s = s_C$. Assume $n_C > n_I$ so that $m = n_I s_I < n_I < n_C$. We study the dependence of investment on s, that is s_C, by looking at the partial derivative.

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\beta_c}{\widehat{\beta_c}} &= \frac{1}{\left(s\,(p-1)-1\right)^2}\frac{1}{\left(m+sp-1\right)^2} \\ &\quad p\left(p^2 s^2 - mps^2 - 2ps + ms^2 + 2ms + 1\right) \end{aligned}$$

Expressed as a function of s, the derivative would change sign at

$$\left(p^2 - mp + m)s^2 + 2(-p+m) + 1\right) = 0$$

We need to investigate the sign of the discriminant of this equation which is

$$-m\left(2m-2p-mp+p^2+1\right)$$

This means there are no roots if

$$\begin{aligned} 2m-2p-mp+p^2+1 &> 0 \Leftrightarrow \\ (n_C-1)^2+1 &> 2(n_C-1)+n_I s_I(n_C-3) \end{aligned}$$

which is automatically true if $s_I < 1-\frac{1}{n_C}$ which is true. This means that the sign of the derivative in respect to s does not changes as s grows. We only need to check it for very small s then. We have that

$$\lim_{s\to 0}\frac{\partial}{\partial s}\frac{\beta_c}{\widehat{\beta}_c}=\frac{1}{(m-1)^2}\left[p+(m-1)(p-1)+(m-1)(1-p)\right]=\frac{p}{(m-1)^2}>0.$$

Investment by the core is increasing in s (that is s_C).

Role of n_C. We can now analyse the role of n_C. With $n_C-1\equiv p$ and $s_C\equiv s$ we see that

$$\frac{\partial}{\partial n}\frac{\beta_c}{\widehat{\beta}_c}=\frac{\partial}{\partial p}\frac{ps^2+(p-1)s-1}{(1-ps-m)\left((p-1)s-1\right)}=\frac{s(s+1)\left(p^2s^2-2ps+ms+1\right)}{\left(s(p-1)-1\right)^2(m+ps-1)^2}$$

One way to evaluate the sign of $p^2s^2-2ps+ms+1$ is to consider this as a polynomial of s and check for roots

$$p^2s^2-2ps+ms+1=0$$

The discriminant of this equation is $-4mp+m^2$ when both $m,p\neq 0$. The relevant value is then

$$-4mp+m^2=n_I s_I\left(-4(n_C-1)+n_I s_I\right)<0$$

So as soon as

$$n_C>\frac{n_I s_I}{4}+1\Leftrightarrow 1>\frac{s_I}{4}+\frac{1}{n_C}$$

as $n_I\leq n_C$ which is clearly true. This means that the sign of the derivative in respect to n_C doesn't change as s grows. We only need to check it for very small s then. We have that

$$\lim_{s\to 0}\frac{\partial}{\partial n}\frac{\beta_c}{\widehat{\beta}_c}=\frac{1}{(1-m)}+\frac{1}{(1-m)^2(-1^2}>0$$

We now prove the results on over-investment. By Proposition 2 core-firms over invest if and only if

$$\gamma_{cc}<\gamma_{cc}^2+\gamma_{Ic}^2+\gamma_{fc}^2$$

but note that $\gamma_{fc}=0$ and so $\gamma_{Ic}=1-\gamma_{cc}$ Thus the above inequality for overinvestment becomes

$$\gamma_{cc}<\gamma_{cc}^2+(1-\gamma_{cc})^2$$

which is like $\gamma_{cc} < 1/2$. This condition is

$$\frac{[1-(n_C-1)s_C-n_Is_I][1-(n_C-2)s_C]}{(s_C+1)[1-(n_C-1)s_C]} < \frac{1}{2}$$

