On the strong chromatic number of random graphs

Po-Shen Loh * Benny Sudakov †

Abstract

Let G be a graph with n vertices, and let k be an integer dividing n. G is said to be strongly k-colorable if for every partition of $V(G)$ into disjoint sets $V_1 \cup \ldots \cup V_r$, all of size exactly k, there exists a proper vertex k-coloring of G with each color appearing exactly once in each V_i. In the case when k does not divide n, G is defined to be strongly k-colorable if the graph obtained by adding $k\lceil \frac{n}{k} \rceil - n$ isolated vertices is strongly k-colorable. The strong chromatic number of G is the minimum k for which G is strongly k-colorable. In this paper, we study the behavior of this parameter for the random graph $G_{n,p}$. In the dense case when $p \gg n^{-1/3}$, we prove that the strong chromatic number is a.s. concentrated on one value $\Delta + 1$, where Δ is the maximum degree of the graph. We also obtain several weaker results for sparse random graphs.

1 Introduction

Let G be a graph, and let $V_1, \ldots, V_r$ be disjoint subsets of its vertex set. An *independent transversal* with respect to $\{V_i\}_{i=1}^r$ is an independent set in G which contains exactly one vertex from each V_i. The problem of finding sufficient conditions for the existence of an independent transversal, in terms of the ratio between the part sizes and the maximum degree Δ of the graph, dates back to 1975, when it was raised by Bollobás, Erdős, and Szemerédi [10]. Since then, much work has been done [1, 3, 5, 14, 15, 17, 18, 22, 26, 27], and this basic concept has also appeared in several other contexts, such as linear arboricity [4], vertex list coloring [23, 24, 8], and cooperative coloring [2, 19]. In the general case, it was proved by Haxell [14] that an independent transversal exists as long as all parts have size at least 2Δ. The sharpness of this bound was shown by Szabó and Tardos [26], extending earlier results of [18] and [27]. On the other hand, we proved in [19] that the upper bound can be further reduced to $(1 + o(1))\Delta$ if no vertex has more than $o(\Delta)$ neighbors in any single part. Such a condition arises naturally in certain applications, e.g., vertex list coloring.

In the case when all of the V_i are of the same size k, it is natural to ask when it is possible to find not just one, but k disjoint independent transversals with respect to the $\{V_i\}$. This is closely related to the following notion of strong colorability. Given a graph G with n vertices and a positive integer k dividing n, we say that G is *strongly k-colorable* if for every partition of $V(G)$ into disjoint sets $V_1 \cup \ldots \cup V_r$, all of size exactly k, there exists a proper vertex k-coloring of G with each color appearing exactly once in each V_i. Notice that G is strongly k-colorable iff the chromatic number of

*Department of Mathematics, Princeton University, Princeton, NJ 08544. E-mail: ploh@math.princeton.edu. Research supported in part by a Fannie and John Hertz Foundation Fellowship, an NSF Graduate Research Fellowship, and a Princeton Centennial Fellowship.

†Department of Mathematics, Princeton University, Princeton, NJ 08544, and Institute for Advanced Study, Princeton. E-mail: bsudakov@math.princeton.edu. Research supported in part by NSF CAREER award DMS-0546523, NSF grant DMS-0355497, USA-Israeli BSF grant, Alfred P. Sloan fellowship, and the State of New Jersey.

any graph obtained from G by adding a union of vertex disjoint k-cliques is k. If k does not divide n, then we say that G is strongly k-colorable if the graph obtained by adding $k\lceil \frac{n}{k} \rceil - n$ isolated vertices is strongly k-colorable. The *strong chromatic number* of G, denoted $s\chi(G)$, is the minimum k for which G is strongly k-colorable.

The concept of strong chromatic number first appeared independently in work by Alon [4] and Fellows [11]. It was also the crux of the longstanding "cycle plus triangles" problem popularized by Erdős, which was to show that the strong chromatic number of the cycle on $3n$ vertices is three. That problem was solved by Fleischner and Stiebitz [12]. The strong chromatic number is known [11] to be monotonic in the sense that strong k-colorability implies strong $(k+1)$-colorability. It is also easy to see that $s\chi(G)$ must always be strictly greater than the maximum degree Δ: simply take V_1 to be the neighborhood of a vertex of maximal degree, and partition the rest of the vertices arbitrarily. The intriguing question of bounding the strong chromatic number in terms of the maximum degree has not yet been answered completely. Alon [5] showed that there exists a constant c such that $s\chi \leq c\Delta$ for every graph. Later, Haxell [15] improved the bound by showing that it is enough to use $c = 3$, and in fact even $c = 3 - \epsilon$ for ϵ up to $1/4$ [16]. On the other hand, Fleischner and Stiebitz [13] observed that the disjoint union of complete bipartite graphs $K_{\Delta,\Delta}$ cannot be strongly $(2\Delta - 1)$-colored. Indeed, put each part of one of the $K_{\Delta,\Delta}$ into the sets V_1 and V_2, respectively. Then these 2Δ vertices should get different colors. It is believed that this lower bound is tight and the strong chromatic number of any graph with maximum degree Δ should be at most 2Δ.

It is natural to wonder what is the asymptotic behavior of the strong chromatic number for the random graph $G_{n,p}$, relative to the maximum degree of the graph. As usual, $G_{n,p}$ is the probability space of all labeled graphs on n vertices, where every edge appears randomly and independently with probability $p = p(n)$. We say that the random graph possesses a graph property $\mathcal{P}$ *almost surely*, or a.s. for brevity, if the probability that $G_{n,p}$ satisfies $\mathcal{P}$ tends to 1 as n tends to infinity. One of the most interesting phenomena discovered in the study of random graphs is that many natural graph invariants are highly concentrated (see, e.g., [21] for the result on the clique number and [25, 20, 6] for the concentration of the chromatic number). In this paper we show that the strong chromatic number is another example of a tightly concentrated graph parameter. For dense random graphs, it turns out that we can concentrate $s\chi(G_{n,p})$ on a single value, and for some smaller values of p we were only able to determine $s\chi(G_{n,p})$ asymptotically. In the statement of our first result, and in the rest of this paper, the notation $f(n) \gg g(n)$ means that $f/g \to \infty$ together with n. Also, all logarithms are in the natural base e.

Theorem 1.1 *Let Δ be the maximum degree of the random graph $G_{n,p}$, where $p < 1 - \theta$ for any arbitrary constant $\theta > 0$.*

(i) *If $p \gg \left(\frac{\log^4 n}{n}\right)^{1/3}$, then almost surely the strong chromatic number of $G_{n,p}$ equals $\Delta + 1$.*

(ii) *If $p \gg \left(\frac{\log n}{n}\right)^{1/2}$, then a.s. the strong chromatic number of $G_{n,p}$ is $(1 + o(1))\Delta$.*

Unfortunately, our approach breaks down completely when $p \ll n^{-1/2}$. However, for this range of p, we have a different argument which shows how to find at least one independent transversal.

Theorem 1.2 *Let Δ be the maximum degree of the random graph $G_{n,p}$. If $p \geq \frac{\log^4 n}{n}$, then almost surely every collection of disjoint subsets $V_1, \ldots, V_r$ of $G_{n,p}$ with all $|V_i| \geq (1 + o(1))\Delta$ has an independent transversal.*

This rest of this paper is organized as follows. In Section 2, we prove both parts of our first theorem concerning the strong chromatic number of relatively dense random graphs. We then shift our attention to the sparser case, proving our second result about transversals in Section 3. The last section of our paper contains some concluding remarks. Throughout this exposition, we will make no attempt to optimize absolute constants, and will often omit floor and ceiling signs whenever they are not crucial, for the sake of clarity of presentation.

