

# Convolution and Concurrency

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## Abstract

We show how concurrent quantales and concurrent Kleene algebras arise as convolution algebras  $Q^X$  of functions from structures  $X$  with two ternary relations that satisfy relational interchange laws into concurrent quantales or Kleene algebras  $Q$ . The elements of  $Q$  can be understood as weights; the case  $Q = \mathbb{B}$  corresponds to a powerset lifting. We develop a correspondence theory between relational properties in  $X$  and algebraic properties in  $Q$  and  $Q^X$  in the sense of modal and substructural logics, and boolean algebras with operators. As examples, we construct the concurrent quantales and Kleene algebras of  $Q$ -weighted words, digraphs, posets, isomorphism classes of finite digraphs and pomsets.

## 1 Introduction

Our initial motivation has been the provision of recipes for constructing graph models for concurrent quantales and concurrent Kleene algebras [HMSW11]. These axiomatise the sequential and parallel compositions  $\cdot$  and  $\parallel$  of concurrent and distributed systems as well as their finite sequential and parallel iterations and impose that these compositions interact via a lax interchange law  $(w \parallel x) \cdot (y \parallel z) \leq (w \cdot y) \parallel (x \cdot z)$ . Two classical models are languages over finite words with respect to sequential and shuffle composition in interleaving concurrency, and languages over partial orders or partial words (pomsets) with respect to serial and parallel composition in true concurrency. The relation  $\leq$  is interpreted as set inclusion in these models.

In both models, the language-level algebras are constructed by lifting structural properties of compositions from single objects—single words, single posets—to power sets. In fact, both liftings are instances of the classical Stone-type duality between  $n + 1$ -ary relations and  $n$ -ary operators (or modalities) on power set boolean algebras [JT51, Gol89]; here for ternary relations and binary operators. In the word model, the ternary relations on words are  $u = v \cdot w$  and  $u \in v \parallel w$ ; the binary operators on power sets are  $X \cdot Y = \{u \cdot v \mid u \in X \wedge v \in Y\}$  and  $X \parallel Y = \bigcup \{u \parallel v \mid u \in Y \wedge v \in X\}$ . In the poset model, the ternary relations on posets are  $P = P_1 \cdot P_2$  provided  $P_1 \cdot P_2$  is defined ( $P_1 \cdot P_2$  being disjoint union with additional arrows from each element of  $P_1$  to each element of  $P_2$ ) and  $P \preceq P_1 \parallel P_2$  provided  $P_1 \parallel P_2$  is defined ( $P_1 \parallel P_2$  being disjoint union and the relation holds if there is a bijective order morphism from the right-hand poset to the left-hand one). The binary operators on power sets are  $X \cdot Y = \{P_1 \cdot P_2 \mid P_1 \in X, P_2 \in Y \text{ and } P_1 \cdot P_2 \text{ is defined}\}$  and  $X \parallel Y = \{P \mid \exists P_1 \in X, P_2 \in Y. P \preceq P_1 \parallel P_2 \text{ and } P_1 \parallel P_2 \text{ is defined}\}$ .

Both constructions generalise further to weighted words and weighted po(m)sets [DKV09] and beyond that—yet ignoring interchange—to arbitrary functions  $X \rightarrow Q$  from partial monoids or ternary relations over  $X$  into quantales  $Q$  [DHS16, DHS17]. The binary operations on function spaces  $Q^X$  then generalise to convolutions of the form  $(f * g) x = \bigvee \{f x \bullet g y \mid R_{yz}^x\}$ , and the algebra on the function space  $Q^X$  is called *convolution algebra*. This raises the more specific question how concurrent quantales and similar structures on  $Q^X$  can be constructed from ternary relations on  $X$  and weight quantales  $Q$ —in particular the above lax interchange law and its variants on  $Q^X$ . This question is not only of structural interest. Operationally, checking relational properties on  $X$  tends to be much simpler than those on  $Q^X$ , and the first activity suffices if the construction of  $Q^X$  from  $X$  and  $Q$  is uniform. The rest of this article investigates this question.

First, in Section 2, we summarise the previous approach to relational convolution in  $Q^X$  [DHS17], where  $X$  is a set equipped with a ternary relation and  $Q$  a quantale, and introduce the basic lifting construction, namely that  $Q^X$  forms a quantale if  $X$  satisfies a relational associativity law and  $Q$  is a quantale, and if a suitable set of relational units is present.

In Section 3 we prove novel correspondence results between relational interchange laws on  $X$  and algebraic interchange laws on  $Q$  and  $Q^X$ . Proposition 3.4 shows that interchange on  $X$  and  $Q$  give rise to those on  $Q^X$ . In addition, under mild non-degeneracy conditions on  $Q$ , interchange laws on  $Q$  and  $Q^X$  give rise to those on  $X$  (Proposition 3.7) and, under mild non-degeneracy conditions on  $X$ , interchange laws on  $X$  and  $Q^X$  give rise to those on  $Q$  (Proposition 3.8). In combination, these results show that interchange laws on  $X$  and  $Q$  are precisely what is needed to obtain such laws on  $Q^X$ .

Additional correspondences are then presented in Section 4. First, we prove such results for sets of relational units in  $X$  and quantalic units on  $Q$  and  $Q^X$  and show how the above degeneracy conditions simplify in the presence of units. Secondly, we show how correspondences for (semi-)associativity and commutativity laws arise from those for interchange.

Equipped with these correspondences we then introduce relational interchange monoids and interchange quantales in Section 5 and package the individual correspondences for these structures in the main theorem of this article (Theorem 5.6): a correspondence result between relational monoids  $X$ , interchange quantales  $Q$  and interchange quantales  $Q^X$ . Interchange quantales are essentially concurrent quantales without commutativity assumptions on the “parallel” composition. In addition we prove a weak Eckmann-Hilton argument that shows that certain small interchange laws are subsumed by the one presented above.

In light of the duality between  $n+1$ -ary relations and boolean algebras with  $n$ -ary operators, the natural question arises how a more general duality between  $X$ ,  $Q$  and  $X^Q$  can be obtained. Partial results are already known [HWW18]. We explain the special case of the power set lifting ( $Q = \mathbb{B}$ ) in Section 6 and relate this results with constructions from Section 3, but leave the general case for future work.

In Section 7 we specialise Theorem 5.6 to interchange Kleene algebras and concurrent Kleene algebras, which requires finiteness and grading assumptions on ternary relations (Theorem 7.6).

Finally, Sections 8-11 apply the constructions from Theorem 5.6 and 7.6 to the examples mentioned above. In Section 8 we construct the concurrent quantale and Kleene algebra of  $Q$ -weighted shuffle languages using an isomorphism between ternary relations and certain multimonooids [GL06]. In Section 10 and 11 we construct the concurrent quantale and Kleene algebra of  $Q$ -weighted digraph languages and those of isomorphism classes of finite digraphs. To prepare for these constructions, Section 9 introduces partial interchange monoids, which form relational interchange monoids under certain restrictions. It then suffices to show that graphs under the operations  $\cdot$  and  $\parallel$  outlined form such monoids. The specialisation to (weighted) partial orders and partial words or pomsets, which are isomorphism classes of labelled partial orders, is then straightforward.

Ultimately these results yield uniform construction principles for (weighted) concurrent quantales and Kleene algebras from simpler structures such as ternary relations, multimonooids and similar ordered monoidal structures: to construct such structures it suffices to know the underlying relational structure, the rest is then automatic. Beyond that, our results provide valuable structural insights that might be stepping stones to future duality results.

## 2 Relational Convolution: a Summary

This section summarises the general approach.

Relational convolution [DHS17] has its origins in Jónsson and Tarski’s boolean algebras with operators [JT51], Rota’s foundations of combinatorics [Rot64], Schützenberger’s approach to language theory [BR84, DKV09] and Goguen’s L-fuzzy maps and relations [Gog67]. It is an operation in the algebra of functions  $X \rightarrow Q$  from a set  $X$  into a complete lattice  $Q$  equipped with an additional operation

- of composition and constrained by a ternary relation on  $X$ , which we identify with a predicate of type

$X \rightarrow X \rightarrow X \rightarrow \mathbb{B}$ . It is defined as

$$(f * g) x = \bigvee_{y,z:R_{yz}^x} f y \bullet g z,$$

where the right-hand side abbreviates  $\bigvee\{f y \bullet g z \mid R_{yz}^x\}$  and  $\bigvee$  denotes the supremum in  $Q$ . It is well known that the function space  $Q^X$  forms a complete lattice when the order and sups in  $Q$  are extended pointwise [AJ94]. Yet the convolution  $*$  need not satisfy any algebraic laws on  $Q^X$ , unless conditions on  $R$  and  $Q$  are imposed.

This is reminiscent of modal correspondence theory, where conditions on relational Kripke frames force algebraic properties of modal operators and vice versa, or more generally to dualities between categories of  $n + 1$ -ary relational structures and those of boolean algebras with  $n$ -ary operators [JT51, Gol89]. In fact,  $R$  is a ternary relational structure and  $*$  a binary modality similar to the product of the Lambek calculus [Lam58], the chop operation of interval temporal logics [MM83] or the separating conjunction of separation logic [ORY01].

**Example 2.1.** Let  $X$  be an incidence algebra of closed intervals (over  $\mathbb{R}$ , say) [Rot64], with interval fusion  $[p, q][r, s]$  equal to  $[p, s]$  if  $q = r$ , and undefined otherwise. Let  $R_{yz}^x$  hold if the fusion of intervals  $y$  and  $z$  is defined and equal to  $x$ . Let  $Q = \mathbb{B}$  be the (complete) lattice of booleans with  $\bullet$  as meet. Functions  $X \rightarrow \mathbb{B}$  are then predicates ranging over intervals in  $X$ . The predicate  $f * g$  holds of an interval  $x$  whenever it can be decomposed into a prefix interval  $y$  and a suffix interval  $z$  by fusion such that  $f y$  and  $g z$  both hold. This captures the semantics of the binary chop modality of interval temporal logics [MM83].  $\square$

It is well known that chop is associative in the convolution algebra  $\mathbb{B}^X$  due to associativity of meet in  $\mathbb{B}$  and associativity of interval fusion in  $X$ —up to definedness.

**Definition 2.2** ([Ros90]). A *quantale*  $Q$  is a complete lattice equipped with an associative composition  $\bullet$  that preserves sups in both arguments: for all  $a, b \in Q$  and  $A, B \subseteq Q$ ,

$$a \bullet \left( \bigvee B \right) = \bigvee \{a \bullet b \mid b \in B\} \quad \text{and} \quad \left( \bigvee A \right) \bullet b = \bigvee \{a \bullet b \mid a \in A\}.$$

A quantale is *unital* if  $\bullet$  has a unit 1.

Convolution is then associative in  $Q^X$  if  $R$  is *relationally associative* [DHS17]: for all  $x, u, v, w \in X$ ,

$$\exists y. R_{uv}^y \wedge R_{yw}^x \Leftrightarrow \exists y. R_{uy}^x \wedge R_{vw}^y.$$

This yields one direction of a correspondence between the ternary relation  $R$  and convolution  $*$  viewed as a binary modality. Similarly, convolution is commutative in  $Q^X$  if  $R$  is *relationally commutative*: for all  $x, u, v \in X$ ,

$$R_{uv}^x \Rightarrow R_{vu}^x.$$

Finally, if the unary relation  $E^x$  is a set of relational units for  $R$  and the quantale  $Q$  unital with unit of composition 1, then convolution has the indicator function  $id_E$  as a left and right unit, where

$$id_E x = \begin{cases} 1, & \text{if } E^x, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.3** ([DHS17]). The set  $E \subseteq X$  is a *set of relational units* for the ternary relation  $R$  over  $X$  if it satisfies, for all  $x, y \in X$ ,

$$\exists e. R_{ex}^x \wedge E^e, \quad R_{ey}^x \wedge E^e \Rightarrow x = y, \quad \exists e. R_{xe}^x \wedge E^e, \quad R_{ye}^x \wedge E^e \Rightarrow x = y.$$

This guarantees that each  $x \in X$  has a unique left unit as well as a unique right one. With the Kronecker delta function  $\delta_x : X \rightarrow \mathbb{B}$  defined as  $\delta_x y$  equal to 1 if  $x = y$  and to 0 otherwise, therefore,

$$id_E = \bigvee_{e: E^e} \delta_e.$$

The convolution algebras on  $Q^X$  can now be described as follows.

**Theorem 2.4** ([DHS17]).

1. If  $R$  is relationally associative and  $Q$  a quantale, then  $Q^X$  is a quantale.
2. If  $R$  is relationally associative and commutative and  $Q$  an abelian quantale, then  $Q^X$  is an abelian quantale.
3. If  $R$  is relationally associative and has relational units, and  $Q$  is a unital quantale, then  $Q^X$  is a unital quantale .

**Example 2.5.** Let  $(X^*, \cdot, \varepsilon)$  be the free monoid over  $X$ . For  $u, v, w \in X^*$  define  $R_{vw}^u \Leftrightarrow u = v \cdot w$ . Then  $R$  is relationally associative with relational unit  $\varepsilon$ . For any quantale  $Q$ , the convolution algebra is the quantale  $Q^{X^*}$  of  $Q$ -weighted languages over  $X$  whereas  $\mathbb{B}^{X^*}$  is the usual language quantale over  $X$ . Convolution is (weighted) language product (the boolean case is also known as complex or Minkowski product). The construction generalises to arbitrary monoids.

