

Improved mixing time bounds for the Thorp shuffle

BEN MORRIS*

Abstract

E. Thorp introduced the following card shuffling model. Suppose the number of cards n is even. Cut the deck into two equal piles. Drop the first card from the left pile or from the right pile according to the outcome of a fair coin flip. Then drop from the other pile. Continue this way until both piles are empty. We show that if n is a power of 2 then the mixing time of the Thorp shuffle is $O(\log^3 n)$. Previously, the best known bound was $O(\log^4 n)$.

Key words: Markov chain, mixing time.

1 Introduction

Card shuffling has a rich history in mathematics, dating back to work of Markov [7] and Poincare [11]. A basic problem is to determine the mixing time, i.e., the number of shuffles necessary to mix up the deck (see Section 3 for a precise definition). In [6], the author found a general method that reduces bounding the mixing time of a card shuffle to verifying a local condition that involves only pairs of cards. This was used to give mixing time bounds for the Thorp shuffle and Durrett's L -reversal chain. In the present paper, we build on the techniques of [6] and get an improved analysis of the Thorp shuffle.

2 Previous work

Thorp [13] introduced the following card shuffling model in 1973. Assume that the number of cards, n , is even. Cut the deck into two equal piles. Drop the first card from the left pile or the right pile according to the outcome of a fair coin flip; then drop from the other pile. Continue this way, with independent coin flips deciding whether to drop LEFT-RIGHT or RIGHT-LEFT each time, until both piles are empty.

Analyzing the Thorp shuffle is an old problem with theoretical roots. However, recently the Thorp shuffle has found applications in applied cryptography. The author, Phil Rogaway and Till Stegers have used the Thorp shuffle as the basis for a practical algorithm for encoding small messages such as social security numbers and credit card numbers (see [9]). In order to analyze the algorithm it is important to have good bounds on the mixing time.

The Thorp shuffle, despite its simple description, has been hard to analyze. Determining its mixing time has been called the “longest-standing open card shuffling problem” [3]. In [10] the

*Department of Mathematics, University of California, Davis. Email: morris@math.ucdavis.edu. Research partially supported NSF grant DMS-0707144.

author obtained the first poly log upper bound, proving a bound of $O(\log^{44} n)$, valid when n is a power of 2. Montenegro and Tetali [8] built on this to get a bound of $O(\log^{29} n)$. In [6] the bound was improved to $O(\log^4 n)$, with no power-of-two assumption. In the present paper we show that if the number of cards is a power of two, then the mixing time is $O(\log^3 n)$.

3 Background

In this section we give some basic definitions and recall some notation from [6]. Let $p(x, y)$ be transition probabilities for a Markov chain on a finite state space V with a uniform stationary distribution. For probability measures μ and ν on V , define the total variation distance $\|\mu - \nu\| = \sum_{x \in V} |\mu(x) - \nu(x)|$, and define the mixing time

$$T_{\text{mix}} = \min\{n : \|p^n(x, \cdot) - \mathcal{U}\| \leq \frac{1}{4} \text{ for all } x \in V\}, \quad (1)$$

where \mathcal{U} denotes the uniform distribution.

For a probability distribution $\{p_i : i \in V\}$, define the (relative) entropy of p by $\text{ENT}(p) = \sum_{i \in V} p_i \log(|V|p_i)$, where we define $0 \log 0 = 0$. The following well-known inequality links relative entropy to total variation distance. We have

$$\|p - \mathcal{U}\| \leq \sqrt{\frac{1}{2} \text{ENT}(p)}. \quad (2)$$

If X is a random variable (or random permutation) taking finitely many values, define $\text{ENT}(X)$ as the relative entropy of the distribution of X . Note that if $\mathbf{P}(X = i) = p_i$ for $i \in V$ then $\text{ENT}(X) = \mathbf{E}(\log(|V|p_X))$. We shall think of the distribution of a random permutation in \mathcal{S}_n as a sequence of probabilities of length $n!$, indexed by permutations in \mathcal{S}_n . If \mathcal{F} is a sigma-field, then we shall write $\text{ENT}(X | \mathcal{F})$ for the relative entropy of the conditional distribution of X given \mathcal{F} . Note that $\text{ENT}(X | \mathcal{F})$ is a random variable. If π is a random permutation in \mathcal{S}_n , then for $1 \leq k \leq n$, define $\mathcal{F}_k = \sigma(\pi^{-1}(k), \dots, \pi^{-1}(n))$, and define $\text{ENT}(\pi, k) = \text{ENT}(\pi^{-1}(k) | \mathcal{F}_{k+1})$ (where we think of the conditional distribution of $\pi^{-1}(k)$ given \mathcal{F}_{k+1} as being a sequence of length k). The standard entropy chain rule (see, e.g., [2]) gives the following proposition.