2 Strong chromatic number

In this section, we prove Theorem 1.1, which determines the value of the strong chromatic number of a rather dense random graph. To this end, we first prove several lemmas that establish certain useful properties of random graphs. We will use these properties to find a partition of $G_{n,p}$ into independent transversals.

2.1 Properties of random graphs

Lemma 2.1 *Let $\theta > 0$ be an arbitrary fixed constant. If $\sqrt{\frac{\log n}{n}} \ll p < 1 - \theta$ then a.s. $G_{n,p}$ has the following properties.*

(i) *No pair of distinct vertices has more than $(1 + o(1))np^2$ common neighbors.*

(ii) *The maximum degree is strictly between np and $1.01np$, and there is a unique vertex of maximum degree.*

(iii) *The gap between the maximum degree and the next largest degree is at least $\frac{\sqrt{np}}{\log n}$.*

Proof. For the first property, fix an arbitrary constant $\delta > 0$ and two distinct vertices u and v. Their codegree X is binomially distributed with parameters $n-2$ and p^2. Thus by the Chernoff bound (see, e.g., Appendix A in [7]), $\mathbb{P}\left[X \geq (1 + \delta)np^2\right] \leq e^{-\Theta(\delta^2 np^2)} = o(n^{-2})$. Taking a union bound over all $O(n^2)$ choices for u and v, we find that the probability that the first property is not satisfied tends to 0 as $n \to \infty$. The second and third claims are special cases of Corollary 3.13 and Theorem 3.15 in [9], respectively. $\square$

Lemma 2.2 *Let $\alpha > 0$ be an arbitrary fixed constant and let $\sqrt{\frac{\log n}{n}} \ll p \leq \frac{3}{5}$. Then almost surely $G_{n,p}$ does not contain a set U of size αnp and $50 \log n$ sets T_i, $|T_i| \leq \left\lceil \frac{1}{p} \right\rceil$, such that all the sets are disjoint and for every i all but at most $\alpha np/50$ vertices in U have neighbors in T_i.*

Proof. Fix sets U and $\{T_i\}$ as specified above. If all but at most $\alpha np/50$ vertices in U have neighbors in T_i, we say for brevity that T_i *almost dominates* U. For a given vertex v, the probability that it has a neighbor in T_i is $1 - (1 - p)^{|T_i|} \leq 1 - (1 - p)^{\lceil 1/p \rceil} < 7/8$ for all $p \leq 3/5$, since $1 - (1 - p)^{\lceil 1/p \rceil}$ is maximal in that range when $p \to 1/2$ from below. Therefore, by a union bound we have

$$\mathbb{P}[T_i \text{ almost dominates } U] \le \binom{\alpha np}{\alpha np - \alpha np/50}\left(\frac{7}{8}\right)^{\alpha np - \alpha np/50} = \binom{\alpha np}{\alpha np/50}\left(\frac{7}{8}\right)^{49\alpha np/50}$$
$$\le \left(50e\left(\frac{7}{8}\right)^{49}\right)^{\alpha np/50} < 3^{-\alpha np/50}.$$

Since all sets T_i are disjoint, the events that T_i and T_j almost dominate U are independent. This implies that

$$\mathbb{P}[\text{every } T_i \text{ almost dominates } U] \le \left(3^{-\alpha np/50}\right)^{50\log n} = 3^{-\alpha np \log n}.$$

Using that $\log n/p = o(np)$ and $\lceil 1/p \rceil \le 2/p$, we can bound the probability that there is a choice of $\{T_i\}$ and U which violates the assertion of the lemma by

$$\mathbb{P} \le \binom{n}{\alpha np}\left[\frac{2}{p}\binom{n}{2/p}\right]^{50\log n} 3^{-\alpha np \log n}$$
$$\le n^{\alpha np}\left(\frac{2}{p}\right)^{50\log n} n^{\frac{100\log n}{p}} 3^{-\alpha np\log n}$$
$$= e^{(1+o(1))\alpha np \log n} \cdot 3^{-\alpha np \log n} = o(1),$$

so we are done. □

Lemma 2.3 *Let $\alpha > 0$ be an arbitrary fixed constant and let $\sqrt{\frac{\log n}{n}} \ll p \le \frac{3}{5}$. Then almost surely every collection of at most $\lceil \frac{1}{p} \rceil$ disjoint subsets of size αnp in $G_{n,p}$ has an independent transversal.*

Proof. Fix a collection of disjoint subsets $V_1, \ldots, V_r$, $r \le \lceil \frac{1}{p} \rceil$, of $G_{n,p}$, each of size αnp. A partial independent transversal T is an independent set with at most one vertex in every V_i, and we say that it almost dominates some part if all but at most $\alpha np/50$ vertices in that part have neighbors in T. For every V_i, let $\{T_{ij}\}$ be a maximal collection of pairwise disjoint partial independent transversals, each of which almost dominates V_i. Then, by Lemma 2.2, a.s. the total number of T_{ij} must be at most $r(50 \log n)$. Delete all the sets T_{ij} from the graph, and let $\{V_i'\}$ be the remaining parts. Clearly, it suffices to find an independent transversal among the $\{V_i'\}$.

Since $\log n/p = o(np)$ and each T_{ij} is a partial transversal, each part loses a total of $\le r(50\log n) \le 50\lceil \frac{1}{p} \rceil \log n = o(np)$ vertices from the deletions. We can now use the greedy algorithm to find an independent transversal. Take v_1 to be any remaining vertex in V_1', and iterate as follows. Suppose that we already have constructed a partial independent transversal $\{v_1, \ldots, v_{\ell-1}\}$ such that $v_i \in V_i'$ for all $i < \ell$. This partial independent transversal does not almost dominate V_ℓ, or else it would contradict the maximality of $\{T_{\ell j}\}$ above. So, there are at least $\alpha np/50$ choices for $v_\ell \in V_\ell$ that would extend the partial independent transversal $\{v_1, \ldots, v_{\ell-1}\}$. Yet V_ℓ lost only $o(np)$ vertices in the deletion process, so there is still a positive number of choices for $v_\ell \in V_\ell'$ as well. Proceeding in this way, we find a complete independent transversal. □

Lemma 2.4 *Let $\sqrt{\frac{\log n}{n}} \ll p \le \frac{3}{5}$. Then the following statement holds almost surely. For every choice of s and t that satisfies $np/2 \le s \le 2np$ and $40\log n \le t \le s - 40\lceil \frac{1}{p} \rceil \log n$, $G_{n,p}$ does not contain a collection of disjoint subsets $U, T_1, \ldots, T_t$ such that $|U| = s$, each of the $|T_i| \le \lceil \frac{1}{p} \rceil$, and at least $s - t$ vertices of U have neighbors in every T_i.*

Proof. Fix some (s, t) within the above range. As we saw in the proof of Lemma 2.2, for a given vertex v the probability that it has a neighbor in T_i is $1-(1-p)^{|T_i|} \le 1-(1-p)^{\lceil 1/p \rceil} < 7/8$, and by disjointness these events are independent for all $1 \le i \le t$. Therefore we can bound the the probability that there is a collection of sets which satisfies the above condition by

$$\begin{aligned} \mathbb{P} &\le \binom{n}{s} \left[\frac{2}{p} \binom{n}{2/p} \right]^t 2^s \left(\frac{7}{8} \right)^{(s-t)t} \\ &\le \frac{n^s}{s!} \left(n^{2/p} \right)^t 2^s \left(\frac{7}{8} \right)^{(s-t)t} \\ &\le n^{s+2t/p} \left(\frac{7}{8} \right)^{(s-t)t} . \end{aligned} \tag{1}$$

Throughout this bound, we use $\lceil \frac{1}{p} \rceil \le \frac{2}{p}$. The first binomial coefficient and the quantity in the square brackets bound the number of ways to choose the sets U and $\{T_i\}$. The 2^s bounds the number of ways to select a subset of size $s-t$ from U, and the final factor bounds the probability that all vertices in this subset have neighbors in every T_i.

