# A note on stability conditions for planar switched systems[1]

Moussa BALDE,
LMDAN-LGDA Département de Mathématiques et Informatique, UCAD, Dakar-Fann, Senegal
mbalde@ucad.sn

Ugo BOSCAIN,
Le2i, CNRS, Université de Bourgogne, B.P. 47870, 21078 Dijon Cedex, France
ugo.boscain@u-bourgogne.fr

Paolo MASON
Istituto per le Applicazioni del Calcolo Mauro Picone - CNR Viale del Policlinico 137 - 00161 Roma, Italy
and
Laboratoire des signaux et systèmes, Université Paris-Sud, CNRS, Supélec, 91192 Gif- Sur-Yvette, France
Paolo.Mason@lss.supelec.fr

**Abstract**

This paper is concerned with the stability problem for the planar linear switched system $\dot{x}(t) = u(t)A_1x(t)+(1-u(t))A_2x(t)$, where the real matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ are Hurwitz and $u(\cdot) : [0,\infty[\to \{0,1\}$ is a measurable function. We give coordinate-invariant necessary and sufficient conditions on $A_1$ and $A_2$ under which the system is asymptotically stable for arbitrary switching functions $u(\cdot)$. The new conditions unify those given in previous papers and are simpler to be verified since we are reduced to study 4 cases instead of 20. Most of the cases are analyzed in terms of the function $\Gamma(A_1, A_2) = \frac{1}{2}(\mathrm{tr}(A_1)\mathrm{tr}(A_2) - \mathrm{tr}(A_1A_2))$.

**Keywords:** planar switched systems, asymptotic stability, quadratic Lyapunov functions

## 1 Introduction

Let $A_1$ and $A_2$ be two $2\times 2$ real Hurwitz matrices. In this paper we are concerned with the problem of finding necessary and sufficient conditions on $A_1$ and $A_2$ under which the switched system

$$\dot{x}(t) = u(t)A_1x(t) + (1-u(t))A_2x(t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{1}$$

is globally asymptotically stable, uniformly with respect to measurable switching functions $u(\cdot) : [0,\infty[\to \{0,1\}$ (**GUAS** for short, see Definition 1 below).

This problem has been studied in [3] in the case in which both $A_1$ and $A_2$ are diagonalizable in $\mathbb{C}$ (*diagonalizable* case in the following) and in [1] in the case in which at least one among $A_1$ and $A_2$ is not (*nondiagonalizable* case in the following). (See also [7] as well as the related work [6].)

In both cases the stability conditions are given in terms of coordinate-invariant parameters. Unfortunately the parameters used in the diagonalizable case become singular in the nondiagonalizable one and therefore the two cases were studied separately.

The purpose of this note is to unify and simplify these conditions, reformulating them in terms of new invariants that permit to treat all cases at the same time.

We have reduced the cases to be studied from 20 (14 in the diagonalizable case[2] and 6 in the nondiagonalizable one) to the following 4 cases (see Theorem 1).

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[2]The stability conditions given in [3] were not correct in the case called **RC.2.2.B**. See [7] for the correction

**S1:** the first one corresponds to the case in which there exists a common quadratic Lyapunov function. The condition of **S1** is indeed equivalent to the condition given in [10] but is simpler to check. Recall however that the existence of a common quadratic Lyapunov function is only a sufficient condition for **GUAS** (i.e. there exist **GUAS** systems not admitting a quadratic Lyapunov function). See [4, 7] for details.

**S2:** the second one corresponds to the situation in which there exists $v \in (0,1)$ such that $vA_1 + (1-v)A_2$ has a positive real eigenvalue. In this case the system is unbounded since it is possible to build a trajectory going to infinity approximating the (non admissible) trajectory corresponding to $u(t) \equiv v$ and having the direction of the unstable eigenvector of $vA_1 + (1-v)A_2$.

**S3:** in the third case there exists a nonstrict common quadratic Lyapunov function. The system is only uniformly stable, but not **GUAS**, since there exists a trajectory not tending to the origin when $t$ goes to infinity.

**S4:** in the fourth case the stability analysis of the system reduces to the study of a single trajectory called *worst trajectory*. If this trajectory tends to the origin then the system is **GUAS** (in this case there exists a polynomial Lyapunov function, but not a quadratic one). If it is periodic then the system is uniformly stable but not **GUAS**. If it is unbounded then the system is unbounded.

For a discussion of various issues related to stability of switched systems, we refer the reader to [4, 5].

