KYP lemma based stability and control law design for differential linear repetitive processes with applications[☆]

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Abstract

Repetitive processes are a class of 2D systems that have physical applications, including the design of iterative learning control laws where experimental validation results have been reported. This paper uses the Kalman-Yakubovich-Popov lemma to develop new stability tests for differential linear repetitive processes that are computationally less intensive than those currently available. These tests are then extended to allow control law design for stability and performance.

Keywords: Kalman-Yakubovich-Popov lemma, differential linear repetitive processes, stability.

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1. Introduction

Many physical systems complete the same finite duration operation over and over again. Repetitive processes have this characteristic where a series of sweeps or passes are made through dynamics defined over a finite duration known as the pass length. Once each pass is complete, the process resets to

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the original location and the next one begins. The output on each pass is termed the pass profile and the notation for scalar or vector valued variables is $y_k(t), 0 \leq t \leq \alpha < \infty, k \geq 0$, where y is the scalar or vector valued variable, the integer k is the pass number and α is the pass length. Also the previous pass profile contributes to dynamics of the next one and the result can be oscillations in the pass profile sequence $\{y\}_k$ that increase in amplitude from pass-to-pass (k) and cannot be controlled by standard systems theory.

The original use of repetitive process models was in the coal mining and metal rolling industries where references to the original papers are given in [1]. In coal mining, the cutting machine rests on the previous pass profile, the height of the stone/coal interface above some datum line, during the production of the current one and the basic geometry confirms that this industrial application is a repetitive process in the sense defined above. The stability problem for this repetitive process is caused, in the main, by the machine's weight and can result in undulations in the previous pass profile of a level that productive work is no longer possible without their removal.

The repetitive process setting can also be used for analysis and control of other systems. Examples include classes of iterative learning control schemes [2, 3] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [4]. Iterative learning control algorithms designed using a repetitive process setting have been experimentally tested [5, 6]. Also there has been recent work on the use of this setting for the analysis of OL-Nash games with a gas pipeline application [7].

This paper considers differential linear repetitive processes where the dynamics along the pass are governed by a linear matrix differential equation and the pass-to-pass dynamics by a discrete linear matrix equation. The stability theory [1] for linear repetitive processes is of the bounded-input bounded-output (BIBO) type and is based on an abstract model in a Banach space setting that includes a large range of examples as special cases.

Stability of linear repetitive processes imposes a BIBO property defined in terms of the linear operator that describes the contribution of the previous pass profile to the next one, either over the finite pass length or independent of this parameter, where this latter case can be analyzed mathematically by letting $\alpha \rightarrow \infty$. It is the latter property that is required in many applications and for the processes considered in this paper it can be expressed in terms of three linear matrix inequalities (LMIs) [8]. Each of these conditions has a well defined physical interpretation and this paper uses the Kalman-Yakubovich-Popov (KYP) lemma to develop new control law design algorithms for stability and performance, where the latter aspect has received relatively little attention in the literature.

Throughout this paper, the null and identity matrices with appropriate dimensions are denoted by 0 and I respectively. The notation $X \succeq Y$ (respectively $X \succ Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). Also sym{M} denotes the symmetric matrix $M+M^T$ and $\rho(\cdot)$ denotes the spectral radius of its argument. The superscript $*$ denotes the complex conjugate transpose of a matrix, the superscript $\perp$ the orthogonal complement of a matrix and $\otimes$ the Kronecker matrix product.

Use will be made of the following results, known as the KYP lemma [9], its generalized version [10] and the Elimination or Projection Lemma [11] respectively.

Lemma 1. *Let matrices $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$ and $\Theta = \Theta^T \in \mathbb{R}^{(p+q)\times(p+q)}$ be given and suppose that $\det(j\omega I - A) \neq 0$ for any $\omega \in \mathbb{R}$. Then the following statements are equivalent*

i) for any $\omega \in \mathbb{R} \cup \infty$

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \tag{1}$$

ii) there exists a symmetric matrix P such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0. \tag{2}$$

Remark 1. As discussed in [10], particular choices of the matrix Θ allows representation of various system properties, including positive-realness and bounded-realness.

Lemma 2. *Let the matrices Θ, F, Φ and Ψ be given and denote by N_ω the null space of $T_\omega F$, where $T_\omega = [I \ \ -j\omega I]$. The inequality*

$$N_\omega^* \Theta N_\omega \prec 0, \ \textit{with} \ \omega \in [\omega_l, \omega_u], \tag{3}$$

holds if and only if there exist $Q \succ 0$ and a symmetric matrix P such that

$$F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta \prec 0, \tag{4}$$

where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_l\omega_u \end{bmatrix}, \ \omega_c = \frac{(\omega_l + \omega_u)}{2}.$$

Lemma 3. *Given a symmetric matrix $\Gamma \in \mathbb{R}^{p\times p}$ and two matrices Λ, Σ of column dimension p, there exists a matrix W such that the following LMI holds*

$$\Gamma + \mathrm{sym}\{\Lambda^T W \Sigma\} \prec 0, \tag{5}$$

if and only if the following two projection inequalities with respect to W are satisfied

$$\Lambda^{\perp T} \Gamma \Lambda^{\perp} \prec 0,\ \Sigma^{\perp T} \Gamma \Sigma^{\perp} \prec 0, \tag{6}$$

where $\Lambda^{\perp}$ and $\Sigma^{\perp}$ are arbitrary matrices whose columns form a basis of nullspaces of Λ and Σ respectively.