The logarithm of (1) is quadratic in t with positive t^2-coefficient. Therefore, the right hand side of (1) is largest when t is minimum or maximum in its range $40 \log n \le t \le s - 40 \lceil \frac{1}{p} \rceil \log n$. Let us begin with the small end, i.e., $t = 40 \log n$. Then, since $\log n/p \ll np$ and $s \ge np/2$, we have that

$$\begin{aligned} n^{s+2t/p} \left(\frac{7}{8} \right)^{(s-t)t} &\le e^{(1+o(1))s \log n} \left(\frac{7}{8} \right)^{(40-o(1))s \log n} \\ &\le e^{(1+o(1))s \log n} \, e^{-(4-o(1))s \log n} = o\big(n^{-2}\big). \end{aligned}$$

Similarly, if $t = s - 40 \lceil \frac{1}{p} \rceil \log n$, the bound is

$$\begin{aligned} n^{s+2t/p} \left(\frac{7}{8} \right)^{(s-t)t} &\le e^{3s \log n/p} \left(\frac{7}{8} \right)^{(40-o(1))s \lceil \frac{1}{p} \rceil \log n} \\ &\le e^{3s \log n/p} \, e^{-(4-o(1))s \lceil \frac{1}{p} \rceil \log n} = o\big(n^{-2}\big). \end{aligned}$$

Since the number of choices for t and s is at most n^2, we conclude that the probability that the assertion of the lemma is violated is $o(1)$. □

2.2 Proof of Theorem 1.1

We start by proving part (i) of Theorem 1.1. If Δ is the maximum degree of $G_{n,p}$, then the strong chromatic number must be at least $\Delta + 1$, as we already mentioned in the introduction. Suppose that G is a graph obtained from $G_{n,p}$ by adding $(\Delta+1)\lceil \frac{n}{\Delta+1} \rceil - n$ isolated vertices, and we have a partition of $V(G)$ into $V_1 \cup \ldots \cup V_r$ with every $|V_i| = \Delta + 1$. By Lemma 2.1, $\Delta \ge np$ almost surely, so this implies that $r \le \lceil \frac{1}{p} \rceil$. Note that if $3/5 \le p < 1 - \theta$, then $r \le 2$ and the theorem is an immediate consequence of the following lemma.

Lemma 2.5 *Let $3/5 \le p < 1 - \theta$, where $\theta > 0$ is an arbitrary fixed constant, and let $V(G) = V_1 \cup V_2$ be a partition of the vertices of G described above, with $|V_1| = |V_2| = \Delta + 1$. Then a.s. V_1 can be perfectly matched to V_2 via non-edges of G.*

Proof. Without loss of generality, we may assume that V_1 contains at most $n/2$ original vertices of $G_{n,p}$. Let $B \subset V_1$ be those original vertices. The rest of V_1 consists of isolated vertices, so any perfect matching of B to V_2 trivially extends to a full perfect matching between V_1 and V_2. Therefore, by Hall's theorem, it suffices to verify that each subset $A \subset B$ has at least $|A|$ non-neighbors in V_2. If $A = \{v\}$ is a single vertex, this is immediate because $|V_2| > \Delta \geq d(v)$. For larger A, the Hall condition translates into checking that $\Delta + 1 - |N(A)| \geq |A|$, where $N(A)$ denotes the set of common neighbors of A in V_2. Since $|A| \geq 2$ we have, by Lemma 2.1(i), that the size of $N(A)$ is at most $(1 + o(1))np^2$. So the Hall condition is satisfied for all A with $2 \leq |A| \leq \theta np/2 < \Delta - (1 + o(1))np^2$.

Let c be a constant for which $p - 2p^c > 1/2$ for all p in the range $[3/5, 1 - \theta)$. One can easily show using a Chernoff bound that a.s. every set of c distinct vertices in $G_{n,p}$ has at most $2np^c$ common neighbors. This implies that the Hall condition is also satisfied for all A of size at least c, since then

$$\Delta + 1 - |N(A)| \;>\; np - 2np^c \;>\; n/2 \;\geq\; |B| \;\geq\; |A|.$$

Together with the previous paragraph, this completes the proof. □

It remains to consider $p < 3/5$, so we will assume that bound on p for the remainder of this section. We use the following strategy to produce a partition of $\cup V_i$ into a disjoint union of independent transversals.

1. Find an independent transversal through the unique vertex of maximum degree Δ, and delete this transversal from the graph.

2. As long as there exists a vertex v which has at least $0.9np$ neighbors in some part V_i, find an independent transversal T through v, and delete T from the graph.

3. As long as there exists a minimal partial independent transversal T such that all but at most $np/100$ vertices in some part V_i have neighbors in T, split T into two nonempty ($|T| \geq 2$ because of Step 2) disjoint partial independent transversals $T_1 \cup T_2$. Note that by minimality of T, each part V_i contains a subset U_i of at least $np/100$ vertices which have no neighbors in T_1. By Lemma 2.3, there is an independent transversal through $\{U_i\}$, which can be used to extend T_1 to a full independent transversal T_1'. Delete T_1' from the graph, and then perform the same completion/deletion procedure for T_2.

4. Finally, we construct the rest of the independent transversals, building them simultaneously from V_1 to V_r using Hall's matching theorem. Our deletions in Steps 1–3, together with the properties of $G_{n,p}$ which we established in the previous subsection, will ensure that this is possible.

The following lemma, which we prove later, ensures that we will indeed find the independent transversals claimed in Steps 1–2.

