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Abstract

This paper deals with Harsanyi power solutions for cooperative games in which partial cooperation is based on specific union stable systems given by the winning coalitions derived from a voting game. This framework allows for analyzing new and real situations in which there exists a feedback between the economic influence of each coalition of agents and its political power. We provide an axiomatic characterization of the Harsanyi power solutions on the subclass of union stable systems arisen from the winning coalitions from a voting game when the influence is determined by a power index. In particular, we establish comparable axiomatizations, in this context, when considering the Shapley-Shubik power index, the Banzhaf index and the Equal division power index which reduces to the Myerson value on union stable systems. Finally, a new characterization for the Harsanyi power solutions on the whole class of union stable systems is provided and, as a consequence, a characterization of the Myerson value is obtained when the equal power measure is considered.

Keywords: Cooperative TU-game, Harsanyi dividend, Harsanyi power solution, Union stable system, Shapley value, Banzhaf value, Voting games, power measures, Myerson value.

1. Introduction

For a firm, it is one thing to have the ability to produce, but it is useless unless the firm is allowed to produce. In other words, the economic power of a firm emerges only if it is accompanied by political or legal power. As an example, Google has a huge worldwide economic power, but cannot exert it in China where its government currently prevents Google's search service to operate without censure. As a more concrete example, let us consider the social cost problems first suggested by Coase (1960) in which are involved a set of victims and a set of polluters. The activity of the latter creates damage that affect the victims. In order to iron those conflicts and to solve the problem of social cost, a negotiation will take place within a coalition of polluters and victims with the objective to sign a binding agreement about how much activity the polluter will be able to undertake. Now, the permission granted to each coalition which wants to sign such binding

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agreements about the level of activity of the polluter is interpreted as the ability for the coalition to control the decision of a committee that assigns these rights (see Gonzalez et al., 2018, for details).

In this article, we provide a model based on cooperative game theory in order to apprehend such situations. We enrich the classical model of a (economic) cooperative game — a set of agents and a characteristic function specifying the economic power of each coalition of agents — with a voting game on the same set of agents. This voting game is modeled by a nonempty set of winning coalitions with the usual monotonicity property: each superset of a winning coalition is also a winning coalition. In this framework, an allocation rule specifies a utility for each agent for participating in each pair of economic and voting games. The latter two structures are likely to influence each other in the design of an allocation rule. On the one hand, as suggested above, the sharing of economic resources can depend on the political power of coalitions of agents. On the other hand, the measure of political power might be impacted by the economic power of coalitions, for example because of their ability to incur lobbying expenses.

Here, we explore the first of the two types of influence. For classical cooperative games a class of well-known allocation rules is the class of Harsanyi solutions introduced by Vasil'ev (1982) and then studied by Derks et al. (2000), among others. Each Harsanyi solution distributes the Harsanyi dividend of each coalition among its members in proportion to exogenously given weights. For voting games, two well-known power indices are provided by Shapley and Shubik (1954) and Banzhaf (1965). The Shapley-Shubik index measures the likelihood that an agent is decisive if the agents are called upon to vote one by one in favor of a decision. The Banzhaf index measures the proportion of coalitions for which a given agent is pivotal (i.e. a winning coalition that is not winning anymore without this agent). We combine both types of allocation rules in order to study a specific class of Harsanyi (power) solutions in which the Harsanyi dividend of each winning coalition is shared among its members in proportion to their relative political power as measured by an arbitrary power index σ in the voting subgame induced by the coalition. The idea to combine economic and political power also appears in Laruelle and Valenciano (2007) where the weights on an asymmetric Nash bargaining solution are specified by the Shapley-Shubik index of a voting game.

Theorem 1 characterizes the Harsanyi power solution induced by the power index σ. In addition to the classical axioms of Efficiency and Additivity, we introduce three other axioms. The first one is a variant of the Null agent out axiom (Derks and Haller, 1999), which requires that an allocation rule is not sensitive to the removal of a null agent in the economic game. The second one requires that if all winning coalitions enjoy a null worth in the economic game, then all agents should be treated equally. The third one is inspired by the axioms of σ-point unanimity (Algaba et al., 2015) and communication ability (Borm et al., 1992). If only the grand coalition has a non null worth in the economic game, the axiom imposes that the agents are paid in proportion to the power index σ.

