Measurable Systems and Behavioral Sciences[*]

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Abstract

Individual choices often depend on the order in which the decisions are made. In this paper, we expose a general theory of measurable systems (an example of which is an individual's preferences) allowing for incompatible (non-commuting) measurements. The basic concepts are illustrated in an example of non-classical rational choice. We conclude with a discussion of some of the basic properties of non-classical systems in the context of social sciences. In particular, we argue that the distinctive feature of non-classical systems translates into a formulation of bounded rationality.

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1 Introduction

In economics, an agent is defined by her preferences and beliefs, in psychology by her values, attitudes and feelings. One also talks about "eliciting" or "revealing" preferences and attitudes. This tacitly presumes that those properties are sufficiently well-defined (determined) and stable. In particular, it is assumed that the mere fact of subjecting a person to an elicitation procedure, i.e., to "measure" her taste does not affect the taste. Yet, psychologists are well aware that simply answering a question about a feeling may modify a person's state

of mind. For instance when asking a person "Do you feel angry?" a "yes" answer may take her from a blended emotional state to an experience of anger. But before answering the question, it may be neither true nor false that the person was angry. It may be a "jumble of emotions"[25]. Similarly, Erev, Bornstein and Wallsten (1993) show in an experiment that simply asking people to state the subjective probability they assign to some event affects the way they make subsequent decisions. The so-called "disjunction effect" (Tversky and Shafir (1992)) may also be viewed in this perspective. In a well-known experiment, the authors find that significantly more students report they would buy a non-refundable Hawai vacation if they knew whether they passed the exam or failed compared to when they don't know the outcome of the examination. In the case they passed, some buy the vacation to reward themselves. In the case they failed, some purchase the vacation to console themselves. When they don't know, a seemingly inconsistent behavior is observed: fewer vacations are being purchased than in any one of the two possible events.

In the examples above, the mere fact of subjecting an agent to a procedure that reveals her feeling, preferences or beliefs seems to affect her. In this paper, we propose to adopt a measurement theoretical approach to behavior: actual behavior reveals preferences (or beliefs) in the sense of being the outcome of a measurement of those preferences. Interestingly, Kahneman and A. Tversky explicitly discuss some behavioral anomalies in terms of measurement theory: *"Analogously, - to classical physical measurement - the classical theory of preference assumes that each individual has a well-defined preference order and that different methods of elicitation produce the same ordering of options"*. But, *"In these situations - of violation of procedural invariance - observed preferences are not simply read off from some master list; they are actually constructed in the elicitation process."* ([12] p. 504). A. Sen [24] also emphasizes that the "act of choice" has implications for preferences. In this work we adopt the view that performing a measurement on a system generally changes its state. In particular, an experiment or a decision situation that reveals a person's preferences affects that person's preferences.

Is it possible to build a predictive model of a system whose state changes as we perform measurements on it? We assert that it is if the interaction between systems and measurement instruments satisfies some natural conditions. We formulate them as axioms and show that the state space is endowed with the structure of an atomistic orthomodular orthospace and the states are realized as probability measures on the state space.

Of course, our formalization does not build on an empty spot. The question of modeling a system that changes when being measured is at the heart of Quantum Mechanics (QM). Birkhoff and von Neumann's seminal article from 1936 initiated a rich literature on the mathematical foundations of QM. For an excellent review of the field see the introductory chapter in Coecke, Moore and Wilce (2000). Recently, the interest for QM has been rapidly expanding to other fields. Partly, this is due to the development of quantum computing, which inspires physicists and more recently economists to investigate the use of quantum information in games (Eisert (1999), La Mura (2004)). Another avenue of research has emerged in response to observations that classical (or macro) objects (e.g. human perception or preferences) can exhibit properties specific to QM-objects. In Lambert-Mogiliansky, Zamir and Zwirn (2003), a Hilbert space model is proposed to describe economic agents' preferences and decision-making. Aerts (1994), Busemeyer and Townsend (2004) and Khrenikov et al. (2003) investigate quantum-like phenomena in psychology. The basic idea is that the mathematical formalism of QM, often referred to as "quantum logic" rather than its physical content, is a suitable model for describing, explaining and predicting human behavioral phenomena in psychology and social sciences.

In this paper we expose the foundations of a general measurement theory. The objective with the proposed formulation is to allow assessing the relevance of this framework for social sciences including for the analysis of individual choice and in particular for modelling bounded rationality.

Section 2 offers a few examples of quantum and quantum-like behavior. In Section 3 we introduce basic notions of measurement theory, namely that of measurement and of state. They are illustrated in models of rational choice in Section 4. Axioms and their consequences are exposed in Sections 5 and 6. Section 7 discusses an interpretation of the basic axioms and properties for behavioral sciences.

2 Examples

Example 1: *The spin of an electron*

An electron is endowed with several characteristics including the spin. The spin is an intrinsic property of any particle and corresponds to a magnetic moment which can be

measured.[1]

It is well-known that the outcome of the measurement is always $\pm 1/2$ (in some units) independently of the orientation of the measurement device. If we measure a concrete electron along some axis x and obtain result $+1/2$, then a new measurement along the same axis will give the same result. Assume we prepare a number of electrons this way. If we, for the second measurement, modify the orientation of the axis, e.g., the measurement device is turned by 90°, the result now shows equal probability for both outcomes. As we anew perform the measurement along the x-axis, we do not recover our initial result. Instead, the outcome will be $-1/2$ with .5 probability.

We limit ourselves to noting that once the spin of the electron along some axis is known, the results of the measurement of the spin of that electron along some other axis has a probabilistic character. This is a central feature. In the classic world, we are used to deal with probabilities. But there the explanation for the random character of the outcome is easily found. We simply do not know the exact state of the system, which we represent by a probability mixture of other states. If we sort out this mixture in the end we obtain a pure state and then the answer will be determinate. In the case with the spin, it is not possible to simultaneously eliminate randomness in the outcome of measurements relative to different axis.

Example 2: *A fly in a box*

Consider a box divided by two baffles into four rooms (left/front (LF), Left/Back (LB), RF and RB. In this box, we hold a fly that flies around. Because of the baffles, it is limited in its movements to the room where it is.

Assume that we only have access to two types of measurements. The first allows answering the question whether the fly is in the Left (L) or the Right (R) half of the box. And, in the process of measurement, the baffle between the Front (F) and the Back (B) half of the box is lifted while the separation between Right and Left is left in place. During that process, the fly flies back and forth from Front to Back. When the measurement operation is over and the baffle between Front and Back put back in place, the position of the fly is therefore quite random (LF or LB). The same applies for the measurement of Front/Back.

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[1]Stern and Gerlagh created an instrument such that the interaction between the magnetic moment of the electron and that of the experimental setup generates the splitting of a beam of electrons. A measure of the deviation can be interpreted of the measurement of spin (along some orientation).

Assume that we have performed the measurement L/R and obtained answer L. Repeating that same measurement even 100 times we will always obtain the same answer L. But if we do, in between, the F/B measurement, we have equal (for the sake of simplicity) chances to obtain R as L. We see that the behavior of our system reminds of that of the spin (when the Stern-Gerlach device is rotated by an angle of 90°). Here the position of the fly cannot be determined with certainty with respect to the two measurements (LR) and (FB) simultaneously. The measurement affects the system in an uncontrollable and unavoidable way. This simple example exhibits all basic features of the non-classical measurement theory developed in this paper.

Example 3: *Attitudes and preferences*

Consider the following situation. We are dealing with a group of individuals and we are interested in their preferences (or attitudes). We dispose of two tests.

The first test is a questionnaire corresponding to a Prisoners' Dilemma against an anonymous opponent. The options are cooperate (C) and defect (D). The second test corresponds to the first mover's choice in an Ultimatum Game (UG). The choice is between making an offer of (9,1) or of (4,6).[2]

The observations we are about to describe cannot be obtained in a world of rational agents whose preferences are fully described by their monetary payoff.[3] But this is not our point. Our point is that such observations exhibit the same patterns as the ones we described in the spin and fly example above.

Suppose that we have the following observations. The respondents who answer C to the first questionnaire repeat (with probability close to one) their answer when asked immediately once more. We now perform the second test (UG) and the first test (PD) again. In that last PD test we observe that not all respondents repeat their initial answers. A (significant) share of those who previously chose to cooperate now chooses to defect.[4]

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[2]In the Ultimatum game the first mover makes an offer. The respondent either accepts the deal and the payoffs are distributed accordingly. Or he refuses in which case no one receives any payoff.

[3]Game Theory uniquely predicts behavior: people defect (D) in the PD and (with common knowledge of rationality) they offer (9,1) in UG. Experimentalists have however taught us to distinguish between monetary payoffs, which can be controlled and preferences, which may include features beside monetary payoffs unknown to the designer of the experiment.