Lemma 2.6 *Let $V_1 \cup \ldots \cup V_r$ be the above partition of $V(G)$, and let x be any vertex in this graph.*

- *If x is the unique vertex of maximum degree Δ, then G contains an independent transversal through x.*

- *If x is not of maximum degree, then for all $k \le \lceil \frac{1}{p} \rceil$ and for any collection of subsets $V_i' \subset V_i$, $|V_i'| = \Delta + 1 - k$, one of which contains x, there exists an independent transversal through x with respect to $\{V_i'\}$.*

Let us bound the number of independent transversals we delete in the first 3 steps. Note that if two vertices have at least $0.9np$ neighbors in the same V_i, since by Lemma 2.1 $|V_i| \le \Delta + 1 \le 1.01np$, their codegree will be at least $0.79np \ge 1.01np^2$, contradicting Lemma 2.1. Therefore, during the first two steps, we will delete at most $r + 1 \le \lceil \frac{1}{p} \rceil + 1$ transversals. Next, suppose that after deleting $O\big(\lceil \frac{1}{p} \rceil \log n\big)$ independent transversals from G, we have that for some set T all but at most $np/100$ vertices of some V_i have neighbors in T. Since $\lceil \frac{1}{p} \rceil \log n \ll np$, this certainly implies that the number of vertices in the original V_i with no neighbors in T was bounded by $np/50$. Together with Lemma 2.2, this ensures that for each fixed V_i, $1 \le i \le r$, we never repeat Step 3 more than $50 \log n$ times. Since each iteration deletes two independent transversals and $r \le \lceil \frac{1}{p} \rceil$, we conclude that by the time we reach Step 4, we have deleted at most $1 + \lceil \frac{1}{p} \rceil + 100 \lceil \frac{1}{p} \rceil \log n < 110 \lceil \frac{1}{p} \rceil \log n$ independent transversals from G.

Let us now describe Step 4 in more detail. At this point, all parts V_i have the same size $|V_i| = s = \Delta + 1 - k$, where $k < 110 \lceil \frac{1}{p} \rceil \log n = o(np)$ is the total number of independent transversals deleted so far. We build the remaining s disjoint independent transversals simultaneously as follows. Start s partial independent transversals $\{T_i\}_{i=1}^{s}$ by arbitrarily putting one vertex of V_1 into each T_i. Now suppose we already have disjoint partial independent transversals $\{T_i\}_{i=1}^{s}$ through $V_1, \dots, V_\ell$. Create an auxiliary bipartite graph H whose right side is $V_{\ell+1}$ and left side has s vertices, identified with the transversals $\{T_i\}$. Join the i-th vertex on the left side with a vertex $v \in V_{\ell+1}$ if and only if v has no neighbors in T_i. Then, a perfect matching in this graph will yield a simultaneous extension of each T_i which covers $V_{\ell+1}$.

We ensure a perfect matching in H by verifying the Hall condition, i.e., we show that for every $t \le s$, every set of t vertices on the left side of H has neighborhood on the right side of size at least t. Observe that after Step 3, for every T_i there are more than $np/100$ vertices in $V_{\ell+1}$ which have no neighbors in T_i. Therefore every vertex on the left side of H has degree greater than $np/100$ and hence the Hall condition is trivially satisfied for all $t \le np/100$. If the Hall condition fails for some $np/100 < t \le s - 40 \lceil \frac{1}{p} \rceil \log n$, then by definition of H, there are t partial independent transversals among $\{T_i\}$ and a subset W of $V_{\ell+1}$ of size greater than $s - t$ such that every vertex of W has neighbors in every one of these transversals (i.e., is not adjacent to them in H). This contradicts Lemma 2.4, so the Hall condition also holds for these t. It remains to check the case when $t > s - 40 \lceil \frac{1}{p} \rceil \log n$. Note that given any vertex v in $V_{\ell+1}$ and any collection of disjoint partial independent transversals, the number of them in which v can have a neighbor is at most the degree of v. However, we deleted the maximum degree vertex in Step 1, so by Lemma 2.1 $d(v) \le \Delta - \frac{\sqrt{np}}{\log n}$. Since $p \gg \left(\frac{\log^4 n}{n}\right)^{1/3}$, this is less than $\Delta - 150 \lceil \frac{1}{p} \rceil \log n \le s - 40 \lceil \frac{1}{p} \rceil \log n$. Therefore, in the auxiliary graph H, any set of $t > s - 40 \lceil \frac{1}{p} \rceil \log n$ vertices on the left side has neighborhood equal to the entire right side. Hence Hall's condition is satisfied for all t and we can extend our transversals. This completes the proof, since one can iterate this extension procedure to convert all T_i into full independent transversals. □

Proof of Lemma 2.6. First, consider the case when x is not the vertex of maximum degree Δ and we have a collection of subsets $V_i' \subset V_i$ of size $\Delta + 1 - k$, where $k \le \lceil \frac{1}{p} \rceil$. Without loss of generality, assume that $x \in V_1'$, and recall that by Lemma 2.1, the maximum degree Δ satisfies $np < \Delta < 1.01np$. If the number of neighbors of x in every set V_i', $i \geq 2$, is at most $0.96np$ then delete them and denote the resulting sets V_i''. Since each V_i'' still has size at least $\Delta + 1 - \lceil \frac{1}{p} \rceil - 0.96np > 0.03np$, by Lemma 2.3 there exists a partial independent transversal through $V_2'', \ldots, V_r''$, which together with x provides a full independent transversal containing x. Next, suppose that x has at least $0.96np$ neighbors in some part, say V_2'. Since the degree of x is less than $\Delta < 1.01np$, it must then have less than $0.05np$ neighbors in every other V_i'. Furthermore, since x is not of maximum degree and $p \gg \left(\frac{\log^4 n}{n}\right)^{1/3}$, Lemma 2.1 implies that $(\Delta + 1) - d(x) \geq \frac{\sqrt{np}}{\log n} \gg 2\lceil \frac{1}{p} \rceil \geq r + k$. Therefore there are more than r vertices in V_2' not adjacent to x. Also by Lemma 2.1, the codegree of every pair of vertices is at most $1.01np^2 < 0.61np$, so in particular no two vertices can both have $\geq 0.9np$ neighbors in any given V_i'. By the pigeonhole principle, there must be a vertex $y \in V_2'$ not adjacent to x with less than $0.9np$ neighbors in each of the other V_i'. That means that every other part has less than $0.05np$ neighbors of x and $0.9np$ neighbors of y. Since $|V_i'| \geq \Delta - \lceil \frac{1}{p} \rceil > 0.99np$, there are still at least $0.04np$ vertices left in each V_i', $i \geq 3$, that are non-adjacent to both x and y. Thus we can apply Lemma 2.3 as above to complete $\{x, y\}$ into an independent transversal.

The case when x is the vertex of maximum degree has a similar proof but involves one more step. As in the previous paragraph, we may assume that $x \in V_1$ and has at least $0.96np$ neighbors in V_2, or else we are done. Let W_2 be the set of vertices in V_2 that are not adjacent to x. Since $|V_2| = \Delta + 1$, we have $W_2 \neq \emptyset$. If there exists some $y \in W_2$ that has $< 0.9np$ neighbors in each of the other $V_i, i \geq 3$, then we can complete $\{x, y\}$ to a full independent transversal as above. Otherwise, by Lemma 2.1 the codegree of every pair of vertices is at most $1.01np^2 < 0.61np$ and hence each $y \in W_2$ is associated with a distinct part in which it has $\geq 0.9np$ neighbors. Yet x has exactly $|W_2| - 1$ neighbors among the other parts $V_i, i \geq 3$, so there must exist $y \in W_2$ such that x has no neighbors in the part (without loss of generality it is V_3) in which y has $\geq 0.9np$ neighbors. Since x is the unique vertex of maximum degree and $p \gg \left(\frac{\log^4 n}{n}\right)^{1/3}$, Lemma 2.1 gives

$$d(y) \;\leq\; \Delta - \frac{\sqrt{np}}{\log n} \;<\; \Delta - \left\lceil \frac{1}{p} \right\rceil \;\leq\; \Delta - r.$$