The paper is organized as follows. In Section 1.1 we recall the fundamental notions of stability and the different types of Lyapunov functions used in the paper. Section 2 contains our main result. In Section 3.1 we define the normal forms that are needed in the proof. In Section 3.2 we give the details of the proof.

## 1.1 Notions of stability

Let us recall some classical notions of stability which will be used in the following.

**Definition 1** *For $\delta > 0$ let $B_\delta$ be the unit ball of radius $\delta$, centered in the origin. Denote by $\mathcal{U}$ the set of measurable functions defined on $[0,\infty[$ and taking values on $\{0,1\}$. Given $x_0 \in \mathbb{R}^2$, we denote by $\gamma_{x_0,u(\cdot)}(\cdot)$ the trajectory of (1) based in $x_0$ and corresponding to the control $u(\cdot)$. We say that the system (1) is*

- **unbounded** *at the origin if there exist $x_0 \in \mathbb{R}^2$ and $u(\cdot) \in \mathcal{U}$ such that $\gamma_{x_0,u(\cdot)}(t)$ goes to infinity as $t$ goes to infinity;*
- **uniformly stable** *at the origin if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\gamma_{x_0,u(\cdot)}(t) \in B_\varepsilon$ for every $t > 0$, for every $u(\cdot) \in \mathcal{U}$ and every $x_0 \in B_\delta$;*
- **globally uniformly asymptotically stable** *at the origin (**GUAS**, for short) if it is uniformly stable at the origin and globally uniformly attractive, i.e., for every $\delta_1, \delta_2 > 0$, there exists $T > 0$ such that $\gamma_{x_0,u(\cdot)}(t) \in B_{\delta_1}$ for every $t \geq T$, for every $u(\cdot) \in \mathcal{U}$ and every $x_0 \in B_{\delta_2}$.*

*Remark* **1** The stability properties of the system (1) do not change if we allow measurable switching functions taking values in $[0,1]$ instead of $\{0,1\}$ (see for instance [7]). More precisely the system (1) with $u(\cdot) : [0,\infty[\to \{0,1\}$ is **GUAS** (resp. uniformly stable, resp. unbounded) if and only the system (1) with $u(\cdot) : [0,\infty[\to [0,1]$ is. In the following we name *convexified system* the switched system with $u(\cdot)$ taking values in $[0,1]$.

Since the stability properties of the system (1) do not depend on the parametrization of the integral curves of $A_1x$ and $A_2x$, we have the following.

**Lemma 1** *If the switched system $\dot{x} = u(t)A_1x + (1-u(t))A_2x$, $\quad u(\cdot) : [0,\infty[\to \{0,1\}$, has one of the stability properties given in Definition 1, then the same stability property holds for the system $\dot{x} = u(t)(A_1/\alpha_1)x + (1-u(t))(A_2/\alpha_2)x$, for every $\alpha_1, \alpha_2 > 0$.*

**Definition 2** *A* common Lyapunov function *(* **LF** *for short) $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ for a switched system of the form (1) is a continuous function such that $V(\cdot)$ is positive definite (i.e. $V(x) > 0$, $\forall x \neq 0$, $V(0) = 0$) and $V(\cdot)$ is strictly decreasing along nonconstant trajectories.*

*A positive definite continuous function $V : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is said to be a* nonstrict common Lyapunov function *if $V(\cdot)$ is nonincreasing along nonconstant trajectories.*

*A* common quadratic Lyapunov function *(quadratic* **LF** *for short) is a function of the form $V(x) = x^T P x$ where $P$ is a positive definite symmetric matrix and the matrices $A_1^T P + PA_1$ and $A_2^T P + PA_2$ are negative definite.*

We recall that, for systems of type (1), the existence of a **LF** is equivalent to **GUAS**[3] (see for instance [4]). Moreover the existence of a nonstrict **LF** guarantees the uniform stability of (1).

## 2 Stability conditions for two-dimensional bilinear switched systems

We start this section by defining the notations and the objects that will be used to state our stability result. In the following the word invariant will indicate any object which is invariant with respect to coordinate transformations. As usual, we denote by $\det(X)$ and $\operatorname{tr}(X)$ the determinant and the trace of a matrix $X$. If $X \in \mathbb{R}^{2\times 2}$ the discriminant is defined as

$$\delta_X = \operatorname{tr}(X)^2 - 4\det(X)\,.$$

Given a pair of matrices $X, Y$ we define the following object:

$$\Gamma(X,Y) := \frac{1}{2}(\operatorname{tr}(X)\operatorname{tr}(Y) - \operatorname{tr}(XY))\,.$$