[4]D. Balkenberg and T. Kapplan (University of Exeter, unpublished) conducted an experiment with those two same games but with two populations of respondents. They investigate the frequency of the choices when the two games are played in one order compared to when they are played in the reverse order. The

How do we understand this kind of behavior? When deciding in the PD our respondent may feel conflicted: she wants to give trust and encourage cooperation, but she does not like to be taken advantage of. Consider the case when her optimistic 'I' takes over: she decides to cooperate. When asked again immediately after, her state of mind is that of the optimistic 'I' so she feels no conflict: she confirms her first choice. Now she considers the UG. The deal (4,6) is very generous but it may be perceived as plain stupid. The (9,1) offer is not generous but given the alternative it should not be perceived as insulting. She feels conflicted again because her optimistic 'I' does not provide clear guidance. Assume she chooses (9,1). Now considering the Prisoners' Dilemma again, she feels conflicted anew. Indeed, her choice of (9,1) is not in line with the earlier optimistic mood so she may now choose to defect.[5]

As in the spin and the fly example, the measurement (elicitation of preferences) affects the agent in an uncontrollable way so the observed behavior (measurement outcomes) may exhibit instances characteristic for quantum-like systems.

3   Measurements and states

In this section we introduce and discuss two basic concepts of the theory, namely the concepts of measurement and of state.

3.1   Measurements

A system is anything that we can perform *measurements* on. A measurement is an interaction between a system and some measurement device, which yields some result, the *outcome of the measurement* that we can observe and record. The set of possible outcomes of a measurement M is denoted $O(M)$. For instance in the case with the Stern-Gerlach experimental setup, we let the electron travel through a non-homogeneous magnetic field and observe deviation either up or down. In the example with the fly we lift up a baffle and observe in which half of the box the fly is located. In our third example, we let people play the Prisoner Dilemma (and the UG) and observe their choice.

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data shows an impact of the first choice on the second which is characteristic of non-classical measurements.

[5]We do not in any manner mean that the proposed description in terms of inner conflict is the only possible one. A variety of psychological stories are consistent with such phenomena of non-commutativity.

First-kindness

Measurements constitute a special class of interactions. We focus on non-destructive measurements, which means that the system is not destroyed in the process of measurement so we can perform new measurements on the system. In particular, we can perform a measurement M twice in a row. If the outcomes of the two measurements always coincide, we say that the measurement M is a *first-kind* measurement.[6]

In other words the results of a first-kind measurement are repeatable (reproducible). This is a very important point that deserves some additional comments. One may wonder why a "measurement" would fail to satisfy the property of first-kindness. There are several reasons for that. A first and most important reason is that the system is evolving. For instance, the thirst of a person running a marathon is not the same from one time to another along the race. In this paper we focus on systems that do not have an own dynamics (or alternatively on situations where measurements are made so close in time that we can disregard the own dynamics). A second reason for failing first-kindness is noise in the measurement instrument itself. We shall assume that measurements do not bring in own uncertainty. A third reason is that the measurement operation actually is a combination of *incompatible* measurements. We return to this point soon.

In what follows, we assume that all measurements are first-kind. Indeed, if a measurement is not first-kind it is unclear what we measure and what the relation is between the outcome of the measurement and our system. Of course, the question about first-kindness of any concrete measurement is an experimental one.

Compatibility

Two measurements are *compatible* if they, roughly speaking, can be performed simultaneously or more precisely, if the performance of one measurement does not affect the result of the other. Suppose that the first measurement gave outcome o; then we perform the second measurement and the first one anew. In case we are dealing with compatible measurements we obtain outcome o with certainty.

Given two compatible measurements M and N we can construct a third finer measurement. We may perform M and thereafter N and view this as a new (compound) measurement

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[6]The term "first-kind" measurement was proposed by W. Pauli. J. von Neumann used the following formulation: "If the physical quantity is measured twice in succession on a system S then we get the same value each time."

$M * N$ with outcome set $O(M) \times O(N)$. Because of compatibility, the measurement $M * N$ is a first-kind measurement.

If all measurements are compatible we can substitute them with a single finest (complete) measurement, which is also first-kind. Performing that measurement we learn everything about the system. Such a system is *classical*.

The existence of incompatible measurements is a distinctive feature of non-classical systems. It is closely related to the impact of measurements on the state and the existence of "dispersed" states (see next subsection).

In the examples of Section 2 all measurements were incompatible.

3.2 States

Measurable systems

As we perform a measurement and observe its result we learn something about a system. All the information that we have about a system is "encapsulated" in the *state* of the system. The state is the result of past measurements and it is the basis for making predictions of future measurements. A theory (or a model) of a system should describe the set of states, the results of any measurement in every state and the change in the state induced by any measurement.[7]

The state of a system predicts the result of any measurement. But we do not assume that it predicts a unique outcome. We only assume that the state determines the probabilities for the outcomes, that is it determines a random outcome.

In order to avoid technical subtleties associated with the notion of probability, we shall in what follows assume that the sets $O(M)$ are finite. In such a case, a probabilistic measure (or a random element) on $O(M)$ is a collection of non-negative numbers (probabilities) $\mu(o)$ for each $o \in O(M)$ subjected to the condition $\sum_{o \in O(M)} \mu(o) = 1$. The set (a simplex indeed) of probabilistic measures on $O(M)$ is denoted $\Delta(O(M))$. In such a way the state s defines a random outcome in $O(M)$, that is a point $\mu_M(s) \in \Delta(O(M))$ for every measurement M.

Of course, the random outcome $\mu_M(s)$ can be degenerated, that is $\mu_M(o|s) = 1$ for some outcome $o \in O(M)$. In the general case, the outcome is random; moreover, we are interested in systems with "intrinsic uncertainty". We return to this central point later, for now we

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[7]If the system has an own dynamic the model should be enriched with a description of its evolution over time.

note that in the general case measurements impact on (change) the state. Indeed, let s be a state such that the outcome of a measurement M is not uniquely determined. After having performed measurement M (and obtained outcome o) the new state s' of the system differs from s because (according to the first-kindness of M) now the result of M is uniquely determined and equal to o.

Definition. A *measurable system* is a system equipped with a set $\mathcal{M}$ of first-kind measurements. A *model* of a measurable system includes the following three collections of data:

1) a set of *states* $\mathbb{S}$;

2) an *outcome mapping*, $\mu_M : \mathbb{S} \to \Delta(O(M))$ for every measurement $M \in \mathcal{M}$;

3) a *transition mapping*, $\tau_{M,o} : \mathbb{S} \to \mathbb{S}$ for every measurement $M \in \mathcal{M}$ and any of its outcome $o \in O(M)$.

The first mapping defines the probabilities for the possible outcomes when performing measurement M in an arbitrary state s. The second mapping $\tau_{M,o}$ points out where the state s goes (transits) as we perform measurement M and obtain outcome $o \in O(M)$. We have to recognize that the mappings $\tau_{M,o}$ are not defined for those states in which the outcome o is impossible.

It is useful at this point to introduce a few notions that we also use later. Let M be a measurement and $A \subset O(M)$. Denote

$$E_M(A) = \{s \in \mathbb{S}, \ \mu_M(A|s) := \sum_{o \in A} \mu_M(o|s) = 1\}.$$

The set $E_M(A)$ consists of the states endowed with the following property: the result of the measurement M belongs to A for sure. The set $E_M(o)$ for $o \in O(M)$ is called the *eigenset* of measurement M corresponding to outcome o.

In these terms the mapping $\tau_{M,o}$ is not defined on the subset $E_M(O(M) \setminus \{o\})$, where o can not be an outcome of M. The image of $\tau_{M,o}$ coincides with the eigenset $E_M(o)$.

Pure states

Although it is not necessary, we shall suppose that two states coincide if all their predictions are the same. (Here we follow Mackey: "A state is a possible simultaneous set of statistical distributions of the observables."). In that case, we can consider the set $\mathbb{S}$ as some subset of the convex set $\times_{M \in \mathcal{M}} \Delta(O(M))$.

This allows to speak about mixtures of states. A state σ is called a (convex or probabilistic) *mixture* of states s and t with (non-negative) weights α and $1 - \alpha$, if

$$\mu_M(o|\sigma) = \alpha\mu_M(o|s) + (1 - \alpha)\mu_M(o|t)$$

for any $M \in \mathcal{M}$ and any $o \in O(M)$. Mixtures of three or more states are defined similarly. A state is said to be *pure* if it is not a non-trivial mixture of other states.

Without loss of generality one can suppose that the set of states $\mathbb{S}$ is convex (as a subset of $\times_{M \in \mathcal{M}} \Delta(O(M))$). The subset $\mathbb{P}$ of pure states is the set of extreme points of the convex set $\mathbb{S}$, $\mathbb{P} = \text{ext}(\mathbb{S})$. In the sequel we assume that $\mathbb{S}$ is the convex hull of $\mathbb{P}$, $\mathbb{S} = \text{co}(\mathbb{P})$. Moreover, it is quite natural to assume that the transition mappings are linear (i.e., are compatible with the convex structure on S). For this reason we can work with the set of pure states $\mathbb{P}$ instead of $\mathbb{S}$. Of course, we should keep in the mind that the transition state $\tau_{M,o}(s)$ can be mixed.

In the classical world, pure states are *dispersion-free*, that is the outcome of any measurement performed on a system in a pure state is uniquely determined. Randomness in the results of a measurement indicates that the system is in a mixed state. One can sort out (or filter) this mixture by making measurements so as to eventually obtain a pure state.