2. Background and stability analysis

The state-space model of a differential linear repetitive process over $0 \leq t \leq \alpha$, $k \geq 0$, is

$$\begin{aligned} \dot{x}_{k+1}(t) =& Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t), \\ y_{k+1}(t) =& Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t), \end{aligned} \tag{7}$$

where $\alpha < +\infty$ denotes the pass length and on pass k $x_k(t) \in \mathbb{R}^n$ is the state vector, $y_k(t) \in \mathbb{R}^m$ is the pass profile vector and $u_k(t) \in \mathbb{R}^r$ is the input vector. The boundary conditions are $x_{k+1}(0) = 0$, $k \geq 0$ and $y_0(t)$, with entries that are known functions of t over $[0, \alpha]$. No further explicit mention of the boundary conditions is made in this paper. Also it is assumed throughout this paper that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ observable.

In this model the state updating is in t and the pass profile updating is in k. The terms $B_0 y_k(t)$ and $D_0 y_k(t)$, respectively, represent the contribution from the previous pass profile. In [1] examples are considered where other representations for the contribution from the previous pass profile are required. As expected, the previous pass terms are critical to the stability properties of these processes.

The stability theory [1] for linear repetitive processes is based on an abstract model in a Banach space setting that includes a wide range of such processes as special cases, including those described by (7). Let E_α be a Banach space, W_α a linear subspace of E_α and L_α a bounded linear operator mapping E_α into itself. Then the dynamics of a linear repetitive process with constant pass length are described by linear recursion relations of the form

$$y_{k+1} = L_\alpha y_k + b_{k+1}, k \geq 0, \tag{8}$$

where y_k is the pass profile on pass k and $b_{k+1} \in W_\alpha, k \geq 0$. The term $L_\alpha y_k$ represents the contribution of pass k to pass $k+1$ and b_{k+1} represents initial conditions, disturbances and control input effects that enter on pass $k+1$.

Stability for repetitive processes requires that, for given any initial profile y_0 and any disturbance sequence $\{b_{k+1}\}_{k\geq 0}$ that converges strongly to b_∞ as $k \to \infty$, the sequence of pass profiles generated $\{y_k\}_{k\geq 1}$ converges strongly to y_∞ as $k \to \infty$. This property is termed asymptotic stability in the pass-to-pass direction, or asymptotic stability for short, of (8) and for the given finite pass length α is equivalent [1] to the existence of finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0,1)$ such that $||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k$, where $||\cdot||$ denotes both the norm on E_α and the induced operator norm as appropriate. This property holds [1] if and only if $\rho(L_\alpha) < 1$. If this property holds let y_∞, termed the limit profile, denote the strong limit of $\{y_k\}_{k\geq 1}$ under asymptotic stability and consider an input sequence $\{u_k\}_{k\geq 1}$ that converges strongly as $k \to \infty$ to u_∞. Then

$$y_\infty = (I - L_\alpha)^{-1} b_\infty,$$

where I denotes the identity operator in E_α.

For processes described by (7) it has been shown [1] that asymptotic stability holds if and only if $\rho(D_0) < 1$. Also if (7) is asymptotically stable and the input sequence applied $\{u_k\}_{k\geq 1}$ converges strongly as $k \to \infty$ to u_∞, the resulting limit profile is described by the state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) =& Ax_\infty(t) + Bu_\infty(t) + B_0 y_\infty(t), \\ y_\infty(t) =& Cx_\infty(t) + Du_\infty(t) + D_0 y_\infty(t), \end{aligned} \tag{9}$$

or, since $I - D_0$ is invertible by asymptotic stability, a differential linear systems state-space model with state matrix $A_{lp} = (A + B_0(I - D_0)^{-1}C)$.

Asymptotic stability of (7) does not guarantee that the limit profile has acceptable along-the-pass dynamics. A simple example is $A = -1$, $B = 1$, $B_0 = 1 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where $\beta > 0$ is a real scalar. In this case $A_{lp} = \beta$ and the standard linear system with this state matrix is unstable in the t-direction, that is, along the pass.

The problem highlighted by this example can be overcome by demanding the BIBO property for any possible value of the pass length, where mathematically this can be analyzed by letting $\alpha \to \infty$. This is the stability along-the-pass property that [1] is equivalent to the existence of finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0,1)$, which are independent α, such that $||L_\alpha^k|| \leq M_\infty \lambda_\infty^k$, $k \geq 0$.

Lemma 4. *[1] A differential linear repetitive process described by (7) is stable along the pass if and only if the following conditions hold:*

i) $\rho(D_0) < 1$,

ii) all eigenvalues of the matrix A lie in $\mathbb{C}_-$, the open left-half of the complex plane $\mathbb{C}$, and

iii) all eigenvalues of $G(s) = C(sI - A)^{-1}B_0 + D_0$, $s = j\omega$, $\forall\omega \geq 0$, have modulus strictly less than unity.