You can see that if $s_c = 0$ then the above condition becomes

$$n_Is_I > \frac{1}{2}$$

To see that this makes sense, consider a hierarchy and suppose that s_I is large. Then C firms takes lot of risk, they over invest and this is absorbed by the I firms. The case of out-section firms is more cumbersome as we need to show that

$$\gamma_{ff} < \gamma_{ff}^2 + \gamma_{cf}^2 + \gamma_{If}^2$$

From Proposition 4 we have

$$\begin{aligned}
\gamma_{cf} &= -\frac{n_F\left(1-(n_C-1)s_C-n_Is_I\right)\left(n_Cs_Cs_F-s_F\left((n_C-1)s_C-1\right)\right)}{(-1)\,s_C^2+(n_C-2)s_C-1} \\
\gamma_{If} &= -\frac{n_F\left(n_C^2s_Cs_Is_F-n_Cs_Is_F\left((n_C-1)s_C-1\right)\right)}{(n_C-1)\,s_C^2+(n_C-2)s_C-1} \\
\gamma_{ff} &= (1-n_Cs_F)
\end{aligned}$$

When $s_c = 0$ we obtain

$$\begin{aligned}
\gamma_{cf} &= n_F\left(1-n_Is_I\right)s_F \\
\gamma_{If} &= n_F\left(n_Cs_Is_F\right) \\
\gamma_{ff} &= (1-n_Cs_F)
\end{aligned}$$

then

$$\gamma_{ff} < \gamma_{ff}^2 + \gamma_{cf}^2 + \gamma_{If}^2$$

becomes

$$\begin{aligned}
(1-n_Cs_F)-(1-n_Cs_F)^2 &< \left(n_F\left(1-n_Is_I\right)s_F\right)^2+n_F^2\left(n_Cs_Is_F\right)^2 \\
n_Cs_F\left(1-n_Cs_F\right) &< n_F^2s_F^2\left(n_C^2s_I^2+n_I^2s_I^2-2n_Is_I+1\right)
\end{aligned}$$

Proof of Proposition 5: Welfare in the fat bow tie with $n_I = 1, n = 2, n_F = 1$

From Proposition 4 we have

$$\Gamma = \frac{1}{\Delta}\begin{bmatrix} \Delta I_{n_I\times n_I} & a\Uparrow_{n_I\times n_C} & b\Uparrow_{n_I\times n_F} \\ 0 & c\Uparrow_{n_C\times n_C}+dI_{n_C\times n_C} & h\Uparrow_{n_C\times n_F} \\ 0 & 0 & (1-n_Cs_F)\,\Delta I_{n_F\times n_F} \end{bmatrix}$$

with

$$\begin{aligned} a &= (-s_C s_I + s_I(-1)) \\ b &= -(s_C s_I s_F - s_I s_F(-1)) \\ c &= -s_C(1 - n_I s_I) \\ d &= (1 - n_I s_I)(-1) \\ h &= -(1 - n_I s_I)(n_C s_C s_F - s_F(-1)) \end{aligned}$$

The diagonal reads

$$\Gamma_D = \frac{1}{\Delta}\begin{bmatrix} \Delta I_{n_I \times n_I} & 0 & 0 \\ 0 & (1 - n_I s_I)((-s_C - 1) I_{n_C \times n_C} & 0 \\ 0 & 0 & (1 - s_F)\Delta I_{n_F \times n_F} \end{bmatrix}$$

$$\Phi = \Gamma\Gamma_D^{-1} = \begin{bmatrix} I_{n_I \times n_I} & W_{IC} \Uparrow_{n_I \times n_C} & W_{IF} \Uparrow_{n_I \times n_F} \\ 0 & W_{CC1} \Uparrow_{n_C \times n_C} + W_{CC2} I_{n_C \times n_C} & W_{CF} \Uparrow_{n_C \times n_F} \\ 0 & 0 & I_{n_F \times n_F} \end{bmatrix}$$

with

$$\begin{aligned} W_{IC} &= \frac{(-n_C s_C s_I + s_I((n_C - 1)s_C - 1))}{(1 - (n_C - 1)s_C - n_I s_I)((n_C - 2)s_C - 1)} \\ W_{CC1} &= \frac{s_C}{1 - (n_C - 2)s_C} \\ W_{CC2} &= \frac{(n_C - 1)s_C - 1}{(n_C - 2)s_C - 1} \\ W_{CF} &= -\frac{(1 - (n_C - 1)s_C - n_I s_I)(n_C s_C s_F - s_F((n_C - 1)s_C - 1))}{((n_C - 1)s_C^2 + (n_C - 2)s_C - 1)(1 - n_F s_F)} \\ W_{IF} &= -\frac{\left(n_C^2 s_C s_I s_F - n_C s_I s_F((n_C - 1)s_C - 1)\right)}{((n_C - 1)s_C^2 + (n_C - 2)s_C - 1)(1 - n_C s_F)} \end{aligned}$$