By means of these invariants we can define the following invariants associated to (1):

$$\tau_i := \begin{cases} \dfrac{\operatorname{tr}(A_i)}{\sqrt{|\delta_{A_i}|}} & \text{if } \ \delta_{A_1} \neq 0\,,\ \delta_{A_2} \neq 0\,, \\ \dfrac{\operatorname{tr}(A_i)}{\sqrt{|\delta_{A_j}|}} & \text{if } \ \delta_{A_1}\delta_{A_2} = 0 \text{ but } \delta_{A_j} \neq 0\,, \\ \dfrac{\operatorname{tr}(A_i)}{2} & \text{if } \ \delta_{A_1} = \delta_{A_2} = 0\,, \end{cases}$$

$$k := \frac{2\tau_1\tau_2}{\operatorname{tr}(A_1)\operatorname{tr}(A_2)}\Big(\operatorname{tr}(A_1A_2) - \frac{1}{2}\operatorname{tr}(A_1)\operatorname{tr}(A_2)\Big)\,,$$

$$\Delta := 4(\Gamma(A_1,A_2)^2 - \Gamma(A_1,A_1)\Gamma(A_2,A_2))\,.$$

$$\mathcal{R} := \frac{2\Gamma(A_1,A_2) + \sqrt{\Delta}}{2\sqrt{\det(A_1)\det(A_2)}} e^{\tau_1 t_1 + \tau_2 t_2}\,, \text{where, for } \ i = 1,2\,,$$

$$t_i := \begin{cases} \frac{\pi}{2} - \arctan\frac{\operatorname{tr}(A_1)\operatorname{tr}(A_2)(k\tau_i + \tau_{3-i})}{2\tau_1\tau_2\sqrt{\Delta}} & \text{if } \ \delta_{A_i} < 0 \\ \operatorname{arctanh}\frac{2\tau_1\tau_2\sqrt{\Delta}}{\operatorname{tr}(A_1)\operatorname{tr}(A_2)(k\tau_i - \tau_{3-i})} & \text{if } \ \delta_{A_i} > 0 \\ \frac{2\sqrt{\Delta}}{(\operatorname{tr}(A_1A_2) - \operatorname{tr}(A_1)\operatorname{tr}(A_2)/2)\tau_i} & \text{if } \ \delta_{A_i} = 0 \end{cases}.$$

Remark **2** Let us define

$$sign(x) := \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Notice that, for every matrix $X \in \mathbb{R}^{2\times 2}$, one has $\Gamma(X,X) = \det(X)$. Also, since the Killing form of $\mathbb{R}^{2\times 2}$ is defined as $\mathcal{B}(X,Y) = 4\operatorname{tr}(XY) - 2\operatorname{tr}(X)\operatorname{tr}(Y)$ one has $sign(k) = sign(\mathcal{B}(A_1,A_2))$. Finally, notice that if $A_1, A_2$ are Hurwitz then $\tau_i < 0$ for $i = 1,2$ and $sign\,\Gamma(A_1,A_2) = sign(\tau_1\tau_2 - k)$.

[3] In [2, 7, 8, 9] it is actually shown that the **GUAS** property is equivalent to the existence of a polynomial **LF**

## 2.1 Statement of the results

In this section we state our main result which characterizes conpletely the stability properties of two-dimensional bilinear switched systems. Our necessary and sufficient conditions apply both to the non-degenarate cases studied in [3] and to the degenarate ones studied in [1].

**Theorem 1** *We have the following stability conditions for the system* (1)

**S1** *If* $\Gamma(A_1,A_2) > -\sqrt{\det(A_1)\det(A_2)}$, *and* $\mathrm{tr}(A_1A_2) > -2\sqrt{\det(A_1)\det(A_2)}$ *then the system admits a quadratic* **LF***.*

*If* $-\sqrt{\det(A_1)\det(A_2)} < \Gamma(A_1,A_2) \leq \sqrt{\det(A_1)\det(A_2)}$ *then the condition* $\mathrm{tr}(A_1A_2) > -2\sqrt{\det(A_1)\det(A_2)}$ *is automatically satisfied. As a consequence the system admits a quadratic* **LF***.*

**S2** *If* $\Gamma(A_1,A_2) < -\sqrt{\det(A_1)\det(A_2)}$, *then the system is unbounded,*

**S3** *If* $\Gamma(A_1,A_2) = -\sqrt{\det(A_1)\det(A_2)}$, *then the system is uniformly stable but not* **GUAS***,*

**S4** *If* $\Gamma(A_1,A_2) > \sqrt{\det(A_1)\det(A_2)}$, *and* $\mathrm{tr}(A_1A_2) \leq -2\sqrt{\det(A_1)\det(A_2)}$ *then the system is* **GUAS***, uniformly stable (but not* **GUAS***) or unbounded respectively if*

$$\mathcal{R} < 1, \mathcal{R} = 1, \mathcal{R} > 1.$$

The following corollary will be derived from item **S1** of the previous theorem.