Therefore V_3 contains a subset W_3 of at least $r + 1$ vertices which are not adjacent to both x and y. Since for every $i \geq 4$ at most one vertex in W_3 can have more than 0.81 neighbors in V_i (by another codegree argument), the pigeonhole principle ensures that there is a vertex $z \in W_3$ such that z has at most $0.81np$ neighbors in each $V_i, i \geq 4$. Also note that x has less than $0.05np$ neighbors in each such V_i, and y has less than $0.11np$. Therefore every V_i, $i \geq 4$, has in total less than $0.05np + 0.11np + 0.81np < (\Delta + 1) - 0.03np$ neighbors of any of $\{x, y, z\}$, so we can apply Lemma 2.3 as before to complete $\{x, y, z\}$ into an independent transversal. □

Proof of Theorem 1.1 (ii). We may assume that $p < n^{-1/4}$ because the case $p \geq n^{-1/4}$ is already a consequence of part (i) of this theorem. Fix an arbitrary $\epsilon > 0$. Suppose that G is a graph obtained from $G_{n,p}$ by adding $(1+\epsilon)\Delta\lceil \frac{n}{(1+\epsilon)\Delta} \rceil - n$ isolated vertices and $V(G)$ is partitioned into $V_1 \cup \ldots \cup V_r$ with every $|V_i| = (1+\epsilon)\Delta$. Since $\Delta \geq np$ a.s., we have that $r \leq \lceil \frac{1}{p} \rceil$. We use the same Steps 1–4 to produce a partition of $\cup V_i$ into a disjoint union of independent transversals. Actually Steps 1–2 can now be made into a single step, since there is no need here to treat the vertex of maximum degree separately. The codegree argument implies again that we perform Steps 1–2 at most $r+1$ times. Moreover, the existence of the independent transversals claimed in these two steps follows easily from Lemma 2.3. Indeed, suppose that we have deleted $O(\lceil \frac{1}{p} \rceil)$ independent transversals from G. Since $p \gg \left(\frac{\log n}{n}\right)^{1/2}$, we have $1/p = o(np)$ and thus every part still has size at least $(1+\epsilon/2)\Delta$. Let x be an arbitrary remaining vertex. Since the degree of x is at most Δ, every part still contains at least $\epsilon\Delta/2$ vertices non-adjacent to x. By Lemma 2.3, we can find an independent transversal through these vertices which will extend $\{x\}$.

There is no change in the analysis of Step 3 and the same argument as in the proof of part (i) shows that the total number of transversals deleted from G in Steps 1–3 is at most $O(\lceil \frac{1}{p} \rceil \log n)$. Since $p \gg \left(\frac{\log n}{n}\right)^{1/2}$, this number is $o(np)$, and therefore in the beginning of Step 4 each part V_i still has size $s \geq (1+\epsilon/2)\Delta$. Recall that in Step 4 we build the remaining s disjoint independent transversals simultaneously, extending them one vertex at time to cover each new part $V_{\ell+1}$. So again we define an auxiliary bipartite graph H whose left part corresponds to the partial independent transversals $\{T_i\}$ on $V_1, \ldots, V_\ell$, right part is $V_{\ell+1}$, and the i-th vertex on the left is adjacent to $v \in V_{\ell+1}$ iff v has no neighbors in transversal T_i. A perfect matching in H gives a simultaneous extension of each T_i.

Hence it is enough to verify the Hall condition for H, i.e., we must show that for all $t \leq s$, every set of t vertices on the left has at least t neighbors on the right. The proof that this holds for all $t \leq s - 40\lceil \frac{1}{p} \rceil \log n$ is exactly the same as in part (i) and we omit it here. So suppose that $t > s - 40\lceil \frac{1}{p} \rceil \log n \geq s - o(np) > (1+\epsilon/3)\Delta$. Since the degree of every vertex $v \in V_{\ell+1}$ is at most Δ, it can have neighbors in at most $\Delta < t$ transversals. Therefore there is at least one transversal in our set of size t which has no neighbors of v, and hence every set of $t > s - 40\lceil \frac{1}{p} \rceil \log n$ vertices on the left has neighborhood equal to entire right side of H. This verifies the Hall condition and completes the proof. □

3 Independent transversals

In this section, we prove our second theorem. We only need to consider here the range $\frac{\log^4 n}{n} \ll p \ll \frac{\log^{3/4} n}{\sqrt{n}}$, since part (ii) of Theorem 1.1 implies Theorem 1.2 for larger values of p. Again, we begin by showing that $G_{n,p}$ satisfies certain properties almost surely.

3.1 Properties of random graphs

Lemma 3.1 *If $\frac{\log n}{n} \ll p \ll \frac{\log^{3/4} n}{\sqrt{n}}$, then a.s. $G_{n,p}$ has the following properties:*

1. *No pair of distinct vertices has more than $3\log^{3/2} n$ common neighbors.*

2. *The maximum degree is strictly between np and $1.01np$.*

Proof. The codegree X of a fixed pair of vertices is binomially distributed with parameters $n-2$ and p^2. Therefore

$$\mathbb{P}\left[X \geq 3\log^{3/2} n\right] \leq \binom{n-2}{3\log^{3/2} n}(p^2)^{3\log^{3/2} n} \leq \left(\frac{enp^2}{3\log^{3/2} n}\right)^{3\log^{3/2} n} \ll (e/3)^{3\log^{3/2} n} = o(n^{-2}).$$

Taking a union bound over all $O(n^2)$ pairs of vertices, we see that the first property holds a.s. The second property is a special case of Corollary 3.13 in [9]. $\square$

Lemma 3.2 *Let $C \geq 20$ and let G be a graph obtained from the random graph $G_{n,p}$ by connecting every vertex to at most $8 \log^2 n$ new neighbors. Then a.s. every subset $S \subset V(G)$ of size $|S| \leq Cp^{-1} \log^2 n$ spans a subgraph with average degree less than $6C \log^2 n$, i.e., contains $< 3C|S| \log^2 n$ edges.*

Proof. Since the edges which we add to the random graph can increase the number of edges inside S by at most $|S|(8 \log^2 n)/2 = 4|S| \log^2 n$, it suffices to show that in $G_{n,p}$ a.s. every subset S as above spans less than $eC|S| \log^2 n$ edges. The probability that this is not the case is at most

$$\begin{aligned}
\sum_{m=1}^{Cp^{-1}\log^2 n} \binom{n}{m} \binom{\binom{m}{2}}{eCm\log^2 n} p^{eCm\log^2 n} &\leq \sum_{m=1}^{Cp^{-1}\log^2 n} n^m \left(\frac{em}{2eC\log^2 n} \cdot p \right)^{eCm\log^2 n} \\
&\leq \sum_{m=1}^{Cp^{-1}\log^2 n} n^m 2^{-eCm\log^2 n} \\
&\leq \sum_{m=1}^{Cp^{-1}\log^2 n} \left(n 2^{-eC\log^2 n} \right)^m = o(1),
\end{aligned}$$

