Published in final edited form as:
Math Comput Simul. 2013 January ; 87: . doi:10.1016/j.matcom.2013.02.003.

CENTER CONDITIONS AND CYCLICITY FOR A FAMILY OF CUBIC SYSTEMS: COMPUTER ALGEBRA APPROACH

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Abstract

Using methods of computational algebra we obtain an upper bound for the cyclicity of a family of cubic systems. We overcame the problem of nonradicality of the associated Bautin ideal by moving from the ring of polynomials to a coordinate ring. Finally, we determine the number of limit cycles bifurcating from each component of the center variety.

Key words and phrases

cyclicity; limit cycles; center problem

1. Introduction

We consider systems of ordinary differential equations on $\mathbb{R}^2$ of the form

$$\dot{u}=\lambda u-v+\sum_{j+k=2}^{N}A_{j,k}u^jv^k=P(u,v),\quad \dot{v}=u+\lambda v+\sum_{j+k=2}^{N}B_{j,k}u^jv^k=Q(u,v), \qquad (1)$$

where λ is arbitrarily close to zero (possibly zero). The *degree* of system (1) is $N = \max\{\deg P, \deg Q\}$. Depending on nonlinear terms the origin of system (1) is either a *center* (every orbit is an oval surrounding the origin), or a *focus* (every trajectory spirals towards or away from the origin). The problem of distinguishing between a center and a focus is called the *center* or the *center-focus* problem, for more details, see e.g. [20].

For system (1) denote by (λ, A, B) the set of its parameters λ, $A_{j,k}$ and $B_{j,k}$, and by $E(\lambda, A, B)$ the associated space of parameters. Let also $n_{(\lambda,A,B),\varepsilon}$ denote the number of limit cycles of system (1) that lie wholly within an ε-neighborhood of the origin. We define the key concept of this article, namely the cyclicity of a singular point.

We say that the singularity at the origin for system (1) with fixed coefficients $(\lambda^*, A^*, B^*) \in E(\lambda, A, B)$ has *cyclicity k* with respect to the space $E(\lambda, A, B)$ if there exist positive constants δ_0 and ε_0 such that for every pair δ and ε satisfying $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$

$$\max\{n_{(\lambda,A,B),\varepsilon}:|(\lambda,A,B)-(\lambda^*,A^*,B^*)|<\delta\}=k.$$

The problem of cyclicity of a center or a focus of a system of the form (1), which we always assume to be located at the origin, is also known as the *local 16th Hilbert problem* [11].

The concept of cyclicity was introduced by Bautin in his seminal paper [1], where he showed that the cyclicity of focus or center in quadratic systems is three. The cyclicity of the quadratic system has been studied by other methods in [14] and [24]. The cyclicity for some Liénard systems has been studied recently in [22, 25]; for some systems of high degrees in [18]; and the relationship between the cyclicity and the center problem in [12].

The cyclicity for systems with quadratic and cubic homogeneous nonlinearities can be established easily using algorithms of computational algebra because the Bautin ideal for these systems is radical. The problem becomes more difficult if the Bautin ideal is not radical (which appears to be the generic case). An approach to finding the cyclicity in the case of nonradical Bautin ideal has recently been proposed in [17].

In this paper we further generalize the method of [17] and apply it to the study of the cyclicity problem for a family of real cubic systems whose expression in the complex form is

$$\dot{x}=\lambda x+ix(1-a_{10}x-a_{20}x^2-a_{11}x\bar{x}-a_{02}\bar{x}^2), \quad (2)$$

where $x = u + iv$ (the connection between system (1) and (2) is explained in detail in section 2). The motivation for studying system (2) is that it is one of the very few 5-parameter cubic systems where the computation of the primary decomposition of the Bautin ideal is feasible (because of computational complexity it is extremely difficult to treat 6-parameter cubic systems with modern tools of computational algebra even using very powerful computers). The center problem for system (2) has been solved in [3].

If we add to (2) the complex conjugate equation and consider $\bar{x}$ as a new unknown function y, and $\bar{a}_{ij}$ as new parameters b_{ji} we obtain the associated complex system

$$\dot{x}=\lambda x+ix(1-a_{10}x-a_{20}x^2-a_{11}xy-a_{02}y^2), \ \dot{y}=\lambda y-iy(1-b_{01}y-b_{02}y^2-b_{11}xy-b_{20}x^2). \quad (3)$$

which is also called the *Lotka-Volterra* system (see [5]). We shall study in detail the structure of the Bautin ideal of system (3) as well as its variety. This will be used to derive bounds for the cyclicity of the origin of the *real* system (2). We will see in section 4 that the proposed method gives an estimation for the cyclicity of an elementary center or focus for "almost all" points of the center variety. If with this approach one can algorithmically obtain a bound for the cyclicity for "almost all" points of the center variety of any polynomial family, then it can be considered as a solution to the cyclicity problem for an elementary center or focus. However to confirm or reject this hypothesis more studies are needed.

The article is organized as follows. In section 2 we discuss a general approach to studying cyclicity of polynomial systems. In section 3 we characterize the existence of the local analytic first integral of system (3). This is a preliminary result needed in section 4 in order to solve the cyclicity problem of system (2). Finally, in section 5 we estimate the cyclicity of each component of the center variety of system (2).

2. Bautin ideal and cyclicity

In this section we briefly review an approach for studying the cyclicity problem. We also discuss a method for bounding the cyclicity of a singular point by moving from the ring of polynomials to a new ring, in which the Bautin ideal becomes radical or has a simple structure.