A distinctive feature of non-classical systems is the existence of dispersed (that is non dispersion-free) pure states. This feature can be called "intrinsic uncertainty". It is closely related to two other properties of non-classical systems: the existence of incompatible measurements and the impact of measurements on states. If a state is dispersion-free i.e., the outcome of every possible measurement is uniquely determined, there is no reason for the state to change. If all pure states are dispersion-free then measurements do not impact on pure states and therefore all measurements are compatible. On the contrary, if a state is dispersed then by necessity it will be modified by an appropriate measurement. On the other hand, the change in a pure state is the reason for incompatibility of measurements. The initial outcome of the first measurement M is not repeated because the system has been modified by the second measurement N.

4   An illustration: non-classical rational choice

Let us illustrate the above introduced notions in (thought) examples of non-classical rational choice behavior.

We shall consider a situation where an agent is making a choice out of a set of alternatives X. A primitive measurement is a choice from a subset $A \subset X$; the set of outcomes of the measurement is A (i.e. we consider single-valued choices). For this reason we denote such a choice-measurement A. A main idea is that a choice out of "small" subsets is well-defined and rational. By well-defined we mean that the corresponding measurement is first-kind. By rational we mean that consecutive choices from "small" subsets satisfy Houthakker's axiom (or the principle of independence of irrelevant alternatives, IIA). Our motivation is that an agent may, in his mind, structure any "small" set of alternatives, i.e., he is capable of simultaneously comparing those alternatives. He may not be able to do that within a "big" set (which we interpret as bounded rationality, see Section 7). This does not means that our agent cannot make a choice from a "big" set. For example, he might use an appropriate sequence of binary comparisons and choose the last winning alternative. However, such a compound choice-measurement would not in general be first-kind.

We formulate Houthakker's axiom in the following way. Let A and B be two "small" subset, and $A \subset B$. In our context, Houthakker's axiom consists of two parts:

1) Suppose the agent chooses from B an element a which also belongs to A. If the consecutive measurement is A then the agent chooses a.

2) Suppose the agent chooses from A an element a. If the consecutive measurement is B then the outcome of the choice is not in $A \setminus \{a\}$.

In order to be more concrete, we shall consider as "small" subsets of the size 3 or less.

We begin with the case when all choice-measurements are compatible. Performing binary choices we obtain some binary relation $\prec$ on X. From ternary choices we see that this relation is transitive so that the relation $\prec$ is a linear order. Therefore, it is natural to identify the set $\mathbb{P}$ of pure states with the set of linear orders on X. We obtain the well-known classical model.

We now relax the assumption of compatibility of all choice-measurements and consider three different models.

Model 1. X includes three alternatives a, b and c. To define a model we need to define the set of states, the outcome and the transition mappings.

Let the set of pure states $\mathbb{P}$ consists of three states denoted $[a]$, $[b]$ and $[c]$. When the agent is asked to choose an item out of X he chooses a in state $[a]$, b in $[b]$, and c in $[c]$.

The choice-measurement X does not change the state. When the agent is asked to choose an item out of $\{a, b\}$, a choice-measurement that we denote by ab, he chooses a in the state $[a]$, b in $[b]$; in the state $[c]$ he chooses a and b with equal chances (and transits into $[a]$ or $[b]$ correspondingly). Symmetrically for the choice-measurements ac and bc.

It is clear that Houthakker's axiom is satisfied in this model. Note that the choice measurements are incompatible. Indeed, let for instance the agent be in the state $[c]$ and choose c out of X. We then ask her to choose out of $\{a, b\}$. After that choice-measurement she chooses a or b out of X but not c.

Model 2. The set of alternatives X is the same as in the previous model. But the set of states differs. Now we identify (pure) states with outcomes of our four choice-measurements ab, ac, bc, abc. That is $\mathbb{P} = \{\underline{a}b, \ a\underline{b}, \ \underline{a}c, \ a\underline{c}, \ \underline{b}c, \ b\underline{c}, \ \underline{a}bc, \ a\underline{b}c, \ ab\underline{c}\}$. Here $\underline{a}c$ denotes the choice of item a out of $\{a, c\}$ and so on.

To define the model we have to specify the outcomes of the measurements and the corresponding state transition.

Let the state be $a\underline{b}c$. The outcome of measurement abc is obvious, as well as that of measurements ab and bc (by Houthakker's axiom). The corresponding new states are $a\underline{b}c$, $a\underline{b}$ and $\underline{b}c$ respectively. But what about choice-measurement ac? We assume that (with equal chances) the new state is $\underline{a}c$ or $a\underline{c}$.

Let now the state be $a\underline{b}$. By definition b is the outcome of measurement ab and the state does not change. Suppose that we perform the choice-measurement . We assume that the new state is $ab\underline{c}$ with probability $1/3$ and is $a\underline{b}c$ with probability $2/3$. Houthakker's axiom says that a cannot be the outcome the abc measurement. Suppose that the measurement ac is performed. The new state is $a\underline{c}$ with probability $2/3$ and $\underline{a}c$ with probability $1/3$. Similarly, if we perform measurement bc we obtain c with probability $1/3$ and b with probability $2/3$.

The outcomes and the transitions into other states are defined symmetrically. This completes the definition of our model which obviously satisfies the Houthakker axiom. Model 2 describes a rational (non-classical) choice behavior as does Model 1. Yet, the pure states cannot be identified with orderings of the alternatives as in the classical model.

Clearly, the eigensets of the measurement abc are the one-element sets $\{\underline{a}bc\}$, $\{a\underline{b}c\}$ and $\{ab\underline{c}\}$. The eigensets of measurement ab look more interesting. They are the two-elements sets $\{\underline{a}b, \underline{a}bc\}$ and $\{a\underline{b}, a\underline{b}c\}$. The eigensets of the measurements bc and ac are defined similarly. Here is the full list of the properties:

a) 3 one-element subsets: $\{\underline{a}bc\}$, $\{a\underline{b}c\}$, $\{ab\underline{c}\}$ (represented by the nodes of the second line from below in the lattice below);

b) 6 two-element subsets: $\{\underline{a}b, \underline{a}bc\}$, $\{a\underline{b}, a\underline{b}c\}$, $\{\underline{a}c, \underline{a}bc\}$, $\{a\underline{c}, ab\underline{c}\}$, $\{\underline{b}c, a\underline{b}c\}$, $\{b\underline{c}, ab\underline{c}\}$ (they are represented by the nodes of the third line from below in the lattice);

c) 3 four-element subsets: $\{\underline{a}bc, a\underline{b}c, \underline{a}c, \underline{b}c\}$, $\{\underline{a}bc, ab\underline{c}, \underline{a}b, b\underline{c}\}$, $\{a\underline{b}c, ab\underline{c}, a\underline{c}, a\underline{b}\}$ (represented by the nodes of the fourth line of the lattice);

d) the empty set $\emptyset$ and the whole set $\mathbb{P}$ (respectively the bottom and the upper node in the lattice).

Note that the intersection of properties is a property as well. The lattice of the properties is drawn below:
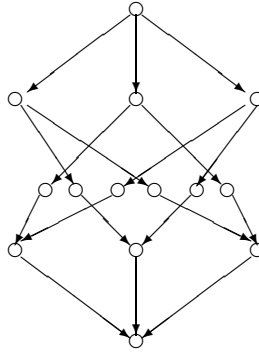

Figure 1

We note also that the lattice is not atomistic (not all elements can be written as the join of atoms).

Model 3. Here the set X consists of four items: a, b, c and d. For any $A \subset X$ with 3 or 2 elements the corresponding choice-measurement A is first-kind. We assume as before that Houthakker's axiom holds in any two consecutive choice-measurements. In addition, we assume that choices out of A and B are compatible if $A \cup B \neq X$. Thus, our agent is "more classical" than in Model 2.

As in the classical case we can perform binary and ternary measurements on each triple of items taken separately and reveal a linear order on that triple. It is therefore natural to identify the set of pure states with the collection of those linear orders. There are 24 such orders: $[a > b > c]$, $[c > d > a]$, and so on.

Suppose that the state is $[a > b > c]$.

1) If $A \subset \{a, b, c\}$ then the outcome of choice-measurement A is determined by the order $a > b > c$; the measurement does not change the state.

2) If we perform measurement A with outcome set $\{a, b, d\}$, the new state will be $[a > b > d]$, $[a > d > b]$ or $[d > a > b]$ with equal chances; the outcomes are a, a and d correspondingly. For $A = \{a, c, d\}$ or $\{b, c, d\}$ the new states are defined similarly.

3) If we perform measurement A with outcome set $\{a, d\}$, the new state can be one of $[a > b > d]$, $[a > c > d]$, $[a > d > b]$, $[a > d > c]$, $[d > a > b]$, and $[d > a > c]$ with equal chances. In the first four cases the outcome is a; in the two last cases the outcome is d. Similarly for $A = \{b, d\}$ and $A = \{c, d\}$.

The eigenset of the measurement abc corresponding to a is $\{[a > b > c], [a > c > b]\}$. The eigenset of the measurement ab corresponding to a is $\{[a > b > c]$, $[a > c > b]$, $[a > b > d]$, $[a > d > b]$, $[c > a > b]$, $[d > a > b]\}$.