Our contribution possesses some similarities with the literature on games played on combinatorial structures. The closest article is perhaps Algaba et al. (2015), where the exogenous structure is a union stable system, *i.e.* a set of feasible coalitions such that the union of two intersecting coalitions is also feasible. The authors provide a similar characterization of a class of Harsanyi power solutions. There are, however, two major differences with our work. Firstly, while the set of winning coalitions in a voting game is a union stable system, the converse is not true. As a consequence, some of the axioms invoked by Algaba et al. (2015) cannot be reused in our case as, for instance, the Inessential support property. Secondly, any set of connected coalitions on a graph is a union stable system, which implies that games played on union stable systems can be seen as a generalization of communication graph games introduced in Myerson (1977). To the contrary, the voting structure that we consider cannot be assimilated to a graph structure. In fact, the set of winning coalitions of a voting game does not always correspond to the set of connected coalitions of a graph on the agent set as pointed by van den Brink (2012). This allows us to replace the classical power measures on graph (such as the degree of each node) by voting power indices.

Another advantage of our contribution is that Theorem 1 is still valid on the larger class of games played on union stable systems subject to a minor adaptation of the axiom of Efficiency (Theorem 3). We also single out the relevant Harsanyi power solutions obtained by using, in the voting game, the Shapley-Shubik power index, the Banzhaf index and the Equal division power index, respectively. The latter one coincides with the Myerson value for games played on union stable systems introduced by Algaba et al. (2001). Theorem 2 provides a comparable axiomatization of these three Harsanyi power solutions.

The rest of the article is organized as follows. Section 2 introduces the material necessary to state our results. The main result is stated in Section 3. Section 4 presents the characterization of three relevant Harsanyi power solutions. Section 5 adapts our main result to the broader class of games on union stable systems. Section 6 concludes.

2. Preliminaries

2.1. Notation

The cardinality of a finite set S will be denoted either by the lower case s or by the symbol $|S|$. The collection of all subsets of S will be denoted by 2^S, and, for notational convenience, we will write singleton $\{i\}$ as i. If a linear space V is the direct sum of the subspaces V^1 and V^2, we write $V = V^1 \oplus V^2$.

2.2. Cooperative games

Let $\mathbb{N}$ be the universe of potential agents and let $N \subseteq \mathbb{N}$ a finite set of n agents. Each subset S of N is called a *coalition* and N is often called the *grand coalition*. A *cooperative transferable utility (TU)-game* is a pair (N, v) where $N \subseteq \mathbb{N}$ and $v : 2^N \longrightarrow \mathbb{R}$ is a *coalition function* such that $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ describes the worth of S when its members cooperate. Denote by G the set of all TU-games. The *subgame* of a TU-game $(N, v) \in G$ with respect to an agent set $S \subseteq N$ is the TU-game $(S, v_S) \in G$ where for each $T \subseteq S$, $v_S(T) = v(T)$.

Two distinct agents $i, j \in N$ are *equal agents* in $(N, v) \in G$ if, for each coalition $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$. Agent $i \in N$ is a *null agent* in $(N, v) \in G$, if, for each $S \subseteq N \setminus i$, $v(S \cup i) = v(S)$.

Given a finite set of agents N and a nonempty coalition $S \subseteq N$, the *unanimity TU-game* $(N, u_S) \in G$ is defined as $u_S(T) = 1$ if $T \supseteq S$ and $u_T(S) = 0$ otherwise. It is well known that the collection of unanimity TU-games u_S, $S \subseteq N$, $S \neq \emptyset$, forms a basis for the linear space of coalition functions v on a fixed agent set $N \subseteq \mathbb{N}$. As a result, each TU-game (N, v) can be written as a linear combination of unanimity TU-games in a unique way, as follows

$$v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) u_S, \tag{1}$$

where the coordinates $\Delta_v(S)$ are called the *Harsanyi dividends* (Harsanyi, 1959) of (N, v), and are computed through the following expression,

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T), \quad \forall S \subseteq N, S \neq \emptyset. \tag{2}$$

Remark 1 From the recursive formula (2), it is satisfied,

1. If agent $i \in N$ is a null agent in $(N, v) \in G$, then, for each $S \ni i$, we have $\Delta_v(S) = 0$.
2. Given a TU-game $(N, v) \in G$ and its subgame (S, v_S), it holds that, for each nonempty $T \subseteq S$, $\Delta_{v_S}(T) = \Delta_v(T)$.