**Corollary 1** *If* $\det([A_1,A_2]) \geq 0$ *then the system admits a quadratic* **LF***.*

<u>*Remark*</u> **3** In the diagonalizable case $\delta_{A_1}\delta_{A_2} \neq 0$ the parameters $\tau_1$, $\tau_2$, and $k$ are invariant under the transformation $(A_1,A_2) \to (A_1/\alpha_1, A_2/\alpha_2)$, for every $\alpha_1,\alpha_2 > 0$. This is no more true in the nondiagonalizable case. Notice however that in any case the stability conditions of Theorem 1 do not depend on coordinate transformations or on rescalings of the type $(A_1,A_2) \to (A_1/\alpha_1, A_2/\alpha_2)$. This is true in particular for the function $\mathcal{R}$.

# 3 Proof of the main results

## 3.1 Normal forms

The aim of this section is to reduce all the possible choices of the matrices $A_1, A_2$ to suitable normal forms, obtained up to coordinates transformations and rescaling of the matrices (see Lemma 1 and Remark 3 above), and depending directly on the coordinate invariant parameters introduced above. The normal forms used here describe all the possible situations for two-dimensional bilinear switched systems, covering at the same time the diagonalizable case studied in [3] and the nondiagonalizable one studied in [1]. They will play a key role in the proof of our results.

**Lemma 2** *We have the following cases depending on the rank of* $[A_1,A_2]$*:*

1. *If* $\det([A_1,A_2]) \neq 0$*, up to a linear change of coordinates and a renormalization according to Lemma 1, we can assume the following.*

$$A_1 = \begin{pmatrix} \tau_1 & 1 \\ sign(\delta_{A_1}) & \tau_1 \end{pmatrix}, \tag{2}$$

   *(a) If* $\det([A_1,A_2]) < 0$ *there exists* $F \in \mathbb{R}$, $|F| \geq 1$ *such that*

$$F + \frac{sign(\delta_{A_1}\delta_{A_2})}{F} = 2k$$

   *and* $A_2$ *has the form*

$$A_2 = \begin{pmatrix} \tau_2 & sign(\delta_{A_2})/F \\ F & \tau_2 \end{pmatrix}. \tag{3}$$

*(b) If* $\det([A_1,A_2])>0$ *then* $\delta_{A_i}>0$ *for* $i=1,2$, $k\in(-1,1)$ *and* $A_2$ *has the form*

$$A_2=\begin{pmatrix} \tau_2+\sqrt{1-k^2} & k \\ k & \tau_2-\sqrt{1-k^2}\end{pmatrix}. \tag{4}$$

2. *Under the hypothesis that* $rank([A_1,A_2])=1$*, it is always possible, up to exchanging* $A_1$ *and* $A_2$*, to find a linear change of coordinates which diagonalizes* $A_1$ *and renders* $A_2$ *upper triangular.*

3. *If* $[A_1,A_2]=0$ *then it must be* $sign(\delta_{A_1})=sign(\delta_{A_2})$*. If* $\delta_{A_i}<0$ *for* $i=1,2$ *or* $\delta_{A_i}>0$ *for* $i=1,2$ *then, up to a linear change of coordinates and a renormalization according to Lemma 1,* $A_1,A_2$ *assume the form* (2) *and* (3)*, respectively, with* $F=k=\pm1$*. If* $\delta_{A_1}=\delta_{A_2}=0$ *then* $A_1,A_2$ *can be put in upper triangular form with the elements of* $A_i$ *equal to* $\tau_i$ *on the diagonal, for* $i=1,2$*.*

**Proof of Lemma 2.** For simplicity we will prove the lemma just in the case $\det([A_1,A_2])\neq 0$, the other case being analogous. Note that Lemma 2 was proven in [3] when $\delta_{A_1}<0$, $\delta_{A_2}<0$ and in [1] in the case $\delta_{A_1}\delta_{A_2}=0$. Therefore, we can assume either $\delta_{A_1}>0$ or $\delta_{A_2}>0$. First consider the case $\delta_{A_1}>0$. In this case we can find a system of coordinates such that

$$A_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\quad A_2=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad a,b,c,d\in\mathbb{R}. \tag{5}$$