Consider now a choice out of the set $\{a, b, c, d\}$. It is not a first-kind choice-measurement. Here many scenarios are possible. We shall assume the agent proceeds by making two (first-kind) measurements. First she chooses from a pair then from the triple consisting of the first selected item and the two remaining items. We can call this behavior "procedural rational" because the agent proceeds as if she had preferences over the 4 items.

Let us consider the following scenario. Assume that the agent just made a choice in abc and that the outcome was $ab\underline{c}$ so the state belongs to $\{[c > b > a], [c > a > b]\}$. Suppose that when confronted with $abcd$ the agent follows the following procedure she performs the measurement ab and then the measurement bcd (this means that the first outcome is $a\underline{b}$). After the first measurement, the state can be anyone of $[c > b > a]$, $[b > a > d]$, $[b > d > a]$ and $[d > b > a]$. Therefore the outcome of bcd can be b. This violates of the principle of independence of irrelevant alternative (IIA) and demonstrates preference reversal.[8]

The examples above demonstrate that there can be many different models of one and same measurable system. Which of them is the correct one? It is an empirical question. We also see from this illustration that as we consider the possibility of incompatible choice-measurements on subsets of X, the behavior of a non-classical rational man differs from that of a classical rational man in ways that can accommodate behavioral anomalies.

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[8]In models 1 and 2 we could also obtain a phenomena of preference reversal but not in two consecutive choices. Model 3 allows for that because the choice out of four items is a compound measurement.

5 Basic structure on the state space

In this section we go further with the formal investigation. We show that if the model of a measurable system $(\mathcal{M}, \mathbb{S}, \mu, \tau)$ is endowed with some additional properties (that we formulate as axioms) then the set of states is equipped with the structure of an orthomodular ortho-separable orthospace.

5.1 Properties

We remind that, for $M \in \mathcal{M}$ and $A \subset O(M)$, we introduced the set $E_M(A)$ as the set of states s such that when performing measurement M on a system in state s, the result of the measurement belongs to A for sure. The sets of the form $E_M(A)$ are called *properties* of our system, namely the property to imply A when performing measurement M.

Different instruments measure different properties of a system. But it may happen that one and the same property can be measured by several instruments. We require that in such a case the probability for the property does not depend on the instrument. We note that Model 1 from Section 4 does not meet that requirement. Indeed, property $[a]$ can be obtained by measurements abc and ab. But in state $[c]$, outcome a never obtains with the first instrument while a obtains with probability $1/2$ with the second instrument. Yet, this is a very reasonable requirement and we impose the following (slightly stronger) monotonicity condition:

Axiom 1 *Let M and M' be two measurements, let $A \subset O(M)$, and $A' \subset O(M')$. Suppose that $E_M(A) \subset E_{M'}(A')$. Then $\mu_M(A|s) \leq \mu_{M'}(A'|s)$ for any state s.*

In particular, if $P = E_M(A) = E_{M'}(A')$ is a property then the probabilities $\mu_M(A|s)$ and $\mu_{M'}(A'|s)$ are equal and depend only on the property P. We denote this number as $s(P)$ and understand it as the probability to obtain property P when performing a suitable measurement of the system in state s. By definition we have

$$s(P) = 1 \Leftrightarrow s \in P.$$

The set of all properties is denoted by $\mathcal{P}$. As a subset of $2^{\mathbb{S}}$, $\mathcal{P}$ is a poset (a partially ordered set). It has a minimal element $\mathbf{0} = \emptyset$ and a maximal element $\mathbf{1} = \mathbb{S}$. Any state $s \in \mathbb{S}$ defines the monotone function $s : \mathcal{P} \to \mathbb{R}_+$, $s(\mathbf{0}) = 0$, $s(\mathbf{1}) = 1$.

We now describe another basic structure on the property poset $\mathcal{P}$. Let $P = E_M(A)$ be a property. The subset $P^{op} = E_M(O(M) \setminus A)$ is a property too. We have $s(P^{op}) = 1 - s(P)$ for any state $s \in \mathbb{S}$. Thus

$$P^{op} = \{s \in \mathbb{S}, \ s(P^{op}) = 1\} = \{s \in \mathbb{S}, \ s(P) = 0\}$$

and the set-property P^{op} depends only on P. We call it the *opposite* property to P.

Lemma 1 *The poset $\mathcal{P}$ equipped with the operation ' is an orthoposet. That is the following assertions holds:*

1) If $P \subset Q$ then $Q^{op} \subset P^{op}$;

2) $P \cap P^{op} = \varnothing$ for every property P;

3) $P'' = P$ for every property P.

Proof. 1) Suppose $s \in Q^{op}$. Then $s\left(Q^{op}\right) = 1$ and $s\left(Q\right) = 1 - s\left(Q^{op}\right) = 0$. Since $P \subset Q$ by Axiom 1 we have that $s\left(P\right) = 0$ as well. Therefore $s\left(P^{op}\right) = 1 - s\left(P\right) = 1$ and $s \in P^{op}$. Assertions 2) and 3) are obvious. $\square$

From Lemma 1 we obtain that $\mathbf{1}$ is the only property which contains both P and P^{op}. Indeed if $P \cup P' \subset Q$ then by 1) $Q^{op} \subset P^{op} \cap P'' = \varnothing$, $Q^{op} = \mathbf{0}$ and $Q = \mathbf{1}$. In other words the supremum $P \vee P^{op}$ equals $\mathbf{1}$ in the *ortho-poset* $\mathcal{P}$. Note that in general $P \cup P^{op} \neq \mathbb{S}$.

5.2 Orthospaces

The set of states $\mathbb{S}$ possesses a similar orthogonality structure. We say that two states s and t are *orthogonal* (and write $s \perp t$) if there exists a property P such that $s(P) = 1$ and $t(P) = 0$. Since, for the opposite property P^{op}, it holds $s(P^{op}) = 0$ and $t(P^{op}) = 1$, we have $t \perp s$, so that $\perp$ is a symmetric relation on the set $\mathbb{S}$. Clearly, $\perp$ is an irreflexive relation.

This lead us to the following general notion.

Definition. A symmetric irreflexive binary relation $\perp$ on a set X is called an *orthogonality relation*. A set X equipped with an orthogonality relation $\perp$ is called an *orthospace*.

Example 4. Consider an Euclidean space H equipped with a scalar product (x, y). We say that vectors x and y are orthogonal if $(x, y) = 0$. The symmetry of the orthogonality relation follows from the symmetry of the scalar product. To obtain the irreflexivity we have to remove the null vector. So that $X = H \setminus \{0\}$ is an orthospace. A Hilbert space over

the field of complex numbers is another example. Such a model is standard for Quantum Mechanic.

When graphically representing an orthospace, one may connect orthogonal elements and obtain a (non-oriented) graph. This representation is often the most "economic", since few edges need to be written. Alternatively, one may connect non-orthogonal (or *tolerant*[9]) elements. The tolerance graph quickly becomes extremely complex. In simple cases, we combine the two representations. Dotted lines depict orthogonality and solid lines depict tolerantness. The graphs of Example 2 (A fly in a box) is in figure 2.

Figure 2

For $A \subset X$, we denote by $A^\perp$ the set of elements that are orthogonal to all elements of A,

$$A^\perp = \{x \in X, \ x \perp A\}.$$

For instance $\emptyset^\perp = X$ and $X^\perp = \emptyset$. If $A \subset B$ then $B^\perp \subset A^\perp$.

Definition. The subset $A^{\perp\perp}$ is called the *ortho-closure* of a subset A. A set F is said to be *ortho-closed* (also called a *flat*) if $F = F^{\perp\perp}$.

It is easily seen that for any $A \subset X$, the set $A^\perp$ is ortho-closed. Indeed, let $F = A^\perp$. Then $F \subset F^{\perp\perp}$. On the other side, $A \subset A^{\perp\perp} = F^\perp$; applying $\perp$ we reverse the inclusion relation $F^{\perp\perp} \subset A^\perp = F$. In particular, the ortho-closure of any A is ortho-closed.

Let $\mathcal{F}(X, \perp)$ denote the set of all flats of orthospace $(X, \perp)$ ordered by the set-theoretical inclusion, which we denote $\leq$. It contains the largest element X, denoted $\mathbf{1}$, and the smallest element $\emptyset$, denoted $\mathbf{0}$. Moreover the poset $\mathcal{F}(X, \perp)$ is a (complete) lattice. The intersection

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[9]The term "tolerant" is used in mathematics to refer to a symmetric and reflexive relation.

of two (or more) flats is a flat implying that $A \wedge B$ exists and equals $A \cap B$. The join $A \vee B$ also exists and is given by the formula

$$A \vee B = (A \cup B)^{\perp\perp}.$$

Definition. An *ortholattice* is a lattice equipped with a mapping $\perp \colon \mathcal{F} \to \mathcal{F}$ such that

i. $x = x^{\perp\perp}$;

ii. $x \leq y$ if and only if $y^\perp \leq x^\perp$;

iii. $x \vee x^\perp = \mathbf{1}$.

Thus the poset $\mathcal{F}(X, \perp)$ is an ortholattice.