We now look at quadratic terms.

$$\Phi^2 = \left[\left\{\Gamma\Gamma_{jj}^{-1}\right\}_{ij}\right]^2 = \begin{bmatrix} I_{n_I \times n_I} & W_{IC}^2 \Uparrow_{n_I \times n_C} & W_{IF}^2 \Uparrow_{n_I \times n_F} \\ 0 & (W_{CC1} \Uparrow_{n_C \times n_C} + W_{CC2} I_{n_V \times n_C})^2 & W_{CF}^2 \Uparrow_{n_C \times n_F} \\ 0 & 0 & I_{n_F \times n_F} \end{bmatrix}$$

where

$$\begin{aligned} & (W_{CC1} \Uparrow_{n_C \times n_C} + W_{CC2} I_{n_V \times n_C})^2 \\ = & \frac{1}{((n_C - 2)s_C - 1)^2}\left(s_C^2 \Uparrow_{n_C \times n_C} + ((n_C - 2)s_C - 1)^2 I_{n_V \times n_C}\right) \\ = & W_{CC1}^2 \Uparrow_{n_C \times n_C} + I_{n_V \times n_C} \end{aligned}$$

Thus

$$\Phi^2 = \left[\left\{\Gamma\Gamma_{jj}^{-1}\right\}_{ij}\right]^2 = \begin{bmatrix} I_{n_I\times n_I} & W_{IC}^2 \Uparrow_{n_I\times n_C} & W_{IF}^2 \Uparrow_{n_I\times n_F} \\ 0 & W_{CC1}^2 \Uparrow_{n_C\times n_C} + I_{n_V\times n_C} & W_{CF}^2 \Uparrow_{n_C\times n_F} \\ 0 & 0 & I_{n_F\times n_F} \end{bmatrix}$$

Therefore

$$\begin{aligned} & \left[\sum_j \left(\Phi_{ij} - \frac{1}{2}\Phi_{ij}^2\right)\right]_{(n_I+n_C+n_F)\times 1} \\ = & \begin{bmatrix} \frac{1}{2}I_{n_I\times n_I} & \left(W_{IC} - \frac{1}{2}W_{IC}^2\right) \Uparrow_{n_I\times n_C} & \left(W_{IF} - \frac{1}{2}W_{IF}^2\right) \Uparrow_{n_I\times n_F} \\ 0 & \left(W_{CC1} - \frac{1}{2}W_{CC1}^2\right) \Uparrow_{n_C\times n_C} + \left(W_{CC2} - \frac{1}{2}\right) I_{n_C\times n_C} & \left(W_{CF} - \frac{1}{2}W_{CF}^2\right) \Uparrow_{n_C\times n_F} \\ 0 & 0 & \frac{1}{2}I_{n_F\times n_F} \end{bmatrix} \Uparrow_{(n_I+n+n_F)\times 1} \end{aligned}$$

Replacing the values $n_I = 1, n_C = 2, n_F = 1$ we find that $\Delta = s_C^2 - 1$ and

$$\Phi = \begin{bmatrix} 1 & \frac{s_I(s_C+1)}{1-s_C-s_I} & \frac{s_I(s_C+1)}{1-s_C-s_I} & \frac{2s_Is_F}{(1-s_C)(1-s_F)} \\ 0 & 1 & s & \frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)} \\ 0 & s & 1 & \frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally

$$\begin{aligned} W_i &= \widehat{\beta}\mu \sum_j \left(\Phi_{ij} - \frac{1}{2}\Phi_{ij}^2\right) \\ &= \widehat{\beta}\mu \begin{bmatrix} 1/2 + 2\frac{s_I(s_C+1)}{1-s_C-s_I} - \left(\frac{s_I(s_C+1)}{1-s_C-s_I}\right)^2 + \frac{2s_Is_F}{(1-s_C)(1-s_F)} - \frac{1}{2}\left(\frac{2s_Is_F}{(1-s_C)(1-s_F)}\right)^2 \\ 1/2 + s_C - \frac{1}{2}s_C^2 + \frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)} - \frac{1}{2}\left(\frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)}\right)^2 \\ 1/2 + s_C - \frac{1}{2}s_C^2 + \frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)} - \frac{1}{2}\left(\frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)}\right)^2 \\ 1/2 \end{bmatrix} \end{aligned}$$