Without any loss of generality we can assume that $\lambda_1<\lambda_2$. The discriminant of $A_1$ is $(\lambda_2-\lambda_1)^2$ and the discriminant of $A_2$ is $\delta_{A_2}=(a-d)^2+4bc$, which can be positive or negative. We have

$$[A_1,A_2]=\begin{pmatrix} 0 & b(\lambda_1-\lambda_2)\\ c(\lambda_2-\lambda_1) & 0\end{pmatrix}. \tag{6}$$

Therefore $\det[A_1,A_2]=bc(\lambda_1-\lambda_2)^2$ and $\delta_{[A_1,A_2]}=-4bc(\lambda_1-\lambda_2)^2$ and $k=\frac{d-a}{\sqrt{|\delta_{A_2}|}}$. If $\det[A_1,A_2]<0$ consider the linear transformation

$$T=\begin{pmatrix} -\sqrt{\frac{-b}{c}} & \sqrt{\frac{-b}{c}}\\ 1 & 1\end{pmatrix},$$

which diagonalizes $[A_1,A_2]$. Then a straightforward computation shows that

$$\frac{2}{\sqrt{|\delta_{A_1}|}}T^{-1}A_1T=\begin{pmatrix}\tau_1 & 1\\ 1 & \tau_1\end{pmatrix}\qquad\text{and}\qquad \frac{2}{\sqrt{|\delta_{A_2}|}}T^{-1}A_2T=\begin{pmatrix}\tau_2 & sign(\delta_{A_2})/F\\ F & \tau_2\end{pmatrix},$$

where $F$ satisfies the equation $F+sign(\delta_{A_2})/F=2k$, and moreover we can assume $|F|\geq 1$ up to eventually exchange the reference coordinates. If $\delta_{A_1}<0$ then it must be $\delta_{A_2}>0$ and, exchanging the roles of $A_1$ and $A_2$, we can repeat the previous procedure obtaining

$$A_1=\begin{pmatrix}\tau_1 & sign(\delta_{A_1})/F\\ F & \tau_1\end{pmatrix},\quad A_2=\begin{pmatrix}\tau_2 & 1\\ sign(\delta_{A_2}) & \tau_2\end{pmatrix}.$$

Then the required normal forms are obtained by exchanging the coordinates and by a dilation along one of the coordinate axis.

Consider now the case $\det[A_1,A_2]=bc(\lambda_1-\lambda_2)^2>0$. We have

$$bc>0\quad\Rightarrow\quad \delta_{A_2}=(a-d)^2+4bc>0\ \text{ and }\ |k|=\frac{|a-d|}{\sqrt{(a-d)^2+4bc}}<1\,.$$

In this case $[A_1,A_2]$ is no more diagonalizable. Using the transformation

$$U=\begin{pmatrix}\frac{1}{c}\sqrt{bc} & -\frac{1}{c}\sqrt{bc}\\ 1 & 1\end{pmatrix}$$

we get

$$\frac{2}{\sqrt{|\delta_{A_1}|}}U^{-1}A_1U=\begin{pmatrix}\tau_1 & 1\\ 1 & \tau_1\end{pmatrix}\qquad\text{and}\qquad \frac{2}{\sqrt{|\delta_{A_2}|}}U^{-1}A_2U=\begin{pmatrix}\tau_2+\sqrt{1-k^2} & k\\ k & \tau_2-\sqrt{1-k^2}\end{pmatrix},$$

which concludes the proof of the lemma. ∎

## 3.2 Proof of Theorem 1 and Corollary 1

To prove our main result we will assume, from now on, that $A_1, A_2$ are under the normal forms given by Lemma 2. The following lemma, which can be proved by direct computation, will be used to take advantage of the conditions of [10] which describe the systems admitting a quadratic **LF**.