5.3 The intersection axiom

We have associated to a measurable system two objects: the orthoposet of properties $\mathcal{P}$ and the ortholattice of flats $\mathcal{F}(\mathbb{S}, \perp)$. It is intuitively clear that these two objects are closely related. But for now we can only assert the following inclusion

$$P^{op} \subset P^\perp$$

for a property P. Indeed, by the definition of $\perp$, any element of P is orthogonal to any element of the opposite property P^{op}.

In order to go further we impose the following *intersection axiom*

Axiom 2 *The intersection of any properties is a property.*

Axiom 2 puts conditions on the set of measurements as it requires that if P and Q are two properties there must exist a measurement such that $P \cap Q$ is one of its eigensets. Axiom 2 is fulfilled in Models 2 and 3 from Section 4. As we shall see this Axiom implies that properties and flats are the same. To prove this we first get a few consequences of Axiom 2.

Lemma 2 $P^{op} = P^\perp$ *for any property P.*

Proof. Since $P^{op} \subset \mathbf{P}^\perp$, we have to check the opposite inclusion $P^\perp \subset P^{op}$. Let t be an arbitrary element of $P^\perp$, and let s be an arbitrary element of P. Since $t \perp s$, then by the definition of $\perp$ there exists a property E_s such that $t \in E_s$ and $s \in E_s^{op}$. Set $E = \cap_{s \in P} E_s$; by Axiom 2 E is a property. Since $E \subset E_s$ for any s, we have $E_s^{op} \subset E^{op}$. Together with $s \in E_s^{op}$ we obtain that $P \subset E^{op}$. Therefore $E \subset P^{op}$, and we have that $t \in \cap_s E_s = E \subset P^{op}$. $\square$

In particular, any property $P = (P^{op})^{op} = (P^{op})^{\perp}$ is orthoclosed. Hence the inclusion $\mathcal{P} \subset \mathcal{F}$ holds. We now prove the inverse inclusion, that is any flat is a property. We first establish this faci for flats of the form $\{s\}^{\perp}$, where s is a state. Let $P(s)$ denote the least property containing the state s, that is intersection of all properties containing the state s.

Lemma 3 $s^{\perp} = P(s)^{op}$.

Proof. Since $s \in P(s)$, we have that $P(s)^{op} = P(s)^{\perp}$ (by Lemma 1) is contained in $\{s\}^{\perp}$. In order to check the reverse inclusion, we consider an arbitrary element t of $\{s\}^{\perp}$. By definition this means that $s \in E$ and $t \in E^{op}$ for some property E. Since $P(s)$ is the minimal property containing s, we have $P(s) \subset E$. Hence $E^{op} \subset P(s)^{op}$ and $t \in P(s)^{op}$. This prove the inclusion $\{s\}^{\perp} \subset P(s)^{op}$. □

In particular, flats of the form $\{s\}^{\perp}$ are properties. Since any flat is an intersection of subsets of the form $\{s\}^{\perp}$, from the intersection axiom we obtain that any flat is a property. Thus, we proved the following important theorem

Theorem 1 $\mathcal{P} = \mathcal{F}(\mathbb{S}, \perp)$.

From Theorem 1 we see that the orthoclosure $A^{\perp\perp}$ of a set A is the least property containing A. It consists of states having the properties that are common to all elements of A. The elements of $A^{\perp\perp}$ are also called *superpositions* of A. The following Proposition implies that any mixture of A is a superposition of A.

Proposition 1 *Suppose that a state s is the convex mixture of states $s_1, ..., s_n$ with strictly positive coefficients α_i. Then the orthoclosure of s is the same as the orthoclosure of $\{s_1, ..., s_n\}$.*

Proof. We have to show that s is endowed with property P if and only if $s_1, ..., s_n$ are endowed with property P. $s \in P \Leftrightarrow s(P) = 1$. But $s(P) = \sum_i \alpha_i s_i(P)$. Since $s_i(P) \leq 1$ and $\alpha_i > 0$ for all i, we have that $\sum_i \alpha_i s_i(P) = 1$ if and only if all $s_i(P) = 1$ i.e., if and only if $s_1, ..., s_n \in P$. □

Corollary. *The natural mapping $\mathcal{F}(\mathbb{P}, \perp) \to \mathcal{F}(\mathbb{S}, \perp)$, where $\mathbb{P}$ is the set of pure states with the induced orthogonality relation, is an isomorphism of ortholattices.*

For this reason we can work with the orthospace $\mathbb{P}$ of pure states holding in mind that mixtures are in principle possible.

5.4 Atomicity and the preparation axiom

A state s is an *atom* if the set $\{s\}$ is orthoclosed. By Proposition 1, any atom is a pure state. Model 2 from Section 4 shows that the inverse is not true. Nevertheless in the sequel we restrict our attention to systems in which any pure state is atom. We formulate this requirement in terms of measurements.

Axiom 3 *For any pure state $s \in \mathbb{P}$, there exists a measurement M such that $\{s\}$ is one of its eigensets.*

In other words, any pure state is fully characterized by its properties. Substantively (or operationally) it means that, given any state s, there exists an experimental set-up which can "prepare" the system in that state s. Axiom 3 is rather reasonable (it is fulfilled in Model 3 from Section 4) and we explore its consequences.

Let s and t be two pure states. Due to Axiom 3, the set $\{t\}$ is a property and therefore we can speak about $s(t) := s(\{t\})$, the probability for a transition from the state s to the state-property t.

Lemma 4 *Suppose Axiom 3 is fulfilled. Then $t(s) = 0$ if and only if $s \perp t$.*

Proof: Let us suppose that $s \perp t$ and let P be a property such that $s \in P$ and $t(P) = 0$. From the inclusion $\{s\} \subset P$ and the monotonicity axiom we have $t(\{s\}) = 0$. The converse assertion is more obvious because s belongs to the property $\{s\}$ on which t vanishes. $\square$

Corollary 1 $s(t) = 0$ if and only if $t(s) = 0$.

In the general case, $s(t)$ can differ from $t(s)$.

Axiom 3 implies the ortho-separability of the orthospace $\mathbb{P}$. Let us remind that an orthospace $(X, \perp)$ is called *ortho-separable* if any single-element subset $\{x\}$ of X is a flat. It is easy to check that $\{x\}$ is a flat if and only if for any $y \neq x$ there exists z orthogonal to x but not to y. For example, the orthospace in figure 3

Figure 3

is not ortho-separable, since $a^{\perp\perp} = \{a, c, d\} \neq \{a\}$.

It is worthwhile noting that any ortho-separable orthospace $(X, \perp)$ can be reconstructed from the ortho-lattice $\mathcal{F}(X, \perp)$. Recall that an *atom* of a lattice $\mathcal{F}$ is a minimal non-zero elements of $\mathcal{F}$. A lattice $\mathcal{F}$ is called *atomistic* if any element of the lattice is the join of atoms. If $(X, \perp)$ is an ortho-separable orthospace then $\mathcal{F}(X, \perp)$ is a complete atomistic ortho-lattice.

Conversely, let $\mathcal{F}$ be a complete atomistic ortho-lattice. Then, there exists an ortho-separable orthospace $(X, \perp)$ (unique up to isomorphism) such that $\mathcal{F}$ is isomorphic to $\mathcal{F}(X, \perp)$. One needs to take the set of atoms of $\mathcal{F}$ as the set X; atoms x and y are orthogonal if $x \leq y^{\perp}$. For more details see, for example [20]. Roughly speaking, ortho-separable orthospaces and atomistic ortholattices are equivalent objects.

5.5 Orthomodularity

It was early recognized that the failure of classical logic to accommodate quantum phenomena was due to the requirement that the lattice of properties should satisfy the distributivity law. Birkhoff and von Neumann [3] proposed to substitute the distributivity law by the modularity law. As it turned out, the weaker notion of orthomodularity proved to be more adequate, see [13].

Definition. An orthospace $(X, \perp)$ is said to be *orthomodular* if, for every two flats F and G such that $F \subset G$ there exists an element $x \in G$ which is orthogonal to F.

In other words, if x does not belong to a flat F then there exists a superposition of F and x which is orthogonal to F. Orthomodularity permits constructing orthogonal bases in

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the same way as in Euclidean spaces. An *orthobasis* of a flat F is a subset B of mutually orthogonal elements such that $F = B^{\perp\perp}$. In is easy to see that (for any orthospace) the maximal flat **1** has an orthobasis. If X is orthomodular then each flat has an orthobasis. More precisely, there holds

Lemma 5 *Let F be a flat in an orthomodular space, and $B' \subset F$ be a subset consisting of mutually orthogonal elements. Then, there exists an orthobasis B of F containing B'.*

Proof. Let B be a maximal (by inclusion) subset in F which contains B' and consists of mutually orthogonal elements. We claim that the orthoclosure of B coincides with F. In opposite case there exists an element x in F orthogonal to B. If we add x to B we extend B which contradicts the assumption of B being maximal. $\square$

In particular, beginning with the empty B' we can construct an orthobasis of any flats. Note, however, that ortobases of the same flat (even the maximal flat **1**) can have different numbers of elements.