By the first condition in Lemma 4 asymptotic stability is a necessary condition for stability along the pass. The second condition demands stability of the state dynamics on any pass and the third has a Nyquist-based interpretation. In the single-input single-output (SISO) case, for simplicity, this condition requires that the Nyquist plot generated by $G(s)$ lies inside the unit circle in the complex plane for all $s = j\omega$, $\forall\omega$. Hence this condition requires that each frequency component of the initial pass profile is attenuated from pass-to-pass.

The tracking control problem for repetitive processes is to specify a reference vector and design a control law to force the pass profile sequence $\{y_k\}_{k\geq 1}$ to converge to this vector in the pass-to-pass direction under stability along the pass. The vast majority of the results on control law design for these processes are for stabilization. This paper develops new results for stabilization plus transient performance along the passes and the first of these is the following result for stability along the pass derived using Lemma 1.

Theorem 1. *A differential linear repetitive process described by (7) is stable along the pass if there exist $P_1 \succ 0$ and $P_2 \succ 0$ such that the LMI*

$$\begin{bmatrix} AP_1 + P_1A^T + C^TP_2C & P_1B_0 + C^TP_2D_0 \\ B_0^TP_1 + D_0^TP_2C & B_0^TP_1B_0 + D_0^TP_2D_0 - P_2 \end{bmatrix} \prec 0, \tag{10}$$

is feasible.

Proof. The LMI (10) can be rewritten as

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \Theta \prec 0, \tag{11}$$

where

$$\Theta = \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} P_2 & 0 \\ 0 & -P_2 \end{bmatrix} \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix}. \tag{12}$$

By Lemma 1, (11) is equivalent to

$$\begin{bmatrix} (j\omega I - A)^{-1}B_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B_0 \\ I \end{bmatrix} \prec 0,$$

or

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} P_2 & 0 \\ 0 & -P_2 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \prec 0, \tag{13}$$

where $G(j\omega)$ is the frequency response matrix obtained from $G(s)$ of $iii)$ in Lemma 4. Moreover, (13) can be written as

$$G(j\omega)^* P_2 G(j\omega) - P_2 \prec 0, \tag{14}$$

and the existence of a $P_2 \succ 0$ satisfying this last condition immediately implies that $\rho(G(j\omega)) < 1 \; \forall \omega \in \mathbb{R} \cup \infty$, that is, feasibility of (10) guarantees that condition $iii)$ of Lemma 4 holds. Furthermore, feasibility of (10) implies

$$\begin{aligned} &AP_1 + P_1A^T + C^T P_2 C \prec 0, \\ &B_0^T P_1 B_0 + D_0^T P_2 D_0 - P_2 \prec 0, \end{aligned}$$

and, since $P_1 \succ 0$ and $P_2 \succ 0$, $C^T P_2 C \succeq 0$ and $B_0^T P_1 B_0 \succeq 0$. Hence $AP_1 + P_1A^T \prec 0$ and $D_0^T P_2 D_0 - P_2 \prec 0$ must hold, respectively. Equivalently, all eigenvalues of the matrix A must have strictly negative real parts and $\rho(D_0) < 1$, respectively, and feasibility of (10) guarantees that all three conditions Lemma 4 are satisfied.

The main drawback of Theorem 1 is that a significant degree of conservativeness can result from the requirement that there exists a constant matrix P_2 for $\forall \omega \in \mathbb{R} \cup \infty$. Previous work [12] has shown that condition iii) of Lemma 4 can be replaced by the existence of a positive definite Hermitian matrix $P_2(j\omega)$ such that

$$G^*(j\omega) P_2(j\omega) G(j\omega) - P_2(j\omega) \prec 0,$$

for all $\omega \in \mathbb{R} \cup \infty$. This is a necessary and sufficient condition for stability along the pass and, using a Kronecker product setting, tests can be developed that involve computations where all matrices involved are frequency independent [13]. This condition is not suitable for extension to control law design.

To remove the need for the matrix P_2 to be constant for all $\omega \in \mathbb{R} \cup \infty$, the method is to extend the theory to allow the use of piecewise constant matrices over a priori chosen frequency ranges. Specifically, the complete frequency range is divided into N intervals, where each of these do not have to include an equal range of values, such that

$$[0, \infty] = \bigcup_{i=1}^{N} [\omega_{i-1}, \omega_i], \ \ \omega_0 = 0, \ \ \omega_N = \infty \tag{15}$$

and then apply the result of Lemma 2 to each interval. This allows the use of a piecewise constant matrix P_{2i} $\forall i = 1, \ldots, N$, together with piecewise constant matrices P_{1i} and Q_{1i}, in the following result with the aim of achieving a less conservative solution than Theorem 1, particularly when applied to control law design in the next section.

Theorem 2. *Suppose that the entire frequency range is arbitrarily divided into N frequency intervals as given in (15). Then a differential linear repetitive process described by (7) is stable along the pass if there exist $P_{1i} \succ 0$, $P_{2i} \succ 0$, $Q_i \succ 0$ and W_1 such that the following LMIs are feasible*

$$\begin{bmatrix} -Q_i & P_{1i} + j\omega_{ci} Q_i - W_1 & 0 & 0 \\ P_{1i} - j\omega_{ci} Q_i - W_1^T & -\omega_{i-1}\omega_i Q_i + A^T W_1 + W_1^T A & W_1^T B_0 & C^T \\ 0 & B_0^T W_1 & -P_{2i} & D_0^T \\ 0 & C & D_0 & -P_{2i} \end{bmatrix} \prec 0, \tag{16}$$

for all $i = 1, \ldots, N$ where $\omega_{ci} = (\omega_{i-1} + \omega_i)/2$.