**Lemma 3** *For any $\sigma \in [0,1]$, we define*

$$\phi(\sigma) := \det(\sigma A_1 + (1-\sigma)A_2), \qquad \psi(\sigma) := \det(\sigma A_1 + (1-\sigma)A_2^{-1}).$$

*We have*

$$\phi(\sigma) = \sigma^2 \det A_1 + 2\sigma(1-\sigma)\Gamma(A_1, A_2) + (1-\sigma)^2 \det A_2 \tag{7}$$

*and*

$$\psi(\sigma) = \frac{1}{\det A_2}(\sigma^2 \det A_1 \det A_2 + \sigma(1-\sigma)\mathrm{tr}(A_1A_2) + (1-\sigma)^2). \tag{8}$$

**Proof of S1.** Recall that the main result in [10] claims that the system (1) admits a quadratic **LF** if and only if $\phi(\sigma) > 0$ and $\psi(\sigma) > 0$ for every $\sigma \in [0,1]$. Notice from Lemma 3 that $\phi(\sigma) > 0$ if and only if either $\Gamma(A_1, A_2) > 0$ or the discriminant $\Delta$ of (7) is negative. Analogously $\psi(\sigma) > 0$ if and only if either $\mathrm{tr}(A_1A_2) > 0$ or the discriminant $\mathrm{tr}(A_1A_2)^2 - 4\det(A_1)\det(A_2)$ of (8) is negative. It is therefore clear that the cases considered in **S1** are those satisfying the conditions of [10]. The last statement of **S1** comes from the following series of inequalities

$$\sqrt{\det(A_1)\det(A_2)} \geq \frac{1}{2}(\mathrm{tr}(A_1)\mathrm{tr}(A_2) - \mathrm{tr}(A_1A_2)) \geq -\frac{1}{2}\mathrm{tr}(A_1A_2).$$

This concludes the proof of **S1**.

**Proof of Corollary 1.** To prove Corollary 1 in the case $\det([A_1, A_2]) > 0$ we use the point *1.(b)* of Lemma 2. In particular we have that $\Gamma(A_1, A_2) = \tau_1\tau_2 - k > 0$ and $\mathrm{tr}(A_1A_2) = 2(\tau_1\tau_2 + k) > 0$ so that the conditions of **S1** are satisfied. In the case $\det([A_1, A_2]) = 0$ the result was already known (see for instance [5]), and it can be easily proved by using the normal forms defined in Lemma 2.

In what follows we will always assume $\det([A_1, A_2]) < 0$.

**Proof of S2 and S3.** Assume that $\Gamma(A_1, A_2) \leq -\sqrt{\det(A_1)\det(A_2)}$. Then a straightforward computation shows that the minimum of $\phi(\sigma)$ is given by

$$\sigma_0 = \frac{\det A_2 - \Gamma(A_1, A_2)}{\det A_1 + \det A_2 - 2\Gamma(A_1, A_2)} \in (0,1) \quad \text{and} \quad \phi(\sigma_0) = \frac{-\Delta}{4(\det A_1 + \det A_2 - \Gamma(A_1, A_2))} \leq 0.$$

In particular in the case described by **S2** we have $\phi(\sigma_0) < 0$ and therefore the matrix $\sigma_0 A_1 + (1-\sigma_0)A_2$ has a positive real eigenvalue, so that the system is unbounded (see Remark 1).

Similarly when $\Delta = 0$ and $\Gamma(A_1, A_2) < 0$ we have $\phi(\sigma_0) = 0$ so that the system cannot be **GUAS**. In this case to prove that the system is uniformly stable it is possible to show that the system admits the following non strict quadratic **LF**:

$$V(x) = V(x_1, x_2) = x_1^2 + \frac{(sign(\delta_{A_1})sign(\delta_{A_2}) - F^2)^2}{4F^2(\tau_{A_1}F - \tau_{A_2}sign(\delta_{A_1}))^2}x_2^2$$

**Proof of S4.** First observe that, under the conditions of **S4**, we have $F < -1$ and $k < 0$ since, when $A_1$ and $A_2$ are in normal form, $\mathrm{tr}(A_1A_2) = F + \frac{sign(\delta_{A_1}\delta_{A_2})}{F} + 2\tau_1\tau_2 = 2(k + \tau_1\tau_2) < 0$ and $|F| > 1$.

To prove **S4** we introduce the set of points where the vector fields $A_1x$ and $A_2x$ are parallel:

$$\mathcal{Z} = \{x \in \mathbb{R}^2 \,:\, Q(x) = 0\},$$

where $Q(x) := \det(A_1x, A_2x)$. The discriminant of the quadratic function $Q(x)$ coincides with $\Delta$. Since $\Delta > 0$ then $\mathcal{Z}$ consists on a pair of noncoinciding straight lines passing through the origin. Take a point $x \in \mathcal{Z} \setminus \{0\}$. We say that $\mathcal{Z}$ is *direct* (respectively, *inverse*) in $x$ if $A_1x$ and $A_2x$ have the same (respectively, opposite) versus. We have the following lemma.