Now we provide the basic definition of an ideal generated by polynomials, its variety and the radical of the ideal. Let k is a field and denote by $k[x_1, x_2, \ldots, x_n]$ the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in the field k. Consider $f_1, f_2, \ldots, f_s \in k[x_1, \ldots, x_n]$ and define the following set

$$I := \left\{ \sum_{j=1}^{s} h_j f_j : h_1, \ldots h_s \in k[x_1, x_2, \ldots, x_n] \right\}.$$

We say that I is the *ideal generated by the polynomials* $f_1, \ldots, f_s$ and denoted by $\langle f_1, f_2, \ldots, f_s \rangle$; the polynomials $f_1, \ldots, f_s$ are called the *generators* of I. By

$$\mathbf{V}(I) = \{(a_1, a_2, \cdots, a_n) \in k^n : f_j(a_1, a_2, \cdots, a_n) = 0 \text{ for every } f_j \in I\}$$

we denote the variety of the ideal I. The radical of I, denoted by $\sqrt{I}$, is the set

$$\sqrt{I} = \{f \in k[x_1, x_2, \ldots, x_n] : \text{there exists } p \in \mathbb{N} \text{ such that } f^p \in I\}.$$

We say that an ideal $J \in k[x_1, x_2, \ldots, x_n]$ is *radical* if $J = \sqrt{J}$. Note that, if I is an ideal, then its radical $\sqrt{I}$ is also an ideal and defines the same variety as I, i.e.

$$\mathbf{V}(\sqrt{I}) = \mathbf{V}(I).$$

2.1. Complexification and focus quantities

The notion of the integrability and existence of a center are closely related. Recall that $\Phi(u, v)$ is a first integral of (1) if

$$P(u, v)\Phi'_u + Q(u, v)\Phi'_v = 0.$$

Using the notion of the first integral Poincaré and Lyapunov characterized the existence of a center of systems (1).

Theorem 2.1 (Poincaré-Lyapunov). *System* (1) *with* $\lambda = 0$ *has a center at* (0, 0) *if and only if it admits a formal first integral of the form*

$$\Phi(u, v) = u^2 + v^2 + \sum_{j+k=3}^{\infty} \Phi_{jk} u^j v^k. \quad (4)$$

In this theorem and the rest of the section j, k denote nonnegative integers. We introduce the complex variable $x = u + iv$, and using $u = (x + \bar{x})/2$ and $v = (x - \bar{x})/(2i)$ obtain from (1)

$$\dot{x} = \lambda x + ix - \sum_{j+k=2}^{N} a_{jk} x^j \bar{x}^k. \quad (5)$$

System (5) is just a complex form of the real system (1). Now adjoining the complex conjugate

$$\dot{x}=\lambda x+ix-\sum_{j+k=2}^{N} a_{jk}x^j\bar{x}^k,\dot{\bar{x}}=\lambda\bar{x}-i\bar{x}+\sum_{j+k=2}^{N}\bar{a}_{jk}x^k\bar{x}^j,$$

and replacing $\bar{x}$ by an independent complex variable y and $\bar{a}_{jk}$ by independent complex coefficient b_{kj}, we obtain

$$\dot{x}=\lambda x+ix-\sum_{j+k=2} a_{jk}x^jy^k=\lambda x+ix+\tilde{P}(x,y),\ \dot{y}=\lambda y-iy+\sum_{j+k=2}^{N} b_{kj}x^ky^j=\lambda y-iy+\tilde{Q}(x,y). \quad (6)$$

We say that system (6) is the *Complexification* of system (1). Following Dulac [9] (see also [20]) we consider the following extension of the concept of a center to system (6). We say that system (6) on $\mathbb{C}^2$ with $\lambda = 0$ has a *center* at the origin if it admits a formal first integral of the form

$$\Psi(x,y)=xy+\sum_{j+k=3}^{\infty}\Psi_{jk}x^jy^k. \quad (7)$$

The definition is justified by Theorem 2.1. One can always find a function Ψ of the form (7) such that

$$[ix+\tilde{P}(x,y)]\Psi_x'+[-iy+\tilde{Q}(x,y)]\Psi_y'=g_{11}(xy)^2+g_{22}(xy)^3+\dots,$$

where g_{qq} is a polynomial in coefficients a_{ij}, b_{kj} of system (6) called the *qth* focus *quantity*. It is clear that the vanishing of all the focus quantities g_{qq} is sufficient for the origin to be a center. It turns out this is also necessary. We let (a, b) to denote the coefficient string $(a_{20}, \dots, a_{0N}, b_{20}, \dots, b_{0N})$. Then system (6) with $\lambda = 0$ and $(a, b) = (a^*, b^*)$ has a center at the origin if and only if $g_{qq}(a^*, b^*) = 0$ for all $q \in \mathbb{N}$ (see e.g. [2], [20]). This explains the importance played by the following ideal

$$\mathcal{B} := \langle g_{kk}:k \in \mathbb{N}\rangle \subset \mathbb{C}[a,b],$$

called the *Bautin ideal* and its variety $\mathbf{V}(\mathcal{B})$ called the *center variety*.

Since every ideal in the polynomial ring is finitely generated there exists $K \in \mathbb{N}$ such that the first K focus quantities generate the Bautin ideal $\mathcal{B}$. In other words if we denote by $\mathcal{B}_K := \langle g_{11}, \dots, g_{KK}\rangle$, then there exists $K \in \mathbb{N}$ such that $\mathcal{B} = \mathcal{B}_K$. In reality to solve the center problem for a specific family (6) is to describe the center variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal.

2.2. Radical Bautin ideal

It is often convenient to use the focus quantities of system (6) to determine the cyclicity of the origin of system (1) when it is a center. If system (6) is a Complexification of the real system (1), then by the change of variables $a_{jk} = A_{jk} + iB_{jk}$ and $b_{kj} = A_{jk} - iB_{jk}$ we can obtain the focus quantities of the real system $g_{kk}^{\mathbb{R}}(A,B)=g_{kk}(a(A,B),\bar{a}(A,B))$. As before A and B denote the parameters $A_{j,k}$ and $B_{j,k}$, respectively, of the real system (1) and $(A, B) = (A_{20}, \dots, A_{0N}, B_{N0}, \dots, B_{20})$.