Proof. For each $i = 1, 2, \ldots, N$, applying the Schur's complement formula to the corresponding LMI of (16) gives an inequality that can be written in the form of (5) where

$$\Gamma = [I \ F_B] \Omega_i [I \ F_B]^T, \tag{17}$$

$$F_B = \begin{bmatrix} 0 \\ C^T \\ D_0^T \end{bmatrix}, \Omega_i = \begin{bmatrix} \Phi \otimes P_{1i} + \Psi \otimes Q_i & 0 \\ 0 & \Pi_i \end{bmatrix}, \Pi_i = \begin{bmatrix} -P_{2i} & 0 \\ 0 & P_{2i} \end{bmatrix}, \tag{18}$$
$$\Sigma = [0 \ I \ 0], \Lambda = \begin{bmatrix} -I & A & B_0 \end{bmatrix}, \omega_{ci} = (\omega_{i-1} + \omega_i)/2,$$

and Ψ and Φ are given in (4). Suppose also that there exist matrices $P_{1i} \succ 0$, $P_{2i} \succ 0$, $Q_i \succ 0$ and W_1 such that LMIs (16) are feasible for all $i = 1, \dots, N$. Then on applying Lemma 3 to (16) the following LMIs are also feasible

$$\begin{bmatrix} A^T & I & 0 \\ B_0^T & 0 & I \end{bmatrix} \begin{bmatrix} -Q_i & P_{1i} + j\omega_{ci}Q_i & 0 \\ P_{1i} - j\omega_{ci}Q_i & -\omega_{i-1}\omega_i Q_i + C^T P_{2i} C & C^T P_{2i} D_0 \\ 0 & D_0^T P_{2i} C & D_0^T P_{2i} D_0 - P_{2i} \end{bmatrix} \begin{bmatrix} A & B_0 \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0, \quad (19)$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -Q_i & P_{1i} + j\omega_{ci}Q_i & 0 \\ P_{1i} - j\omega_{ci}Q_i & -\omega_{i-1}\omega_i Q_i + C^T P_{2i} C & C^T P_{2i} D_0 \\ 0 & D_0^T P_{2i} C & D_0^T P_{2i} D_0 - P_{2i} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \prec 0. \quad (20)$$

Also, (20) holds if and only if (19) holds and therefore it remains to establish the feasibility of the LMI (19), which can be written as

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^T (\Phi \otimes P_{1i} + \Psi \otimes Q_i) \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} P_{2i} & 0 \\ 0 & -P_{2i} \end{bmatrix} \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix} \prec 0. \quad (21)$$

Defining Θ as in (12) and applying Lemma 2 to (21) gives

$$\begin{bmatrix} (j\omega I - A)^{-1} B_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B_0 \\ I \end{bmatrix} \prec 0, \quad \forall \omega \in [\omega_{i-1}, \omega_i],$$

or, equivalently,

$$G^*(j\omega) P_{2i} G(j\omega) - P_{2i} \prec 0, \quad \forall \omega \in [\omega_{i-1}, \omega_i], \quad (22)$$

for $i = 1, \dots, N$. Hence stability along the pass holds in each frequency interval and given (15) this property is guaranteed for the entire frequency range.

Remark 2. Following [10], for $i = 0$, the low frequency range starting from $\omega = 0$, that is, $\omega_0 = 0$, the matrix Ψ in (4) can be taken as

$$\Psi = \begin{bmatrix} -1 & 0 \\ 0 & \omega_1^2 \end{bmatrix}.$$

For $i = N$, the high frequency range ending with $\omega = \infty$, that is, $\omega_N = \infty$, the matrix Ψ in (4) can be taken as

$$\Psi = \begin{bmatrix} 1 & 0 \\ 0 & -\omega_{N-1}^2 \end{bmatrix}, \quad (23)$$

and hence Σ and Λ in (18) must be replaced by

$$\Sigma = [-I\ \beta I\ 0] \text{ and } \Lambda = \left[-I\ A+BK_1\ B_0+BK_2\right],$$

where $\beta > 0$ is an arbitrary real scalar.

Currently, there is no systematic procedure for dividing the entire frequency range into sub-ranges but, in general, the conservativeness decreases as the number of sub-ranges increase. This is due to the introduction of additional auxiliary slack matrix variables (introduced for every subrange) and hence more degrees of freedom are obtained. When control law design procedure is considered, the availability of frequency sub-ranges allows the design to be specialized to the particular requirements of the application under consideration. Examples include suppression of narrow-band frequency disturbances or in design to enforce tracking of areference signal where the dominant frequencies in this signal can be determined by inspection and the frequency sub-ranges accordingly determined. This design is a practical alternative to demanding frequency attenuation over the complete frequency range.