**Lemma 4** *If $\mathcal{Z}$ is direct (resp. inverse) in $x_0 \in \mathcal{Z} \setminus \{0\}$ then $\mathcal{Z}$ is direct (resp. inverse) in every point of $\mathcal{Z} \setminus \{0\}$. Moreover in the case* **S4** *we have that $\mathcal{Z}$ is always direct.*

**Proof of Lemma 4.** Let $\mathcal{Z} = D_1 \cup D_2$ where $D_1$, $D_2$ are straight lines passing through the origin. Let us observe that, if $x \in D_i$

$$\exists\, \alpha_i \in \mathbb{R} \quad s.t. \quad A_2 A_1^{-1} A_1 x = A_2 x = \alpha_i A_1 x\,,$$

i.e. $\alpha_i$ is an eigenvalue of $A_2 A_1^{-1}$ and $A_1 x$ belongs to the eigenspace associated to it. So $\alpha_1 \alpha_2 = \det(A_2 A_1^{-1}) = \frac{\det(A_2)}{\det(A_1)} > 0$ which implies that $sign(\alpha_1) = sign(\alpha_2)$ i.e. $\mathcal{Z}$ is either direct in every point or inverse in every point. On the other hand it is easy to verify that $A^{-1} = \frac{1}{\det A_1}(2\tau_1 Id - A_1)$, where $Id$ denotes the identity matrix, which, in the case **S4**, implies

$$\alpha_1 + \alpha_2 = \mathrm{tr}(A_2 A_1^{-1}) = \mathrm{tr}\Big(\frac{1}{\det A_1}(2\tau_1 A_2 - A_2 A_1)\Big) = \frac{2\Gamma(A_1, A_2)}{\det(A_1)} > 0$$

so that $\mathcal{Z}$ is direct. □

Let $m_i$ be the slope of $D_i$, for $i = 1, 2$. Then, if $v_i$ is a vector spanning $D_i$, the orientation of the vector $A_1 v_i$ with respect to the radial direction is determined by the quantity

$$sign(\det(A_1 v_i, v_i)) = sign(m_i^2 - sign(\delta_{A_1})).$$

Similarly, the orientation of the vector $A_2 v_i$ with respect to the radial direction is given by

$$sign(\det(A_2 v_i, v_i)) = sign\left(\frac{m_i^2 sign(\delta_{A_2}) - F^2}{F}\right) = sign(F^2 - m_i^2 sign(\delta_{A_2})).$$

**Lemma 5** *If $\mathcal{Z}$ is direct, i.e. if $\Gamma(A_1, A_2) > 0$, it must be*

$$sign(m_i^2 - sign(\delta_{A_1})) = sign(F^2 - m_i^2 sign(\delta_{A_2})) = +1$$

**Proof of Lemma 5.** Since $\Gamma(A_1, A_2) > 0$ by the previous equalities we get that $\varepsilon := sign(m_i^2 - sign(\delta_{A_1})) = sign(F^2 - m_i^2 sign(\delta_{A_2}))$. If $\varepsilon = 0$ we are in the conditions of **S1**, since $[A_1, A_2] = 0$. If $\varepsilon = -1$ then it must be $sign(\delta_{A_1}) = sign(\delta_{A_2}) = 1$. In this case we have $F^2 < m_i^2 < 1$ which is impossible since $|F| > 1$. □

As a consequence the vectors $A_i x$ point in the clockwise sense for every $x \in \mathcal{Z}$. This property allows to define the main tool for checking the stability of (1) under the conditions of **S4**.

**Definition 3** *Assume that we are in the conditions of* **S4** *and under the normal forms of Lemma 2. Fix $x_0 \in \mathbb{R}^2 \setminus \{0\}$. The worst-trajectory $\gamma_{x_0}$ is the trajectory of (1), based at $x_0$, and having the following property. At each time $t$, $\dot{\gamma}_{x_0}(t)$ forms the smallest angle in clockwise sense with the exiting radial direction.*

Figure 1 expresses graphically the meaning of the previous definition.