The next theorem reveals how the concept of minimality of the Bautin ideal is related to the cyclicity of the center at the origin. Given a Noetherian ring R and an ordered set $\mathscr{V} = \{\nu_1, \nu_2, \dots\} \subset R$, we construct a basis MinBasis(I) of the ideal $I = \langle \upsilon_1, \upsilon_2, \dots \rangle$ as follows:

- **a.** initially set MinBasis(I) = $\{\nu_p\}$, where υ_p is a first non-zero element of $\mathscr{V}$
- **b.** sequentially check successive elements υ_j, starting with $j = p + 1$, adding υ_j to MinBasis(I) if and only if $\upsilon_j \notin \langle \text{MinBasis(I)} \rangle$.

The basis MinBasis(I) constructed as above is called the *minimal basis* of the ideal I with respect to the ordered set $\mathscr{V}$ The cardinality of MinBasis(I) is called the Bautin depth of I [15]. The proof of the following theorem can be found, for example, in [20] and [17].

Theorem 2.2 (Radical Ideal Cyclicity Bound). *Suppose that for* (6) *with* $\lambda = 0$ *the following two conditions hold:*

- **a.** $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_K)$,
- **b.** $\mathscr{B}_K$ *is a radical ideal.*

Then the cyclicity of the origin of system (1) *is at most the cardinality of* MinBasis($\mathscr{B}_K$), that is the Bautin depth of $\mathscr{B}$.

2.3. Nonradical Bautin ideal

While there always exists K such that the first condition in Theorem 2.2 holds, the second condition does not always hold. In some cases it is possible to overcome this difficulty caused by the nonradicality of the ideal $\mathscr{B}_K$ [21]. The idea is to move the problem to a different ring in which the image of the Bautin ideal becomes radical [17] or has a simple structure (for more details see section 4).

The following result, which is a reformulation of Theorem 6.2.9 of [20].

Theorem 2.3. *Suppose that for* (6) *with* $\lambda = 0$ *we have* $\mathscr{B} = \mathscr{B}_m$. *Then the cyclicity of the origin of system* (1) *is at most* m.

In order to estimate the cyclicity of system (2) using Theorem 2.3 we will also need the following result.

Proposition 2.4. *Let* $I = \langle g_1, \dots, g_t \rangle$ *be an ideal in* $\mathbb{C}[x_1, \dots, x_n]$ *such that the primary decomposition of I is given as* $I = P_1 \cap \dots \cap P_k \cap Q_1 \cap \dots \cap Q_m$ *where* P_i *and* Q_i *are primary ideals such that* $P_i = \sqrt{P_i}$ *for* $i = 1, \dots, k$, *and* $Q_j \neq \sqrt{Q_j}$ *for* $j = 1, \dots, m$. *Let* $Q = Q_1 \cap \dots \cap Q_m$ *and g be a polynomial vanishing on* $\mathbf{V}(I)$. *Let* $x^* = (x_1^*, \dots, x_n^*)$ *be an arbitrary point of* $\mathbf{V}(I) \setminus \mathbf{V}(Q)$. *Then in a small neighborhood of* x^* *we have* $g = g_1 f_1 + \dots + g_t f_t$, *where* $f_1, \dots, f_t$ *are power series convergent at* x^*.

The proposition can be proved similarly as the corresponding result in [16].

3. Center variety of cubic system

Here we determine the conditions for the origin of family (3) to be a center, which is our first main result. This will be used, in the following section, to establish another main result of the paper, namely the determination of the cyclicity of the origin of system (2).

We shall need the following lemma, for a proof see [5].

Lemma 3.1. *If the system* (6) with $\lambda = 0$ *has a local inverse integrating factor*

$$V=(xy)^{\alpha}\prod_{i=1}^{m}F_i^{\beta_i},$$

with F_i *analytic in x and y,* $F_i(0, 0) \neq 0$ for $i = 1, \ldots, m$, $\alpha \neq 0$, *and* α *not an integer greater than 1, then it has a first integral of the form* (7).

The following theorem, is our main result for this section. It generalizes the result obtained in [3] for real system (2).

Theorem 3.2. *Let* $\mathbf{V}(\mathscr{B})$ *be the variety of the Bautin ideal of family* (3) *with* $\lambda = 0$ *and let* $\mathscr{B}_4 := \langle g_{11}, g_{22}, g_{33}, g_{44}\rangle$. *Then we have* $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_4)$. *Moreover,* $\mathbf{V}(\mathscr{B})$ *consists of the following five irreducible components*

$$\mathbf{V}(\mathscr{B})=\mathbf{V}(J_1)\cup\mathbf{V}(J_2)\cup\mathbf{V}(J_3)\cup\mathbf{V}(J_4)\cup\mathbf{V}(J_5),$$

where

$$\begin{aligned} J_1&=\langle a_{10}^2b_{02}-a_{20}b_{01}^2, a_{02}a_{20}-b_{20}b_{02}, a_{02}a_{10}^2-b_{20}b_{01}^2, a_{11}-b_{11}\rangle\\ J_2&=\langle a_{20}+b_{20}, a_{11}-b_{11}, a_{02}+b_{02}\rangle\\ J_3&=\langle a_{11}, a_{02}, b_{02}b_{11}\rangle\\ J_4&=\langle a_{20}, a_{11}, b_{11}, b_{20}\rangle\\ J_5&=\langle a_{11}, a_{02}, b_{11}, b_{20}\rangle. \end{aligned}$$

Proof. Necessity. Using, for example, the algorithm given in [20] we compute the first seven focus quantities $g_{11}, \ldots, g_{77}$ of system (3). The first four of them are given in Appendix A. The other three polynomials are too long to be presented in the paper however the interested reader can compute them easily with the help of, e.g. Mathematica [23] (see [20, p.308] an algorithm; and [10, ?] for a generalization). By the Hilbert Basis Theorem $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_k)$ for some $k \in \mathbb{N}$. Using the Radical Membership Test one can verify that

$$g_{44}\notin\sqrt{\langle g_{11}, g_{22}, g_{33}\rangle}\text{but}g_{55}, g_{66}, g_{77}\in\sqrt{\langle g_{11}, g_{22}, g_{33}, g_{44}\rangle},$$

which leads us to expect that $\mathbf{V}(\mathscr{B}_4) = \mathbf{V}(\mathscr{B})$. It is straightforward to check (using any specialized computer algebra system, e.g. Singular) that the irreducible decomposition of $\mathbf{V}(\mathscr{B}_4)$ consists of five components given in the statements of the theorem.