Proposition 1 *For any $i \leq n$ we have*

$$\text{ENT}(\pi) = \mathbf{E}\left(\text{ENT}(\pi | \mathcal{F}_i)\right) + \sum_{k=i}^n \mathbf{E}(\text{ENT}(\pi, k)).$$

To compute the relative entropy in first term on the right hand side, we think of the distribution of π given \mathcal{F}_i as a sequence of probabilities of length $(i-1)!$.

Remark: Substituting $i = 1$ into the formula gives $\text{ENT}(\pi) = \sum_{k=1}^n \mathbf{E}(\text{ENT}(\pi, k))$. \square

If we think of π as representing the order of a deck of cards, with $\pi(i) =$ location of card i , then this allows us to think of $\mathbf{E}(\text{ENT}(\pi, k))$ as the portion of the overall entropy $\text{ENT}(\pi)$ that is attributable to the location k . We will also need the following proposition.

Proposition 2 *Let ν_1 and ν_2 be random permutations on $\{0, \dots, n-1\}$. Suppose that there is a set $W \subset \{0, 1, \dots, n-1\}$ such that $\nu_1^{-1}(x) = \nu_2^{-1}(x)$ for all $x \in W$. Let $\mathcal{F} = \sigma(\nu_1^{-1}(x) : x \in W)$. Then*

$$\text{ENT}(\nu_1) - \text{ENT}(\nu_2) = \mathbf{E}(\text{ENT}(\nu_1 | \mathcal{F}) - \text{ENT}(\nu_2 | \mathcal{F})).$$

Proof: By the chain rule for entropy, for $i = 1, 2$ we can write

$$\text{ENT}(\nu_i) = \text{ENT}(\nu_i^{-1}(x) : x \in W) + \mathbf{E}(\text{ENT}(\nu_i | \mathcal{F}))$$

Since the first term doesn't depend on i the proposition follows. \square

Definition 3 For $p, q \geq 0$, define $d(p, q) = \frac{1}{2}p \log p + \frac{1}{2}q \log q - \frac{p+q}{2} \log\left(\frac{p+q}{2}\right)$.

We will need the following proposition, which is easily verified using calculus.

Proposition 4 ([6]) Fix $p \geq 0$. The function $d(p, \cdot)$ is convex.

Observe that $d(p, q) \geq 0$, with equality iff $p = q$ by the strict convexity of the function $x \rightarrow x \log x$. If $p = \{p_i : i \in V\}$ and $q = \{q_i : i \in V\}$ are both probability distributions on V , then we can define the ‘‘distance’’ $d(p, q)$ between p and q , by $d(p, q) = \sum_{i \in V} d(p_i, q_i)$. (We use the term *distance* loosely and don't claim that $d(\cdot, \cdot)$ satisfies the triangle inequality.) Note that $d(p, q)$ is the difference between the average of the entropies of p and q and the entropy of the average (i.e. an even mixture) of p and q .

We will use the following projection lemma.

Lemma 5 ([6]) Let X and Y be random variables with distributions p and q , respectively. Fix a function g and let P and Q be the distributions of $g(X)$ and $g(Y)$, respectively. Then $d(p, q) \geq d(P, Q)$.

Let \mathcal{U} denote the uniform distribution on V . Note that if μ is an arbitrary distribution on V , then $\text{ENT}(\mu)$ and $d(\mu, \mathcal{U})$ are both notions of a distance from μ to \mathcal{U} . The following lemma relates the two.

Lemma 6 ([6]) For any distribution μ on V we have

$$d(\mu, \mathcal{U}) \geq \frac{c}{\log |V|} \text{ENT}(\mu),$$

for a universal constant $c > 0$.

A card shuffle can be described as a random permutation chosen from a certain probability distribution. If we start with the identity permutation and each shuffle has the distribution of π , then after t steps the cards are distributed like $\pi_1 \cdots \pi_t$, where the π_i are i.i.d. copies of π .