□

2.3. Union stable systems

A set system is a pair $(N, \mathcal{F})$ where N represents a finite set of agents and $\mathcal{F} \subseteq 2^N$ is a collection of feasible coalitions. A set system $(N, \mathcal{F})$ is *union stable* if for $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$ it holds that $S \cup T \in \mathcal{F}$. Denote by US the set of union stable systems $(N, \mathcal{F})$ where $N \subseteq \mathbb{N}$ and $\mathcal{F} \subseteq 2^N$.

In most cases, it is more transparent to represent a union stable system $(N, \mathcal{F})$ by its *supports*, being a subset of $\mathcal{F}$ from which, with some specified operations, we can generate the full union stable system. Consider the set

$$E(\mathcal{F}) = \left\{R \in \mathcal{F}: \text{ there are } S, T \in \mathcal{F},\ S \neq R,\ T \neq R,\ S \cap T \neq \emptyset, \text{ with } R = S \cup T\right\},$$

consisting of those feasible coalitions, which can be written as the union of two distinct feasible coalitions with a nonempty intersection. The complement of the set $E(\mathcal{F})$ with respect to $\mathcal{F}$, denoted by $B(\mathcal{F})$, is called the *basis* of $(N, \mathcal{F})$ and its elements are the *supports* of $(N, \mathcal{F})$. Therefore, each support cannot be expressed as the union of two distinct feasible coalitions with a nonempty intersection. In Algaba et al. (2000), it is shown that $\mathcal{F}$ can be generated inductively from the basis $B(\mathcal{F})$. Precisely, this inductive process defines a closure operator $^{-}\colon 2^{\mathcal{F}} \longrightarrow 2^{\mathcal{F}}$, and $B(\mathcal{F})$ is the minimal subset of $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$.

Given a set system $(N, \mathcal{F})$, a coalition $T \subseteq S \subseteq N$ is called a *$\mathcal{F}$-component* of S if $T \in \mathcal{F}$ and there exists no $R \in \mathcal{F}$ such that $T \subsetneq R \subseteq S$. Therefore, the $\mathcal{F}$-components of S are the maximal feasible coalitions that belong to $\mathcal{F}$ and are contained in S. We denote by $C_{\mathcal{F}}(S)$ the collection, possibly empty, of the $\mathcal{F}$-components of S. Union stable systems can be characterized in terms of the $\mathcal{F}$-components of a coalition in the following way: The set system $(N, \mathcal{F})$ is union stable if and only if for any $S \subseteq N$ such that $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a collection of pairwise disjoint subsets of S (see Algaba et al., 2000).

For a union stable system $(N, \mathcal{F}) \in US$ and coalition $S \subseteq N$, we denote by $(S, \mathcal{F}_S)$ the *subsystem induced by* S and given by $\mathcal{F}_S = \{T \in \mathcal{F} : T \subseteq S\}$. It is easy to see that $(S, \mathcal{F}_S) \in US$.

2.4. Cooperative games on union stable systems and Harsanyi power solutions

In a TU-game $(N, v) \in G$, any coalition $S \subseteq N$ is assumed to be able to form and earn the worth $v(S)$. However, in most applications not every set of participants can form a feasible coalition. Therefore, cooperative game-theoretic models have been developed that take into account restrictions on coalition formation. They consider a set of feasible coalitions $\mathcal{F} \subseteq 2^N$ that need not contain all subsets of the agent set N. Formally, a *TU-game on a union stable system* is a triplet $(N, v, \mathcal{F})$ where (N, v) is TU-game in G and $(N, \mathcal{F})$ is a union stable set system in US representing the set of feasible coalitions. The possible gains from cooperation as modeled by (N, v) and the restrictions on cooperation reflected by the union stable system $(N, \mathcal{F})$ are incorporated in a *$\mathcal{F}$-restricted TU-game* $(N, v^{\mathcal{F}})$ defined as

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T), \quad \forall S \subseteq N. \tag{3}$$

The definition of the $\mathcal{F}$-restricted game can be understood as follows. If coalition $S \in \mathcal{F}$, then the members are allowed to cooperate to obtain $v(S)$. If S is not a feasible coalition, i.e. $S \in 2^N \setminus \mathcal{F}$, then not all agents belonging to S are allowed to cooperate. Then, coalition S splits into components (if any) and the best that members of S can accomplish is to cooperate within each of these components $T \in C_{\mathcal{F}}(S)$ to obtain $v(T)$.