The orthomodularity of space $(X, \perp)$ implies the orthomodularity of the corresponding ortho-lattice of flats $\mathcal{F}(X, \perp)$. Recall that an ortho-lattice $\mathcal{F}$ is called *orthomodular* if, for every its elements a and b such that $a \leq b$, the following equality holds

$$b = a \vee (b \wedge a^{\perp}).$$

Lemma 6 *An orthospace $(X, \perp)$ is orthomodular if and only if its ortholattice $F(X, \perp)$ is orthomodular.*

Proof. Let $(X, \perp)$ be orthomodular space and let F and G be two flats such that $F \subset G$. We have to show that $G = F \vee (G \wedge F^{\perp})$. Denote by G' the right hand side of the expression; it is clear that $G' \subset G$. If the inclusion is strict then G consists an element x orthogonal to G'. In particular, x is orthogonal to F, that is x belongs to $F^{\perp}$. Since x belongs to G as well then x belongs to $G \cap F^{\perp}$ and all the more to G'. But we obtain that x is orthogonal to x which contradicts the irreflexivity of the orthogonality relation $\perp$. Conversely, let the lattice $\mathcal{F}(X, \perp)$ be orthomodular and $F \subset G$ be two different flats. Since $G = F \vee (G \cap F^{\perp})$ then $G \cap F^{\perp}$ is nonempty. Every element of $G \cap F^{\perp}$ belongs to G and is orthogonal to F. $\square$

We now impose the following requirement

Axiom 4 *Let P and Q be comparable properties (that is either $P \subset Q$ or $Q \subset P$). Then there exists a measurement $M \in \mathcal{M}$ such that $P = E_M(A)$ and $Q = E_M(B)$ for some A, B in $O(M)$.*

This is a serious restriction. For example, it is violated in Model 2 of Section 3. A first consequence of Axiom 4 is the orthomodularity of the state space.

Proposition 2 *If Axiom 4 is fulfilled then the state space S (or P) is orthomodular.*

Indeed, suppose that $P \subset Q$ are two different properties. Let M be a measurement such that $P = E_M(A)$ and $Q = E_M(B)$ for $A, B \subset O(M)$. Obviously, $A \subset B$ and this inclusion is strict. If $b \in B \setminus A$ then every element of $E_M(b)$ belongs to Q and is orthogonal to P. $\square$

Another important consequence of Axiom 4 is that any state $s \in \mathbb{S}$ can be considered as a probability measure on the orthospace $\mathbb{P}$.

Definition. A *probability measure* on an orthospace $(X, \perp)$ is a function $p : \mathcal{F}(X, \perp) \to \mathbb{R}_+$ satisfying the following two requirements:

1) if F and F' are orthogonal flats then $p(F \vee F') = p(F) + p(F')$;
2) $p(\mathbf{1}) = 1$.

By induction we obtain the equality $p(F_1 \vee ... \vee F_n) = p(F_1) + ... + p(F_n)$ for any mutually orthogonal flats $F_1, ..., F_n$. It is natural to call this property *ortho-additivity*. The requirement 2) is simply a normalization. Note that 1) implies $p(\mathbf{0}) = 0$. If the orthospace $(X, \perp)$ is orthomodular (what we shall assume) then p is monotone (that is $p(F) \leq p(Q)$ for $F \subset Q$). When all elements of X are orthogonal each to other (and X is a finite set) we come to conventional notion of a probability measure on X, see 5.1.

We already (see 5.1) represented an arbitrary state s as a function on $\mathcal{F}(\mathbb{P}, \perp)$. We now assert that this function is ortho-additive.

Proposition 3 *Axiom 4 implies that any state s (as a function on $\mathcal{F}(\mathbb{P}, \perp)$) is a probability measure.*

Proof. Since $s(\mathbf{1}) = 1$ we have to check the ortho-additivity of s. Let F and F' be two orthogonal flats, and $G = F \vee F'$. Since $F \subset G$ then, by Axiom 4, there exists a measurement M such that $F = E_M(A)$ and $G = E_M(B)$ for $A \subset B \subset O(M)$. Set $A' = B \setminus A$.

We claim that $F' = E_M(A')$. We begin with inclusion $\subset$. Let us consider an arbitrary state t from F'. The outcome of the measurement M in the state t cannot belong to A according to orthogonality of t and F. Hence the outcome of the measurement belongs to A' with certainty, that is $t \in E_M(A')$. Thus, $F' \subset E_M(A')$.

Suppose now that t is a state from $E_M(A')$, but not from F'. Due to orthomodularity, we can assume that t is orthogonal to F'. Since t is orthogonal to $F = E_M(A)$ as well, we conclude that t is orthogonal to $F \vee F' = G$ and therefore cannot belong to $E_M(A')$. A contradiction. The claim is proven.

Now $s(F \vee F') = \mu_M((A \cup A')|s) = \mu_M(B|s) = s(G)$. $\square$

In particular, if $\{b_1, ..., b_n\}$ is an orthobasis of a property F then $s(F) = s(b_1) + ... + s(b_n)$ for any state s.

6 Impact of measurements

Here we assume that $(\mathcal{M}, \mathbb{S}, \mu, \tau)$ is a model of some measurable system satisfying Axioms 1-4. In the preceding Section we have shown that the state space P is an ortho-separable orthomodular space. In this section we show that measurements act as orthogonal projections in this orthospace.

6.1 Ideal measurements

We know that measurements impact on the state, that is the state of a system is modified by the performance of measurements. Here we investigate measurements that "minimally" impact on the state. These measurements are called *ideal*. Let us give a precise definition. Let $M \in \mathcal{M}$ be a measurement with eigensets $F(o) = E_M(o), \ o \in O(M)$.

Definition. A measurement M is *ideal* if, for every state s, the new state $\tau_{M,o}(s)$ belongs to the convex hull of the flat $F(o) \wedge (s \vee F(o)^\perp)$.

Note that we earlier said that the transition mapping $\tau_{M,o}$ is undefined for states belonging to $F(o)^\perp$. This is in agreement with the fact that, for $s \in F(o)^\perp$, the flat $F(o) \wedge (s \vee F(o)^\perp) = F(o) \wedge F(o)^\perp$ is empty. On the contrary, if s does not belong to $F(o)^\perp$ then the flat $s \vee F(o)^\perp$ is strictly larger than $F(o)^\perp$. According to orthomodularity it contains

an element orthogonal to $F(o)^\perp$, that is an element belonging to $F(o)$. Therefore the flat $F(o) \wedge (s \vee F(o)^\perp)$ is nonempty indeed.

Proposition 4 *Let M be an ideal measurement. Suppose that a pure state s belong to one of the eigensets of M. Then the performance of measurement M leaves the state s unaffected.*

In that sense an ideal measurement minimally impacts on states or produces "a least perturbation".

Proof. Let us suppose that s belongs to the eigenset $F = F(o)$. By the definition of ideality, the new state is in $F \wedge (s \vee F^\perp)$. Since $s \in F$ we have the dual inclusion $F^\perp \subset \{s\}^\perp$. By force of orthomodularity $s^\perp = F^\perp \vee (s^\perp \vee F)$. Applying $\perp$ we obtain the equality $F \wedge (s \vee F^\perp) = s^{\perp\perp}$. By Axiom 3, that last set is made out of a single element s. Therefore the new state coincides with s. $\square$

Let us mention one more property of ideal measurements. When performing an ideal measurement the new state s' is not orthogonal to the old state s. This follows from the fact that $F \wedge (s \vee F^\perp)$ and $s^\perp$ do not intersect. In fact,

$$s^\perp \wedge F \wedge (s \vee F^\perp) = (s \vee F^\perp)^\perp \wedge (s \vee F^\perp) = 0.$$

Strengthening Axiom 4, we postulate that there exists sufficiently many ideal measurements. Let $M \in \mathcal{M}$ be a measurement. As we know, different flats $E_M(o)$ are orthogonal to each other. Moreover, the join of all $E_M(o)$ is equal to $\mathbf{1}$. Indeed, if a state s is orthogonal to $E_M(o)$ then $E_M(o) \subset s^\perp$ and consequently $s(E_M(o)) \le s(s^\perp) = 1 - s(s) = 0$; on the other hand, $\sum_o s(E_M(o)) = 1$. This leads us to the following definition

Definition. An *Orthogonal Decomposition of the Unit (ODU)* is a finite family of flats $(F_i, \ i \in I)$ such that

    a. F_i and F_j are orthogonal if $i \ne j$;

    b. $\vee_{i \in I} F_i = \mathbf{1}$.

Thus, for a measurement M, the family of eigensets $(E_M(o), \ o \in O(M))$ is an ODU. The next *ideality* axiom asserts that any ODU may be obtained as the collection of the eigensets of some ideal measurement.

Axiom 5 *For any ODU $(F_i, \ i \in I)$, there exists an ideal measurement $M \in \mathcal{M}$ with the outcome set $O(M) = I$ and the eigensets $E_M(i) = F_i$.*

Axiom 5 connects ideal measurements with ODUs. This allows to investigate the central issue of compatibility of measurements which we do next.

6.2 Compatible measurements

In Section 3 we informally discussed the notion of compatible measurements. In order to consider this issue more formally we need to introduce a notion of commutativity in the orthomodular space $\mathbb{P}$. Two (or more) flats commute (or are compatible) if they possess a common orthobasis (see 5.5). More precisely, a family $(F_i, \ i \in I)$ of flats *commute* if there exists an orthobasis B of $\mathbb{P}$ and a family $(A_i, \ i \in I)$ of subsets of B such that A_i is an orthobasis of the flat F_i.