4 Thorp shuffle

Recall that the Thorp shuffle has the following description. Assume that the number of cards, n , is even. Cut the deck into two equal piles. Drop the first card from the left pile or the right pile according to the outcome of a fair coin flip; then drop from the other pile. Continue this way, with independent coin flips deciding whether to drop LEFT-RIGHT or RIGHT-LEFT each time, until both piles are empty.

We will actually work with the time reversal of the Thorp shuffle, which has the same mixing time (since the Thorp shuffle is a random walk on a group; see [12]). For convenience, we assume

that $n = 2^d$ is a power of two. By writing the position of each card, from the bottom card (0) to the top card ($2^d - 1$), in binary, we can view the positions as elements of the d -dimensional unit hypercube $\{0, 1\}^d$. The reverse Thorp (RT) shuffle can then be constructed in the following way (see, e.g., [9]). Let $Z = \{Z(l, t) : l \in \{0, 1\}^{d-1}, t \in \{0, 1, \dots\}\}$ be a collection of i.i.d., Bernoulli(1/2) random variables. Note that $x \in \{0, 1\}^d$ can be written as $x = (L(x), R(x))$, where $L(x)$ and $R(x)$ are the leftmost $d - 1$ and rightmost bit, respectively, of x . The transition rule for the RT shuffle is as follows. At time t , suppose that the current state $X_t = \pi$. Then the new state $X_{t+1} = \nu \circ \pi$, where ν is the permutation that sends

$$(L, R) \rightarrow (R \oplus Z(L, t), L).$$

We are now ready to state the technical result of this paper.

Lemma 7 *Let X_t be the reverse Thorp shuffle with 2^d cards. There is a universal constant c such that if μ is a random permutation which is independent of $\{X_t\}$ then*

$$\text{ENT}(X_d \circ \mu) \leq (1 - c/d)\text{ENT}(\mu).$$

Before proving this lemma we show how it gives the desired mixing time bound.

Theorem 8 *The mixing time of the reverse Thorp shuffle with 2^d cards is $O(d^3)$.*

Proof: Repeated applications of Lemma 7 give

$$\begin{aligned} \text{ENT}(X_{kd}) &\leq (1 - c/d)^k \text{ENT}(\text{id}) \\ &\leq e^{-ck/d} d^{2d}. \end{aligned}$$

Now let α be large enough so that $\left(\frac{2}{e^\alpha}\right)^d \leq 1/8$ for all d . Then if $k = \lceil \alpha d^2 / c \rceil$ we have

$$\text{ENT}(X_{kd}) \leq e^{-ck/d} d^{2d} \leq \frac{1}{8}$$

and hence $\|X_{kd} - \mathcal{U}\| \leq \frac{1}{4}$ by equation 2. The theorem follows since k is $O(d^2)$. \square

We now give the proof of lemma 7.

Proof of Lemma 7: Fix an integer $T \geq 1$. For integers $j < n$, define $T_j = \lfloor \log_2 j \rfloor + 1 - T$. Note that $T_0 \leq T_1 \leq \dots \leq T_{n-1}$. Let \tilde{Z} be obtained from Z by flipping the value of $Z(L(X_{T_j}), T_j)$ for all j . More precisely, define

$$\tilde{Z}(l, t) = \begin{cases} 1 - Z(l, t) & \text{if for some } j \text{ we have } L(X_{T_j}(j)) = l \text{ and } T_j = t; \\ Z(l, t) & \text{otherwise.} \end{cases}$$

Let $\{\tilde{X}_t : t \geq 0\}$ be the reverse Thorp shuffle process defined by using \tilde{Z} instead of Z . For j with $0 \leq j < n$, define $\Gamma_j(X) = (X_1(j), \dots, X_d(j))$, with a similar definition for $\Gamma_j(\tilde{X})$. For k with $0 \leq k \leq n$, define

$$\begin{aligned} \mathcal{F}_k &= \sigma(\Gamma_j(X), \Gamma_j(\tilde{X}) : j \geq k) \\ &= \sigma(X_t(j), \tilde{X}_t(j) : j \geq k, 0 \leq t \leq d) \end{aligned}$$

Since \mathcal{F}_n is trivial and X_d is \mathcal{F}_0 -measurable, we have

$$\text{ENT}(X_d \circ \mu) - \text{ENT}(\mu) = \text{ENT}(X_d \circ \mu | \mathcal{F}_n) - \text{ENT}(X_d \circ \mu | \mathcal{F}_0) \quad (3)$$

$$= \sum_{j=0}^{n-1} \text{ENT}(X_d \circ \mu | \mathcal{F}_{j+1}) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j) \quad (4)$$