Sufficiency. Now, we verify that each point of $\mathbf{V}(J_k)$, $k = 1, 2, 3, 4, 5$ corresponds to a system with a center at the origin.

Component $\mathbf{V}(J_1)$. Using the algorithm from [20, Table 5.1, p.235], see also [19]) we find that the Zariski closure of all time-reversible systems in the family (3), denoted by $\mathscr{R}$, is the variety of the ideal J_1. Thus every system from $\mathbf{V}(J_1)$ admits a first integral of the form (7) and, therefore has a center.

Component $\mathbf{V}(J_2)$. System (3) that corresponds to component $\mathbf{V}(J_2)$ can be written as

$$\dot{x}=x(1-a_{10}x+b_{20}x^2-b_{11}xy+b_{02}y^2),\ \dot{y}=-y(1-b_{20}x^2-b_{01}y-b_{11}xy-b_{02}y^2). \quad (8)$$

This system has two invariant algebraic curves $l_1 = x$ and $l_2 = y$ yielding the integrating factor $\mu = x^{-2}y^{-2}$ and the first integral

$$H=\frac{1}{xy}(1-a_{10}x-b_{01}y+b_{20}x^2-b_{02}y^2)+b_{11}\log(xy).$$

Thus H is of the form

$$H=\frac{R}{xy}+c\log(xy),$$

where $c = b_{11}$ and $R = 1 + h.o.t.$ ($h.o.t.$ stands for higher order terms). We claim that there exists an analytic function $S = 1 + h.o.t.$ such that

$$\frac{R}{xy}+c\log(xy)=\frac{S}{xy}+c\log\left(\frac{xy}{S}\right).$$

To prove the claim we consider

$$K(x,y,S):=(R-S)+cxy\log(S)=0. \quad (9)$$

We note that $K(0, 0, 1) = 0$ and $\frac{dK}{dS}(0,0,1)\neq 0$. Therefore, by the implicit function theorem there is a function $S = 1 + h.o.t.$, which gives the solution to (9) in a neighborhood of the origin. This proves the claim. Now it is easy to see that $\bar{H} = xy/S$ is an analytic first integral of system (8).

Component $\mathbf{V}(J_3)$. We shall show that the system

$$\dot{x}=x(1-a_{10}x-a_{20}x^2),\dot{y}=-y(1-b_{20}x^2-b_{01}y) \quad (10)$$

has a center at the origin, i.e. it admits a first integral of the form (7). Equivalently, we show that it admits a first integral of the form

$$\Psi=\sum_{k=1}^{\infty}f_k(y)x^k,$$

where, for $k \in \mathbb{N}$, $f_k(y)$ is an analytic function in the neighborhood of 0. Let $\bar{f}_{k-1}=b_{20}yf'_{k-2}(y)-(k-1)a_{10}f_{k-1}(y)$. It is easy to see that $\dot{\Psi} = 0$ if and only if

$$f_1(y)+y(b_{01}y-1)f'_1(y)=0 \quad (11)$$

$$-a_{10}f_1(y)+2f_2(y)+y(b_{01}y-1)f'_2(y)=0 \quad (12)$$

$$-(k-2)a_{20}f_{k-2}(y)+\bar{f}_{k-1}(y)+kf_k(y)+y(b_{01}y-1)f_k'(y)=0 \quad (13)$$

for $k \geq 3$.

We claim, and show by induction that $f_k(y)$ in (11)–(13) are of the form

$$f_k=\frac{P_k(y)}{(b_{01}y-1)^k}, \text{for} k=1,2,\ldots, \quad (14)$$

where $P_k(k) \in \mathbb{C}[y]$ of degree at most k. Integrating (11) and (12) we get

$$f_1(y)=\frac{yc_1}{1-b_{01}y}, \text{and} f_2(y)=\frac{y(a_{10}+yc_2)}{(1-b_{01}y)^2}, c_1, c_2 \in \mathbb{C},$$

which proves our claim for $k = 1, 2$. Assuming our claim is true for $k = 1, 2, \ldots, j - 1$, we show that it is also true for $k = j$. Consider (13) for $k = j$ and note that $\bar{f}_{j-1}(y)$ is of the form (14) with $k = j - 1$. Therefore for $k = j$, equation (13) is equivalent to

$$y(b_{01}y-1)f_j'+kf_j=\frac{Q_{j-1}(y)}{(b_{01}y-1)^{j-1}},$$

where $Q_{j-1}(y) \in \mathbb{C}[y]$ of degree at most $j - 1$. Equivalently

$$y(b_{01}y-1)^j f_j'+j(b_{01}y-1)^{j-1}f_j=Q_{j-1}(y). \quad (15)$$

To conclude, we need to show that $f_j(y)$ in (15) has the form (14). If

$$f_j(y)=\frac{P_j(y)}{(b_{01}y-1)^j}, \text{then} f_j'(y)=\frac{P_j'(y)(b_{01}y-1)-jb_{01}P_j(y)}{(b_{01}y-1)^{j+1}}.$$

Substituting the above expressions to (15), we get

$$yP_j'(y)-jP_j(y)=Q_{j-1}(y).$$

Note that this equation can be easily solved for P_j. Therefore, function $f_j(y)$ is of the form (14), which proves the claim.

Component $\mathbf{V}(J_4)$. This case is similar to the previous one under the involution $a_{kj} \leftrightarrow b_{kj}$.