Words are *finitely decomposable* in that each word can only be split into finitely many prefix/suffix pairs. All sups in convolutions therefore remain finite and  $Q$  can be replaced by an arbitrary semiring. This yields the usual weighted languages formalised as rational power series in the sense of Schützenberger and Eilenberg [BR84, DKV09].  $\square$

**Example 2.6.** Define a composition  $\cdot : X \times X \rightarrow X$  on a set  $X$  such that  $(a, b) \cdot (c, d)$  is equal to  $(a, d)$  if  $b = c$  and undefined otherwise. For  $x, y, z \in X \times X$ , let  $R_{yz}^x$  hold if and only if  $y \cdot z$  is defined and equal to  $x$ , and let  $E = \{(a, a) \mid a \in X\}$ . Then  $R$  is relationally associative and the elements of  $E$  are the relational units. For any quantale  $Q$ , the convolution algebra  $Q^{X \times X}$  is the quantale of  $Q$ -weighted binary relations over  $X$ , while  $\mathbb{B}^{X \times X}$  is simply the quantale of binary relations over  $X$ . Convolution is (weighted) relational composition. This specialises to quantales of weighted closed intervals in linear orders, as in Example 2.1, which can be represented by ordered pairs  $(a, b)$  in which  $a \leq b$ .  $\square$

**Example 2.7.** Define a composition  $\oplus$  on the set of partial functions of type  $X \rightarrow Y$  such that  $f \oplus g$  is  $f \cup g$  if  $\text{dom } f$  and  $\text{dom } g$  are disjoint and undefined otherwise. The set  $Y^X$  can model the heap memory area with addresses in  $X$ , values in  $Y$  and  $\oplus$  as heaplet addition. For  $f, g, h : X \rightarrow Y$  let  $R_{gh}^f$  hold if and only if  $g \oplus h$  is defined and equal to  $f$ . Then  $R$  is relationally associative and commutative; the empty partial function (which is undefined everywhere) is its relational unit. For any abelian quantale  $Q$ , the convolution algebra is the abelian quantale  $Q^{(Y^X)}$  of  $Q$ -weighted assertions of separation logic over the set  $Y^X$  of heaps. Convolution is separating conjunction [DHS16]. The standard assertion algebra of separation logic is formed by  $\mathbb{B}^{(Y^X)}$ .  $\square$

### 3 Correspondences for Interchange Laws

Theorem 2.4 generalises to correspondences between quantales with two compositions  $\bullet$  and  $\blacklozenge$  related by seven interchange laws and relational structures with suitable relational constraints. The choice of the interchange laws is explained in Section 5; the six small interchange laws are precisely those that can be derived from the seventh in the presence of suitable units. We start with the relational structures.

**Definition 3.1.**

1. A *relational magma*  $(X, R)$  is a set  $X$  equipped with a ternary relation  $R : X \rightarrow X \rightarrow X \rightarrow \mathbb{B}$ . It is *unital* if there is a set  $E$  of relational units satisfying the axioms in Definition 2.3.
2. A *relational bi-magma*  $(X, \mathbf{R}, \mathbf{R})$  is a set  $X$  equipped with two ternary relations  $\mathbf{R}$  and  $\mathbf{R}$ . It is *unital* if  $\mathbf{R}$  has a set of relational units  $\mathbf{E}$  and  $\mathbf{R}$  a set of relational units  $\mathbf{E}$ .

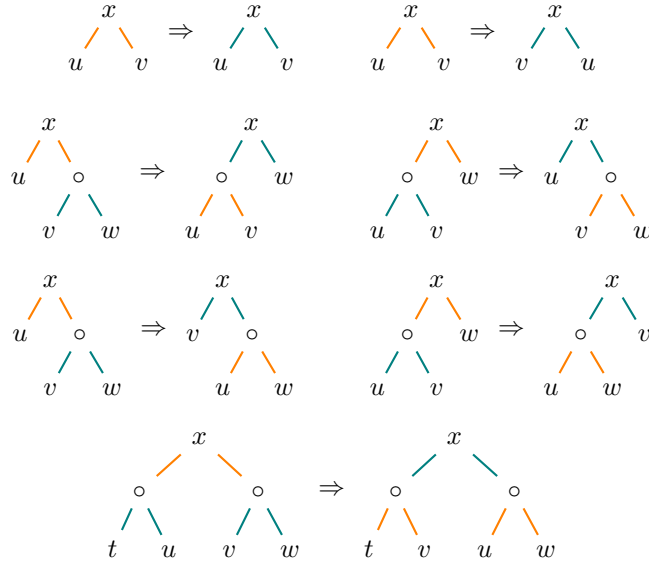


Figure 1: Trees memoised by the relational interchange laws.

The constraints considered on a bi-magma  $X$  are, for  $t, u, v, w, x, y, z \in X$ , the *relational interchange laws*

$$R_{uv}^x \Rightarrow R_{uv}^x, \quad (\text{RI}_1)$$

$$R_{uv}^x \Rightarrow R_{vu}^x, \quad (\text{RI}_2)$$

$$\exists y. R_{uy}^x \wedge R_{vw}^y \Rightarrow \exists y. R_{uv}^y \wedge R_{yw}^x, \quad (\text{RI}_3)$$

$$\exists y. R_{uv}^y \wedge R_{yw}^x \Rightarrow \exists y. R_{uy}^x \wedge R_{vw}^y, \quad (\text{RI}_4)$$

$$\exists y. R_{uy}^x \wedge R_{vw}^y \Rightarrow \exists y. R_{vy}^x \wedge R_{uw}^y, \quad (\text{RI}_5)$$

$$\exists y. R_{uv}^y \wedge R_{yw}^x \Rightarrow \exists y. R_{uv}^x \wedge R_{yw}^y, \quad (\text{RI}_6)$$

$$\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z \Rightarrow \exists y, z. R_{tv}^y \wedge R_{yz}^x \wedge R_{uw}^z. \quad (\text{RI}_7)$$

They memoise the relationships between the trees in  $X$  shown in Figure 1.

Next we turn to quantales. As their monoidal structure emerges in the constructions, we generalise.

**Definition 3.2.**

1. A *prequantale* [Ros90] is a structure  $(Q, \leq, \bullet)$  such that  $(Q, \leq)$  is a complete lattice and the binary operation  $\bullet$  on  $Q$  preserves sups in both arguments. It is *unital* if  $\bullet$  has unit  $1$ .
2. A *bi-prequantale* is a structure  $(Q, \leq, \bullet, \circ)$  such that  $(Q, \leq, \bullet)$  and  $(Q, \leq, \circ)$  are both prequantales. It is unital if  $\bullet$  has unit  $1$  and  $\circ$  unit  $1$ .

A quantale is thus a prequantale with associative composition.

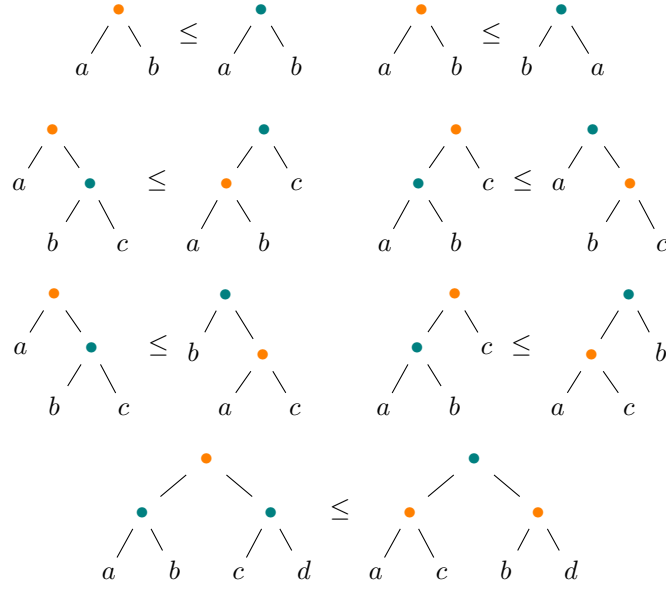


Figure 2: Syntax trees of the algebraic interchange laws.

For  $a, b, c, d \in Q$  we define the *algebraic interchange laws*

$$a \bullet b \leq a \bullet b, \quad (\text{I}_1)$$

$$a \bullet b \leq b \bullet c, \quad (\text{I}_2)$$

$$a \bullet (b \bullet c) \leq (a \bullet b) \bullet c, \quad (\text{I}_3)$$

$$(a \bullet b) \bullet c \leq a \bullet (b \bullet c), \quad (\text{I}_4)$$

$$a \bullet (b \bullet c) \leq b \bullet (a \bullet c), \quad (\text{I}_5)$$

$$(a \bullet b) \bullet c \leq (a \bullet c) \bullet b, \quad (\text{I}_6)$$

$$(a \bullet b) \bullet (c \bullet d) \leq (a \bullet c) \bullet (b \bullet d). \quad (\text{I}_7)$$

Interestingly, the syntax trees of these laws in  $Q$ , as shown in Figure 2, have the structure of the trees in  $X$  in Figure 1. The following example provides some intuition.

**Example 3.3.** Let  $X = Q$ ,  $R_{ab}^x \Leftrightarrow x \leq a \bullet b$  and  $R_{ab}^x \Leftrightarrow x \leq a \bullet b$ . Then, for instance,

$$\exists y, z. R_{ab}^y \wedge R_{yz}^x \wedge R_{cd}^z \Leftrightarrow x \leq (a \bullet b) \bullet (c \bullet d),$$

and likewise for the other terms in interchange laws. The relational and algebraic interchange laws then translate into each other. For instance, for (RI<sub>7</sub>) and (I<sub>7</sub>),

$$\begin{aligned} (\exists y, z. R_{ab}^y \wedge R_{yz}^x \wedge R_{cd}^z \Rightarrow \exists y, z. R_{ac}^y \wedge R_{yz}^x \wedge R_{bd}^z) &\Leftrightarrow (x \leq (a \bullet b) \bullet (c \bullet d) \Rightarrow x \leq (a \bullet c) \bullet (b \bullet d)) \\ &\Leftrightarrow (a \bullet b) \bullet (c \bullet d) \leq (a \bullet c) \bullet (b \bullet d), \end{aligned}$$

that is,

$$\left( \begin{array}{c} \begin{array}{c} x \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ t \quad u \quad v \quad w \end{array} \Rightarrow \begin{array}{c} x \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ t \quad v \quad u \quad w \end{array} \end{array} \right) \Leftrightarrow \left( \begin{array}{c} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ t \quad u \quad v \quad w \end{array} \leq \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ t \quad v \quad u \quad w \end{array} \end{array} \right).$$

With this particular encoding, the relational interchange laws simply memoise the algebraic ones.  $\square$

In general, however, the relationship between relational and algebraic convolution is more complex. The left-to-right translation in Example 3.3 can therefore fail.

For functions  $f, g : X \rightarrow Q$  we define the convolutions

$$(f * g)x = \bigvee_{y, z: R_{yz}^x} f y \bullet g z \quad \text{and} \quad (f \ast g)x = \bigvee_{y, z: R_{yz}^x} f y \bullet g z.$$

They can be represented using trees in  $X$ ,  $Q$  and  $Q^X$  as

$$f \overset{*}{/} g = \lambda x. \bigvee \left\{ \begin{array}{c} \bullet \\ / \quad \backslash \\ f y \quad g z \end{array} \middle| \begin{array}{c} x \\ / \quad \backslash \\ y \quad z \end{array} \right\} \quad \text{and} \quad f \overset{\ast}{/} g = \lambda x. \bigvee \left\{ \begin{array}{c} \bullet \\ / \quad \backslash \\ f y \quad g z \end{array} \middle| \begin{array}{c} x \\ / \quad \backslash \\ y \quad z \end{array} \right\}.$$

Convolution thus translates trees with the same structure in  $X$  and  $Q$  into trees in  $Q^X$ .

One can then prove correspondences between relational and algebraic interchange laws. First we show that relational interchange laws in  $X$  and algebraic interchange laws in  $Q$  force algebraic interchange laws in the convolution algebra on  $Q^X$ .

**Proposition 3.4.** *Let  $X$  be a relational bi-magma and  $Q$  a bi-prequantale. Then  $(RI_k)$  in  $X$  and  $(I_k)$  in  $Q$  imply  $(I_k)$  in  $Q^X$ , for each  $1 \leq k \leq 7$ .*

*Proof.* Let  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z \Rightarrow \exists y, z. R_{tv}^y \wedge R_{yz}^x \wedge R_{uw}^z$  in  $X$  and  $(w \bullet x) \bullet (y \bullet z) \leq (w \bullet y) \bullet (x \bullet z)$  in  $Q$ . Then

$$\begin{aligned} ((f * g) * (h * k)) x &= \bigvee \left\{ \bigvee \{ f t \bullet g u \mid R_{tu}^y \} \bullet \bigvee \{ h v \bullet k w \mid R_{vw}^z \} \mid R_{yz}^x \right\} \\ &= \bigvee \{ (f t \bullet g u) \bullet (h v \bullet k w) \mid \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z \} \\ &\leq \bigvee \{ (f t \bullet h v) \bullet (g u \bullet k w) \mid \exists y, z. R_{tv}^y \wedge R_{yz}^x \wedge R_{uw}^z \} \\ &= \bigvee \left\{ \bigvee \{ f t \bullet h v \mid R_{tv}^y \} \bullet \bigvee \{ g u \bullet k w \mid R_{uw}^z \} \mid R_{yz}^x \right\} \\ &= ((f * h) * (g * k)) x \end{aligned}$$

proves  $(I_7)$  in  $Q^X$ . Alternatively, using trees,

$$\begin{aligned} \begin{array}{c} \bullet \\ / \quad \backslash \\ / \quad \backslash \\ f \quad g \quad h \quad k \end{array} &= \lambda x. \bigvee \left\{ \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ f t \quad g u \quad h v \quad k w \end{array} \middle| \begin{array}{c} x \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \quad / \quad \backslash \\ t \quad u \quad v \quad w \end{array} \right\} \\ &\leq \lambda x. \bigvee \left\{ \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ f t \quad h v \quad g u \quad k w \end{array} \middle| \begin{array}{c} x \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \quad / \quad \backslash \\ t \quad v \quad u \quad w \end{array} \right\} \\ &= \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ f \quad h \quad g \quad k \end{array}. \end{aligned}$$

The proofs for the small interchange laws are similar, and left to the reader. In particular, the proof of  $(I_3)$  from  $(RI_3)$  and that of  $(I_4)$  from  $(RI_4)$  are related by *opposition*: one can be obtained from the other by swapping the operands of  $\ast$ ,  $\ast$ ,  $\bullet$  and  $\bullet$  and the lower indices of  $R$  and  $R$ , that is, by reversing the algebraic syntax trees in  $Q$  and the trees in  $X$  memoised in the relational interchange laws. The same holds for the proof of  $(I_5)$  from  $(RI_5)$  and that of  $(I_6)$  from  $(RI_6)$ .  $\square$

Next we show that, under mild nondegeneracy conditions on  $Q$ , algebraic interchange laws in  $Q^X$  force relational interchange laws in  $X$ , and that under mild nondegeneracy conditions on  $X$ , algebraic interchange laws in  $Q^X$  force algebraic interchange laws in  $Q$ . Yet we begin with a definition and some lemmas.