The condition of Theorem 1 can be rewritten (by applying transformations) as a special case of the result in Theorem 2. In particular, set $Q_i = Q$, $P_{1i} = P_1$ and $P_{2i} = P_2$ (there are no frequency sub-ranges and hence the matrix variables do not vary) and then (16) can be written as

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^T (\Phi \otimes P_1 + \Psi \otimes Q) \begin{bmatrix} A & B_0 \\ I & 0, \end{bmatrix} + \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} P_2 & 0 \\ 0 & -P_2 \end{bmatrix} \begin{bmatrix} C & D_0 \\ 0 & I \end{bmatrix} \prec 0. \quad (24)$$

Taking taking $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\Psi = 0$ (no frequency sub-ranges are considered) gives the condition of Theorem 1. Consequently, Theorem 2 introduces additional free matrix variables, which, in turn, gives greater freedom in selection and the possibility of less conservative results.

In the LMI condition of Theorem 2, the matrix variable W_1 is the same for all frequency sub-ranges but it is routine to show that a different matrix can be chosen in each sub-range, that is, W_{1i} for $i = 1, \ldots, N$, with resulting LMI

$$\begin{bmatrix} -Q_i & P_{1i} + j\omega_{ci}Q_i - W_{1i} & 0 & 0 \\ P_{1i} - j\omega_{ci}Q_i - W_{1i}^T & -\omega_{i-1}\omega_i Q_i + A^T W_{1i} + W_{1i}^T A & W_{1i}^T B_0 & C^T \\ 0 & B_0^T W_{1i} & -P_{2i} & D_0^T \\ 0 & C & D_0 & -P_{2i} \end{bmatrix} \prec 0. \quad (25)$$

This option should also reduce the level of conservativeness through the introduction of more free matrix variables. Moreover, in this case the product terms between process state-space model matrices and the P_{1i} in the LMI of Theorem 2 are removed and this may again lead to a reduction of conservativeness. In the case of control law design, is not possible to chose a different W_1 in each sub-range since the computed control law matrices may not be equal. A simple way to avoid this problem, used in the next section, is to set is to set $W_{1i} = W_1,\ i = 1, \dots, N$.

3. Control law design

The control of differential linear repetitive processes requires the use of control laws that combine current and previous pass action. One control law is of the form

$$u_{k+1} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}, \tag{26}$$

where K_1 and K_2 are compatibly dimensioned matrices to be found. This control law is formed as a weighted sum of current pass state feedback and feedforward of the previous pass profile, see [1] for further background on this form of control action. Moreover, the algorithms for control law design currently available mainly focus on achieving stability along the pass for the controlled process and it is necessary to expand the number of algorithms for stability along the pass plus performance specifications. In this context, the eigenvalues of the state matrix A govern the dynamics produced along each pass and the open question addressed in this section is one of control law design in the presence of constraints on the locations of the eigenvalues of the current pass state matrix of the controlled process.

The result of Theorem 2 cannot be easily extended to control law design since it would involve bilinear terms arising from products of the matrices W_1 and the control law matrices K_1 and K_2 (and this would also be the case if the matrix W_1 was replaced by a different choice for each frequency subrange). The following new result avoids this difficulty for stabilizing control law design.

Theorem 3. *Suppose that a control law of the form (26) is applied to a differential linear repetitive process described by (7). Suppose also that the entire frequency range is divided into N frequency intervals given in (15). Then* *the resulting controlled process is stable along the pass if there exist matrices $\widehat{P}_{1i} \succ 0$, $\widehat{P}_{2i} \succ 0$, $\widehat{Q}_i \succ 0$, N_1, N_2, $\widehat{W}_1$ and $\widehat{W}_2$ such that the following LMIs*

$$\begin{aligned}
&\begin{bmatrix}
-\widehat{Q}_i & \widehat{P}_{1i}+j\omega_{ci}\widehat{Q}_i-\widehat{W}_1^T \\
\widehat{P}_{1i}-j\omega_{ci}\widehat{Q}_i-\widehat{W}_1 & -\omega_{i-1}\omega_i\widehat{Q}_i+\mathrm{sym}\left\{A\widehat{W}_1+BN_1\right\} \\
0 & \widehat{W}_2^TB_0^T+N_2^TB^T \\
0 & C\widehat{W}_1+DN_1
\end{bmatrix}\\
&\begin{bmatrix}
0 & 0 \\
B_0\widehat{W}_2+BN_2 & \widehat{W}_1^TC^T+N_1^TD^T \\
-\widehat{P}_{2i} & \widehat{W}_2^TD_0^T+N_2^TD^T \\
D_0\widehat{W}_2+DN_2 & \widehat{P}_{2i}-\widehat{W}_2^T-\widehat{W}_2
\end{bmatrix}\prec 0,
\end{aligned} \tag{27}$$

are feasible for all $i = 1, \ldots, N$, where $\omega_{ci} = (\omega_{i-1}+\omega_i)/2$. If these LMIs are feasible, stabilizing control law matrices K_1 and K_2 of (26) can be computed using