Fix a union stable set system $(N, \mathcal{F})$. The set of coalition functions

$$G^N = \big\{v : 2^N \longrightarrow \mathbb{R}, v(\emptyset) = 0\big\},$$

forms a linear space G^N of dimension $2^n - 1$. The subspaces $G^N_{\mathcal{F}}$ and $G^N_{\overline{\mathcal{F}}}$ of G^N are defined as follows

$$G^N_{\mathcal{F}} = \big\{v \in G^N : \forall S \subseteq N, v(S) = v^{\mathcal{F}}(S)\big\} \text{ and } G^N_{\overline{\mathcal{F}}} = \big\{v \in G^N : \forall S \in \mathcal{F},\ v(S) = 0\big\}.$$

Obviously, it is satisfied

$$G^N = G^N_{\mathcal{F}} \oplus G^N_{\overline{\mathcal{F}}}.$$

Notice that $v^{\mathcal{F}}$ is the projection of v on $G^N_{\mathcal{F}}$ along $G^N_{\overline{\mathcal{F}}}$.

Denote by GUS the set of games on union stable systems $(N, v, \mathcal{F})$ where $(N, v) \in G$ and $(N, \mathcal{F}) \in US$. An *allocation rule* on GUS is a map Φ that assigns to each $(N, v, \mathcal{F}) \in GUS$ a payoff vector $\Phi(N, v, \mathcal{F}) \in \mathbb{R}^n$. Each coordinate $\Phi_i(N, v, \mathcal{F}) \in \mathbb{R}$ represents agent's i payoff for his or her participation in $(N, v, \mathcal{F})$.

In Algaba et al. (2015), the class of Harsanyi power solutions on GUS is introduced as follows. A power measure for union stable systems is a function σ defined on US which assigns to each union stable system $(N, \mathcal{F}) \in US$ a nonnegative vector $\sigma(N, \mathcal{F}) \in \mathbb{R}^n_+$. Given a *power measure* σ, the corresponding Harsanyi power solution Φ^σ on US is defined as

$$\forall i \in N, \quad \Phi^\sigma_i(N, v, \mathcal{F}) = \sum_{\substack{S \subseteq N, S \ni i \\ \sum_{j \in S} \sigma_j(S, \mathcal{F}_S) > 0}} \frac{\sigma_i(S, \mathcal{F}_S)}{\sum_{j \in S} \sigma_j(S, \mathcal{F}_S)} \Delta_{v^{\mathcal{F}}}(S) \tag{4}$$

Hence, the Harsanyi power solution Φ^σ_i assigns to each game on union stable systems $(N, v, \mathcal{F})$ the Harsanyi solution (see Vasil'ev, 2003) of the corresponding $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$, where any Harsanyi dividend $\Delta_{v^{\mathcal{F}}}(S)$ of coalition S in the restricted game is distributed to the agents in S proportional to their power in $(S, \mathcal{F}_S)$.

Lemma 1 *(Algaba et al., 2015, Lemma 1)*
Let $(N, v, \mathcal{F}) \in US^N$ be a union stable structure. Then $\Delta_{v^{\mathcal{F}}}(S) = 0$, for each $S \notin \mathcal{F}$.

Thanks to Lemma 1, each Harsanyi power solution Φ^σ can be rewritten as follows

$$\forall i \in N, \quad \Phi^\sigma_i(N, v, \mathcal{F}) = \sum_{\substack{S \in \mathcal{F}, S \ni i \\ \sum_{j \in S} \sigma_j(S, \mathcal{F}_S) > 0}} \frac{\sigma_i(S, \mathcal{F}_S)}{\sum_{j \in S} \sigma_j(S, \mathcal{F}_S)} \Delta_{v^{\mathcal{F}}}(S). \tag{5}$$

2.5. Cooperative games on political structure

Games on union stable systems extend communication graph games as considered in Myerson (1977) and Meessen (1988) and have a close relationship with hypergraphs (Algaba et al., 2004). A communication graph game is a triplet (N, v, L) where (N, v) is a TU-game and (N, L) is a communication graph. It is easy to see that the set system $(N, \mathcal{F})$ defined by those coalitions $S \subseteq N$ which induce connected subgraphs $(S, L(S))$, where $L(S)$ denotes the subset of links which are incident to the elements of S, is a union stable system. The Harsanyi power solutions on GUS as defined in (4) also extend the Harsanyi power solutions on communication graph games introduced by van den Brink et al. (2011).