For all  $x, y \in X$  and  $a \in Q$ , we define the function  $\delta_x^a : X \rightarrow Q$  by

$$\delta_x^a y = \begin{cases} a, & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases}$$

and the function  $(- | -) : Q \rightarrow \mathbb{B} \rightarrow Q$ , for all  $a \in Q$  and  $P : \mathbb{B}$ , by

$$(a | P) = \begin{cases} a, & \text{if } P, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously,  $\delta_x^a y = (a | x = y)$ .

**Lemma 3.5.** *Let  $X$  be a relational bi-magma and  $Q$  a bi-prequantale. For all  $a, b, c, d \in Q$  and  $x, t, u, v, w \in X$ ,*

- (1)  $(\delta_u^a * \delta_v^b) x = (a \bullet b | R_{uv}^x)$ ,
- (2)  $(\delta_u^a * (\delta_v^b * \delta_w^c)) x = (a \bullet (b \bullet c) | \exists y. R_{uy}^x \wedge R_{vw}^y)$ ,
- (3)  $((\delta_u^a * \delta_v^b) * \delta_w^c) x = ((a \bullet b) \bullet c | \exists y. R_{uv}^y \wedge R_{yw}^x)$ ,
- (4)  $((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x = ((a \bullet b) \bullet (c \bullet d) | \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z)$ ,
- (5) *properties (1)–(4) hold with colours interchanged.*

*Proof.* For (4), we use the proof of Proposition 3.4 to calculate

$$\begin{aligned} ((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x &= \bigvee \{ (\delta_t^a x_1 \bullet \delta_u^b x_2) \bullet (\delta_v^c x_3 \bullet \delta_w^d x_4) \mid \exists y, z. R_{x_1 x_2}^y \wedge R_{yz}^x \wedge R_{x_3 x_4}^z \} \\ &= ((a \bullet b) \bullet (c \bullet d) \mid \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z) \\ &\leq (a \bullet b) \bullet (c \bullet d). \end{aligned}$$

Alternatively, using trees,

$$\begin{aligned} & \begin{array}{c} \text{*} \\ / \quad \backslash \\ \text{*} \quad \text{*} \\ / \quad \backslash \quad / \quad \backslash \\ \delta_t^a \quad \delta_u^b \quad \delta_v^c \quad \delta_w^d \end{array} = \lambda x. \bigvee \left\{ \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \delta_t^a x_1 \quad \delta_u^b x_2 \quad \delta_v^c x_3 \quad \delta_w^d x_4 \end{array} \left| \begin{array}{c} x \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \quad / \quad \backslash \\ x_1 \quad x_2 \quad x_3 \quad x_4 \end{array} \right. \right\} \\ &= \lambda x. \left( \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ a \quad b \quad c \quad d \end{array} \left| \begin{array}{c} x \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \quad / \quad \backslash \\ t \quad u \quad v \quad w \end{array} \right. \right) \\ &\leq \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ a \quad b \quad c \quad d \end{array}. \end{aligned}$$

All other proofs are similar, and left to the reader. In particular, (3) follows from (2) by opposition.  $\square$



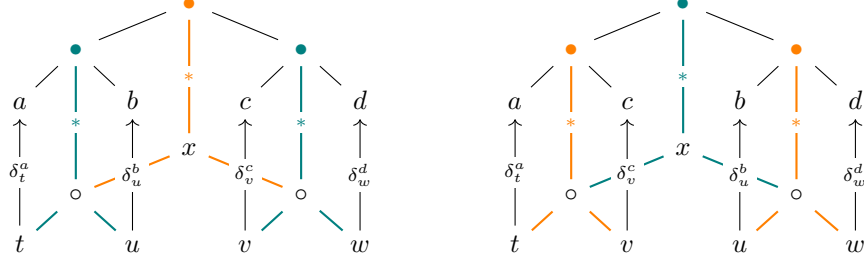


Figure 3:  $(\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d) x$  representing  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z$  using  $(a \bullet b) \bullet (c \bullet d)$  in Lemma 3.6(4) together with its dual.

The next lemma shows that the trees memoised by the relational interchange laws can be expressed in terms of deltas and convolution in the presence of the following mild nondegeneracy conditions on  $Q$ :

$$\exists a, b \in Q. a \bullet b \neq 0, \quad (D_1)$$

$$\exists a, b, c \in Q. a \bullet (b \bullet c) \neq 0, \quad (D_2)$$

$$\exists a, b, c \in Q. (a \bullet b) \bullet c \neq 0, \quad (D_3)$$

$$\exists a, b, c, d \in Q. (a \bullet b) \bullet (c \bullet d) \neq 0. \quad (D_4)$$

**Lemma 3.6.** *Let  $X$  be a relational bi-magma and  $Q$  a bi-prequantale. Then*

- (1)  $R_{yz}^x \Rightarrow (\delta_y^a * \delta_z^b) x = a \bullet b$ , and the converse implication follows from  $(D_1)$ ,
- (2)  $\exists y. R_{uy}^x \wedge R_{vw}^y \Rightarrow (\delta_u^a * (\delta_v^b * \delta_w^c)) x = a \bullet (b \bullet c)$ , and the converse implication follows from  $(D_2)$ ,
- (3)  $\exists y. R_{uv}^y \wedge R_{yw}^x \Rightarrow ((\delta_u^a * \delta_v^b) * \delta_w^c) x = (a \bullet b) \bullet c$ , and the converse implication follows from  $(D_3)$ ,
- (4)  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z \Rightarrow ((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x = (a \bullet b) \bullet (c \bullet d)$ , and the converse implication follows from  $(D_4)$ ,
- (5) properties (1)–(5) hold with colours interchanged, including in the degeneracy conditions.

*Proof.* For (4), suppose  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z$ . Then

$$((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x = ((a \bullet b) \bullet (c \bullet d) \mid \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z) = (a \bullet b) \bullet (c \bullet d)$$

by Lemma 3.5(4). For the converse implication,

$$0 \neq (a \bullet b) \bullet (c \bullet d) = ((\delta_t^a * \delta_u^b) \bullet (\delta_v^c * \delta_w^d)) x = ((a \bullet b) \bullet (c \bullet d) \mid \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z)$$

by Lemma 3.5(4) and therefore  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z$ .

All other proofs are similar, and left to the reader. (3) follow from (2) by opposition.  $\square$

Intuitively, convolutions of delta functions represent trees in  $X$  in the function space  $Q^X$  by creating their “shadows” in  $Q$ —which requires nondegeneracy. The case of Lemma 3.6(4) and its dual are shown in Figure 3.

We are now prepared to prove our second correspondence result, namely that algebraic interchange laws in  $Q^X$  force relational interchange laws in  $X$  subject to mild nondegeneracy conditions on  $Q$ .

**Proposition 3.7.** *Let  $X$  be a relational bi-magma and  $Q$  a bi-prequantale. Then  $(D_{\lceil \frac{k}{2} \rceil})$  in  $Q$  and  $(I_k)$  in  $Q^X$  imply  $(RI_k)$  in  $X$ , for each  $1 \leq k \leq 7$ .*

*Proof.* Suppose  $(a \bullet b) \bullet (c \bullet d) \neq 0$  for some  $a, b, c, d \in Q$  and  $(\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d) \leq (\delta_t^a * \delta_v^c) * (\delta_u^b * \delta_w^d)$ . Then, using Lemma 3.6(4),

$$\begin{aligned} \exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z &\Leftrightarrow 0 \neq ((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x \\ &\Rightarrow 0 \neq ((\delta_t^a * \delta_v^c) * (\delta_u^b * \delta_w^d)) x \\ &\Leftrightarrow \exists y, z. R_{tv}^y \wedge R_{yz}^x \wedge R_{uw}^z \end{aligned}$$

proves (RI<sub>7</sub>). The remaining proofs are similar. Those for (RI<sub>3</sub>) and (RI<sub>4</sub>) and those for (RI<sub>5</sub>) and (RI<sub>6</sub>) are related by opposition.  $\square$

Finally, we prove the third correspondence result for interchange laws, namely that algebraic interchange laws on  $Q^X$  force those on  $Q$ , subject to the following mild degeneracy conditions on  $X$ :

$$\exists x, u, v. R_{uv}^x, \tag{RD_1}$$

$$\exists x, y, u, v, w. R_{uy}^x \wedge R_{vw}^y, \tag{RD_2}$$

$$\exists x, y, u, v, w. R_{uv}^y \wedge R_{yw}^x, \tag{RD_3}$$

$$\exists x, y, z, t, u, v, w. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z. \tag{RD_4}$$

**Proposition 3.8.** . Let  $X$  be a relational bi-magma and  $Q$  a bi-prequantale. Then  $(RD_{\lceil \frac{k}{2} \rceil})$  in  $X$  and  $(I_k)$  in  $Q^X$  imply  $(I_k)$  in  $Q$ , for each  $1 \leq k \leq 7$ .

*Proof.* Suppose  $(\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d) \leq (\delta_t^a * \delta_v^c) * (\delta_u^b * \delta_w^d)$  for some  $a, b, c, d \in Q$  and let  $\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z$  for some  $t, u, v, w \in X$ . Then, using Lemma 3.6(4) and 3.5(4),

$$(a \bullet b) \bullet (c \bullet d) = ((\delta_t^a * \delta_u^b) * (\delta_v^c * \delta_w^d)) x \leq ((\delta_t^a * \delta_v^c) * (\delta_u^b * \delta_w^d)) x \leq (a \bullet c) \bullet (b \bullet d)$$

proves (I<sub>7</sub>) in  $Q$ . The remaining proofs are similar. Those for (RD<sub>3</sub>) and (RD<sub>4</sub>) and those for (RD<sub>5</sub>) and (RD<sub>6</sub>) are related by opposition.  $\square$

It may be helpful to check the proofs of Propositions 3.7 and 3.8 with the diagrams in Figure 3. The degeneracy conditions are necessary. Indeed, if  $Q$  is a singleton set, then so is  $Q^X$  and hence will obey all axioms independently of  $X$ . Similarly, if all products on  $Q$  vanish, then  $Q^X$  will automatically satisfy many axioms as all convolutions will be trivial.

## 4 Further Correspondences

When the relational bi-magma  $X$  and the bi-prequantale  $Q$  are both unital, units can be defined in  $Q^X$  as in Section 2:

$$id_E x = \begin{cases} 1, & \text{if } E^x, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad id_E x = \begin{cases} 1, & \text{if } E^x, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4 then shows that  $Q^X$  is a unital quantale if  $Q$  is a unital quantale and both compositions are associative and unital in  $X$ . We restate the three kinds of correspondences for units in the weaker setting of relational magmas and prequantales.

**Proposition 4.1.** Let  $X$  be a relational magma and  $Q$  a prequantale.

1. If  $X$  and  $Q$  are unital, then so is  $Q^X$ .
2. If  $Q^X$  is unital and  $1 \neq 0$  in  $Q$ , then so is  $X$ .
3. If  $Q^X$  is unital and  $E \neq \emptyset$  in  $X$ , then so is  $Q$ .

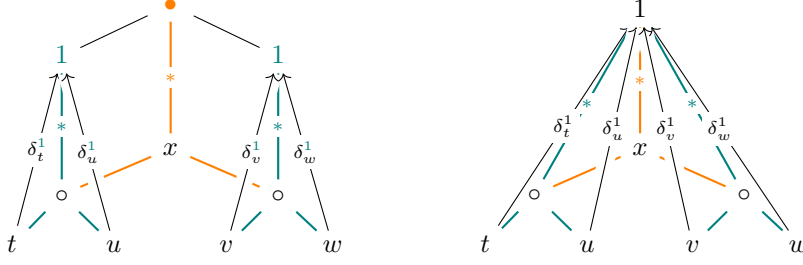


Figure 4: Degeneracy condition  $(D_4)$  in the presence of units.

*Proof.*

1. If  $X$  and  $Q$  are unital, then

$$(f * id_E)x = \bigvee \{f y \bullet \delta_e z \mid R_{yz}^x \wedge E^e\} = \bigvee \{f y \bullet 1 \mid \exists e. R_{ye}^x \wedge E^e\} = (f x \mid \exists e. R_{xe}^x \wedge E^e) = f x,$$

where the last two steps use the relational unit axioms from Definition 2.3. The proof for left units follows by opposition.

2. If  $id_E$  is the right unit in  $Q^X$ , then

$$(1 \mid (y = x)) = \delta_y x = (\delta_y * id_E)x = (1 \mid \exists e. R_{ye}^x \wedge E^e).$$

Suppose  $1 \neq 0$ . Then  $x = y$  implies  $\exists e. R_{xe}^x \wedge E^e$ , the existence axiom for right relational units, and  $\exists e. R_{ye}^x \wedge E^e$  implies that  $x = y$ , the uniqueness axiom. The proofs for left units follow by opposition.

3. If  $id_E$  is the right unit in  $Q^X$ , then

$$a \bullet 1 = (a \bullet 1 \mid \exists e. R_{xe}^x \wedge E^e) = \bigvee \{\delta_x x \mid \exists e. R_{xe}^x \wedge E^e\} = (\delta_x^a * id_E)x = \delta_x^a x = a$$

proves that  $1$  is a right unit in  $Q$ . The left unit law follows by opposition.  $\square$

In the presence of non-trivial units in  $X$  and  $Q$ , the nondegeneracy conditions for interchange laws in Proposition 3.7 and 3.8 simplify. Condition  $(D_1)$  becomes trivial with  $1 \bullet 1 = 1 \neq 0$ , condition  $(D_2)$  with  $1 \bullet (1 \bullet 1) = 1 \neq 0$  and condition  $(D_3)$  by opposition. Condition  $(D_4)$  reduces to  $(1 \bullet 1) \bullet (1 \bullet 1) = 1 \bullet 1 \neq 0$ , but remains non-trivial. It becomes trivial when  $1 = 1$ . It is easy to check that the nondegeneracy conditions  $(RD_1)$ – $(RD_3)$  become trivial in a similar way, using the fact that  $R_{ee}^e$  holds for all  $e \in E$  and  $R_{ee}^e$  for all  $e \in E$ . Once again,  $(RD_4)$  becomes trivial when  $E = E$ .