$$K_1 = N_1\widehat{W}_1^{-1},\ K_2 = N_2\widehat{W}_2^{-1}. \tag{28}$$

Proof. Suppose that set of LMIs (27) are feasible in $\widehat{P}_{1i} \succ 0$, $\widehat{P}_{2i} \succ 0$, $\widehat{Q} \succ 0$, N_1, N_2, $\widehat{W}_1$ and $\widehat{W}_2$ where, without loss of generality, it is also assumed that $\widehat{W}_1$ and $\widehat{W}_2$ are invertible. Next, apply the congruence transformation defined by $\mathrm{diag}\left\{\widehat{W}_1^{-1}, \widehat{W}_1^{-1}, \widehat{W}_2^{-1}, \widehat{W}_2^{-1}\right\}$, introduce the change the variables

$$Q_i=\widehat{W}_1^{-T}\widehat{Q}_i\widehat{W}_1^{-1}, P_{1i}=\widehat{W}_1^{-T}\widehat{P}_{1i}\widehat{W}_1^{-1}, W_1=\widehat{W}_1^{-1},\ W_2=\widehat{W}_2^{-1},\ P_{2i}=\widehat{W}_2^{-T}\widehat{P}_{2i}\widehat{W}_2^{-1},$$

and apply (28) to obtain

$$\begin{aligned}
&\begin{bmatrix}
-Q_i & P_{1i}+j\omega_{ci}Q_i-W_1 & 0 & 0 \\
P_{1i}-j\omega_{ci}Q_i-W_1^T & -\omega_{i-1}\omega_iQ_i+\mathrm{sym}\left\{W_1^TA+W_1^TBK_1\right\} & W_1^TB_0+W_1^TBK_2 & 0 \\
0 & B_0^TW_1+K_2^TB^TW_1 & -P_{2i} & 0 \\
0 & 0 & 0 & P_{2i}
\end{bmatrix}\\
&+\mathrm{sym}\left\{\begin{bmatrix} 0 \\ C^T+K_1^TD^T \\ D_0^T+K_2^TD^T \\ -I \end{bmatrix}\begin{bmatrix}0 \ 0 \ 0 \ W_2\end{bmatrix}\right\}\prec 0.
\end{aligned} \tag{29}$$

Introducing the substitutions

$$\Sigma=\begin{bmatrix}0 \ 0 \ 0 \ I\end{bmatrix},\ \Lambda=\begin{bmatrix}0 \ C+DK_1 \ D_0+DK_2 \ -I\end{bmatrix}$$

into (29) and then applying Lemma 3 gives

$$\begin{bmatrix} -Q_i & P_{1i}+j\omega_{ci}Q_i-W_1 & 0 \\ P_{1i}-j\omega_{ci}Q_i-W_1^T & -\omega_{i-1}\omega_iQ_i+\text{sym}\left\{W_1^TA+W_1^TBK_1\right\} & W_1^TB_0+W_1^TBK_2 \\ 0 & B_0^TW_1+K_2^TB^TW_1 & -P_{2i} \end{bmatrix}$$
$$+\begin{bmatrix} 0 \\ C^T+K_1^TD^T \\ D_0^T+K_2^TD^T \end{bmatrix} P_{2i} \begin{bmatrix} 0 & C+DK_1 & D_0+DK_2 \end{bmatrix} \prec 0.$$

Application of the Schur's complement formula to this last inequality gives (16) for the controlled process state-space model and by Theorem 2 stability along the pass holds for each frequency interval and hence for the entire range.

3.1. Design for stability and performance

In most practical situations, it will be required to design a control law for stability and performance and/or disturbance rejection. The vast majority of the currently available control law design algorithms for linear repetitive processes are for stabilization and there is a need for algorithms that ensure stability plus satisfactory transient response along the passes. An obvious application area with this requirement is iterative learning control as discussed in the conclusions' section of this paper.

Given that the eigenvalues of the current pass state matrix A in (7) govern the dynamics produced along the trial, the problem considered in this section is the design of the control law (26) for stability along the pass plus the assignment of the eigenvalues of $A_{cl} = A+BK_1$, inside a pre-specified region of the open left-half of the complex plane.

The assignment of the eigenvalues of A_{cl} is by choice of Φ in (4) or (18), where [10] this matrix must satisfy $\det(\Phi) < 0$ and hence the region of interest for the eigenvalues of A_{cl} is either a circle or a half-plane. Consider, therefore, the circle of radius r with center at c given by

$$C(c,r) := \{x + jy \in \mathbb{C} : |x + jy - c| = r\}. \tag{30}$$

Then, to guarantee that the interior of this circle is located in open left-half of the complex plane, requires $c < 0$ and $|c| > r$ and Φ of the form

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & -c \\ -c & |c|^2 - r^2 \end{bmatrix}. \tag{31}$$

Also if the eigenvalues of A_{cl} are located in $C(c, r)$ then the lower bound on the exponential-decay rate and the damping ratio of the corresponding modes are $|c| - r$ and $\sqrt{1 - \frac{r^2}{c^2}}$, respectively, for $c < 0$ and $|c| > r$. If the modes are required to have a minimum decay rate of $a > 0$ then the eigenvalues of A_{cl} must be located to the left of the line $x = -a$ in the complex plane. The region of the complex plane whose boundary is a straight line in the complex plane with normal vector $a + jb \neq 0$ is the collection of points given by

$$L(a, b, d) = \{x + jy \in \mathbb{C} : \; 2(ax + by + d) = 0\},$$

and the matrix Φ must be chosen as

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} = \begin{bmatrix} 0 & a + jb \\ a - jb & 2d \end{bmatrix}. \tag{32}$$