Component $\mathbf{V}(J_5)$. Now we consider

$$\dot{x}=x(1-a_{10}x-a_{20}x^2), \dot{y}=-y(1-b_{01}y-b_{02}y^2). \quad (16)$$

It has the following invariant curves $l_1 = x$, $l_2 = y$, $l_3 = 1 - b_{01}y - b_{02}y^2$, $l_4 = 1 - a_{10}x - a_{20}x^2$, with the corresponding cofactors $k_1 = 1 - a_{10}x - a_{20}x^2$, $k_2 = -1 + b_{01}y + b_{02}y^2$, $k_3 = -y(-b_{01} -2b_{02}y)$, $k_4 = x(-a_{10} -2a_{20}x)$. We can construct an inverse integrating factor of the form $V = xyl_3l_4$. Therefore, by Lemma 3.1 system (16) admits a first integral Ψ of the form (7).

4. Cyclicity of a cubic system

In this section we prove the main result of the paper, namely a bound for the cyclicity of the origin of system (2).

Theorem 4.1. *Assume that* $|a_{20}|+|a_{11}|+|a_{02}|\neq 0$. *Then the cyclicity of the center at the origin of system* (2) *is at most four.*

To prove Theorem 4.1 we shall make use of the following lemma.

Lemma 4.2. *The focus quantities* g_{kk} *of system* (3) *belong to the polynomial sub-algebra* $\mathbb{C}[h_1, \ldots, h_{12}] \subset \mathbb{C}[a, b]$, *where*

$$\begin{array}{llll} h_1=a_{11} & h_2=b_{11} & h_3=a_{20}a_{02} & h_4=b_{20}b_{02} \\ h_5=a_{10}^2b_{02} & h_6=a_{20}b_{01}^2 & h_7=a_{10}^2a_{02} & h_8=b_{20}b_{01}^2 \\ h_9=a_{10}b_{01} & h_{10}=a_{20}b_{02} & h_{11}=a_{11}b_{11} & h_{12}=a_{02}b_{20}. \end{array} \quad (17)$$

Proof. Define a set $S := \{(1, 0), (2, 0), (1, 1), (0, 2)\}$. Consistently with this we order the eight coefficients as $(a_{10}, a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02}, b_{01})$ so that any monomial, denoted by $[\nu]$, appearing in g_{kk} has the form

$$[\nu] := a_{10}^{\nu_1}a_{20}^{\nu_2}a_{11}^{\nu_3}a_{02}^{\nu_4}b_{20}^{\nu_5}b_{11}^{\nu_6}b_{02}^{\nu_7}b_{01}^{\nu_8} \quad (18)$$

for some $(\nu_1, \ldots, \nu_8) \in \mathbb{N}_0^8$. Let $\mathscr{L}:\mathbb{N}_0^8 \to \mathbb{Z}^2$ be the map defined by

$$\mathscr{L}(\nu)=\nu_1(1,0)+\nu_2(2,0)+\nu_3(1,1)+\nu_4(0,2)+\nu_5(2,0)+\nu_6(1,1)+\nu_7(0,2)+\nu_8(0,1).$$

By $\hat{\nu}$ we denote the involution of ν, i.e. $\hat{\nu} = (\nu_8, \ldots, \nu_1)$. It is shown in [20] that the focus quantities of system (3) have the form

$$g_{kk}=\sum_{\nu:\mathscr{L}(\nu)=(k,k)} g_{(\nu)}([\nu]-[\hat{\nu}]). \quad (19)$$

Define the set $\mathscr{M} \subset \mathbb{N}_0^8$, which has a structure of a monoid by

$$\mathscr{M} := \{\nu \in \mathbb{N}_0^8 : \mathscr{L}(\nu)=(k,k) \text{for some} k \in \mathbb{N}_0\}.$$

First, using the Algorithm in Table 5.1 of [20] we compute a Hilbert basis of $\mathscr{M}$, obtaining a 12-element set

$$\begin{array}{lll} (0010 \quad 0000) & (0200 \quad 0001) & (0100 \quad 0010) \\ (0000 \quad 0100) & (1000 \quad 0020) & (1000 \quad 0001) \\ (1001 \quad 0000) & (1001 \quad 0000) & (0010 \quad 0100) \\ (0000 \quad 1001) & (0000 \quad 1020) & (0001 \quad 1000). \end{array}$$

We denote by ν_j the *jth* element of this list and let $h_j = [\nu_j] \in \mathbb{C}[a, b]$. Hence we obtain (17). Therefore we get

$$g_{kk} = \sum \tilde{g}_{kk}^{(\alpha)} h_1^{\alpha_1} \cdots h_{12}^{\alpha_{12}},$$

where $\tilde{g}_{kk}^{(\alpha)} \in \mathbb{C}$, $k \in \mathbb{N}$, that is, the focus quantities g_{kk} of system (2) with $\lambda = 0$ belong to the polynomial subalgebra $\mathbb{C}[h_1, \ldots, h_{12}]$, which proves our lemma.

Proof of Theorem 4.1. To prove our cyclicity bound we first attempt to use Theorem 2.2. The equality $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_4)$ was established in Theorem 3.2. Thus condition (a) of Theorem 2.2 holds. Now we show that $\mathscr{B}_4$ is not radical and thus the condition (*b*) (of Theorem 2.2) is not fulfilled.

We claim that $\mathscr{B}_4$ is not radical. To see that we denote by $\mathscr{B}_4^r$ the radical of $\mathscr{B}_4$ (in practice we compute the generators of $\mathscr{B}_4^r$ using e.g. the command `radical` of SINGULAR [7]). Now we compute the reduced Gröbner bases $\mathscr{B}_4$ and $\mathscr{B}_4^r$ of the ideals $\mathscr{B}_4$ and $\mathscr{B}_4^r$, respectively. It is clear that $\mathscr{B}_4$ is radical if and only if $\mathscr{B}_4$ and $\mathscr{B}_4^r$ are the same (see [6]). It turns out that (using e.g. `reduce` command of SINGULAR [7]) some of the elements of $\mathscr{B}_4^r$ do not reduce to zero modulo $\mathscr{B}_4$.Therefore $B_4^r \not\subseteq B_4$ which finishes the proof of the claim.