so we are done. $\square$

3.2 Proof of Theorem 1.2

Fix $\epsilon > 0$, and suppose we have disjoint subsets $V_1, \ldots, V_r$ of $G_{n,p}$, with all $|V_i| = (1+\epsilon)\Delta$. By Lemma 3.1, $r < n/\Delta < 1/p$. If a vertex v has more than $\frac{\Delta}{\log n}$ neighbors in some V_i, say that v is *locally big* with respect to V_i. If it has more than $\frac{\Delta}{2\log n}$, call it *almost locally big*. For each i, let B_i be the set of v that are almost locally big with respect to V_i. We claim that $|B_i| < 4 \log n$. Indeed, if $|B_i| \geq 4 \log n$, then Lemma 3.1 together with $\Delta \geq \log^4 n$ and the Jordan-Bonferroni inequality would imply that the union of neighborhoods in V_i of vertices from B_i is at least

$$(4\log n)\frac{\Delta}{2\log n} - \binom{4\log n}{2} 3\log^{3/2} n \;\geq\; \frac{3}{2}\Delta \;>\; |V_i|,$$

contradiction. Next, make each B_i a clique by adding all the missing edges. However, Δ will still refer to the maximum degree of the original graph. Since each vertex is almost locally big with respect to less than $2 \log n$ sets V_i, this operation increases the degree of each vertex by less than $2\log n \cdot 4 \log n = 8 \log^2 n \ll \frac{\Delta}{2\log n}$. Thus every vertex that is locally big after the additions was almost locally big before. In particular, there is now an edge between every pair of vertices that are locally big with respect to the same V_i, and there are less than $r(4 \log n) < 4p^{-1} \log n$ locally big vertices in total.

Let $I_1 \subset [r]$ be the set of indices i such that V_i contains more than $\frac{\epsilon}{4}\Delta$ locally big vertices, and define the notation V_S to represent $\bigcup_{i \in S} V_i$. Note that

$$|V_{I_1}| \;<\; (1+\epsilon)\Delta \cdot \left(\frac{\epsilon}{4}\Delta\right)^{-1} 4p^{-1}\log n \;<\; 20\epsilon^{-1}p^{-1}\log n$$

(we can assume here and in the rest of the proof that ϵ is sufficiently small). As long as there exist $i \notin I_1$ such that there are more than $(240\epsilon^{-1}\log^2 n)|V_i|$ crossing edges between V_i and V_{I_1}, add i to I_1. Note that each such index which we add to V_{I_1} increases the number of edges in this set by more than $(240\epsilon^{-1}\log^2 n)|V_i|$. Therefore if in this process I_1 doubles in size we obtain a set of size at most $40\epsilon^{-1}p^{-1}\log n$ with average degree more than $240\epsilon^{-1}\log^2 n$, which contradicts Lemma 3.2. Thus at the end of the process we have $|I_1| \le 40\epsilon^{-1}p^{-1}\log n$.

Given I_1, for $t \ge 1$ we recursively define $I_{t+1} \subset I_t$ as follows. By Lemma 3.2, V_{I_t} induces less than $(120\epsilon^{-1}\log^2 n)|V_{I_t}|$ edges. Thus, there are less than $2\big(\frac{\Delta}{\log\Delta}\big)^{-1}\cdot(120\epsilon^{-1}\log^2 n)|V_{I_t}|$ vertices in V_{I_t} with $> \frac{\Delta}{\log\Delta}$ neighbors in this set. To define I_{t+1} we consider the following process. Start with I_{t+1} to be the set of all $i \in I_t$ for which V_i has more than $\frac{\epsilon}{4}\Delta$ vertices that have $> \frac{\Delta}{\log\Delta}$ neighbors in V_{I_t}. As long as there exist $i \in I_t \setminus I_{t+1}$ such that there are more than $(240\epsilon^{-1}\log^2 n)|V_i|$ edges between V_i and $V_{I_{t+1}}$, add i to I_{t+1}. As above, Lemma 3.2 ensures that this process must stop before I_{t+1} doubles in size. Therefore in the end we have

$$\begin{aligned}
|I_{t+1}| &\le 2\left(\frac{\epsilon}{4}\Delta\right)^{-1}\cdot 2\left(\frac{\Delta}{\log\Delta}\right)^{-1}\cdot(120\epsilon^{-1}\log^2 n)|V_{I_t}| \\
&\le O\left(\frac{\log^2 n\,\log\Delta}{\Delta^2}|V_{I_t}|\right) \le O\left(\frac{\log^2 n\,\log\Delta}{\Delta}|I_t|\right) \\
&\ll \frac{1}{\log n}|I_t|.
\end{aligned}$$

Clearly, $|I_1| \le r \le n$. Therefore, when $t \ge \frac{2\log n}{\log\log n}$, I_t will be empty. Let σ be the smallest index such that $I_\sigma = \emptyset$. We now recursively build partial independent transversals $T_\sigma, \ldots, T_1$, where T_t is an independent transversal on V_{I_t}. Let us say that T_t satisfies property $\mathbf{P}_t$ if for every $i \notin I_t$, all the vertices in T_t that are not locally big with respect to V_i have together at most $300(\sigma - t)\frac{\Delta}{\log n}$ neighbors in V_i. It is clear that $T_\sigma = \emptyset$ satisfies $\mathbf{P}_\sigma$, so we can apply the following lemma inductively to construct T_1, an independent transversal on V_{I_1} satisfying $\mathbf{P}_1$.

Lemma 3.3 *Suppose $t > 1$, and T_t is an independent transversal on V_{I_t} which satisfies $\mathbf{P}_t$. Then we can extend T_t to T_{t-1}, an independent transversal on $V_{I_{t-1}}$ which satisfies $\mathbf{P}_{t-1}$.*

We postpone the proof of this lemma until Section 3.4. Suppose that we have T_1 as described above. Let J_1 be the set of all indices $j \notin I_1$ such that some $v \in T_1$ is locally big with respect to V_j. Then, as we did with I_1, as long as there exist $\ell \notin I_1 \cup J_1$ such that more than $(600\epsilon^{-1}\log^2 n)|V_\ell|$ edges cross between V_ℓ and V_{J_1}, add ℓ to J_1. Since $|T_1| = |I_1|$ and each vertex can be locally big with respect to at most $(1 + o(1))\log n$ sets V_i, we have that initially $|J_1| \le (1 + o(1))|I_1|\log n \le 50\epsilon^{-1}p^{-1}\log^2 n$. Therefore as before, Lemma 3.2 ensures that this process stops before J_1 doubles in size, so the final set J_1 has size at most $100\epsilon^{-1}p^{-1}\log^2 n$.

As before, we construct a sequence of nested index sets $J_1 \supset \cdots \supset J_\tau = \emptyset$, where for $t \ge 1$, define J_{t+1} in terms of J_t as follows. Let $J_{t+1} \subset J_t$ be the set of all $j \in J_t$ for which V_j contains more than $\frac{\epsilon}{4}\Delta$ vertices that have $> \frac{\Delta}{\log\Delta}$ neighbors in V_{J_t}. Next, as long as there exist $j \in J_t \setminus J_{t+1}$ such that more than $(600\epsilon^{-1}\log^2 n)|V_j|$ edges cross between V_j and $V_{J_{t+1}}$, add j to J_{t+1}. Lemma 3.2 again ensures that we stop before J_{t+1} doubles in size, and the same computation as we did for I_{t+1} shows that $|J_{t+1}| \ll \frac{1}{\log n}|J_t|$. Thus when $t \geq \frac{2\log n}{\log\log n}$, J_t is empty. Let τ be the smallest index for which $J_\tau = \emptyset$.