We claim that for all j with $0 \leq j < n$ we have

$$\mathbf{E}\left(\text{ENT}(X_d \circ \mu | \mathcal{F}_{j+1}) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j)\right) \leq -\text{ENT}(\mu, j) \frac{c}{d}, \quad (5)$$

where $c > 0$ is a universal constant. Note that combining this with equation (4) gives

$$\text{ENT}(X_d \circ \mu) - \text{ENT}(\mu) \leq \frac{c}{d} \sum_{j=0}^{n-1} \leq \text{ENT}(\mu, j) = \frac{c}{d} \text{ENT}(\mu), \quad (6)$$

which proves the lemma. It remains to verify equation (5).

For j with $0 \leq j < n$, define $\widehat{\mathcal{F}}_j = \sigma(\mathcal{F}_{k+1}, \{\Gamma_j(X), \Gamma_j(\tilde{X})\})$. Note that this is the sigma field generated by \mathcal{F}_{k+1} and the *unordered set* $\{\Gamma_j(X), \Gamma_j(\tilde{X})\}$. Note that $\widehat{\mathcal{F}}_j \supset \mathcal{F}_{j+1}$. Hence for all j with $0 \leq j < n$ we have

$$\mathbf{E}(\text{ENT}(X_d \circ \mu | \mathcal{F}_{j+1})) \leq \mathbf{E}(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j)), \quad (7)$$

by Jensen's inequality applied to $x \rightarrow x \log x$. Let $W = \{X_d(j+1), \dots, X_d(n-1)\}$ and let \mathcal{G}_{j+1} denote the sigma-field generated by $(X_d \circ \mu)^{-1}(x)$ for $x \in W$. Let $\mathcal{G}'_{j+1} = \sigma(\mu^{-1}(j+1), \dots, \mu^{-1}(n-1))$. Then

$$\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j) = \mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j, \mathcal{G}_{j+1}) \quad (8)$$

$$- \text{ENT}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1})\right) \quad (9)$$

$$= \mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j, \mathcal{G}'_{j+1}) \quad (10)$$

$$- \text{ENT}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}'_{j+1})\right), \quad (11)$$

where the first equality holds by Proposition 2. Note that $\mathcal{F}_{j+1} = \sigma(S, Z(l, t) : (l, t) \in S)$, where

$$S = \{(r, t) : L(X_t(i)) = l \text{ or } L(\tilde{X}_t(i)) = l \text{ for some } i > j\},$$

that is, S is the collection of bits used to generate $\Gamma_i(X)$ and $\Gamma_i(\tilde{X})$ for $i > j$.

We shall refer to indices i with $0 \leq i < n$ as *cards*. Say that cards i and j are *adjacent at time* t if $L(X_t(i)) = L(X_t(j))$. If $T_j \geq 0$, let $m(j)$ be the card adjacent to j at time T_j .

Note that $\mathcal{F}_j = \sigma(\widehat{\mathcal{F}}_{j+1}, Z(L(X_{T_j}(j)), T_j))$. Therefore, on the event that $(L(X_{T_j}(j)), T_j) \in S$ the expression on the lefthand-side of (8) is 0. However, we now show that if $m(j) < j$, then $(L(X_{T_j}(j)), T_j) \notin S$.

Note that if $(l, t) \in S$, then either $L(X_t(i)) = l$ for some $i > j$, or $t > T_i$ for some $i > j$ (and hence $t > T_j$). Thus if $m(j) < j$, then $(L(X_{T_j}(j)), T_j) \notin S$. So on the event that $m(j) < j$ and $\{\Gamma_j(X), \Gamma_j(\tilde{X})\} = \{\Gamma, \Gamma'\}$, the conditional distribution of $(\Gamma_j(X), \Gamma_j(\tilde{X}))$ given $\widehat{\mathcal{F}}_j$ is an even mixture of (Γ, Γ') and (Γ', Γ) , according to the value of $Z(L(X_{T_j}(j)), T_j)$.