It is possible to guarantee that the eigenvalues of

$$G_{cl}(s) = (C + DK_1)(sI - (A + BK_1))^{-1}(B_0 + BK_2) + (D_0 + DK_2), \tag{33}$$

lie in the interior of the circle of radius $\gamma \leq 1$ with center at the origin. In this case Φ must be of the form (31) where $c = 0$ and $r = \gamma \leq 1$. Moreover, the inequality

$$\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \Phi \otimes P_2 \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} \prec 0,$$

guarantees that $\rho(G_{cl}(j\omega)) < \gamma \; \forall \omega \in \mathbb{R} \cup \infty$. The frequency response matrix $G_{cl}(j\omega)$ governs the pass-to-pass attenuation of the initial pass profile and it is possible to choose different $\gamma_i \leq 1$ for each frequency interval $i = 1, \ldots, N$. This, in turn, allows the rate of pass-to-pass attenuation of the initial pass profile frequency content to vary over the frequency intervals and this property is of particular importance in iterative learning control as discussed in the conclusions section of this paper.

Theorem 4. *Suppose that a control law of the form (26) is applied to a differential linear repetitive process described by (7). Suppose also that the entire frequency range is arbitrarily divided into intervals satisfying (15). Then the resulting controlled process is stable along the pass with eigenvalues of the state matrix* $A_{cl} = A + BK_1$ *located inside the region described by (31) or (32) and* $\rho(G_{cl}(j\omega)) < \gamma_i$ *if there exist matrices* $\widehat{P}_{1i} \succ 0$, $\widehat{P}_{2i} \succ 0$, $\widehat{Q}_i \succ 0$, N_1, N_2,

$\widehat{W}_1$ *and* $\widehat{W}_2$ *such that the following LMIs*

$$\begin{bmatrix} \Phi_{11}\widehat{P}_{1i} - \widehat{Q}_i & \Phi_{12}\widehat{P}_{1i} + j\omega_{ci}\widehat{Q}_i - \widehat{W}_1^T \\ \Phi_{12}^T\widehat{P}_{1i} - j\omega_{ci}\widehat{Q}_i - \widehat{W}_1 & \Phi_{22}\widehat{P}_{1i} - \omega_{i-1}\omega_i\widehat{Q}_i + \mathrm{sym}\left\{A\widehat{W}_1 + BN_1\right\} \\ 0 & \widehat{W}_2^T B_0^T + N_2^T B^T \\ 0 & C\widehat{W}_1 + DN_1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ B_0\widehat{W}_2 + BN_2 & \widehat{W}_1^T C^T + N_1^T D^T \\ -\gamma_i\widehat{P}_{2i} & \widehat{W}_2^T D_0^T + N_2^T D^T \\ D_0\widehat{W}_2 + DN_2 & \widehat{P}_{2i} - \widehat{W}_2^T - \widehat{W}_2 \end{bmatrix} \prec 0, \tag{34}$$

are feasible $i = 1, \ldots, N$, *where* $\omega_{ci} = (\omega_{i-1} + \omega_i)/2$. *If these LMIs are feasible, the required control law matrices* K_1 *and* K_2 *of (26) can be calculated using (28).*

Proof. The LMI (34) is established in an identical manner to that of (27) in Theorem 3 where the regional eigenvalue location constraints are imposed by choosing the appropriate form of Φ.

4. Numerical example

The dynamics of certain types of material rolling can be approximated by a differential linear repetitive process and in this section the linearized model from [14] is considered where in (7)

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -a_0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ c_0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 \\ -b_0 + a_0 b_2 \end{bmatrix}, \\ &C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0,\ D_0 = -b_2, \end{aligned} \tag{35}$$

with

$$a_0 = \frac{\lambda_1\lambda_2}{M(\lambda_1+\lambda_2)}, b_0 = \frac{-\lambda_1\lambda_2}{M(\lambda_1+\lambda_2)}, b_2 = \frac{-\lambda_2}{\lambda_1+\lambda_2}, c_0 = \frac{-\lambda_1}{M(\lambda_1+\lambda_2)},$$

where M is the lumped mass of the roll-gap adjusting mechanism, λ_1 is the stiffness of the adjustment mechanism spring, λ_2 is the hardness of the metal strip and $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ is the composite stiffness of the metal strip and the roll mechanism.