**Corollary 4.2.** *Let  $X$  be a unital relational bi-magma satisfying  $E = E \neq \emptyset$  and  $Q$  a unital bi-prequantale satisfying  $1 = 1 \neq 0$ . Then  $(I_k)$  holds in  $Q^X$  if and only if  $(I_k)$  holds in  $Q$  and  $(RI_k)$  holds in  $X$ , for each  $1 \leq k \leq 7$ .*

In the only-if directions, Functions  $\delta_x$  can now be used. This leads to a simpler relationship between deltas and ternary relations than in Lemma 3.6.

**Corollary 4.3.** *Let  $X$  be a relational magma and  $Q$  a unital prequantale with  $1 \neq 0$ . Then*

$$R_{yz}^x \Leftrightarrow (\delta_y * \delta_z)x = 1.$$

It is therefore compelling to see  $\mathbb{B}$  as the sublattice over  $\{0, 1\}$  of  $Q$  and simply write  $R_{yz}^x = (\delta_y * \delta_z)x$  or even  $(f * g)x = \bigvee_{y,z} f y \bullet g z \bullet R_{yz}^x$ . Figure 4 shows how the presence of units affects the right-hand term in  $(RI_7)$ .

Next we present a correspondence result for relational units that is useful in Section 5.

**Lemma 4.4.** *Let  $X$  be a unital bi-magma and  $Q$  a unital bi-prequantale.*

1. *If  $E \subseteq E$  in  $X$   $1 \leq 1$  in  $Q$ , then  $id_E \leq id_E$  in  $Q^X$ .*
2. *If  $id_E \leq id_E$  in  $Q^X$  and  $1 \neq 0$  in  $Q$ , then  $E \subseteq E$  in  $X$ .*
3. *If  $id_E \leq id_E$  in  $Q^X$  and  $E \neq \emptyset$  in  $X$ , then  $1 \leq 1$  in  $Q$ .*

*Proof.*

1. Let  $E \subseteq E$  and  $1 \leq 1$ . Then  $id_E x = 0 \Leftrightarrow \neg E^x \Rightarrow \neg E^x \Leftrightarrow id_E x = 0$  and therefore  $id_E \leq id_E$ .
2. Let  $id_E \leq id_E$ . If  $E^x$ , then  $0 \neq 1 = id_E x \leq id_E x$  and therefore  $E^x$ .
3. Let  $id_E \leq id_E$  and  $E^x$ . Then  $1 = id_E x \leq id_E x \leq 1$ . □

**Corollary 4.5.** *Let  $X$  be a unital bi-magma with  $E \neq \emptyset$  and  $Q$  a unital bi-prequantale with  $1 \neq 0$ . Then  $id_E \leq id_E$  in  $Q^X$  if and only if  $E \subseteq E$  in  $X$  and  $1 \leq 1$  in  $Q$ .*

Because of the symmetry in the definitions of unital bi-magmas and bi-prequantales, Lemma 4.4 and Corollary 4.5 hold with colours swapped. We do not spell them out explicitly.

The correspondences between interchange laws can be specialised to obtain the associativity laws for a quantale. The relational interchange law (RI<sub>3</sub>),  $\exists y. R_{uy}^x \wedge R_{vw}^y \Rightarrow \exists y. R_{uv}^y \wedge R_{yw}^x$ , becomes the relational semi-associativity law  $\exists y. R_{uy}^x \wedge R_{vw}^y \Rightarrow \exists y. R_{uv}^y \wedge R_{yw}^x$  when colours are switched off; (RI<sub>4</sub>) translates into the opposite implication. Similarly, the interchange laws (I<sub>3</sub>) and (I<sub>4</sub>),  $a \bullet (b \bullet c) \leq (a \bullet b) \bullet c$  and  $(a \bullet b) \bullet c \leq a \bullet (b \bullet c)$ , become the semi-associativity laws  $a \bullet (b \bullet c) \leq (a \bullet b) \bullet c$  and  $(a \bullet b) \bullet c \leq a \bullet (b \bullet c)$ . This yields the following corollary to Proposition 3.4, 3.7 and 3.8.

**Corollary 4.6.** *Let  $X$  be a relational magma and  $Q$  a prequantale.*

1. *If  $X$  is relationally associative and  $Q$  associative, then  $Q^X$  is associative.*
2. *If  $Q^X$  is associative and some  $a, b, c \in Q$  satisfy  $a \bullet (b \bullet c) \neq 0 \neq (a \bullet b) \bullet c$ , then  $X$  is relationally associative.*
3. *If  $Q^X$  is associative and some  $x, y, z, u, v, w \in X$  satisfy  $R_{uz}^x, R_{yw}^x, R_{vw}^z$  and  $R_{uv}^y$ , then  $Q$  is associative.*

Similar correspondences between semi-associativity laws are straightforward, but not as interesting for our purposes. In the presence of units, Corollary 4.6 simplifies further.

**Corollary 4.7.** *Let  $X$  be a unital relational magma satisfying  $E \neq \emptyset$  and  $Q$  a unital prequantale satisfying  $1 \neq 0$ . Then  $Q^X$  is associative if and only if  $X$  is relationally associative and  $Q$  is associative.*

The correspondences between interchange laws can also be specialised to the commutativity law for a quantale. The relational interchange law (RI<sub>2</sub>),  $R_{uv}^x \Rightarrow R_{vu}^x$ , specialises to  $R_{uv}^x \Rightarrow R_{vu}^x$  when colours are switched off while the interchange law (I<sub>2</sub>),  $a \bullet b \leq b \bullet a$ , becomes  $a \bullet b \leq b \bullet a$ . This yields another corollary to Proposition 3.4, 3.7 and 3.8.

**Corollary 4.8.** *Let  $X$  be a relational magma and  $Q$  a prequantale.*

1. *If  $X$  is relationally commutative and  $Q$  abelian, then  $Q^X$  is abelian.*
2. *If  $Q^X$  is abelian and there exist  $a, b \in Q$  with  $a \bullet b \neq 0$ , then  $X$  is relationally commutative.*
3. *If  $Q^X$  is abelian and there exist  $x, y, z \in X$  with  $R_{xz}^x$ , then  $Q$  is abelian.*

In the presence of units, this corollary simplifies further.

**Corollary 4.9.** *Let  $X$  be a unital relational magma satisfying  $E \neq \emptyset$  and  $Q$  a unital quantale satisfying  $1 \neq 0$ . Then  $Q^X$  is abelian if and only if  $X$  is relationally commutative and  $Q$  abelian.*

## 5 Relational Interchange Monoids and Interchange Quantaes

We now start shifting the focus from correspondence theory to construction recipes for quantaes with interchange laws. To avoid nondegeneracy conditions, we assume that relational magmas and quantaes are unital and impose an order between units:  $\emptyset \neq E \subseteq E$  and  $0 \neq 1 \leq 1$ .

Yet first we prove a weak variant of the classical Eckmann-Hilton argument [EH62]. It shows that if a unital bi-magma, a set equipped with composition  $\bullet$  and unit  $1$  and composition  $\bullet$  with unit  $1$ , satisfies the strong interchange law  $(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d)$ , then  $1 = 1$ ,  $\bullet$  and  $\bullet$  coincide, and they are associative and commutative. We show how these properties change if strong interchange is weakened to  $(I_7)$ . This of course requires ordered bimagmas, in which the underlying set is partially ordered and the two compositions preserve the order in both arguments.

**Lemma 5.1** (weak Eckmann-Hilton). *Let  $(S, \leq, \bullet, \bullet, 1, 1)$  be an ordered bimagma in which  $(I_7)$  holds. Then  $1 \leq 1$ , and  $(I_1)$ – $(I_6)$  hold whenever  $1 \leq 1$ .*

The proofs, like the classical Eckmann-Hilton ones, substitute  $1$  and  $1$  in  $(I_7)$  and are straightforward. Analogous results hold for relational bimagmas because of the various correspondence results in the previous section and Lemma 5.1.

**Lemma 5.2.** *Let  $(X, R, R, E, E)$  be a unital relational bimagma in which  $(RI_7)$  holds. Then  $E \subseteq E$ , and  $(RI_1)$ – $(RI_6)$  hold whenever  $E \subseteq E$ .*

*Proof.* First, for all  $e \in S$ , and with  $(RI_7)$  in the fourth step,

$$\begin{aligned}
 E^e &\Leftrightarrow E^e \wedge R_{ee}^e \\
 &\Leftrightarrow E^e \wedge \exists x, y, e', e''. E^{e'} \wedge R_{e'e}^x \wedge R_{xy}^e \wedge R_{ee''}^y \\
 &\Rightarrow \exists e', e''. E^e \wedge E^{e'} \wedge R_{e'e}^e \wedge R_{ee}^e \wedge R_{ee''}^e \\
 &\Rightarrow \exists e', e''. E^e \wedge E^{e'} \wedge R_{e'e}^e \wedge R_{ee}^e \wedge R_{ee''}^e \\
 &\Rightarrow E^e \wedge E^e \wedge R_{ee}^e \wedge R_{ee}^e \wedge R_{ee}^e \\
 &\Rightarrow E^e.
 \end{aligned}$$

Second, let  $E \subseteq E$  and assume  $(RI_7)$ . Then

$$\begin{aligned}
 \exists y. R_{uv}^y \wedge R_{yw}^x &\Leftrightarrow \exists e, y. R_{uv}^y \wedge R_{yw}^x \wedge R_{ew}^w \wedge E^e \\
 &\Leftrightarrow \exists e, y, z. R_{uv}^y \wedge R_{yz}^x \wedge R_{ew}^z \wedge E^e \\
 &\Rightarrow \exists e, y, z. R_{ue}^y \wedge R_{yz}^x \wedge R_{vw}^z \wedge E^e \\
 &\Leftrightarrow \exists e, z. R_{ue}^u \wedge R_{uz}^x \wedge R_{vw}^z \wedge E^e \\
 &\Leftrightarrow \exists z. R_{uz}^x \wedge R_{vw}^z
 \end{aligned}$$

proves  $(RI_6)$ . The proofs of  $(RI_1)$ ,  $(RI_2)$ ,  $(RI_3)$  and  $(RI_5)$  from  $(RI_7)$  are similar, and left to the reader.  $\square$

**Definition 5.3.**

1. A *relational semigroup*  $(X, R)$  is a set  $X$  equipped with a relationally associative ternary relation  $R$ .
2. A *relational monoid* is a relational semigroup  $(X, R)$  with a set  $E \subseteq X$  of relational units for  $R$ .
3. A *relational interchange monoid* is a structure  $(X, R, R, E)$  such that  $(X, R, E)$  and  $(X, R, E)$  are relational monoids and the relational interchange law  $(RI_7)$  holds.

**Definition 5.4.** A (unital) *interchange quantale* is a structure  $(Q, \leq, \bullet, \bullet, 1)$  such that  $(Q, \leq, \bullet, 1)$  and  $(Q, \leq, \bullet, 1)$  are (unital) quantaes, and the interchange law  $(I_7)$  holds.

In light of Lemma 5.1 and 5.2, we always assume that relational interchange monoids and interchange quantales have one single unit that is shared between the relations and compositions, respectively. The following result then summarises these two lemmas.

**Corollary 5.5.**

1. In every relational interchange monoid,  $(RI_1)$ – $(RI_6)$  are derivable.
2. In every unital interchange quantale,  $(I_1)$ – $(I_6)$  are derivable.

The correspondence results from Section 3 and 4 can now be summarised in terms of interchange monoids and interchange quantales as well.

**Theorem 5.6.**

1. If  $X$  is a relational interchange monoid and  $Q$  an interchange quantale, then  $Q^X$  is an interchange quantale.
2. If  $Q^X$  is an interchange quantale and  $1 \neq 0$ , then  $X$  is a relational interchange monoid.
3. If  $Q^X$  is an interchange quantale and  $E \neq \emptyset$ , then  $Q$  is an interchange quantale.

*Proof.* The correspondence for associativity and units in the subquantales is given by Corollary 4.6 and Proposition 4.1; that for interchange between the subquantales by Propositions 3.4, 3.7 and 3.8.  $\square$

Theorem 5.6 shows that, up to mild nondegeneracy assumptions, all interchange quantales of type  $X \rightarrow Q$  are obtained from a relational interchange monoid on  $X$  and an interchange quantale  $Q$ . To build such quantales, one should therefore look for relational interchange monoids, and the advantage is that fewer properties need to be checked.

Interchange quantales generalise concurrent quantales and are strongly related to concurrent Kleene algebras [HMSW11]. The difference is that here we do not assume “parallel composition”  $\bullet$  to be commutative. Yet Theorem 5.6 adapts easily to the commutative case. For a concurrent quantale in  $Q^X$ , an interchange monoid  $X$  with relationally commutative  $R$  and an interchange quantale  $Q$  with commutative  $\bullet$  is needed. A variant of Theorem 5.6 then follows from Corollary 4.9 and 4.2. In particular, the nondegeneracy assumptions simplify to non-triviality assumptions for units and unit sets.

## 6 Duality for Powerset Quantales

An interesting specialisation of Theorem 5.6 is the case of  $Q = \mathbb{B}$ , which forms an interchange quantale with both compositions being meet and both units of composition 1. In particular, in the booleans,  $0 \neq 1$ . The interchange law  $(I_7)$  holds trivially because  $(w \wedge x) \wedge (y \wedge z) = (w \wedge y) \wedge (x \wedge z)$  in any semilattice by associativity and commutativity of meet.

**Corollary 6.1.**  $\mathbb{B}^X \cong \mathcal{P} X$  is an interchange quantale if and only if  $X$  is a relational interchange monoid.

In this case, by Corollary 4.3 and Lemma 3.6,  $R_{yz}^x \Leftrightarrow (\delta_y * \delta_z) x = 1$ ,  $R_{yz}^x \Leftrightarrow (\delta_y * \delta_z) x = 1$  and likewise for the other relational nondegeneracy conditions.

More interestingly, as a powerset boolean algebra,  $\mathbb{B}^X$  is complete and atomic—a CABA—and a well known duality holds. The work of the Tarski school [JT51] and Goldblatt [Gol89] shows that categories of CABAs with  $n$ -ary operators and relational structures with  $n + 1$ -ary relations are dually equivalent. Atomic boolean prequantales are CABAs with a binary operator; relational magmas are relational structures with a ternary relation. Morphisms in the category ABP of atomic boolean (pre)quantales preserve sups, complementation and composition. A morphism  $\rho$  between relational magmas  $(X, R)$  and  $(X', S)$  must satisfy

$$R_{yz}^x \Rightarrow S_{(\rho y)(\rho z)}^{(\rho x)}$$

for all  $x, y, z \in X$ . A morphism is *bounded* if, for all  $x, y, z \in X$ ,

$$S_{yz}^{(\rho x)} \Rightarrow \exists u, v \in X. \rho u = y \wedge \rho v = z \wedge R_{uv}^x.$$

The morphisms in the category  $\mathbf{RM}$  of relational magmas are assumed to be bounded.