Therefore we seek to apply Theorem 2.3 instead. By Lemma 4.2 focus quantities of system (2) belong to the polynomial subalgebra $\mathbb{C}[h_1, \ldots, h_{12}]$, where $h_j = h_j(a, b)$ for $j = 1, \ldots, 12$ are given in (17). We also define the polynomial mapping

$$F:\mathbb{C}^8 \to \mathbb{C}^{12}:(a,b) \mapsto (c_1,\ldots,c_{12}) = (h_1(a,b),\ldots,h_{12}(a,b)),$$

which induces the $\mathbb{C}$-algebra homomorphism

$$F^*:\mathbb{C}[c] \to \mathbb{C}[a,b]: \sum d^{(\alpha)} c_1^{\alpha_1} \cdots c_{12}^{\alpha_{12}} \mapsto \sum d^{(\alpha)} h_1^{\alpha_1}(a,b) \cdots h_{12}^{\alpha_{12}}(a,b),$$

where $d^{(\alpha)} \in \mathbb{C}$.

Now we shall compute $g_{11}^F, \ldots, g_{44}^F$, i.e. we seek to express the focus quantities $g_{11}, \ldots, g_{44} \in \mathbb{C}[a, b]$ in the new variables $c_1, \ldots, c_{12}$. Since the focus quantities are lengthy polynomials, we shall use a convenient algorithmic procedure (see [6]) for finding those expressions. Choose any elimination ordering with $\{a, b\} > \{c\}$: for instance, lexicographical ordering with $a_{20} > a_{10} > a_{11} > a_{02} > b_{20} > b_{11} > b_{01} > b_{02} > c_1 > \ldots > c_{12}$: Let $c = (c_1, \ldots, c_{12})$ and denote by $J \subset \mathbb{C}[a, b, c]$ the ideal generated by $c_j - h_j(a, b)$, that is $J = \langle c_j - h_j(a, b) : j = 1, \ldots, 12\rangle$. Computing a Gröbner basis J_G of the ideal J in $\mathbb{C}[a, b, c]$ with respect to the elimination ordering as above and forming the Gröbner basis $R = J_G \cap \mathbb{C}[c]$ of the ideal $J \cap \mathbb{C}[c]$ yields a 10-element set $\{r_1, \ldots, r_{10}\}$ given by

$$\begin{array}{llll} c_3c_8 - c_6c_{12} & c_4c_7 - c_5c_{12} & c_4c_6 - c_8c_{10} & c_3c_5 - c_7c_{10} \\ c_3c_4 - c_{10}c_{12} & c_1c_2 - c_{11} & c_9^2c_{12} - c_7c_8 & c_9^2c_{10} - c_5c_6 \\ c_4c_9^2 - c_5c_8 & c_3c_9^2 - c_6c_7. & & \end{array}$$

By the Elimination Theorem (see e.g. [6] or [20]) the set given above is a Gröbner basis of R: Finally, the division of each $g_{11}, \ldots, g_{44}$ by J_G yields polynomials $g_{11}^F, \ldots, g_{44}^F$ that are given in Appendix B.

Geometrically equations (17) define a graph in $\mathbb{C}^{8+12}$ and the projection of the graph to $\mathbb{C}^{12}$ denoted by W is computed using the Elimination Theorem. The Zariski closure of the projection is the variety of the ideal R, $\bar{W} = \mathbf{V}(R)$: Hence, to show that

$$\bar{\mathbf{V}}_c = \mathbf{V}(N_c), \text{where} N_c = (\mathscr{B}_4 + J) \cap \mathbb{C}[c].$$

we compute the reduced Gröbner basis of the ideal N_c, and find that it is the same as the reduced Gröbner basis of the ideal $\langle g_{11}^F, \ldots, g_{44}^F, R \rangle \subset \mathbb{C}[c] \subset \mathbb{C}[c]$. Therefore $N_c = \langle g_{11}^F, \ldots, g_{44}^F, R \rangle$ which in turn, gives

$$\bar{\mathbf{V}}_c = \mathbf{V}(N_c) = \mathbf{V}(\langle g_{11}^F, \ldots, g_{44}^F, r_1, \ldots, r_{10} \rangle).$$

By the natural isomorphism of $\mathbb{C}[\bar{W}]$ with $\mathbb{C}[c]/R$ the ideal $g_{11}^F, \ldots, g_{44}^F$ is radical if and only if the ideal $\langle g_{11}^F + R, \ldots, g_{44}^F + R \rangle$ in $\mathbb{C}[c]/R$ is radical. Its easy to see that this is true if the ideal $K = \langle g_{11}^F, \ldots, g_{44}^F, r_1 \ldots, r_{10} \rangle$ is radical in $\mathbb{C}[c]$.

Using the routine `primdecGTZ` of SINGULAR [8, 13] we compute the primary decomposition of K and obtain

$$K = P_1 \cap \cdots \cap P_5 \cap Q, \quad (20)$$

where $P_1, \ldots, P_5$ are prime ideals but unfortunately Q is not prime. Thus K is not a radical ideal in $\mathbb{C}[c]/R$. Therefore, the method of [17] cannot be directly applied.

We note, though, that the Q in (20) is a primary ideal such that $\sqrt{Q} = \langle c_{11}, c_{10} - c_{12}, c_6 + c_8, c_5 + c_7, c_4 + c_{12}, c_3 + c_{12}, c_2, c_1, c_9^2 c_{12} - c_7 c_8 \rangle$. Thus K has the structure as in Proposition 2.4. In the parameter space $E(a, b)$, the variety $\mathbf{V}(Q)$ is defined by $a_{20} b_{01}^2 + b_{20} b_{01}^2 = a_{10}^2 a_{02} + a_{10}^2 b_{02} = a_{02} b_{20} + b_{20} b_{02} = b_{11} = a_{11} = a_{20} b_{02} + b_{20} b_{02} = a_{20} a_{02} - b_{20} b_{02} = 0$ The intersection $\mathbf{V}(Q)$ with the parameter space $E(a)$ is the set $a_{20} = a_{11} = a_{02} = 0$. Let now $(a^*, \bar{a}^*)$ be a point from $E(a, b)$ corresponding to system (2). If $|a_{20}|+|a_{11}|+|a_{02}| \neq 0$ then $F(a^*, \bar{a}^*) \notin \underline{\mathbf{V}}(Q)$. Therefore, by Proposition 2.4, there exist rational functions $f_{j,k}$, $s_{j,k}$ such that for $c \in W$ in a neighborhood of $F(a^*, \bar{a}^*)$ with $|a_{20}|+|a_{11}|+|a_{02}| \neq 0$ we have

$$g_{kk}^F = g_{11}^F f_{1,k} + \cdots + g_{44}^F f_{4,k} + \sum_{j=1}^{10} r_j s_{j,k}. \quad (21)$$

Applying F^* to (21) we get that

$$g_{kk} = g_{11} f_1 + g_{22} f_2 + g_{33} f_3 + g_{44} f_4$$

holds for all $k > 4$ in a neighborhood of $(a^*, \bar{a}^*)$.