Next, delete all neighbors of T_1 in V_{J_1} and all vertices in V_{J_1} that are locally big with respect to any V_k with $k \notin I_1$. Denote the resulting sets V'_j, $j \in J_1$. We claim that each V'_j still has size at least $\frac{\epsilon}{2}\Delta$. Indeed, at most one $v \in T_1$ can be locally big with respect to V_j, because T_1 is an independent set and all vertices that are locally big with respect to the same part were connected by our construction. Thus deleting neighbors of this v can decrease the size of V_j by at most $d(v) < \Delta + 8\log^2 n = (1+o(1))\Delta$. As for the remaining vertices in T_1, which are not locally big with respect to V_j, $\mathbf{P}_1$ ensures that together they have at most $O\big(\sigma\frac{\Delta}{\log n}\big) = o(\Delta)$ neighbors in V_j, since $\sigma \leq \frac{2\log n}{\log\log n}$. Also, by construction of I_1, every part whose index is not in I_1 has at most $\frac{\epsilon}{4}\Delta$ locally big vertices. Hence the size of V'_j is at least $|V_j| - (1+o(1))\Delta - \frac{\epsilon}{4}\Delta \geq \frac{\epsilon}{2}\Delta$, as claimed.

Let us say that a set U_t satisfies property $\mathbf{Q}_t$ if for every $k \notin I_1 \cup J_t$, all the vertices in U_t that are not locally big with respect to V_k have together at most $300(\tau - t)\frac{\Delta}{\log n}$ neighbors in V_k. We need the following analogue of Lemma 3.3.

Lemma 3.4 *Suppose $t > 1$, and U_t is an independent transversal on V'_{J_t} which satisfies $\mathbf{Q}_t$. Then we can extend U_t to U_{t-1}, an independent transversal on $V'_{J_{t-1}}$ which satisfies $\mathbf{Q}_{t-1}$.*

We also postpone the proof of this lemma until Section 3.4. Starting with $U_\tau = \emptyset$, we iterate this lemma until we obtain U_1, an independent transversal on V'_{J_1} which satisfies $\mathbf{Q}_1$. Since $\tau \leq \frac{2\log n}{\log\log n}$, this property implies that each V_k with $k \notin I_1 \cup J_1$ has $O\big(\tau\frac{\Delta}{\log n}\big) = o(\Delta)$ vertices with neighbors in U_1.

Finally, let $K = [r] \setminus (I_1 \cup J_1)$. Delete all neighbors of $T_1 \cup U_1$ and all locally big vertices from every V_k with $k \in K$, and denote the resulting sets by V'_k. All V'_k will still have size at least $\big(1 + \frac{\epsilon}{2}\big)\Delta$, but now no vertex there has more than $\frac{\Delta}{\log n}$ neighbors in any single set V'_k. Thus, the following result from [19] implies that for sufficiently large n, there is an independent transversal on V'_K, which completes $T_1 \cup U_1$ into an independent transversal through all parts.

Theorem 3.5 *(Loh, Sudakov [19]) For every $\epsilon > 0$ there exists $\gamma > 0$ such that the following holds. If G is a graph with maximum degree at most Δ whose vertex set is partitioned into r parts $V_1, \ldots V_r$ of size $|V_i| \geq (1+\epsilon)\Delta$, and no vertex has more than $\gamma\Delta$ neighbors in any single part V_i, then G has an independent transversal.*

This completes the proof of Theorem 1.2, modulo two remaining lemmas. □

3.3 Probabilistic tools

We take a moment to record two results which we will need for the proofs of the remaining lemmas. The first is the symmetric version of the Lovász Local Lemma, which is typically used to show that with positive probability, no "bad" events happen.

Theorem 3.6 *(Lovász Local Lemma [7]) Let $E_1, \ldots, E_n$ be events. Suppose that there exist numbers p and d such that all $\mathbb{P}[E_i] \leq p$, and each E_j is mutually independent of all but at most d of the other events. If $ep(d+1) \leq 1$, then $\mathbb{P}\big[\bigcap \overline{E_i}\big] > 0$.*

The following result is a short consequence of this lemma, and we sketch its proof for completeness.

Proposition 3.7 *(Alon [4]) Let G be a multipartite graph with maximum degree Δ, whose parts $V_1, \ldots, V_r$ all have size at least $2e\Delta$. Then G has an independent transversal.*

Proof. Independently and uniformly select one vertex from each V_i, which we may assume is of size exactly $\lceil 2e\Delta \rceil$. For each edge f of G, let the event A_f be when both endpoints of f are selected. The dependencies are bounded by $2\lceil 2e\Delta \rceil \Delta - 2$, and each $\mathbb{P}[A_f] \le \lceil 2e\Delta \rceil^{-2}$, so the Local Lemma implies this statement immediately. □

3.4 Proofs of remaining lemmas

Since the proofs of Lemmas 3.3 and 3.4 are very similar, we only prove Lemma 3.3. We will simply indicate the two places where the proofs differ.

Proof of Lemma 3.3. Fix some t as in the statement of the lemma. To extend an independent transversal T_t on the set V_{I_t}, satisfying $\mathbf{P}_t$, to one on the larger set $V_{I_{t-1}}$, satisfying $\mathbf{P}_{t-1}$, we will use the following key properties of our construction.

(i) For every $i \in I_{t-1} \setminus I_t$, the set V_i contains at most $\frac{\epsilon}{4}\Delta$ vertices that have $> \frac{\Delta}{\log \Delta}$ neighbors in $V_{I_{t-1}}$.

(ii) Each set V_i has size $(1+\epsilon)\Delta$.

(iii) For every $i \notin I_{t-1}$, there are at most $(\beta \log^2 n)|V_i|$ edges between V_i and $V_{I_{t-1}}$, where we define the constant β to be $240\epsilon^{-1}$.

In the case of Lemma 3.4, property **(ii)** is that each set V_j' has size at least $\frac{\epsilon}{2}\Delta$, and the constant β in property **(iii)** is $\beta = 600\epsilon^{-1}$.

Let $D = I_{t-1} \setminus I_t$. From every V_i with $i \in D$, delete all vertices that have $> \frac{\Delta}{\log \Delta}$ neighbors in $V_{I_{t-1}}$, and all neighbors of vertices in T_t. Denote the resulting sets by V_i^*. Note that now all degrees in the subgraph on $V_D^* = \bigcup_{i \in D} V_i^*$ are at most $\frac{\Delta}{\log \Delta}$. Furthermore, we claim that every $|V_i^*| \ge \frac{\epsilon}{6}\Delta$. To see this, recall that at most one vertex $v \in T_t$ can be locally big with respect to V_i, because T_t is independent and all vertices that are locally big with respect to the same part are connected by our construction. Deleting neighbors of such v can decrease the size of V_i by at most $d(v) < \Delta + 8\log^2 n = (1+o(1))\Delta$. The rest of the vertices in T_t are not locally big with respect to V_i, so $\mathbf{P}_t$ implies that they have less than $O\big(\sigma \frac{\Delta}{\log n}\big) = o(\Delta)$ neighbors in V_i since $\sigma \le \frac{2\log n}{\log\log n}$. Finally, by property **(i)** above, in V_i we will delete at most $\frac{\epsilon}{4}\Delta$ vertices that have $> \frac{\Delta}{\log \Delta}$ neighbors in $V_{I_{t-1}}$, so property **(ii)** implies that $|V_i^*| \ge (1+\epsilon)\Delta - (1+o(1))\Delta - \frac{\epsilon}{4}\Delta \ge \frac{\epsilon}{6}\Delta$, as claimed.