Next we summarise this duality between categories. With every atomic boolean prequantale  $Q$  one associates its dual relational structure—its atom structure— $Q_+ = \text{At } Q$  by defining the ternary relation  $R$  in  $Q$ , as in Example 3.3, by

$$R_{\beta\gamma}^\alpha \Leftrightarrow \alpha \leq \beta \bullet \gamma$$

for all  $\alpha, \beta, \gamma \in Q_+$ . With every morphism  $\varphi : Q \rightarrow Q'$  in  $\mathbf{ABP}$  one associates the map  $\varphi_+ : \text{At } Q' \rightarrow Q$  defined by

$$\varphi_+ \beta = \bigwedge \{a \in Q \mid \beta \leq \varphi a\}.$$

It is easy to check that  $\varphi_+$  maps atoms in  $Q'$  to atoms in  $Q$ .

Conversely, with every relational magma  $(X, R)$  one associates its dual convolution prequantale—its complex algebra— $X^+ = (\mathcal{P} X, \subseteq, *)$ . With every bounded morphism  $\rho : X \rightarrow X'$  one associates the contravariant powerset (or preimage) functor  $\rho^+ : \mathcal{P} X' \rightarrow \mathcal{P} X$ . It is defined, for all  $B \in X'$ , by

$$\rho^+ B = \{x \in X \mid \rho x \in B\}.$$

In this context, our function  $\delta : X \rightarrow X \rightarrow \mathbb{B}$  is isomorphic to the function  $\eta : X \rightarrow \mathcal{P} X$  defined by  $\eta = \{-\}$ . Then  $(\delta_y * \delta_z)x = 1 \Leftrightarrow x \in \{y\} * \{z\}$  and hence  $R_{yz}^x \Leftrightarrow x \in \{y\} * \{z\}$  by Corollary 4.3.

**Proposition 6.2** ([JT51, HMT71]). *Let  $Q$  be an atomic boolean prequantale and  $X$  a relational magma.*

1.  $Q \cong (Q_+)^+$  with isomorphism  $\sigma : Q \rightarrow \mathcal{P}(\text{At } Q)$  given by  $\sigma a = \{\alpha \mid \alpha \leq a\}$ .
2.  $X \cong (X^+)_+$  with isomorphism  $\eta : X \rightarrow \text{At}(\mathcal{P} X)$  given by  $\eta x = \{x\}$ .

To prove (1) one can use that any bijection  $\varphi$  between the atoms of two atomic boolean prequantales  $Q$  and  $Q'$  extends to an isomorphism if and only if  $\alpha \leq \beta \bullet \gamma \Leftrightarrow \varphi \alpha \leq \varphi \beta \bullet \varphi \gamma$  for all  $\alpha, \beta, \gamma \in \text{At } Q$ . The bijection  $\eta$  between atoms in  $Q$  and atoms in  $\mathcal{P} Q$  satisfies this condition, and it turns out that  $\sigma$  is its extension. For (2) it is easy to check that the bijection  $\eta$  is a relational magma morphism.

**Proposition 6.3** ([Gol89]). *The maps  $(-)^+ : \mathbf{RM} \rightarrow \mathbf{ABP}$  and  $(-)_+ : \mathbf{ABP} \rightarrow \mathbf{RM}$  are contravariant functors.*

For  $(-)^+$ , one must show that  $\rho^+$  preserves sups, complementation and composition, for any bounded morphism  $\rho$ . The first two properties follow from Stone's theorem for CABAs. Proving  $\rho^+ B_1 * \rho^+ B_2 \subseteq \rho^+(B_1 * B_2)$  for  $B_1, B_2 \in X'$  requires that  $\rho$  is a morphism, the converse inclusion that it is bounded. Proving  $(-)_+$  requires checking functoriality.

**Theorem 6.4** ([Gol89]). *The composites  $(-)_+ \circ (-)^+$  and  $(-)^+ \circ (-)_+$  are naturally isomorphic to the identity functors on the categories  $\mathbf{ABP}$  and  $\mathbf{RM}$ , respectively. The two categories are dually equivalent.*

The isomorphisms are  $\sigma_Q : Q \mapsto (Q_+)^+$  and  $\eta_X : X \mapsto (X^+)_+$ . Showing that these are components of a natural transformations requires checking that the following diagrams commute.

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & Q' \\ \downarrow \sigma_Q & & \downarrow \sigma_{Q'} \\ (Q_+)^+ & \xrightarrow{(\varphi_+)^+} & (Q'_+)^+ \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\rho} & X' \\ \downarrow \eta_X & & \downarrow \eta_{X'} \\ (X^+)_+ & \xrightarrow{(\rho^+)_+} & (X'^+)_+ \end{array}$$

**Question 6.5.** Is there a Stone-type duality for non-atomic (boolean) quantales and arbitrary convolution algebras?

## 7 Interchange Kleene Algebras

We mentioned in Section 2 that in many classical convolution algebras, including Rota’s incidence algebras and the formal power series of Schaützenberger and Eilenberg’s approach to formal languages, the underlying set  $X$  has a finite decomposition property (Rota calls partial orders with this property *locally finite* [Rot64]). As infinite sups are then not needed to express convolutions, one can specialise quantales to semirings and Kleene algebras, and in particular concurrent Kleene algebras. This is the purpose of this section.

**Definition 7.1.** A *dioid* is a semiring  $(S, +, \bullet, 0, 1)$  in which addition is idempotent.

Hence  $(S, +, 0)$  is a sup-semilattice ordered by  $a \leq b \Leftrightarrow a + b = b$  and least element 0. Moreover  $\bullet$  preserves  $\leq$  in both arguments. A quantale can thus be seen as a complete dioid.

**Definition 7.2.** An *interchange semiring* is a structure  $(S, +, \bullet, \blacklozenge, 0, 1)$  such that  $(S, +, \bullet, 0, 1)$  and  $(S, +, \blacklozenge, 0, 1)$  are dioids, and the interchange law (I<sub>7</sub>) holds.

The six small interchange laws are of course derivable in this setting.

**Definition 7.3.**

1. A *Kleene algebra* is a dioid with a unary star operation  $*$  that satisfies the unfold and induction axioms

$$1 + a \bullet a^* \leq a^*, \quad c + a \bullet b \leq b \Rightarrow a^* \bullet b \leq b, \quad 1 + a^* \bullet a \leq a^*, \quad c + b \bullet a \leq b \Rightarrow b \bullet a^* \leq b.$$

2. An *interchange Kleene algebra* is a structure  $(K, +, \bullet, \blacklozenge, 0, 1, *, \blackstar)$  such that  $(K, +, \bullet, 0, 1, *)$  and  $(K, +, \blacklozenge, 0, 1, \blackstar)$  are Kleene algebras and (I<sub>7</sub>) holds.

We write  $(-)^*$  instead of the usual  $(-)^*$  to distinguish the Kleene star from the convolution operation.

To translate Theorem 5.6 into the Kleene algebra setting all sups must be guaranteed to be finite. This can be achieved by imposing that all functions have finite support, or that the relations  $R_{yz}^x$  and  $R_{yz}^x$  are *finitely decomposable*, that is, for each  $x$  the sets  $\{(y, z) \mid R_{yz}^x\}$  and  $\{(y, z) \mid R_{yz}^x\}$  are finite. If this is the case we call the relational interchange monoid *finitely decomposable* as well.

**Theorem 7.4.** *If  $X$  is a finitely decomposable relational interchange monoid and  $S$  an interchange semiring, then  $S^X$  is an interchange semiring.*

*Proof.* In the construction of the convolution algebra on  $S^X$  it can be checked that all sups remain finite.  $\square$

It is easy to generalise these results from dioids to proper semirings that are ordered. We do not spell out the details. Beyond that it seems interesting to extend Theorem 7.4 to interchange Kleene algebras. First of all, every interchange quantale is an interchange Kleene algebra, because  $*$  and  $\blackstar$  can be defined explicitly in this setting using Kleene’s fixpoint theorem:  $x^* = \bigvee_{k \in \mathbb{N}} x^k$  and  $x^{\blackstar} = \bigvee_{k \in \mathbb{N}} x^k$  satisfy the star axioms, with powers defined recursively in the standard way as  $x^0 = 1$  and  $x^{i+1} = x \bullet x^i$  and likewise for  $x^{\blackstar}$ .

When infinite sups and the sup-preservation properties required for Kleene’s fixpoint theorem are not available, a different approach is needed. We have already shown [ASW14] that formal power series—functions of type  $\Sigma^* \rightarrow K$ , where  $\Sigma^*$  is the free monoid over the finite alphabet  $\Sigma$  and  $K$  a Kleene algebra—form Kleene algebras. In this setting, the star of a power series can be defined recursively [KS86] as

$$f^* \varepsilon = (f \varepsilon)^*, \quad f^* x = (f \varepsilon)^* \bullet \sum_{y, z: x=y \cdot z, y \neq \varepsilon} f y \bullet f^* z,$$

where  $\sum$  indicates a finite sup. The verification of the star axioms for power series uses structural induction over finite words. Yet this is not applicable for general ternary relations. Instead we use a notion of grading that been used for arbitrary monoids by Sakarovitch [Sak03].

The function  $|\cdot| : X \rightarrow \mathbb{N}$  is a *grading* on the relational monoid  $(R, X, E)$  if



- $|x| > 0$  for all  $x \in X$  such that  $x \notin E$ ,
- $|x| = |y| + |z|$  whenever  $R_{yz}^x$ .

Then  $(X, R, E)$  is *graded* if there is a grading on  $X$ . Thus, in a graded monoid  $|e| = 0$  if and only if  $e \in E$ .

**Proposition 7.5.** *If  $(X, R, \{e\})$  is a graded, finitely decomposable, relational monoid and  $K$  a Kleene algebra, then  $K^X$  is a Kleene algebra with*

$$f^* e = (fe)^*, \quad f^* x = (fe)^* \bullet \sum_{y,z:R_{yz}^x, y \neq e} f y \bullet f^* z.$$

*Proof.* We need to check the unfold and induction axioms of Kleene algebra. First, it is well known that the unfold axiom  $1 + a^* \bullet a \leq a^*$  is implied by the other axioms in any Kleene algebra, and can therefore be ignored. Second, the axiom  $c + a \bullet b \leq b \Rightarrow a^* \bullet c \leq b$  follows from the simpler formula  $a \bullet b \leq b \Rightarrow a^* \bullet b \leq b$  and, by opposition,  $c + b \bullet a \leq b \Rightarrow c \bullet a^* \leq b$  follows from  $b \bullet a \leq b \Rightarrow b \bullet a^* \leq b$ , in any Kleene algebra [Koz94]. It thus remains to check that

$$id_e + f * f^* \leq f^*, \quad f * g \leq g \Rightarrow f^* * g \leq g, \quad g * f \leq g \Rightarrow g * f^* \leq g$$

hold in the convolution algebra  $K^X$ .

1.  $id_e + f * f^* = f^*$ . If  $x = e$ , then  $(id_e + f * f^*) e = 1 + (fe) \bullet (fe)^* = (fe)^* = f^* e$ .

Otherwise, if  $x \neq e$ ,

$$\begin{aligned} (id_e + f * f^*) x &= \sum \{f y \bullet f^* z \mid R_{yz}^x\} \\ &= f e \bullet f^* x + \sum \{f y \bullet f^* z \mid R_{yz}^x \wedge y \neq e\} \\ &= f e \bullet f^* e \bullet \sum \{f y \bullet f^* z \mid R_{yz}^x \wedge y \neq e\} + \sum \{f y \bullet f^* z \mid R_{yz}^x \wedge y \neq e\} \\ &= (f e \bullet f^* e + id_E e) \bullet \sum \{f y \bullet f^* z \mid R_{yz}^x \wedge y \neq e\} \\ &= (f e)^* \bullet \sum \{f y \bullet f^* z \mid R_{yz}^x \wedge y \neq e\} \\ &= f^* x. \end{aligned}$$

2.  $(\forall x. (f * g) x \leq g x) \Rightarrow (\forall x. (f^* * g) x \leq g x)$ . We proceed by induction on  $|x|$ .

- (a) Let  $|x| = 0$  and hence  $x = e$ . Then  $(f^* * g) e = (f e)^* \bullet g e \leq g e$  follows from the assumption  $f e \bullet g e \leq g e$  and the first induction axiom of Kleene algebra.
- (b) Let  $|x| > 0$  and therefore  $x \neq e$ . Then, by the induction hypothesis,  $(f * g) y \leq g y$  holds for all  $y$  with  $|y| < |x|$ . In addition, the assumption implies that  $\forall x, y, z. R_{yz}^x \Rightarrow f y \bullet g z \leq g x$ , from which  $(f e)^* \bullet g x = f^* e \bullet g x \leq g x$  follows by star induction in  $K$ . With this property,

$$\begin{aligned} (f^* * g) x &= f^* e \bullet g x + \sum \left\{ f^* e \bullet \sum \{f u \bullet f^* v \mid R_{uv}^y \wedge u \neq e\} \bullet g z \mid R_{yz}^x \wedge y \neq e \right\} \\ &= f^* e \bullet \left( g x + \sum \{ (f u \bullet f^* v) \bullet g z \mid \exists y. R_{uv}^y \wedge R_{yz}^x \wedge u \neq e \wedge y \neq e \} \right) \\ &= f^* e \bullet \left( g x + \sum \{ f u \bullet (f^* v \bullet g z) \mid \exists y. R_{uy}^x \wedge R_{vz}^y \wedge u \neq e \wedge y \neq e \} \right) \\ &\leq f^* e \bullet \left( g x + \sum \{ f u \bullet (f^* * g) y \mid R_{uy}^x \wedge u \neq e \} \right) \\ &\leq f^* e \bullet \left( g x + \sum \{ f u \bullet g y \mid R_{uy}^x \} \right) \\ &\leq f^* e \bullet (g x + (f * g) x) \\ &= f^* e \bullet (g x + g x) \\ &\leq g x. \end{aligned}$$

The first step unfolds the definition of convolution and the Kleene star in  $K^X$ . The second step applies distributivity laws in  $K$ ; the third one associativity in  $X$  and  $K$ . The fourth step introduces a convolution as an upper bound, thus dropping the constraint  $y \neq e$ . The fifth step applies the induction hypothesis to  $y$ . The condition  $u \neq e$  guarantees that  $|y| < |x|$ . The sixth step applies the assumption; the last step the derived property.