In short, $\{g_{11}, g_{22}, g_{33}, g_{44}\}$ is the minimal basis of $\mathscr{B}$ and therefore by Theorem 2.3 the cyclicity of the center at the origin of system (2) with $|a_{20}|+|a_{11}|+|a_{02}| \neq 0$ is at most four.

5. Cyclicity of the components

In this section we study the number of limit cycles that can bifurcate from each component of the center variety of system (2). Our approach is based on a result by Christopher [4], which relates these numbers to the dimension of each component.

First we shall introduce some notations. Denote by $g_{kk}^{\mathbb{R}}$ the polynomials obtained by replacing in g_{kk} the variable b_{qp} with $\bar{a}_{pq}$. Then the center variety of the real system (2) is the variety $\mathbf{V}^{\mathbb{R}}$ in $E(\bar{a})$ of the ideal $\mathscr{B}^{\mathbb{R}}=\langle g_{11}^{\mathbb{R}}, g_{22}^{\mathbb{R}}, \ldots\rangle$. We denote by $J_p(\mathscr{B}_k^{\mathbb{R}})$ the Jacobian of the polynomials $g_{11}^{\mathbb{R}}, g_{22}^{\mathbb{R}}, \ldots, g_{kk}^{\mathbb{R}}$ evaluated at the point p and by $\text{rank}J_p^k$ the rank of $J_p(\mathscr{B}_k^{\mathbb{R}})$.

Theorem 5.1 ([4]). *Assume that for system* (5) *with* $\lambda = 0$ *and a point* $p \in \mathbf{V}^{\mathbb{R}}$, $\text{rank}J_p^k=k$ *Then the codimension of* $\mathbf{V}^{\mathbb{R}}$ *is at least k and there are bifurcations of* (5), *which produce locally k limit cycles from the center corresponding to the parameter value p.*

Moreover, if p lies on a component C of $\mathbf{V}^{\mathbb{R}}$ *of codimension k, then p is a smooth point of the center variety, and the cyclicity of p and any generic point of C is exactly k.*

In order apply this theorem, first we find all the irreducible components of $\mathbf{V}^{\mathbb{R}}$. This can be obtained from the components of the complex center variety of system (3) described in Theorem 3.2 by setting

$$\begin{array}{ll} a_{10}=A_{10}+iB_{10} & a_{20}=A_{20}+iB_{20} \\ a_{11}=A_{11}+iB_{11} & a_{02}=A_{02}+iB_{02} \\ b_{02}=A_{20}-iB_{20} & b_{01}=A_{10}-iB_{10} \\ b_{11}=A_{11}-iB_{11} & b_{20}=A_{02}-iB_{02}, \end{array} \quad (22)$$

where $i=\sqrt{-1}$. The capital letters indicate the coefficients of the real system written in the complex form as (2).

The following theorem describes in details the variety $\mathbf{V}^{\mathbb{R}}$.

Theorem 5.2. *The center variety in* $\mathbb{R}^8$ *of the real system* (2) with $\lambda = 0$ *consists of the following three irreducible components*

$$\mathbf{V}^{\mathbb{R}}=\mathbf{V}(G_1)\cup\mathbf{V}(G_2)\cup\mathbf{V}(G_3),$$

where

$$\begin{aligned} G_1&=\langle B_{11}, A_{20}B_{02}+A_{02}B_{20}, A_{10}^2B_{02}+2A_{02}A_{10}B_{10}-B_{02}B_{10}^2, A_{10}^2B_{20}-2A_{10}A_{20}B_{10}-B_{10}^2B_{20}\rangle \\ G_2&=\langle B_{11}, A_{02}+A_{20}, B_{02}-B_{20}\rangle \\ G_3&=\langle A_{11}, B_{11}, A_{02}, B_{02}\rangle, \end{aligned}$$

of dimension 5, 5, *and* 4, *respectively.*

Proof. We obtain the ideals G_1, G_2 and G_3 by applying the change of variables (22) respectively to the generators of the ideals J_1, J_2 and J_5 of Theorem 3.2. We note that applying (22) to J_3 and J_4 of Theorem 3.2 results in the same ideal $\langle A_{11}, B_{11}, A_{02}, B_{02}, A_{20}, B_{20}\rangle$, which is a subvariety of G_3.

It is straightforward to see that the dimension of $\mathbf{V}(G_2)$ and $\mathbf{V}(G_3)$ is 5 and 4, respectively. Now we show that dim $\mathbf{V}(G_1) = 5$.