For example, flats F and G commute if they are comparable or are orthogonal. One can show that a family $(F_i, \ i \in I)$ of flats commute if every two member of the family commute.

Lemma 7 *Let flats F and G commute. Then $F \wedge (G \vee F^{\perp}) = F \wedge G$.*

Indeed, let B be a common orthobasis of F and G. That is F and G are the orthoclosure of some subsets A and C of B. Then $C \cup (B \setminus A)$ is an orthobasis of the flat $G \vee F^{\perp}$ and $A \cap (C \cup (B \setminus A)) = A \cap C$ is an orthobasis of the flat $F \wedge (G \vee F^{\perp})$. On the other hand, $A \cap C$ is an orthobasis of the flat $F \wedge G$. $\square$

Let M and M' be two ideal measurements with eigensets $E_M(o)$, $o \in O(M)$, and $E_{M'}(o')$, $o' \in O(M')$.

Definition. The ideal measurements M and M' are *compatible* (or *commute*) if every $E_M(o)$ commutes with every $E_{M'}(o')$.

We assert that compatible measurements are compatible in the previously mentioned informal sense, that is performing one of the measurements does not affect the results from the other measurement. Indeed, suppose that a state s is in an eigenset $F := E_M(o)$ and therefore performing M gives outcome o. Suppose further that we perform measurement M' and obtain an outcome o'. Then the new state s' is in the flat $G \wedge (s \vee G^{\perp})$, where $G = E_{M'}(o')$. All the more, the new state s' is in the flat $G \wedge (F \vee G^{\perp}) = F \wedge G$ according to according to Lemma 7. Therefore s' remain in F, and if we perform the measurement M again we obtain the same outcome o.

We next show that two ideal measurements are compatible if and only if they are "coarsening" of a third (finer) measurement. First a definition

Definition. A measurement M' is *coarser* than a measurement M (and M is *finer* than M') if every eigenset of M is contained in some eigenset of M'.

In other words, outcomes of M' can be obtained from outcomes of M by means of a mapping $f : O(M) \to O(M')$. In this case the eigensets $E_{M'}(o')$ have the form $E_M(f^{-1}(o'))$.

If M' and M'' both are coarsening of M then they are compatible. For this aim we have to take an orthobasis common to all eigensets of the measurement M.

Conversely, let M and M' be two compatible measurements. Then there exists an orthobasis B common for eigensets of M and M'. If B is a finite set, then we can take as M'' the (complete) measurement corresponding to the ODU B. In the general case, we have to consider the family of flats $(E_M(o) \wedge E_{M'}(o'))$, where o runs $O(M)$ and o' runs $O(M')$. Because of compatibility, these flats form an ODU. And we have to take as M'' the corresponding measurement which exists by Axiom 5.

Thus, we proved the following

Theorem 2 *Ideal measurements M and M' are compatible if and only if there exists an ideal measurement refining both M and M'.*

6.3   Canonical decomposition of a model

We now show that a model of a measurable system satisfying Axioms 1 to 5 can be written as the direct sums of its irreducible submodels. The argument below holds for any orthospace but is of largest interest for ortho-separable orthospaces.

Let $(X, \perp)$ be an orthospace. We say that two elements of X are *connected* if they can be linked be a chain of pairwise non-orthogonal (tolerant) elements. This relation is an equivalence relation and therefore divides the set X into classes of connected elements which we denote $X(\omega)$, $\omega \in \Omega$. Elements from different connected components are orthogonal to each other; therefore $X(\omega)$ are flats. These flats are called *central* or *classical*.

If F is a flat in X then, for any ω, the set $F \cap X(\omega)$ is a flat in the orthospace $X(\omega)$ equipped with the induced orthogonality relation. Conversely, suppose we have a collection of flats $F(\omega)$ in $X(\omega)$, $\omega \in \Omega$. Then the union $F = \cup_\omega F_\omega$ is a flat in X. In other words, the ortholattice $\mathcal{F}(X, \perp)$ is the direct (orthogonal) product of ortholattices $\mathcal{F}(X(\omega), \perp)$.

Let us go back to a model of a measurable system with orthospace $(\mathbb{P}, \perp)$. As any orthospace, $\mathbb{P}$ decomposes into connected (or irreducible) components $\mathbb{P}(\omega)$, $\omega \in \Omega$. Since these components form an ODU then, by Axiom 4, there exists a corresponding ideal measurement C. Since any state is in some component, Proposition 4 implies that the measurement C affects no state. It is natural to call the measurement C *classical* and to call the corresponding components $X(\omega)$ *classical super-states*. The classical measurement C commutes with any ideal measurement. For this reason, we can without loss of generality, consider only irreducible models.

6.4 Axiom of Purity

The property of ideality of measurements significantly narrowed down the range of the possible impact of a measurement on the state. We know that as we obtain the result o the system moves from state s to state s' belonging to the convex hull of the flat $E_M(o) \wedge (s \vee E_M(o)^\perp)$. If this flat is an atom, the new state is uniquely determined. But if this flat is not an atom (and that is fully possible) the new state may be a probabilistic mixture of states in $E_M(o) \wedge (s \vee E_M(o)^\perp)$.

To see this, let us consider the example of a fly in a 3×2 box. There are two measurements: LR and FCB. Suppose the state F is realized as a probability measure $F(L) = F(C) = F(R) = 1/3$ (of course, $F(F) = 1$ and $F(B) = 0$). If we, in the state F, perform a measurement with eigensets $\{L\}$ and $\{L\}^\perp = \{C, R\}$ and obtain outcome "not L", we may conclude that the image of F is not a pure state but the equiprobable mixture of states C and R. Such a conclusion is in agreement with the ideality of the measurement $\{C, R\}$ which sends the state F into $\{C, R\} \wedge (F \vee \{C, R\}^\perp) = \{C, R\} \wedge (F \vee L) = \{C, R\} \wedge \mathbf{1} = \{C, R\}$.

We introduce a last axiom guaranteeing that under the impact of a measurement any pure state jumps into another pure state. Namely, we consider the following *axiom of purity*

Axiom 6 *For any pure state $s \in \mathbb{P}$ and any flat F the flat $F \wedge (s \vee F^\perp)$ is an atom of the lattice $\mathcal{F}(\mathbb{P}, \perp)$.*

We have introduced above a number of non-trivial axioms. We assert that they are all compatible with each other. Indeed, Example 2 (A fly in a box) gives a model satisfying to all axioms. Another example is the so-called Hilbert space model of Quantum Mechanics.

Example 4 (continued). Let H be a (finite-dimensional) Euclidean space, as in Example 4. And let $\mathbb{P}$ be the set of all one-dimensional vector subspaces of H. The orthogonality relation is clear from Example 4. Any flat is given by a vector subspace V and consists of one-dimensional subspaces in V. Measurements are identified with ODUs. Suppose that $(V_i, i \in I)$ is a family of pairwise orthogonal vector subspaces in H and $\sum_i V_i = H$, and let v be a (non-zero) vector in H representing some state s. Denote by v_i the orthogonal projections of v on subspace V_i. Then under impact of the corresponding measurement the state s moves into the state v_i with probability $\cos^2(\varphi)$, where φ is the angle between the vectors v and v_i (the probability is 0, if $v_i = 0$), and give the outcome i. By the construction, this measurement is ideal. It is easy to check that all other axioms also are fulfilled.

Remark. In some sense Example 4 is not only a special case. If the height of $\mathcal{F}$ is more than 3 then the lattice $\mathcal{F}$ can be realized as a (ortho)lattice of vector subspaces of some Hermitian space over some $*$-field K.[10] The details can be found in [2] or in [11]. If we additionally require that the orthospace $\mathbb{P}$ is compact and connected (as a topological subspace of $\Delta(\mathbb{P}, \perp)$) then the field K is the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions.

7 Non-classical models in social sciences: A discussion

In the last section we want to discuss some of the key properties of general measurable systems in order to help the reader assess their relevance for social sciences. We wish to emphasize that this section is highly explorative and should be viewed as a first step that only aims at opening the discussion.

When applying the theory of measurable system exposed in this paper to behavioral and social sciences, the general idea is to view an individual as a measurable system. She is characterized by her type which encapsulates information about her preferences, attitudes, beliefs, feelings etc. A decision situation (a situation such that she must choose an alternative out of set of alternatives) or a questionnaire is a device that measures her type. Actual behavior, e.g. the choice made in a decision situation, the actions taken in a game or the

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[10]The case of the height 1 is trivial: $\mathcal{F} = \{0, 1\}$ and $\mathbb{P}$ consists of single state. The case of the height 2 is of more interest. The (ortho)lattice $\mathcal{F} = \{0, 1\} \cup \mathbb{P}$, and the mapping $s \mapsto s^\perp$ acts on the set $\mathbb{P}$ as an involution without fixed points. The case of the height 3 is very intricate and unclear.

answer given to a questionnaire are measurement outcomes.

In the Introduction we formulated a question as to whether it is possible to build an interesting theory about a system that changes when being measured. We answered by the affirmative when imposing a series of properties on measurements and on their interaction with the system. The state space representing the system is then endowed with the structure of an atomistic orthomodular orthospace and the states are realized as probability measures on the state space. We next propose a psychological and behavioral interpretation of some of those properties.