Let $\mathcal{L}(W | \mathcal{F})$ denote the conditional distribution of random variable (or random permutation) W given the sigma field \mathcal{F} . Note that

$$\mathcal{L}(X_d \circ \mu | \widehat{\mathcal{F}}_j, \mathcal{G}_{j+1}) = \frac{1}{2} \mathcal{L}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}) + \frac{1}{2} \mathcal{L}(\tilde{X} \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}).$$

Therefore,

$$\begin{aligned} & \text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_{j+1}, \mathcal{G}_{j+1}) \\ & \leq \frac{1}{2} \text{ENT}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}) + \frac{1}{2} \text{ENT}(\tilde{X} \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}) - d(\mathcal{L}(X_d \circ \mu | \mathcal{F}_j), \mathcal{L}(\tilde{X} \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1})) \\ & = \text{ENT}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}) - d(\mathcal{L}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}), \mathcal{L}(\tilde{X} \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1})). \end{aligned}$$

But by the projection lemma,

$$\begin{aligned} d(\mathcal{L}(X_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1}), \mathcal{L}(\tilde{X}_d \circ \mu | \mathcal{F}_j, \mathcal{G}_{j+1})) & \geq d(\mathcal{L}((X_d \circ \mu)^{-1}(X_d(j)) | \mathcal{F}_j, \mathcal{G}_{j+1}), \mathcal{L}((\tilde{X}_d \circ \mu)^{-1}(X_d(j)) | \mathcal{F}_j, \mathcal{G}_{j+1})) \\ & = d(\mathcal{L}(\mu^{-1}(j) | \mathcal{F}_j, \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{F}_j, \mathcal{G}'_{j+1})). \end{aligned}$$

Since μ is independent of X_d , this last quantity is $d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))$. Combining this with equation (8) gives

$$\mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j)\right) \leq -\mathbf{E}\left(d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))\right). \quad (12)$$

Let $j_{d-1}j_{d-2}\cdots j_0$ be the binary representation of j . For cards k and j , write $D(k, j) = \max\{i : k_i \neq j_i\}$. Note that $D(k, j)$ is the minimum value of t such that there is positive probability that k and j are adjacent after t steps. For $t \geq 0$, let $B(j, t) = \{k : D(k, j) = t\}$. For convenience, let $B(j, t) = \emptyset$ if $t < 0$. Let $I = \{0, 1, \dots, j-1\}$. Note that if $k \in B(j, T_j) \cap I$, then $\mathbf{P}(m(j) = k) = \left(\frac{1}{2}\right)^{T_j}$. Equation (12) implies that

$$\begin{aligned} & \mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j)\right) \\ & \leq -\sum_{k \leq j} \mathbf{P}(m(j) = k) \mathbf{E}\left(d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))\right) \\ & = -\sum_{k \in B(j, T_j) \cap I} \left(\frac{1}{2}\right)^{T_j} \mathbf{E}\left(d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))\right) \\ & = -\sum_{k \in B(j, T_j) \cap I} \left(\frac{1}{2}\right)^{r+1-T} \mathbf{E}\left(d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))\right), \end{aligned}$$

where $r = \lceil \log_2 j \rceil$. It follows that if T is a *random variable* and $T_j = r + 1 - T$, then

$$\begin{aligned} & \mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_j) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j)\right) \\ & \leq -\mathbf{E}\left(\sum_{k \in B(j, T_j) \cap I} \left(\frac{1}{2}\right)^{r+1-T} d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1}))\right). \end{aligned}$$

In particular, if T is geometric(1/2), we have

$$\mathbf{E}\left(\text{ENT}(X_d \circ \mu | \widehat{\mathcal{F}}_{j+1}) - \text{ENT}(X_d \circ \mu | \mathcal{F}_j)\right)$$

$$\begin{aligned}
&\leq -\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^{r+1-t} \sum_{k \in B(j, r+1-t) \cap I} d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1})) \\
&= -\left(\frac{1}{2}\right)^{r+1} \sum_{k \leq j} d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1})).
\end{aligned}$$

Since $j \geq 2^{r-1}$, this is at most

$$\begin{aligned}
&-\frac{1}{4} \left(\frac{1}{j} \sum_{k \leq j} d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1})) \right) \\
&\leq -\frac{1}{4} \left(d(\mathcal{L}(\mu^{-1}(j) | \mathcal{G}'_{j+1}), \frac{1}{j} \sum_{k \in I} \mathcal{L}(\mu^{-1}(m(j)) | \mathcal{G}'_{j+1})) \right) \\
&\leq -c\text{ENT}(\mu, j),
\end{aligned}$$

for a universal constant c , where the first inequality follows from Proposition 4 and the second inequality follows from Proposition 6 (since the second argument of d is the uniform distribution). Combining this with equation (7) verifies equation (5), which completes the proof. \square

The above analysis extends to the non power-of-two case and we intend to handle this in the final version of this paper.

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