We begin by finding the rational parametrization of $\mathbf{V}(G_1)$. We claim that it is given by

$$\begin{array}{llll} A_{11}=f_1 & A_{02}=f_2 & A_{10}=f_3 & B_{10}=f_4 \\ A_{20}=f_5 & B_{11}=f_0 & B_{02}=\frac{f_6}{g_1} & B_{20}=\frac{f_7}{g_1}, \end{array} \quad (23)$$

where $f_0 = 0$, $f_1 = t$, $f_2 = u_1$, $f_3 = u_2$, $f_4 = u_3$, $f_5 = u_4$, $f_6 = -2u_1u_2u_3$, $f_7 = 2u_2u_3u_4$ and $g_1=u_2^2-u_3^2$. To prove our claim we construct an ideal I_e by eliminating the variables w, t, u_1, u_2, u_3, u_4 from the ideal

$$\langle 1-wg_1, A_{11}-f_1, A_{02}-f_2, A_{10}-f_3, B_{10}-f_4, A_{20}-f_5, g_1B_{02}-f_6, g_1B_{20}-f_7\rangle$$

in the ring

$$\mathbb{R}[w, t, u_1, u_2, u_3, u_4, A_{02}, A_{10}, A_{11}, A_{20}, B_{20}, B_{10}, B_{02}].$$

We note that $I_e = G_1$. Therefore, by Theorem 2 of [6] we get that (23) is a rational parametrization of G_1; and dim $\mathbf{V}(G_1) \le 5$, which proves our claim. Now, since the rank of the Jacobian of $f_0, \ldots, f_7$ at the point $t = 0$, $u_1 = 1$, $u_2 = -1$, $u_3 = 2$, $u_4 = 3$ is five, we obtain that, in fact, dim $\mathbf{V}(G_1) = 5$, which completes the proof of our theorem.

We define the following polynomials:

$$\begin{aligned} F_1 &= A_{11}(B_{02}-B_{20})(A_{10}^2+B_{10}^2) \\ F_2 &= -A_{11}(2A_{02}A_{10}B_{10}+(B_{10}^2-A_{10}^2)B_{20}) \\ F_3 &= (2A_{10}A_{20}B_{10}-A_{10}^2B_{20}+B_{10}^2B_{20})^2(A_{20}^2+B_{20}^2). \end{aligned}$$

Theorem 5.3. *The cyclicity of a generic point p of* $\mathbf{V}(G_1)$ *with* $F_1(p) \ne 0$ *and of point* p' of $\mathbf{V}(G_2)$ *with* $F_2(p') \ne 0$ *is three. The cyclicity of a point* p'' *of* $\mathbf{V}(G_3)$ *with* $F_3(p'') \ne 0$ *is four.*

Proof. We find that for $\mathbf{V}(G_1)$, rank $J_p^3=3$ at p with $F_1(p) \ne 0$ and similarly for $\mathbf{V}(G_2)$ we obtain rank $J_{p'}^3=3$ at the point p' with $F_2(p') \ne 0$. Therefore, by Theorem 5.1 three limit cycles bifurcate from the origin for the systems corresponding to p and p', respectively. For $\mathbf{V}(G_3)$ we have rank $J_{p''}^4=4$ at the point p'' with $F_3(p'') \ne 0$: Since the codimension of $\mathbf{V}(G_3)$ is 4, by Theorem 5.1 the cyclicity for the system corresponding to p'' is 4.

Acknowledgments

B. Ferčec is supported by the Slovenian Research Agency; A.M. is partially supported by NIH-NIGMS grant 1P50GM094503-01A0. The authors thank Prof. Colin Christopher who suggested how to prove the second case of Theorem 3.2 as well as Prof. Valery Romanovsky and Prof. Douglas Shafer for fruitful discussions. Most of the paper was written when B.F. was visiting the Department of Mathematics at UNC–Charlotte. The authors are also grateful to the referees for their useful comments, which helped to improve the presentation of this work.

Appendix A

The first four focus quantities for the system (3) are as follows:

$g_{11} = b_{11} - a_{11},$

$g_{22} = -a_{02}a_{20} + 6a_{10}a_{11}b_{01} - 6a_{10}b_{01}b_{11} + b_{02}b_{20},$

$$\begin{aligned} g_{33}= & -15a_{02}a_{10}^2a_{11} \\ & -7a_{02}a_{11}a_{20} \\ & +24a_{10}a_{11}^2b_{01} \\ & +20a_{02}a_{10}a_{20}b_{01} \\ & -108a_{10}^2a_{11}b_{01}^2 \\ & -4a_{11}a_{20}b_{01}^2 \\ & -3a_{10}^2a_{11}b_{02} \\ & +4a_{11}a_{20}b_{02} \\ & +16a_{02}a_{10}^2b_{11} \\ & +5a_{02}a_{20}b_{11} \\ & +108a_{10}^2b_{01}^2b_{11} \\ & +3a_{20}b_{01}^2b_{11} \\ & +4a_{10}^2b_{02}b_{11} \\ & -4a_{20}b_{02}b_{11} \\ & -24a_{10}b_{01}b_{11}^2 \\ & +8a_{02}a_{11}b_{20} \\ & -16a_{11}b_{01}^2b_{20} \\ & -5a_{11}b_{02}b_{20} \\ & -20a_{10}b_{01}b_{02}b_{20} \\ & -8a_{02}b_{11}b_{20} \\ & +15b_{01}^2b_{11}b_{20} \\ & +7b_{02}b_{11}b_{20} \qquad , \end{aligned}$$

.

Appendix B

The first four focus quantities (each reduced modulo the ideal generated by the previous ones and up to a constant) are as follows:

$g_{11}^F = -c_1 + c_2,$

$g_{22}^F = 6c_1c_9 - 6c_2c_9 - c_3 + c_4 \; g_{22}^F = 6c_1c_9 - 6c_2c_9 - c_3 + c_4,$

$$
\begin{aligned}
g_{33}^{F}=12c_1^2c_9 \\
& -12c_2^2c_9 \\
& -54c_1c_9^2 \\
& +54c_2c_9^2 \\
& -7/2c_1c_3 \\
& +5/2c_2c_3 \\
& -5/2c_1c_4 \\
& +7/2c_2c_4 \\
& -3/2c_1c_5 \\
& +2c_2c_5 \\
& -2c_1c_6 \\
& +3/2c_2c_6 \\
& -15/2c_1c_7 \\
& +8c_2c_7 \\
& -8c_1c_8 \\
& +15/2c_2c_8 \\
& +10c_3c_9 \\
& -10c_4c_9 \\
& +2c_1c_{10} \\
& -2c_2c_{10} \\
& +4c_1c_{12}-4c_2c_{12},
\end{aligned}
$$

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