For the in-firms we have

$$W_I = \widehat{\beta}\mu\left(1/2 + 2\frac{s_I\left(s_C+1\right)}{1-s_C-s_I}\left(1 - \frac{1}{2}\left(\frac{s_I\left(s_C+1\right)}{1-s_C-s_I}\right)\right) + \frac{2s_Is_F}{\left(1-s_C\right)\left(1-s_F\right)}\left(1 - \frac{1}{2}\left(\frac{2s_Is_F}{\left(1-s_C\right)\left(1-s_F\right)}\right)\right)\right)$$

First term increases in s, s_I, s_F. The term $\frac{s_I(s_C+1)}{1-s_C-s_I}\left(1 - \frac{1}{2}\frac{s_I(s_C+1)}{1-s_C-s_I}\right)$ increases in $\frac{s_I(s_C+1)}{1-s_C-s_I}$ until $\frac{s_I(s_C+1)}{1-s_C-s_I} = 1$ and then decreases. The last term increases in $\frac{2s_Is_F}{(1-s_C)(1-s_F)}$ until $\frac{2s_Is_F}{(1-s_C)(1-s_F)} = 1$ and then decreases. For fixed $0 < s_C < 1/2, s_F = 0$ and large enough R we see that W_I is initially increasing and then decreasing in s_I. For fixed $0 < s_I < 1, s_F = 0$, we see that W_I is initially increasing and then decreasing in s_C. The question is whether $\frac{s_I(s_C+1)}{1-s_C-s_I} > 1$ happens in the range $0 < s_C < 1/2$? The bound is given by $\frac{s_I(3/2)}{1/2-s_I} > 1$ that is $3/2s_I > 1/2 - s_I$ or $5/2s_I > 1/2$ or $s_I > 1/5$. The general pattern is true for small s_F. What happens for larger s_F ? The crucial value for non monotonicity of this term is $\frac{2s_Is_F}{(1-s_C)(1-s_F)} = 1$ which involves large enough

s_I and s_F and possibly s. As s_F tends to 1 we have non monotonicy. Note however, that the contribution of this term is small for small values of $\frac{2s_I s_F}{(1-s_C)(1-s_F)}$. To conclude, for R sufficiently large, s_I sufficiently large, W_I is non monotonous in s and s_F. For s_F sufficiently and small and large W_I is non monotonous in s_I. In the mid range of s_F, as s_I rises the first term is non monotonous and the second term becomes non monotonous only after $\frac{2s_I s_F}{(1-s)(1-s_F)} = 1$ which might never happen. So it is a question of comparing the second and third term, which is hard. For the core firms we have

$$W_C = \widehat{\beta}\mu\left(1/2 + s_C - \frac{1}{2}s_C^2 + \frac{(1-s_C-s_I)\,s_F}{(1-s_C)\,(1-s_F)}\left(1 - \frac{1}{2}\left(\frac{(1-s_C-s_I)\,s_F}{(1-s_C)\,(1-s_F)}\right)\right)\right)$$

The term $\frac{s_F+1}{s_C-1}\,(s_C+s_I-1)$ is increasing in all shares. The term $s_C - \frac{1}{2}s_C^2$ changes the sign of the derivative at $s_C = 1$, so it is always increasing. In addition, the next term becomes decreasing as soon as $\frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)} > 1$ which mean $s_C\,(1-s_C-s_I)\,s_F > (1-s_C)\,(1-s_F)$. In terms of s_C this means

$$s_C > \frac{1-2s_F+s_I s_F}{1-2s_F} > 1$$

As we need s_C smaller then $1/2$ we see that this is not possible. So W_C is increasing in s_C. Concerning the role of s_I note that $\frac{(1-s_C-s_I)s_F}{(1-s_C)(1-s_F)}$ is decreasing in s_I the expression. Finally social welfare in the fat bow tie is given by $W = \sum_i W_i = W_I + 2W_C + W_F$.