3.  $g * f \leq g \Rightarrow g * f^* \leq g$  follows by opposition from (2).  $\square$

The following theorem is then immediate.

**Theorem 7.6.** *If  $X$  is a graded relational interchange monoid with unit  $e$  and  $K$  an interchange Kleene algebra, then  $K^X$  is an interchange Kleene algebra.*

We have already discussed the relationship between interchange quantales and concurrent quantales in Section 5, namely that concurrent quantales are interchange quantales in which  $\bullet$  is commutative and  $\mathbf{1} = \mathbf{1}$ . Similarly, concurrent semirings and concurrent Kleene algebras are interchange semirings and interchange Kleene algebras satisfying these two conditions. It is then a trivial consequence of Theorem 7.4, Corollary 4.9 and Corollary 4.2 that  $S^X$  is a concurrent semiring if  $S$  is a concurrent semiring and  $X$  a finitely decomposable relational monoid. Similarly, by Theorem 7.6 and these corollaries,  $K^X$  is a concurrent Kleene algebra if  $K$  is a concurrent Kleene algebra and  $X$  a graded relational monoid.

## 8 Weighted Shuffle Languages

This extended example shows how weighted shuffle languages [DKV09] can be constructed with our approach. Yet an alternative view on relational interchange monoids is helpful. Obviously, the sets  $\mathcal{P}(X \times X \times X)$  and  $X \times X \rightarrow \mathcal{P}X$  are isomorphic. A ternary relation  $R$  can thus be seen as a *multioperation*  $\odot : X \times X \rightarrow \mathcal{P}X$  defined by

$$x \in y \odot z \Leftrightarrow R_{yz}^x.$$

It can be extended Kleisli-style to the operation  $\odot : \mathcal{P}S \rightarrow \mathcal{P}S \rightarrow \mathcal{P}S$  defined, for all  $A, B \subseteq X$ , by

$$A \odot B = \bigcup \{x \odot y \mid x \in A \wedge y \in B\}.$$

It follows that  $(A \odot B)x = \bigvee \{Ay \wedge Bz \mid R_{yz}^x\}$  if the sets  $A$  and  $B$  are identified with their indicator functions, which turns  $\odot$  into a convolution.

The overloading of the multioperation  $\odot$  and its extension allows rewriting the relational interchange laws more compactly in algebraic form. Relational associativity becomes  $\{x\} \odot (x \odot z) = (x \odot y) \odot \{z\}$ ; the relational interchange law (RI<sub>7</sub>) becomes  $(w \odot x) \odot (y \odot z) \subseteq (w \odot y) \odot (x \odot z)$ .

Multisemigroups, multimonoids and other multialgebras have been studied in mathematics for many decades [Mar34, Kra83, CC10, KM15]. In computer science they appear in the semantics of separation logic [GL06].

The shuffle of two words from an alphabet  $\Sigma$  is obviously a multioperation  $\parallel : \Sigma^* \rightarrow \Sigma^* \rightarrow \mathcal{P}\Sigma^*$ . It can be defined recursively as

$$v \parallel \varepsilon = \{v\} = \varepsilon \parallel v, \quad (av) \parallel (bw) = \{a\} \parallel (v \parallel (bw)) \cup \{b\} \parallel ((av) \parallel w),$$

where  $a$  and  $b$  are letters,  $v$  and  $w$  words and the extension  $\parallel : \mathcal{P}X^* \rightarrow \mathcal{P}X^* \rightarrow \mathcal{P}X^*$  of  $\parallel$  has been tacitly used. It yields the shuffle or Hurwitz product

$$A \parallel B = \bigcup \{x \parallel y \mid x \in A \wedge y \in B\}$$

for  $A, B \subseteq \Sigma^*$  at language level.

To construct the quantale of  $Q$ -weighted shuffle languages using Theorem 5.6(1) it remains to check that the structure  $M = (\Sigma^*, R, R, \{\varepsilon\})$  is a relational interchange monoid with shared unit  $E = E = \{\varepsilon\}$ , where  $R_{yz}^x \Leftrightarrow x = y \cdot z$ , for word concatenation  $\cdot$  and  $R_{yz}^x \Leftrightarrow x \in y \parallel z$  for shuffle.

It is of course straightforward to check that  $(\Sigma^*, R, \{\varepsilon\})$  is a relational monoid: it is in fact isomorphic to the free monoid  $(\Sigma^*, \cdot, \varepsilon)$  and checking the relational associativity and relational unit laws in the first monoid amounts to checking their algebraic counterparts in the second one. Checking that  $(\Sigma^*, R, \{\varepsilon\})$  is a relational monoid—or  $(\Sigma^*, \parallel, \varepsilon)$  a multimonoid—and that the relational interchange law (RI<sub>7</sub>) holds—or the interchange law  $(w\parallel x) \cdot (y\parallel z) \subseteq (w \cdot y)\parallel(x \cdot z)$  with language product  $\cdot$  in the left-hand term and word concatenation  $\cdot$  in the right-hand one—is a surprisingly tedious exercise and requires nested inductions.

The result of this verification is summarised as follows.

**Lemma 8.1.**  *$M$  is a relational interchange monoid with unit  $\varepsilon$  and relationally commutative  $R$ .*

The following corollary to Theorem 5.6(1) is then automatic.

**Corollary 8.2.** *If  $Q$  is an interchange quantale with unit  $1 = \mathbf{1} = 1$  and  $\bullet$  commutative, then  $Q^M$  is an interchange quantale with  $*$  commutative and*

$$(f * g)x = \bigvee_{y, z: x=y \cdot z} f y \bullet g z, \quad (f * g)x = \bigvee_{y, z: x \in y \parallel z} f y \bullet g z, \quad id x = \delta_\varepsilon x.$$

The operation  $*$  is similar to the standard convolution of formal power series, a  $Q$ -weighted generalisation of the standard language product. The operation  $*$  generalises the standard shuffle product  $\parallel$  of languages to the  $Q$ -weighted setting. Yet semirings or at least Kleene algebras are normally used as weight-algebras. A grading on words is needed, and in this particular case the length of words can be used. It is then obvious that  $\Sigma_n^*$ —the size of words of length  $n$  is finite whenever  $\Sigma$  is finite. This yields the following corollary to Theorem 7.6.

**Corollary 8.3.** *If  $K$  is an interchange Kleene algebra with unit  $1$  and  $\bullet$  commutative, then  $K^M$  is an interchange Kleene algebra with unit  $\delta_\varepsilon$  and  $*$  commutative.*

As we have shared units and a commutative shuffle operation, the convolution algebras of weighted shuffle form in fact concurrent Kleene algebras.

Weighted languages are usually defined over semirings instead of dioids. Instead of Kleene algebras one can then use star semirings [DKV09]. The Kleene star can then be defined on  $Q^M$  as before. We conjecture that Corollary 8.3 still holds for ordered star semirings, though we have not checked the details.

Shuffle languages are widely used in the interleaving semantics of concurrent programs. The finite transition and Aczel traces of the rely-guarantee calculus [dRdBH<sup>+</sup>01], in particular, form concurrent quantales, which suffices at least for the analysis of safety and invariant properties.

## 9 Partial Interchange Monoids

Next we prepare for our second example, namely of digraphs under serial and parallel composition. It is then natural to consider these compositions not as ternary relations, but as partial operations on graphs. This leads to more general notions of partial semigroups and monoids. An approach to convolution with partial semigroups and monoids has already been developed in [DHS16].

**Definition 9.1** ([DHS16]). A *partial monoid* is a structure  $(S, \otimes, D, E)$  where  $S$  is a set,  $D \subseteq S \times S$  the domain of definition of the composition  $\otimes : D \rightarrow S$ , which is associative in the sense that

$$D x y \wedge D(x \otimes y) z \Leftrightarrow D y z \wedge D x(y \otimes z), \quad D x y \wedge D(x \otimes y) z \Rightarrow x \otimes (y \otimes z) = (x \otimes y) \otimes z,$$

and  $E \subseteq X$  is a set of units, which satisfy

$$\exists e \in E. D e x \wedge e \otimes x = x, \quad \exists e \in E. D x e \wedge x \otimes e = x, \quad \forall e_1, e_2 \in E. D e_1 e_2 \Rightarrow e_1 = e_2.$$

This definition captures the intuition of partiality that the left-hand side of  $x \otimes (y \otimes z) = (x \otimes y) \otimes z$  is defined if and only if the right-hand side is, and, if either side is defined, then the two sides are equal. This notion of equality is sometimes called *Kleene equality*. Categories, monoids and the interval algebras in Example 2.1, ordered pair algebras in Example 2.6 and heaplet algebras in Example 2.7 all form partial monoids. Instead of the unit axioms presented here we could equally use those of object-free categories [ML98]. The precise relationship between partial monoids and object-free categories is discussed in [CDS20].

The relationship between partial and relational monoids is straightforward. A relational monoid  $(X, R, E)$  is *functional* if  $R_{yz}^x = R_{yz}^{x'} \Rightarrow x = x'$  holds for all  $x, x', z \in X$ . With every functional relational monoid  $(X, R, E)$  one can then associate a partial monoid  $(X, \otimes, D, E)$  with  $Dyz \Leftrightarrow \exists x. R_{yz}^x$  and  $y \otimes z$  being the unique  $x \in X$  that satisfies  $R_{yz}^x$ —if  $Dyz$  is defined. We are particularly interested in the converse construction.

**Lemma 9.2** ([DHS17]). *If  $(S, \otimes, D, E)$  is a partial monoid, then  $(S, R, E)$  is a (functional) relational monoid with*

$$R_{yz}^x \Leftrightarrow x = y \otimes z \wedge Dyz.$$

Next we relate partial monoids with relational interchange monoids. Expressing a variant of the interchange law (I7) in the context of partial monoids requires an ordering on  $S$ . This motivates the following definition.

**Definition 9.3.** A *preordered partial monoid* is a structure  $(S, \preceq, \otimes, D, E)$  such that  $(S, \preceq)$  is a preorder,  $(S, \otimes, D, E)$  a partial monoid, and  $\otimes$  is order preserving in the sense that

$$x \preceq y \wedge Dzx \Rightarrow z \otimes x \preceq z \otimes y \wedge Dzy, \quad x \preceq y \wedge Dxz \Rightarrow x \otimes z \preceq y \otimes z \wedge Dyz.$$

Lemma 9.2 can then be generalised.

**Lemma 9.4.** *Let  $(S, \preceq, \otimes, D)$  be a preordered partial monoid. Then  $(S, R)$  is a relational semigroup with*

$$R_{yz}^x \Leftrightarrow x \preceq y \otimes z \wedge Dyz.$$

*Proof.* For relational associativity,

$$\begin{aligned} \exists y. R_{uy}^x \wedge R_{vw}^y &\Leftrightarrow \exists y. x \preceq u \otimes y \wedge D u y \wedge y \preceq v \otimes w \wedge D v w \\ &\Leftrightarrow x \preceq u \otimes (v \otimes w) \wedge D v w \wedge D u (v \otimes w) \\ &\Leftrightarrow x \preceq (u \otimes v) \otimes w \wedge D u v \wedge D (u \otimes v) w \\ &\Leftrightarrow \exists y. D y w \wedge y \preceq u \otimes v \wedge D u v \wedge x \preceq y \otimes w \\ &\Leftrightarrow \exists y. R_{vw}^y \wedge R_{yw}^x. \end{aligned}$$

□

However the unit laws of preordered partial monoids need not translate to relational semigroups.

**Lemma 9.5.** *Let  $(S, \preceq, \otimes, D)$  be a preordered partial monoid and  $R_{yz}^x \Leftrightarrow x \preceq y \otimes z \wedge Dyz$ . Then*

1.  $\exists e \in E. R_{ex}^x$  and  $\exists e \in E. R_{xe}^x$ ,
2.  $\exists e \in E. R_{ey}^x \Rightarrow x \preceq y$  and  $\exists e \in E. R_{ye}^x \Rightarrow x \preceq y$ .

In (2), it cannot generally be expected that  $x = y$ . The relationship  $x \preceq y$  cannot be captured directly within relational semigroups or monoids.

**Definition 9.6.** A *partial interchange monoid* is a structure  $(S, \preceq, \otimes, \otimes, D, D, E, E)$  such that  $(S, \preceq, \otimes, D, E)$  and  $(S, \preceq, \otimes, D, E)$  are preordered partial monoids,  $E \subseteq E$  and the following interchange law holds:

$$Dwx \wedge D(w \otimes x)(y \otimes z) \wedge Dyz \Rightarrow Dwy \wedge D(w \otimes y)(x \otimes z) \wedge Dxz \wedge (w \otimes x) \otimes (y \otimes z) \preceq (w \otimes y) \otimes (x \otimes z). \quad (\text{pi7})$$

In light of Lemma 9.4 and 9.5 we cannot not expect to relate partial interchange monoids directly with relational interchange monoids. But the relationship is straightforward if we forget relational units and restrict to a single monoidal unit.

**Lemma 9.7.** *Let  $(S, \preceq, \otimes, \otimes, D, D, E, E)$  be a partial interchange monoid in which  $E = \{e\} = E$ . Then the following small interchange laws hold.*

1.  $Dxy \Rightarrow Dxy \wedge x \otimes y \preceq x \otimes y$ ,
2.  $Dxy \Rightarrow Dyx \wedge x \otimes y \preceq y \otimes x$ ,
3.  $Dx(y \otimes z) \wedge Dyz \Rightarrow Dxy \wedge D(x \otimes y)x \wedge x \otimes (y \otimes z) \preceq (x \otimes y) \otimes z$ ,
4.  $Dxy \wedge D(x \otimes y)z \Rightarrow Dx(y \otimes z) \wedge Dyz \wedge (x \otimes y) \otimes z \preceq x \otimes (y \otimes z)$ ,
5.  $Dx(y \otimes z) \wedge Dyz \Rightarrow Dy(x \otimes z) \wedge Dyz \wedge x \otimes (y \otimes z) \preceq y \otimes (x \otimes z)$ ,
6.  $Dxy \wedge D(x \otimes y)z \Rightarrow Dxz \wedge D(x \otimes z)y \wedge (x \otimes y) \otimes z \preceq (x \otimes z) \otimes y$ .