It is clear that the worst trajectory always rotates clockwise around the origin when $\delta_{A_i} \leq 0$ for some $i \in \{0, 1\}$. If $\delta_{A_i} > 0$ for $i = 1, 2$ then the eigenvectors of $A_1$ are $(1, 1)^T$ and $(1, -1)^T$, while the eigenvectors of $A_2$ are $(1, F)^T$ and $(1, -F)^T$. In this case it is easy to check that $m_1 m_2 < 0$ and therefore, from Lemma 5, without loss of generality we can assume

$$F < m_2 < -1 < 1 < m_1 < -F.$$

As a consequence $D_1$ and $D_2$ divide the space into four connected components, each one intersecting the eigenspace of exactly one among $A_1$ and $A_2$. This implies that also in this case the worst trajectory rotates clockwise around the origin. This trajectory is the concatenation of integral curves of $A_2 x$ from points of $D_1$ to points of $D_2$ and integral curves of $A_1 x$ from points of $D_2$ to points of $D_1$ (see Figure 2).

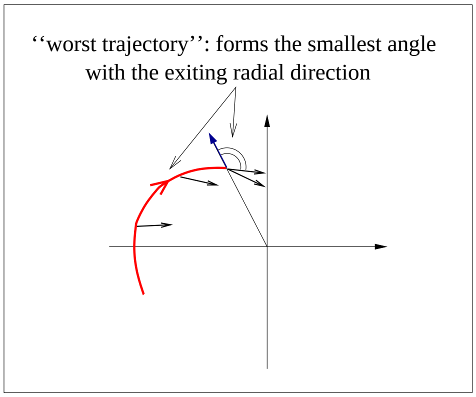


Figure 1: The worst trajectory

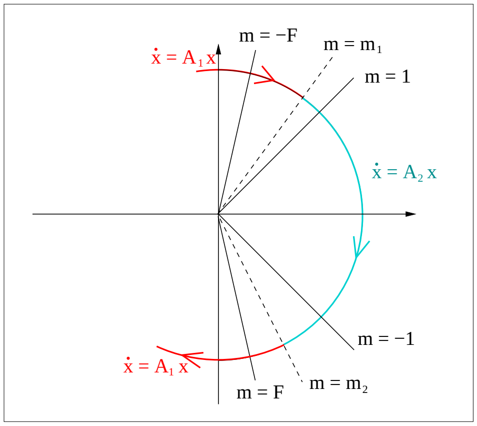


Figure 2: Construction of the worst trajectory in the case $\delta_{A_1} > 0,\ \delta_{A_2} > 0$

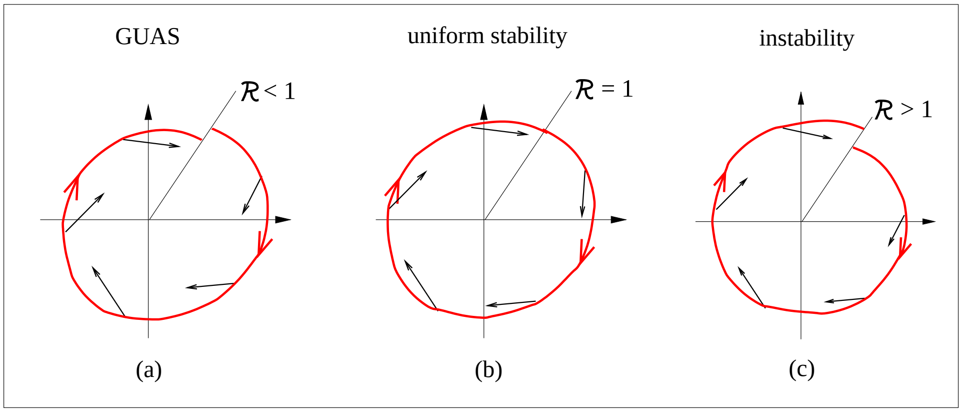


Figure 3: The worst trajectory: meaning of the function $\mathcal{R}$

As explained in the previous papers [1, 3] the behaviour of the worst trajectory is sufficient to derive the stability properties of (1). Let us analyse the worst trajectory $\gamma_{x_0}(\cdot)$ where $x_0 \in D_i$. Assume that $T > 0$ is such that $x_1 = \gamma_{x_0}(T)$ is the first intersection point between the worst trajectory and $D_i$. The worst trajectory tends to the origin as time goes to infinity if and only if $\mathcal{R} := |x_1|/|x_0| < 1$, and in this case the system is **GUAS** (see Figure 3 (a)). It is periodic if and only if $\mathcal{R} = 1$, and in this case the system is uniformly stable but not **GUAS** (see Figure 3 (b)). It blows up if and only if $\mathcal{R} > 1$, and in this case the system is unbounded (see Figure 3 (c)).

The computation of $\mathcal{R}$ was done in details in [1, 3]. The formula which is given in Section 2 is a simpler reformulation of the ones previously obtained, in terms of our invariants. This concludes the proof of **S4**.

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