Proof of Proposition 6. Truncating all series at the third order terms we obtain for $i, j, i \neq j$

$$\begin{aligned}
\rho_{ij} &= \frac{(1-\sum_{p\neq i} s_{pi})\left(s_{ij} + \sum_{p\neq i,j} s_{ip}s_{pj} + \sum_{p\neq i,q\neq j,p\neq q} s_{ip}s_{pq}s_{qj} + ...\right)}{(1-\sum_{p\neq j} s_{pj})\left(1 + \sum_{p\neq j} s_{jp}s_{pj} + \sum_{p\neq j,q\neq j,p\neq q} s_{jp}s_{pq}s_{qj} + ...\right)} \simeq \frac{s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pi}}{1 - \left(\sum_p s_{pj} - \sum_p s_{jp}s_{pj}\right)} \\
&\simeq \left(s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pi}\right)\left(1 + \sum_p s_{pj} - \sum_p s_{jp}s_{pj} + \left(\sum_p s_{pj} - \sum_p s_{jp}s_{pj}\right)^2\right) \\
&\simeq s_{ij} + \sum_p s_{ip}s_{pj} - s_{ij}\sum_p s_{pi} + s_{ij}\sum_p s_{pj} \simeq s_{ij} + \sum_p s_{ip}s_{pj} + s_{ij}\left(\eta_j^{in} - \eta_i^{in}\right)
\end{aligned}$$

Then

$$\bar{\rho}_i = \frac{1}{n}\sum_j \rho_{ij} \simeq \frac{1}{n}\left(\eta_i^{out} + \sum_p s_{ip}\left(\eta_p^{out} + \eta_p^{in}\right) - \eta_i^{out}\,\eta_i^{in} + 1\right)$$

and

$$\bar{\rho} = \frac{1}{n}\sum_i \bar{\rho}_i \simeq \frac{1}{n^2}\sum_i \eta_i^{out} + \frac{1}{n^2}\sum_j \left(\eta_j^{in}\right)^2 + \frac{1}{n}$$

Looking at welfare

$$\begin{aligned}\frac{\sigma^2}{\mu^2}W_i &\simeq \sum_{j\neq i}\left(s_{ij}+\sum_p s_{ip}s_{pj}+s_{ij}\left(\eta_j^{in}-\eta_i^{in}\right)\right)+1\\ &\quad -\frac{1}{2}\left(\sum_{j\neq i}\left(s_{ij}+\sum_p s_{ip}s_{pj}+s_{ij}\left(\eta_j^{in}-\eta_i^{in}\right)\right)^2+1\right)\\ &\simeq \left(\frac{1}{2}+\eta_i^{out}+\sum_p s_{ip}\eta_p^{out}-\eta_i^{out}\eta_i^{in}+\sum_j s_{ij}\eta_j^{in}-\frac{1}{2}\sum_j s_{ij}^2\right)\end{aligned}$$

and the aggregate

$$\begin{aligned}\frac{\sigma^2}{\mu^2}\sum_i W_i &\simeq \frac{n}{2}+\sum_i \eta_i^{out}+\sum_p \eta_p^{in}\eta_p^{out}-\sum_i \eta_i^{out}\eta_i^{in}+\sum_j\left(\eta_j^{in}\right)^2-\frac{1}{2}\sum_{ij}s_{ij}^2\\ &\simeq \frac{n}{2}+\sum_i \eta_i^{out}+\sum_i\left(\eta_i^{in}\right)^2-\frac{1}{2}\sum_{ij}s_{ij}^2\end{aligned}$$

However, the variance of investment of firm i in firm j, i.e. s_{ij}, is

$$\sum_i\left(s_{ij}-\overline{s}_j\right)^2=\sum_i s_{ij}^2-\left(\sum_i s_{ij}\right)^2=\sum_i s_{ij}^2-\left(\eta_j^{in}\right)^2$$

Summing all these variances we obtain

$$\sum_{ij}\left(s_{ij}-\overline{s}_j\right)^2=\sum_{i,j}s_{ij}^2-\sum_j\left(\eta_j^{in}\right)^2$$