*Proof.* We show (3) as an example. Suppose  $Dx(y \otimes z)$  and  $Dyz$ . Then  $Dxe$  and  $D(x \otimes e)(y \otimes z)$  and therefore  $Dxy$ ,  $D(x \otimes y)(e \otimes z)$ ,  $Dez$  and  $(x \otimes e) \otimes (y \otimes z) \preceq (x \otimes y) \otimes (e \otimes z)$  by (pi7). Hence  $x \otimes (y \otimes z) \preceq (x \otimes y) \otimes z$  by the unit laws of partial monoids. The other proofs are similar and left to the reader.  $\square$

With multiple units it seems necessary to require that parallel units are sequential units for all elements, which is artificial.

From now on we call *relational interchange semigroup* a relational interchange monoid in which units may be absent, and the small interchange laws (RI<sub>1</sub>)-(RI<sub>6</sub>) hold in addition to (RI<sub>7</sub>).

**Lemma 9.8.** *If  $(S, \preceq, \otimes, \otimes, D, D, \{e\})$  is a partial interchange monoid, then  $(S, R, R)$  is a relational interchange semigroup with  $R_{yz}^x \Leftrightarrow x = y \otimes z \wedge Dyz$  and  $R_{yz}^x \Leftrightarrow x \preceq y \otimes z \wedge Dyz$ .*

*Proof.* We need to check that (pi7) implies (RI<sub>7</sub>).

$$\begin{aligned}
\exists y, z. R_{tu}^y \wedge R_{yz}^x \wedge R_{vw}^z &\Leftrightarrow \exists y, z. y \preceq t \otimes u \wedge Dtu \wedge x = y \otimes z \wedge Dyz \wedge z \preceq v \otimes w \wedge Dvw \\
&\Leftrightarrow x \preceq (t \otimes u) \otimes (v \otimes w) \wedge Dtu \wedge D(t \otimes u)(v \otimes w) \wedge Dvw \\
&\Rightarrow x \preceq (t \otimes v) \otimes (u \otimes w) \wedge Dtv \wedge D(t \otimes v)(u \otimes w) \wedge Duw \\
&\Leftrightarrow \exists y, z. y = t \otimes v \wedge Dtv \wedge x \preceq y \otimes z \wedge Dyz \wedge z = u \otimes w \wedge Duw \\
&\Leftrightarrow \exists y, z. R_{tv}^y \wedge R_{yz}^x \wedge R_{uw}^z. \quad \square
\end{aligned}$$

This calculation does not depend on the presence of units. Small interchange laws hold in  $S$  by Lemma 9.7. These allow deriving the small relational interchange laws (RI<sub>1</sub>)-(RI<sub>6</sub>) as in the proof of (RI<sub>7</sub>). Hence  $S$  is a relational interchange semigroup.

It is easy to check that we could have used  $R_{yz}^x \Leftrightarrow x \preceq y \otimes z \wedge Dyz$  instead of  $R_{yz}^x \Leftrightarrow x = y \otimes z \wedge Dyz$  in the proof of Lemma 9.8. Using an equational encoding for  $R$ , however, would have broken the proof. The following example shows that even (RI<sub>1</sub>) would break if two equational encodings were used.

**Example 9.9.** Consider the partial monoid over  $\{a, b\}$  with  $D = \{(a, a)\}$ ,  $D = S \times S$ , order and compositions defined by  $b \otimes b = a \otimes b = b \otimes a = a \otimes a = b \prec a = a \otimes a$ , and a suitable unit adjoined. The small interchange law  $Dxy \Rightarrow Dxy \wedge x \otimes y \preceq x \otimes y$  holds for all  $x$  and  $y$ . Let  $R_{yz}^x \Leftrightarrow x = y \otimes z \wedge Dyz$  and  $R_{yz}^x \Leftrightarrow x = y \otimes z \wedge Dyz$ . Then  $R \not\subseteq R$ , that is,  $R_{aa}^b$  and  $\neg R_{aa}^b$ , because  $Daa$ ,  $b = a \otimes a$  and  $b \neq a = a \otimes a$ .  $\square$

Lemma 9.8 yields the following corollary to Theorem 5.6(1).

**Corollary 9.10.** *If  $S$  is a partial interchange monoid with unit  $e$  and  $Q$  an interchange quantale, then  $Q^S$  is a non-unital interchange quantale with convolutions*

$$(f * g)x = \bigvee_{y,z:x=y \otimes z} f y \bullet g z, \quad (f * g)x = \bigvee_{y,z:x \preceq y \otimes z} f y \bullet g z$$

that satisfies the small interchange laws  $(I_1)$ – $(I_6)$  in addition to  $(I_7)$ .

Unitality fails in general because the unit  $id$  of  $*$  need not be the unit of  $*$ :

$$(f * id)x = \bigvee \{f y \bullet 1 \mid R_{ye}^x\} = \bigvee \{f y \mid x \preceq y\} \geq f x,$$

but not necessarily  $f * id = f$ , and similarly for  $id * f = f$ . The retract  $(Q, \leq, *)$  has unit  $id$ ; only the retract  $(Q, \preceq, *)$  does not have  $id$  as a unit. To obtain equality, and hence unital interchange quantales, conditions on  $f$  are needed.

A partial interchange monoid  $(S, \otimes, \otimes, \{e\})$  is *positive* if  $e$  is a minimal element of  $S$  with respect to  $\preceq$ . It is *serially-decomposable* if  $x \preceq y_1 \otimes y_2$  implies that there exists  $x_1, x_2$  such that  $x = x_1 \otimes x_2$ , and  $x_1 \preceq y_1$  and  $x_2 \preceq y_2$ .

**Lemma 9.11.** *Let  $f$  be antitone, that is,  $x \preceq y \Rightarrow f y \leq f x$ . Then  $f * id = f = id * f$ .*

*Proof.*  $(f * id)x = \bigvee \{f y \bullet 1 \mid R_{ye}^x\} = \bigvee \{f y \mid x \preceq y\} = f x$ . The  $\leq$ -direction holds by antitonicity, the  $\geq$ -direction by the above calculation. The proof of  $id * f = f$  is similar.  $\square$

To make  $id$  antitone it seems appropriate to require that  $e$  is minimal with respect to  $\preceq$  and hence that the partial interchange monoid is positive. We also need to check that  $*$  and  $*$  preserve antitonicity.

**Proposition 9.12.** *Let  $(S, \otimes, \otimes, \{e\})$  be a positive serially-decomposable partial interchange monoid and  $Q$  and interchange quantale. Then the antitone functions in  $Q^S$  form a (unital) interchange sub-quantale.*

*Proof.* Unitality follows from Lemma 9.11. It remains to show that  $id$  is antitone and that  $*$  and  $*$  preserve antitonicity. The first fact follows from positivity. For preservation of  $*$ , suppose  $x \preceq y$ . Then

$$(f * g)y = \bigvee \{f y_1 \bullet f y_2 \mid y \preceq y_1 \otimes y_2\} \leq \bigvee \{f x_1 \bullet f x_2 \mid x \preceq x_1 \otimes x_2\} = (f * g)x.$$

For preservation of  $*$ , suppose once again  $x \preceq y$ . Then

$$(f * g)y = \bigvee \{f y_1 \bullet f y_2 \mid y = y_1 \otimes y_2\} \leq \bigvee \{f x_1 \bullet f x_2 \mid x = x_1 \otimes x_2\} = (f * g)x$$

by  $\otimes$ -decompositionality.  $\square$

## 10 Weighted Graph Languages

Our second extended example shows how weighted graph languages can be constructed with our approach. A partial interchange monoid structure can be imposed on graphs in various ways. Partiality arises because, typically, the vertices of the graph operands are supposed to be disjoint. Henceforce, we mean digraph when we say graph. Graphs with undirected edges can be obtained from these in the obvious way.

Formally, we view graphs as binary relations on some set  $X$ . Let graphs  $G_1$  and  $G_2$  be disjoint, that is, they have disjoint vertex sets:  $V_{G_1} \cap V_{G_2} = \emptyset$ . Their *serial composition (complete join) and disjoint union (parallel composition)* are defined as

$$G_1 \cdot G_2 = G_1 \sqcup G_2 \sqcup (V_{G_1} \times V_{G_2}), \quad G_1 \parallel G_2 = G_1 \sqcup G_2,$$

where  $\sqcup$  denotes disjoint union. Both operations are standard [CE12]. This turns graphs under serial composition into partial monoids, and graphs under parallel composition into partial abelian monoids.

A *graph morphism*  $\varphi : G_1 \rightarrow G_2$  between graphs  $G_1$  and  $G_2$  satisfies  $(x, y) \in G_1 \Rightarrow (f x, f y) \in G_2$ . A morphism  $f$  is *faithful*, or a *graph embedding*, if  $(f x, f y) \in G_2$  implies  $(x, y) \in G_1$ . A *graph isomorphism* is a bijective (on vertices) graph embedding. We write  $G_1 \cong G_2$  if there exists a graph isomorphism between  $G_1$  and  $G_2$ . We say that  $G_1$  and  $G_2$  are *isomorphic* or have the same *graph type* if  $G_1 \cong G_2$  and call  $G/\cong$  the *isomorphism class* or *graph type* of  $G$ .

The *subsumption relation*  $\preceq$  between graphs, which is defined by  $G_1 \preceq G_2$  if and only if there exists a bijective (on vertices) graph morphism  $\varphi : G_2 \rightarrow G_1$ , is a preorder. The associated subsumption equivalence  $\simeq$  need not coincide with  $\cong$ , as will be explained in Section 11. We now fix any set  $\mathcal{G}$  of (di)graphs that contains the empty graph  $\varepsilon$  and is closed under serial and parallel composition.

**Proposition 10.1.** *The structure  $(\mathcal{G}, \cdot, \parallel, \{\varepsilon\})$  forms a partial interchange monoid with commutative parallel composition and shared unit  $\varepsilon$ .*

*Proof.* First of all, the partial associativity and unit laws, partial commutativity of disjoint union as well as partial isotonicity of the two compositions must be shown. This is routine. In the presence of a shared unit  $\varepsilon$  it then remains to verify (pi7). For this we need the following isotonicity property of cartesian products:  $A \subseteq B$  implies  $A \times C \subseteq B \times C$  and  $C \times A \subseteq C \times B$ .

We only show that the weak interchange law  $(G_1 \parallel G_2) \cdot (G_3 \parallel G_4) \preceq_r (G_1 \cdot G_2) \parallel (G_3 \cdot G_4)$  holds and leave the remaining laws to the reader. We use the identity function on the  $G_i$  to construct the bijective morphism. We need to show that  $V_{(G_1 \parallel G_2) \cdot (G_3 \parallel G_4)} = V_{(G_1 \cdot G_2) \parallel (G_3 \cdot G_4)}$  and  $(G_1 \cdot G_2) \parallel (G_3 \cdot G_4) \subseteq (G_1 \parallel G_2) \cdot (G_3 \parallel G_4)$  as a relation. First,  $V_{G_i \cdot G_j} = V_{G_i} \cup V_{G_j} = V_{G_i \parallel G_j}$  and therefore

$$V_{(G_1 \parallel G_2) \cdot (G_3 \parallel G_4)} = V_{G_1} \cup V_{G_2} \cup V_{G_3} \cup V_{G_4} = V_{(G_1 \cdot G_2) \parallel (G_3 \cdot G_4)}.$$

Second,

$$\begin{aligned} (G_1 \cdot G_2) \parallel (G_3 \cdot G_4) &= (G_1 \cup G_2 \cup V_{G_1} \times V_{G_2}) \parallel (G_3 \cup G_4 \cup V_{G_3} \times V_{G_4}) \\ &= G_1 \cup G_2 \cup V_{G_1} \times V_{G_2} \cup G_3 \cup G_4 \cup V_{G_3} \times V_{G_4} \\ &\subseteq G_1 \cup G_3 \cup G_2 \cup G_4 \cup (V_{G_1} \cup V_{G_3}) \times (V_{G_2} \cup V_{G_4}) \\ &= (G_1 \parallel G_3) \cup (G_2 \parallel G_4) \cup V_{G_1 \parallel G_3} \times V_{G_2 \parallel G_4} \\ &= (G_1 \parallel G_3) \cdot (G_2 \parallel G_4). \end{aligned}$$

□

Lemma 9.8 and Corollary 9.10 then imply that weighted graph languages form interchange quantales up-to unitality of the parallel quantale retract. But one can do better.

**Lemma 10.2.** *The partial interchange monoid  $(\mathcal{G}, \cdot, \parallel, \{\varepsilon\})$  is positive and serially decomposable.*

*Proof.* It is clear that  $\varepsilon$  is an isolated point with respect to  $\preceq$  and hence minimal. This proves positivity. The proof of serial decomposability is intuitive, but somewhat tedious to spell out formally. Suppose  $G \preceq G_1 \cdot G_2$ . Then the vertices of  $G_1$  and  $G_2$  are disjoint and in addition to the arrows of  $G_1$  and  $G_2$  we have  $V_{G_1} \times V_{G_2}$ . Hence if  $G \preceq G_1 \cdot G_2$ , then the arrows added by the bijective graph morphism  $\varphi : G_1 \cdot G_2 \rightarrow G$  must either be added to  $G_1$  or to  $G_2$ , while  $V_{G_1} \times V_{G_2}$  stays the same. There must thus be  $G'_1 \preceq G_1$  and  $G'_2 \preceq G_2$  such that  $G = G'_1 \cdot G'_2$ . □

Proposition 9.12 then specialises as follows.

**Corollary 10.3.** *If  $Q$  is an interchange quantale with unit 1 and  $\bullet$  commutative, then  $Q^{\mathcal{G}}$  is a (generally non-unital) interchange quantale with  $*$  commutative and*

$$(f * g) x = \bigvee_{y, z: x=y \cdot z} f y \bullet g z, \quad (f \bullet g) x = \bigvee_{y, z: x \preceq y \parallel z} f y \bullet g z.$$

*The subquantale of antitone functions in  $Q^{\mathcal{G}}$  is unital.*

Labels can be added to vertices ad libitum, which yields proper weighted graph languages. Both the serial and the parallel composition preserve order properties. Corollary 10.3 thus specialises immediately to weighted partial orders.