First kindness

The first key property of a measurable system is that measurements satisfy first-kindness. Classical measurement theory (including revealed preference theory) also relies on such an assumption of repeatability. Some reservation may be in place. We do propose that a choice be viewed as a measurement outcome that reveals or more precisely actualizes preferences. But in many settings the repeated character of a choice changes the decision situation. Clearly, a repeated interaction in a game situation is not equivalent to a repetition of one and the same decision situation. The prolific theory of repeated games amply illustrates this. So the repeatedness we have in mind pertains to elementary situations.

Compared with standard revealed preference theory, the requirement of repeatability is limited in two respects. The property applies to a smaller set of choice experiments. As we illustrated in section 4 not all choice sets can be associated with a first-kind measurement unless we are dealing with a fully classical agent. Moreover repeatability is only requested in two *consecutive* identical measurements. If another (incompatible) measurement is performed on the system in-between, the initial result may not be repeated. Therefore although the property of first-kindness is somehow restrictive it is still far less demanding than the standard classical assumption. Yet, it is not an innocuous assumption and in particular it precludes stochastic preferences.

Invariance

Axiom 1 is an axiom of invariance. In the context of choice theory, it is related to the principle of procedure invariance assumed in classical rational choice theory. This principle states that a preference relation should not depend on the procedure of elicitation. Numerous experimental studies were made on choice versus pricing to exhibit examples of violation

of procedure invariance. It is beyond the scope of this short comment to systematically compare the classical concept of procedure invariance with Axiom 1. We confine ourselves to remarking that Axiom 1 applies to systems in the same state and to remind that it concerns first-kind measurements only. In particular, we do not take for granted that the pricing or the matching procedure which are considered as computationally relatively demanding (compared with a choice procedure) can be performed as a single measurement rather than as a sequence of (possibly incompatible) measurements.

Axioms 2, 3, 4 and 5 are axioms which all formulate requirements on the richness of the set of measurements. Another way to look at them that lends itself to an interpretation in choice theory is that when the set of primitive measurements is actually limited - in Model 2 there could only be 4 measurements - the axioms imply that choice-measurements may not all be incompatible with each other. Indeed, it is immediate to see that these axioms are fulfilled for compatible measurements as these can be combined into new measurements satisfying the axioms. For this very reason these axioms do seem very natural to a classically minded person. In a non-classical world they do not follow naturally, which is demonstrated by the fact that these axioms are violated in Model 2. Hence, in a choice theoretic context, these axioms put a limit on how "non-classical" agent (a behavioral system) is allowed to be. Model 3 satisfies all these axioms and still allows for so-called "behavioral anomalies".

States and types

The notion of state is closely related to Harsanyi's classical notion of type which is why we use this term when referring to the state of agents. The Harsanyi type of an agent is a complete description of her preferences, beliefs and private information such that it allows predicting the agent's behavior. By observing past behavior, we learn about an agent's type and can make finer predictions of her future behavior. In Harsanyi's classical world an information about past behavior is used to predict future behavior relying on Bayesian updating. The same holds for any compatible choice-measurements made on a non-classical agent. But generally (when some measurements are incompatible) learning is not Bayesian. In Model 3 of section 4, we saw that the performance of the ab measurement on the agent in state $s \in \{[c > b > a], [c > a > b]\}$ erases information about her preferences what concerns the ordering between b and c so the agent can choose b in $\{b, c, d\}$.

As in Harsanyi's model, a non-classical pure type is maximal information about the agent.

But in contrast with Harsanyi, a pure type may still be dispersed (cf Section 3.4) so knowing the pure type does not allow to predict behavior with certainty. This is reflected in the structure of the type space. In Harsanyi's type space types are orthogonal to each other. In the non-classical model not all (pure) type are orthogonal. Instead, non-orthogonal states are connected with each other in the sense that under the impact of a measurement the state of the system can transit from one state to another. This strongly contrasts with Harsanyi's static model where the act of choosing (i.e. a measurement of the type) has no impact on the type only on payoffs. The non-classical type space models a changing agent, we return to this aspect below.

Incompatible measurements

In the models of rational choice of Section 4 measurements corresponds to sets of alternatives from which the agent makes a choice. Whichever model we choose, the existence of incompatible choice-measurements implies that the agent cannot have a preference order on all items *simultaneously*. Our theory gives a precise meaning to this impossibility. It means: i) if the ordering over some subset of items is known (possibly only to the agent) then his preferences over another incompatible subset is random (dispersed pure states); ii) as the agent makes a choice in a given choice set his type (preferences) is being modified (measurements affect the state). A non-classical agent does not have a fixed type (preferences). The non-classical model is consistent with the hypothesis that an agent's preferences are shaped in the process of elicitation as proposed by Kahneman and Tversky (see Introduction).

Thus, the distinctive feature of the non-classical model namely the existence of incompatible measurements (or alternatively the existence of dispersed states) delivers a formulation of bounded rationality as the impossibility to compare and order all items simultaneously. We view this formulation as particularly interesting because it is also linked to the idea that an agent's preferences are "context-dependent" (see [12], part 6). Both these themes: the issue of comparability in the universal set of items and intrinsic contextuality of preferences are central themes in behavioral and experimental economics.

As for today there is no consensus in Physics about the reasons for the non-classicality of quantum physical phenomena. There exists however a huge literature on the subject in epistemology. We do not wish to speculate on reasons for such phenomena in human behavior. Instead, we note that the (possible) non-classicality of agents invites social scientists to making interactions the central object of their investigation. Clearly, game theory is all

about interactions but it retains that the type of agents is exogenous. This cannot be maintained if agents are non-classical systems. Instead we ought to make agents' type as (partly) endogenous to interaction. [11] We are currently investigating simpler game situations along these lines and we trust that it is a promising avenue of research.

The Stability of the state

The minimal perturbation principle (ideality) means that a coarse measurement leaves unperturbed the uncertainty not sorted out by its set of outcomes. Axiom 5 demands that there exists sufficiently many such ideal measurements.

In applications to behavioral sciences, an interpretation is that when asked to choose out of an initial "state of hesitation" (a dispersed pure state), this hesitation is only resolved so as to be able to produce an answer but no more. The remaining uncertainty is left "untouched". One way to understand this is to see that when we assume that a choice measurement is ideal it is as if we assumed that the individual proceeds by taking the "shortest way" to resolve uncertainty. For instance, suppose as in Model 3 that we ask an individual to make a choice out of $\{a, b, c\}$ and that her initial state is a superposition (see section 5.3) of all six orders. The minimal perturbation principle entails that the individual will proceed so as to find the most preferred item without ordering the 3 items. Uncertainty not resolved by an ideal choice-measurement is left untouched. Therefore we say that ideality implies a certain stability of the type of a non-classical agent. This can be contrasted with the classical assumption of complete stability, i.e. revealing preferences does not affect them at all. Nevertheless ideality may turn out a rather demanding property, which should be viewed as an approximation.

Axiom 6 further precises the impact of measurement. It tells us exactly were a measurement takes the state. We focus on an informational interpretation. In a social science context it implies that whatever choice the individual makes that changes her (pure) type, the new behavioral type encapsulates maximal information about behavior as did the initial type and by ideality we know the new state.[12] In a classical context such a pure state corresponds to a state of complete information. In a non-classical context we know that

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[11]Geanakoplos et al. (1989) pioneered an approach in psychological games where agent's motivation (utility) depends on other's beliefs and therefore are endogenous to the interaction.

[12]**Information in a pure state is maximal in the sense that no new information can be obtained from any measurement without losing some other information, i.e. information that was true in the initial state but is no longer true in the new state**.

there exist dispersed pure states and so a maximal information type (a pure type) does not uniquely predict behavior in all circumstances.

Caveat

In applications to behavioral sciences a less attractive feature of our framework is that a measurement erases information about the previous type. We should however recall that a person is expected to be composed of a number of irreducible systems. The loss of memory only applies locally, within one (irreducible) sub-system. Even within such a system, when assuming ideality, memory is fully lost only in the case the measurement is complete (not coarse). Yet, our approach implies that to some extent an individual's previous choices are not relevant to her current type. She may recall them but she experiences that she has changed. Implicitly, we assume that at a higher cognitive level, the individual accepts changes in, e.g. her tastes, which are not motivated by new cognition.

8   Concluding remarks

In this paper, we have described the basic structure of non-classical measurement theory. The objective has been to investigate, from a theoretical point of view, whether this framework could be suitable for describing, explaining and predicting human behavior.

As a non-classical measurable system, an agent is characterized by her type (preferences, attitudes, beliefs etc...) which changes when she makes a choice actualizing her type. As a consequence behavior exhibits an irreducible uncertainty. Yet, as we impose some axioms on the interaction between measurements and the system, behavior is characterized by sufficient regularity to allow for predictions. We have argued that some of the basic axioms and properties that underline the theory can be given a meaningful interpretation consistent with central themes addressed in psychology, behavioral and social sciences. We also argued that the distinctive feature of non-classical measurement theory, i.e. the existence of incompatible measurements, provides an appealing formulation of bounded rationality.

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