In the case of Lemma 3.4, recall that by construction all V_j' with $j \in J_1$ contain no locally big vertices with respect to any part (we deleted all of them). Thus, the partial transversal U_t contains no locally big vertices with respect to V_j'. Property $\mathbf{Q}_t$ then implies that the total number of neighbors that vertices of U_t have in V_j' is only $O\big(\tau \frac{\Delta}{\log n}\big) = o(\Delta)$. Hence when we reduce V_j' to V_j^* by deleting all neighbors of U_t, and all vertices that have $> \frac{\Delta}{\log \Delta}$ neighbors in $V_{J_{t-1}}$, the total effect of U_t is $o(\Delta)$, not $(1+o(1))\Delta$ as above. Combining this with properties **(i)** and **(ii)**, we see that $|V_j^*| \ge |V_j'| - o(\Delta) - \frac{\epsilon}{4}\Delta \ge \frac{\epsilon}{6}\Delta$, so the claim is still true. This is the second and final place in which the proofs of the two lemmas differ, and explains why Lemma 3.4 holds with part sizes of only $\frac{\epsilon}{2}\Delta$, while Lemma 3.3 requires part sizes of $(1+\epsilon)\Delta$.

Returning to the proof of Lemma 3.3, randomly select a subset $W_i \subset V_i^*$ for each $i \in D$ by independently choosing each remaining vertex of V_i^* with probability $\frac{\log^3 \Delta}{\Delta}$, and let $W = \bigcup_{i \in D} W_i$. Define the following families of bad events. For each $i \in D$, let A_i be the event that $|W_i| < \frac{\epsilon}{8}\log^3 \Delta$, and for each $v \in V_D^*$, let B_v be the event that v has more than $2\log^2 \Delta$ neighbors in W. Also, for each $j \notin I_{t-1}$, let C_j be the event that the collection of vertices in W that are not locally big with respect to V_j has neighborhood in V_j of size $> 300\frac{\Delta}{\log n}$. We use the Lovász Local Lemma to show that with positive probability, none of these events happen.

Let us begin by bounding the dependencies. Say that A_i *lives on* V_i^*, B_v *lives on* the neighborhood of v in V_D^*, and C_j *lives on* the neighborhood of V_j in V_D^*. Note that each of our events is completely determined by the outcomes of the vertices in the set that it lives on. Hence events living on disjoint sets are independent. A routine calculation shows that for any given event, at most $O(\Delta^3)$ other events can live on sets overlapping with its set; the worst case is that an event of C-type can live on a set that overlaps with the sets of $\leq (1+\epsilon)\Delta^3$ other C-type events.

It remains to show that each of $\mathbb{P}[A_i]$, $\mathbb{P}[B_v]$, and $\mathbb{P}[C_j]$ are $\ll \Delta^{-3}$. The size of W_i is distributed binomially with expectation $\geq \frac{\epsilon}{6}\log^3 \Delta$, so by a Chernoff bound, $\mathbb{P}[A_i] < e^{-\Omega(\log^3 \Delta)} \ll \Delta^{-3}$. Similarly, for each $v \in V_D^*$ the expected value of the degree of v in W is at most $\frac{\Delta}{\log \Delta} \cdot \frac{\log^3 \Delta}{\Delta} = \log^2 \Delta$ so $\mathbb{P}[B_v] < e^{-\Omega(\log^2 \Delta)} \ll \Delta^{-3}$. For $\mathbb{P}[C_j]$, we proceed more carefully. For each $0 \leq k \leq 8$, let Y_k be the set of vertices in V_D^* that have between $\frac{\Delta}{\Delta^{(k+1)/8}\log n}$ and $\frac{\Delta}{\Delta^{k/8}\log n}$ many neighbors in V_j. By property **(iii)**, the number of edges between $V_{I_{t-1}}$ and V_j is at most $(\beta \log^2 n)|V_j| \leq 2\beta\Delta\log^2 n$. Therefore, $|Y_k| \leq 2\beta\Delta^{(k+1)/8}\log^3 n$. However, since $\Delta \geq np \geq \log^4 n$, the probability that at least $30\Delta^{k/8}$ vertices in Y_k are selected to be in W is bounded by

$$\begin{aligned}
\mathbb{P} \;\leq\; \binom{2\beta\Delta^{(k+1)/8}\log^3 n}{30\Delta^{k/8}}\left(\frac{\log^3 \Delta}{\Delta}\right)^{30\Delta^{k/8}} &\leq \left(\frac{e \cdot 2\beta\Delta^{1/8}\log^3 n}{30} \cdot \frac{\log^3 \Delta}{\Delta}\right)^{30\Delta^{k/8}} \\
&\leq \left(\frac{e\beta}{15} \cdot \frac{\log^3 \Delta}{\Delta^{1/8}}\right)^{30\Delta^{k/8}} \ll \Delta^{-3}.
\end{aligned}$$

Therefore, with probability $1 - o(\Delta^{-3})$, the collection of vertices in W that are not locally big with respect to V_j has neighborhood in V_j of size less than $\sum_{k=0}^{8} 30\Delta^{k/8}\frac{\Delta}{\Delta^{k/8}\log n} < 300\frac{\Delta}{\log n}$, and hence $\mathbb{P}[C_j] \ll \Delta^{-3}$.

By the Lovász Local Lemma, there exist subsets $W_i \subset V_i^*$ for each $i \in D$ such that none of the A_i, B_v, or C_j hold. In particular, every $|W_i|$ is greater than $2e$ times the maximum degree in the subgraph induced by W, so Proposition 3.7 implies that there exists an independent transversal T' there. Letting $T_{t-1} = T_t \cup T'$, we obtain an independent transversal on $V_{I_{t-1}}$. Since $T' \subset W$ and no C_j hold, we have that for every $j \notin I_{t-1}$, the vertices in $T_t \cup T'$ which are not locally big with respect to V_j have together at most $300(\sigma - t)\frac{\Delta}{\log n} + 300\frac{\Delta}{\log n} = 300(\sigma - (t-1))\frac{\Delta}{\log n}$ neighbors in V_j, i.e., $T_t \cup T'$ satisfies $\mathbf{P}_{t-1}$. $\square$

4 Concluding remarks

A simple modification of our argument yields a slight improvement of Theorem 1.2, and shows that the theorem is in fact true for all $p \gg \frac{\log^{3+\alpha}}{n}$, for any fixed $\alpha > 0$. We decided not to prove that result here in such generality for the sake of clarity of presentation. Also, it is not very difficult, using our approach, to prove a statement similar to Theorem 1.2 for the sparse case, when $p \sim \frac{c}{n}$ for some constant c. However, these extensions are not as interesting as the main problem that remains open, which is to study the behavior of the strong chromatic number of random graphs when $p \leq n^{-1/2}$. We are certain that the strong chromatic number of the random graph $G_{n,p}$ is a.s. $(1+o(1))\Delta$ for every $p \geq \frac{c}{n}$ for some constant c. It would also be very interesting to determine all the values of the edge probability p for which almost surely $s\chi(G_{n,p})$ is precisely $\Delta + 1$.

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