Next we consider convolution algebras that are powerset liftings, that is,  $Q = \mathbb{B}$ . Then  $f : \mathcal{G} \rightarrow \mathbb{B}$  is a set indicator function and we may write  $x \in f$  instead of  $f x$ , identifying the indicator function with the set it represents. Then  $(f * g) x = \bigvee \{f y \bullet g z \mid x = y \cdot z\}$  rewrites as  $x \in f * g \Leftrightarrow \exists y, z. x = y \cdot z \wedge y \in f \wedge z \in g$  and hence

$$f * g = \{y \cdot z \mid y \in f \wedge z \in g\}.$$

Similarly,

$$f * g = \{x \mid x \preceq y \parallel z \wedge y \in f \wedge z \in g\} = \{y \parallel z \mid y \in f \wedge z \in g\} \downarrow,$$

where  $\downarrow$  denotes the down-closure with respect to  $\preceq$ . Moreover, the antitonicity condition rewrites as  $x \preceq y \wedge f y \Rightarrow f x$ , which is precisely  $f = f \downarrow$ , that is,  $f$  is a down-set with respect to  $\preceq$ .

**Corollary 10.4.** *The down-sets in  $\mathcal{P}\mathcal{G}$  form a unital interchange quantale.*

Finally we consider the finite case, and obtain the following corollary of Theorem 7.6.

**Corollary 10.5.** *If  $K$  is an interchange Kleene algebra with unit 1 and  $\mathcal{G}$  a partial interchange monoid of finite graphs, then the antitone functions in  $K^{\mathcal{G}}$  form an interchange Kleene algebra.*

This holds because any finite graph can be decomposed in finitely many ways serially or parallelly into subgraphs. Once again, all results specialise to partial orders, and in particular to labelled partial orders, where vertices are labelled with letters from some alphabet. Sets of partial orders in general, and labelled partial orders in particular, are widely used in concurrency theory [Gra81, Vog92] and the theory of distributed systems [Lam78].

## 11 Weighted Languages of Types of Finite Graphs

Many applications, including those in concurrency and distributed systems, require isomorphism classes and hence types of graphs or (labelled) partial orders. Lifting the results from Section 10 to these is not entirely straightforward. This is well known [Ési02], but we spell out details for the sake of completeness.

**Example 11.1** ([Ési02]). Consider the infinite poset  $(P, \leq_P)$  with  $P = \{p_{i,j} \mid i, j \in \mathbb{N} \wedge (i = 0 \vee j = 0)\}$  and  $p_{i,j} \leq_P p_{k,l}$  if and only if  $i = k = 0$  and  $j \leq l$ , and the infinite poset  $(Q, \leq_Q)$  with  $Q = \{q_{i,j} \mid i, j \in \mathbb{N} \wedge (i = 0 \vee i = 1 \vee j = 0)\}$  and  $q_{i,j} \leq_Q q_{k,l}$  if and only if  $i = k = 0$  or  $i = k = 1$ , and  $j \leq l$ .

Intuitively,  $P$  consists of the disjoint union of the infinite chain formed by the  $p_{0,j}$  and the elements  $p_{i,0}$  with  $i > 0$ , whereas  $Q$  consists of the disjoint union of the infinite chain formed by the  $p_{0,j}$ , the infinite chain formed by the  $p_{1,j}$  and the elements  $p_{i,0}$  with  $i \geq 1$ .

Define the functions  $\varphi : P \rightarrow Q$  and  $\psi : Q \rightarrow P$  by

$$\varphi p_{i,j} = \begin{cases} q_{0,j} & \text{if } i = 0, \\ q_{1,k} & \text{if } i > 0 \wedge j = 2k + 1, \\ q_{k,0} & \text{if } i > 0 \wedge j = 2k, \end{cases} \quad \psi q_{i,j} = \begin{cases} p_{0,2k} & \text{if } i = 0, \\ p_{1,2k+1} & \text{if } i = 1, \\ p_{i-1,0} & \text{if } i > 1. \end{cases}$$

Intuitively,  $\varphi$  maps the chain in  $P$  onto the first chain in  $Q$  and the isolated elements in  $P$  alternately onto the second chain and the isolated elements in  $Q$ , whereas  $\psi$  maps the elements of the two chains in  $Q$  alternately onto the chain in  $P$ , and isolated points in  $Q$  onto isolated points in  $P$ . The morphisms are shown in Figure 5.

By construction,  $\varphi$  and  $\psi$  are both bijective and order-preserving. Hence  $P \preceq Q$  and  $Q \preceq P$ , but of course neither  $P = Q$  nor  $P \cong Q$ .  $\square$

At least in the finite case the situation is simpler. An explanation requires two simple facts about groups.



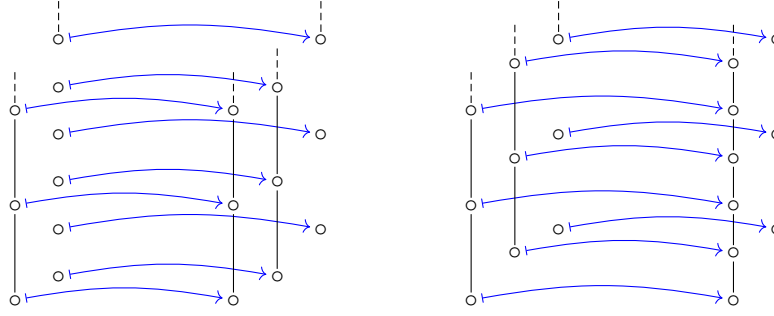


Figure 5: Posets  $P$  and  $Q$  with bijective morphisms  $\varphi$  in left diagram and  $\psi$  in right diagram.

**Lemma 11.2.** *Let  $G$  be the cyclic group generated by  $x$  and let  $x^i = x^j$  for some integers  $i < j$ . Then  $G = \{1, x, x^2, \dots, x^{k-1}\}$ , where  $k = j - i$ .*

*Proof.* The assumption implies that  $x^i = x^i x^k$  for some  $k$ , and thus  $x^k = 1$  by cancellation. By cyclicity, every  $g \in G$  is of the form  $g = x^n$  for some  $n \in \mathbb{N}$  that can be written  $n = pk + q$  for some  $p, q \in \mathbb{N}$  with  $q \leq k - 1$ . Hence  $g = (x^k)^p x^q = 1^p x^q = x^q$  for some  $0 \leq q \leq k - 1$  or, equivalently,  $g \in \{1, x, x^2, \dots, x^{k-1}\}$ . Since every  $x^n \in G$ , this shows that  $G = \{1, x, x^2, \dots, x^{k-1}\}$ .  $\square$

**Lemma 11.3.** *Let  $G$  be a finite cyclic group of order  $n$  generated by  $x$ . Then  $G = \{1, x, x^2, \dots, x^{n-1}\}$  and  $x^n = 1$ .*

*Proof.* By the pigeonhole principle, there must be a minimal  $j \in \mathbb{N}$ ,  $j \leq n$ , such that  $x^j = x^i$  for some  $i \in \mathbb{N}$  with  $i < j$ . Hence the elements  $1, x, x^2, \dots, x^{j-1}$  are pairwise distinct. Then, by Lemma 11.2,  $j = n$ ,  $i = 0$  and  $x^n = x^0 = 1$ .  $\square$

**Lemma 11.4.** *Let  $G_1$  and  $G_2$  be finite graphs such that  $G_1 \preceq G_2$  and  $G_2 \preceq G_1$ . Then  $G_1 \cong G_2$ .*

*Proof.* By assumption there exists order preserving bijections  $\varphi : G_2 \rightarrow G_1$  and  $\psi : G_1 \rightarrow G_2$ , hence  $\chi = \psi \circ \varphi$  is an order preserving bijection on  $G_1$ . As  $\chi$  can be seen as a group action on the finite set  $V_1$ , it generates a finite cyclic group. Hence there is some  $k \in \mathbb{N}$  such that  $\chi^k = id_{V_1}$  by Proposition 11.3. It then follows that  $\varphi$  is faithful: Suppose  $(\varphi x, \varphi y) \in G_2$ . Then  $x = \chi^k x = \chi^{k-1}(\psi(\varphi x)) \rightarrow_R \chi^{k-1}(\psi(\varphi y)) = \chi^k y = y$ . It follows that  $\varphi$  is a graph isomorphism and  $G_1 \cong G_2$ .  $\square$

A similar fact has been proved by Ésik [Ési02]. We henceforth restrict our attention to finite graphs.

Let  $[G] = \{G' \mid G' \cong G\}$  denote the type of  $G$ . We extend the subsumption preorder  $\preceq$  to equivalence classes by  $[G_1] \preceq [G_2] \Leftrightarrow G_1 \preceq G_2$ , overloading notation. This relation is well defined.

**Lemma 11.5.** *Let  $G'_1 \cong G_1$ ,  $G_1 \preceq G_2$  and  $G_2 \cong G'_2$ . Then  $G'_1 \preceq G'_2$ .*

*Proof.* Let  $\varphi_1$  be the graph isomorphism of type  $G_1 \rightarrow G'_1$ ,  $\varphi_2$  the graph isomorphism of type  $G'_2 \rightarrow G_2$  and  $\psi$  the bijective graph morphism of type  $G_2 \rightarrow G_1$ . Then  $\varphi_1 \circ \psi \circ \varphi_2 : G'_2 \rightarrow G'_1$  is a bijective graph morphism as well. Hence  $G'_1 \preceq G'_2$ .  $\square$

**Lemma 11.6.** *The relation  $\preceq$  is a partial order on  $\mathcal{G}/\cong$  if all graphs in  $\mathcal{G}$  are finite.*

*Proof.* Reflexivity and transitivity for  $\preceq$  on  $\mathcal{G}/\cong$  follows from reflexivity and transitivity of  $\preceq$  on  $\mathcal{G}$ . For antisymmetry,  $[G_1] \preceq [G_2]$  and  $[G_2] \preceq [G_1]$  imply  $[G_1] = [G_2]$  for all  $G_1, G_2 \in \mathcal{G}$  by Lemma 11.4.  $\square$

Extending serial and parallel composition of graphs is standard:  $[G_1] \cdot [G_2] = \{G'_1 \cdot G'_2 \mid G'_1 \cong G \wedge G'_2 \cong G\}$  and likewise for  $[G_1] \parallel [G_2]$ . It is also well known that both compositions are well defined: if  $G_1 \cong G'_1$  and  $G_2 \cong G'_2$ , then  $[G_1] \cdot [G_2] = [G'_1] \cdot [G'_2]$  and  $[G_1] \parallel [G_2] = [G'_1] \parallel [G'_2]$ . By contrast to serial and parallel compositions of graphs, those of graph types are total. Finally, equivalence classes are closed with respect to serial and parallel composition.

**Lemma 11.7.** *For all  $G_1, G_2 \in \mathcal{G}$ ,*

1.  $[G_1 \cdot G_2] = [G_1] \cdot [G_2]$ ,
2.  $[G_1 \parallel G_2] = [G_1] \parallel [G_2]$ .

*Proof.*  $H \in [G_1 \cdot G_2]$  if and only if  $H \cong G_1 \cdot G_2$ . This is the case if and only if there are graphs  $G'_1$  and  $G'_2$  such that  $H = G'_1 \cdot G'_2$  and  $G'_1 \cong G_1$  and  $G'_2 \cong G_2$ , which holds if and only if  $H \in [G_1] \cdot [G_2]$ . The proof for  $\parallel$  is similar.  $\square$

**Proposition 11.8.** *The structure  $(\mathcal{G}/\cong, \cdot, \parallel, [\varepsilon])$  is an interchange monoid in which  $\parallel$  is commutative, if all graphs in  $\mathcal{G}$  are finite.*

*Proof.* The associativity, commutativity and unit laws are easy to check, noting that  $[\varepsilon] = \{\varepsilon\}$ . For the interchange law  $[(G_1 \parallel G_2) \cdot (G_3 \parallel G_4)] \preceq [(G_1 \cdot G_3) \parallel (G_2 \cdot G_4)]$ , by definition of  $\preceq$  on equivalence classes, it suffices to show that  $(G_1 \parallel G_2) \cdot (G_3 \parallel G_4) \preceq (G_1 \cdot G_3) \parallel (G_2 \cdot G_4)$ , which holds by Proposition 10.1.  $\square$

Proposition 11.8 specialises immediately to types of finite partial orders with serial and parallel composition, which are known as partial words or pomsets in concurrency theory—when vertex labels are added [Gra81, Gis88]. The instance of Proposition 11.8 for pomsets is due to Gischer [Gis88].

Because some compositions in  $\mathcal{G}/\cong$  may result in the empty set, the interchange monoid can have an annihilator 0, that is, an element for which  $x \cdot 0 = 0 = 0 \cdot x$  and  $x \parallel 0 = 0$  holds for any element  $x$ .

The lifting to convolution algebras—interchange quantales, unital interchange quantales, interchange Kleene algebras—then follows the results of the previous section. The result that the powerset lifting of finite pomsets yields concurrent semirings, interchange semirings in which  $\bullet$  is commutative and  $Q = \mathbb{B}$ , is due to Gischer [Gis88]. Extensions to concurrent Kleene algebras and concurrent quantales have been proved more recently [HMSW11].

## 12 Conclusion

The results in this article support the construction of concurrent quantales and Kleene algebras from relational structures, multimonooids and partial monooids. They can be formalised easily in proof assistants and applied in concurrency verification. In fact, the lifting from ternary relations and partial monooids to quantalic convolution algebras—without interchange laws—has already been formalised with Isabelle/HOL [DGHS17]. Extending this to concurrency is left for future work.

Another interesting avenue for research is the extension of Stone-type duality to our constructions, building on work of Harding, Walker and Walker for lattice-valued functions [HWW18]. Moreover, a categorification of our approach will be published in a successor paper.

Finally, we hope that our results will benefit the construction of real-word graph-based models for concurrent and distributed systems, and ultimately the design of programming languages and verification tools for such systems.

**Acknowledgement:** The authors would like to thank Tony Hoare for discussions on models of concurrent Kleene algebras, and to Brijesh Dongol and Ian Hayes for their collaboration on convolution algebras and comments on early versions of this work. The second and third author have been partially supported by EPSRC grant EP/R032351/1.

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