In this paper the data used is $\lambda_1 = 600$ N/m, $\lambda_2 = 2000$ N/m, $M = 100$ kg and $\lambda = 461.54$, resulting in the state-space model matrices

$$\left[\begin{array}{c|c|c} A & B & B_0 \\ \hline C & D & D_0 \end{array}\right] = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ -4.6154 & 0 & -0.0023 & 1.0651 \\ \hline 1 & 0 & 0 & 0.7692 \end{array}\right].$$

Suppose also that the entire frequency range $[0, \infty]$ is divided into 4 frequency intervals (frequencies are in [rad/sec])

$$[0, \infty] = [0, 1] \cup [1, 2] \cup [2, 3] \cup [3, \infty].$$

Setting $\beta = 10$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$, the LMIs (34) for the regional constraint $C(-5, 2)$ on the eigenvalues of the matrix $A_{cl} = A + BK_1$ are feasible and give

$$K_1 = \begin{bmatrix} 7516.5 \ 4018.9 \end{bmatrix}, \quad K_2 = 440.1.$$

The plot generated by $|G_{cl}(j\omega)|$ computed from (33) for this SISO example is shown in Fig. 1 and confirms that $|G_{cl}(j\omega)| < 1$ for all the frequency intervals as required. Also a routine computation confirms that the eigenvalues

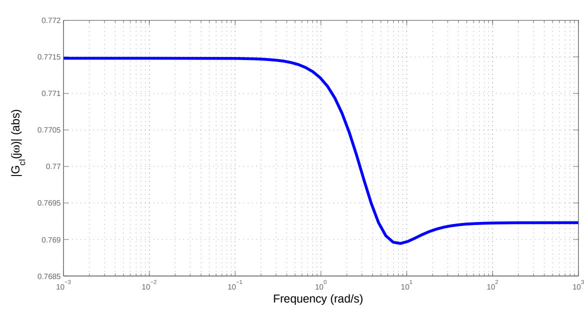

Figure 1: The magnitude of $G_{cl}(j\omega)$ for the controlled process.

of $A_{cl} = A + BK_1$ are $-4.6372 \pm j0.6763$ and clearly lie inside the specified region $C(-5, 2)$.

5. Conclusions

This paper has developed new LMI based conditions for stability of differential linear repetitive processes, leading to new control law design algorithms. These new algorithms allow control law design in the presence of practically motivated design specifications such as regional constraints on the location of the eigenvalues of the state matrix of the controlled process. There is much further research required to fully exploit this approach to control law design with application examples and, in particular, iterative learning control as discussed next.

Iterative learning control (ILC) has been developed for systems that repeat the same finite duration operation over and over. The exact sequence is that an execution, known as a trial is completed and then the system resets to the original location ready for the start of the next one. On competition of each trial all information generated during its execution is available for use in updating the control law for the next trial [2, 3]. Given a reference vector, the error on each trial can be computed and used, together with other previous trial information if required, to compute the control signal to be used on the next trial. Let $y_{ref}(t),\ 0 \leq t \leq \alpha$, be the reference signal and $e_k(t) = y_{ref}(t) - y_k(t)$ the error on trial k. Then the design problem is to enforce error convergence in k under some suitable norm but in many applications it is also required to regulate the dynamics produced along the trials. If the along the trial dynamics are discrete, one way to proceed is to design a feedback control loop for the system and then enforce pass-to-pass error convergence based on the lifted model of the iterative learning control dynamics, see many references in the survey papers [2, 3]. This option is not available if the along the trial dynamics are differential and direct digital control is not possible. Using a repetitive process setting for analysis it is possible to consider control law for along the pass performance and pass-to-pass error convergence simultaneously as outlined next.

Consider a differential linear system whose dynamics are described in the ILC setting as

$$\begin{aligned} \dot{x}_{k+1}(t) =& Ax_{k+1}(t) + Bu_{k+1}(t), \\ y_{k+1}(t) =& Cx_{k+1}(t),\ 0 \leq t \leq \alpha,\ k \geq 0, \end{aligned} \tag{36}$$

where the subscript k denotes the trial number, α the trial length, $x_k(t) \in \mathbb{R}^n$ is the state vector, $y_k(t) \in \mathbb{R}^m$ is the pass profile vector, $u_k(t) \in \mathbb{R}^r$ is the input vector and no loss of generality arises from assuming zero state initial vector on each trial. Also introduce, for analysis purposes, the following vector defined in terms of the difference between the current and previous pass state vector in the system state-space model

$$\eta_{k+1}(t) = \int_0^t (x_{k+1}(\tau) - x_k(\tau))\, d\tau.$$

Suppose that the ILC law to be applied computes the input on trial $k+1$ as $u_{k+1}(t) = u_k(t) + \Delta u_{k+1}(t)$, where this latter term is the correction to the input used on the previous trial and one possible choice is $\Delta u_{k+1}(t) = K_1\dot{\eta}_{k+1}(t) + K_2\dot{e}_k(t)$. In this case, the controlled system dynamics can be written in the form of (7) as

$$\begin{aligned}\dot{\eta}_{k+1}(t) =&(A+BK_1)\eta_{k+1}(t)+(BK_2)e_k(t),\\ e_{k+1}(t) =&-C(A+BK_1)\eta_{k+1}(t)+(I-CBK_2)e_k(t).\end{aligned}$$

This is a differential linear repetitive process state-space model of the form (7) with no input terms, state vector $\eta_{k+1}(t)$ and output (pass profile) vector $e_{k+1}(t)$. Hence both the asymptotic and along the pass stability properties can be applied. If the former is present, that is, $\rho(I-CBK_2) < 1$, then the trial-to-trial error $\{e_k\}$ converges but, due to the finite trial length, the dynamics produced along the trials can be unacceptable. In such cases, stability along the pass (or trial in ILC terminology) is an option and further research effort should be directed towards full exploitation of this approach with onward transfer to applications. Robust control law design should also be addressed.

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