Then

$$\begin{aligned}\frac{\sigma^2}{\mu^2}\sum_i W_i &\simeq \frac{n}{2}+\sum_i \eta_i^{out}+\sum_i\left(\eta_i^{in}\right)^2-\frac{1}{2}\left(\sum_{ij}\left(s_{ij}-\overline{s}_j\right)^2+\sum_j\left(\eta_j^{in}\right)^2\right)\\ &\simeq \frac{n}{2}+\sum_i \eta_i^{out}+\frac{1}{2}\left(\sum_i\left(\eta_i^{in}\right)^2-\sum_{ij}\left(s_{ij}-\overline{s}_j\right)^2\right)\end{aligned}$$

Therefore, In thin networks, $\sum_i U_i(S)>\sum_i U_i(S')$ if

$$\sum_i \eta_i^{out}+\frac{1}{2}\left(\sum_i\left(\eta_i^{in}\right)^2-\sum_{ij}\left(s_{ij}-\overline{s}_j\right)^2\right)>\sum_i \eta_i'^{out}+\frac{1}{2}\left(\sum_i\left(\eta_i'^{in}\right)^2-\sum_{ij}\left(s'_{ij}-\overline{s}'_j\right)^2\right)$$

and using the definition of η_i^{in} and η_i^{out} we obtain the condition in the Proposition. The "only if" part also follows. If S is more integrated than S' then $\eta_i^{out}\geq\eta_i'^{out}$ and the inequality is strict for some i. This implies that moving from S' to S there is a positive first order effect in aggregate utilities. Therefore, for $\bar{s}$ small

enough, aggregate utility is higher in S and than S'.

Proof of Proposition 7. Let $\circ$ be the Hadamard product and $\hat{\Gamma}$ be a $n \times n$ matrix where $\hat{\gamma}_{ii} = 0$ for all i and $\hat{\gamma}_{ij} = \gamma_{ij}/\gamma_{ii}$ for all $i \neq j$. We first prove the statement of existence and uniqueness. Note that our game belongs to the class of games analysed by Rosen 1965. Indeed, recall that

$$U_i(\beta_i, \beta_{-i}) = \sum_j \gamma_{ij}(wr + \beta_j(\mu - r)) - \frac{\alpha}{2} \sum_j \sum_{j'} \gamma_{ij}\beta_j\gamma_{ij'}\beta_{j'}\sigma^2_{jj'}. \tag{12}$$

It is easy to see that $U_i(\beta_i, \beta_{-i})$ is continuous in (β_i, β_{-i}) and it is concave in β_i. Moreover, strategy space is from a convex and bounded support, so our game belongs to the class of games of Rosen 1965. This implies existence. Rosen 1965 also provides a sufficient condition for uniqueness. For some positive vector r, let $g(\beta, r)$ be a vector where element i is $r_i \frac{\partial U_i}{\partial \beta_i}$. Let $G(\beta, r)$ be the Jacobian of $g(\beta, r)$. A sufficient condition for uniqueness is that: there exists a positive vector r such that for every β and β' the following holds

$$(\beta - \beta'^T g(\beta', r) + (\beta'^T g(\beta, r) > 0.$$

Moreover, a sufficient condition for this above condition to hold is that there exists a positive vector r such that the symmetric matrix $G(\beta, r) + G(\beta, r)^T$ is negative definite. In our case, by fixing r to the unit vector, we have that

$$G(\beta, 1) + G(\beta, 1)^T = -\alpha[\Gamma + \Gamma^T] \circ \Omega$$

So, it would be sufficient to show that

$$[\Gamma + \Gamma^T] \circ \Omega$$

is positive definite. It is well known that the Hadamard product of two positive definite matrix is also a positive definite matrix. Since Ω is positive definite, it is sufficient to show that $[\Gamma + \Gamma^T]$ is positive definite. Since the sum of positive definite matrix is a positive definite matrix, it is sufficient to show that Γ is positive definite. The condition that $\sum_j s_{ij} < 1/2$, implies that Γ is a strictly diagonally dominant, and therefore positive definite. We now turn to characterize interior equilibrium. In an interior equilibrium we must have that $\sum_j \gamma_{ij}\sigma^2_{ij}\beta_j = \frac{\mu - r}{\alpha}$, for all $i \in N$. That is

$$[\Gamma \circ \Omega]\, \beta = \frac{\mu - r}{\alpha} \mathbf{1}.$$

Third, assume that $\sigma^2_i = \sigma^2$ for all i and that $\sigma^2_{ij} = \delta\sigma^2$ for all $i \neq j$ and $\delta \in [-1, 1]$. Denote by $D(\Gamma)$ a $n \times n$ diagonal matrix, where the elements in the diagonal are the same as the elements in the diagonal of Γ. Note that $\Gamma \circ \Omega = \sigma^2 \left[\delta \left(\Gamma - D(\Gamma)\right) + D(\Gamma)\right]$ but as $\Gamma - D(\Gamma) = D(\Gamma)\hat{\Gamma}$ we obtain

$$\begin{aligned} \Gamma \circ \Omega &= \sigma^2 \left[\delta D(\Gamma)\hat{\Gamma} + D(\Gamma))\right] \\ &= D(\Gamma) \left[I + \delta\hat{\Gamma}\right] \end{aligned}$$

It then follows that

$$\begin{aligned}\beta &= \frac{\mu - r}{\alpha\sigma^2}\left[D(\Gamma)[I+\delta\hat{\Gamma}]\right]^{-1}\mathbf{1} \\ &= \frac{\mu - r}{\alpha\sigma^2}\left[I+\delta\hat{\Gamma}\right]^{-1}D(\Gamma)^{-1}\mathbf{1},\end{aligned}$$

and therefore, for every i, we have that:

$$\beta_i = \frac{\mu - r}{\sigma^2\alpha}\sum_j\{[I+\delta\hat{\Gamma}]^{-1}\}_{ij}\frac{1}{\gamma_{jj}}$$

Note that if $\delta = 0$ then $[\Gamma\circ\Omega]^{-1} = \frac{1}{\sigma^2}[D(\Gamma)]^{-1}$; if $\delta = 1$ then $[\Gamma\circ\Omega]^{-1} = \frac{1}{\sigma^2}[I-S]D^{-1}$. Indeed $\Gamma\circ\Omega = \sigma^2 D(\Gamma)[I+\hat{\Gamma}] = \sigma^2\Gamma = \sigma^2 D[I-S]^{-1}$. Using this expression we may deduce the equilibrium portfolios. We note that $\sum_j\{[I+\rho\hat{\Gamma}]^{-1}\}_{ij}\frac{1}{\gamma_{jj}}$ is a form of centrality of agent i in network $\hat{\Gamma}$: $\hat{\gamma}_{ij} = \gamma_{ij}/\gamma_{ii}$ and therefore is the ratio between the share that i has on j directly and indirectly and the share that i has of herself directly and indirectly.The limiting case of small correlations is obtained as follows

$$\begin{aligned}\beta_i &= \frac{\mu}{\sigma^2\alpha}\sum_j[I+\delta\hat{\Gamma}]^{-1}_{ij}\frac{1}{\gamma_{jj}} = \frac{\mu}{\sigma^2\alpha}\sum_j\sum_{k=0}[(-\delta)^k\left\{\hat{\Gamma}^k\right\}_{ij}\frac{1}{\gamma_{jj}} \\ &\simeq \frac{\mu}{\sigma^2\alpha}\sum_j\left((\delta_{ij}-\delta\hat{\gamma}_{ij})\frac{1}{\gamma_{jj}}\right) \simeq \frac{\mu}{\sigma^2\alpha}\sum_j\left(\frac{\delta_{ij}}{\gamma_{jj}} - \delta\frac{\gamma_{ij}}{\gamma_{ii}}\frac{1}{\gamma_{jj}}\right) \\ &\simeq \beta_i^*\left(1-\delta\sum_j\left(\frac{\gamma_{ij}}{\gamma_{jj}}\right)\right) \simeq \beta_i^*\left(1-\delta\sum_j\rho_{ij}\right) \\ &\simeq \beta_i^*\left(1-n\delta\overline{\rho}_i\right)\end{aligned}$$

Note that Investment when correlation is infinitesimal is given by $\beta_i \simeq_{\delta\to 0}\beta_i^*\left(1-\delta\sum_j\frac{\gamma_{ij}}{\gamma_{jj}}\right) \simeq_{\delta\to 0}\beta_i^*\left(1-n\delta\overline{\rho}_i\right)$.