On GUS, Algaba et al. (2015) provide axiomatic characterizations of the Harsanyi power solutions and of two well-known Harsanyi power solutions, namely the Myerson value (Myerson, 1977) and the position value (Meessen, 1988). As the latter two Harsanyi power solutions have been initially defined for communication graph games, union stable systems are mainly viewed as generalizations of communication graphs or networks (see also Algaba et al., 2018). In the same way, the proposed power measures are generalizations of power measures for graphs (the degree measure, the equal power measure, etc).

Our main objective is to propose another application of the union stable systems. Assume that a coalition of agents may have the capacity to produce $v(S)$ but does not have the legal authority or political power to implement this activity. It such a situation, it is necessary to separate the economic and political structures. The economic structure is modeled by a classical cooperative game. In order to model the political structure in a population, we consider voting games. A *voting game* is a pair $(N, \mathcal{W})$ where $N \subseteq \mathbb{N}$ is a finite set of agents and $\mathcal{W}$ is a collection of elements of 2^N satisfying the following properties:

1. $\mathcal{W} \neq \emptyset$;
2. $\emptyset \notin \mathcal{W}$;
3. If $T \supseteq S$ and $S \in \mathcal{W}$, then $T \in \mathcal{W}$.

Coalitions in $\mathcal{W}$ are called *winning coalitions*. Properties 1 and 2 indicate that the voting game admits at least one winning coalition and that the empty set is not a winning coalition. Property 3 is monotonicity property ensuring that if a coalition is winning, then any superset inherits this quality. Combining Properties 1 and 3, one obtains that the full set of agents N is a winning coalition, i.e $N \in W$. Intuitively a voting game represents a committee where the coalition N is the set of members of this committee and $\mathcal{W}$ is the set of coalitions that can control the decision of the committee. The subset $M(\mathcal{W}) \subseteq \mathcal{W}$ denotes the (nonempty) subset of minimal winning coalitions in $\mathcal{W}$ with respect to set inclusion, i.e. a coalition is a minimal winning coalition if it does not include any other winning coalition. By V is denoted the set of voting games.

It is easy to check that $V \subseteq US$, (joining to the set of winning coalitions, the empty set). Indeed, for each voting game $(N, \mathcal{W}) \in V$ and each $S, T \in \mathcal{W}$ such that $S \cap T \neq \emptyset$, $S \cup T \in \mathcal{W}$ by point 3 of the definition of a voting game. The subset of minimal winning coalition $M(\mathcal{W})$ is the basis of $(N, \mathcal{W})$ viewed as a union stable set system, and each $S \in M(\mathcal{W})$ is a support.

The *subgame* of a voting game $(N, \mathcal{W}) \in V$ with respect to a winning coalition $S \subseteq \mathcal{W}$ is the voting $(S, \mathcal{W}_S) \in V$ such that $\mathcal{W}_S = \{T \in \mathcal{W} : T \subseteq S\}$.

For each nonempty coalition $S \in N$, we denote by $(N, \mathcal{W}^*_S)$ the voting game on N such that S is the unique minimal winning coalition of $\mathcal{W}^*_S$,

$$\mathcal{W}^*_S = \big\{T \subseteq N : S \subseteq T \subseteq N\big\}.$$

By definition, each $(N, \mathcal{W}) \in V$ can be represented as the union of the voting games $(N, \mathcal{W}_S^*)$, with $S \in M(\mathcal{W})$, i.e.,

$$\mathcal{W} = \bigcup_{S \in M(\mathcal{W})} \mathcal{W}_S^*. \tag{6}$$

A power measure on V is usually named a *power index*. Formally, a power index σ is a map on V that assigns a numerical voting power to each agent in each voting game, that is, for each $(N, \mathcal{W})$ and each $i \in N$, $\sigma_i(N, \mathcal{W}) \in \mathbb{R}_+$ is a numerical measure of i's voting power in the committee N where the winning coalitions are represented by $\mathcal{W}$. For convenience, we do not consider here the null power index σ^0, which distributes a null power to every member of a committee, i.e., $\sigma^0(N, \mathcal{W}) = (0, \ldots, 0)$.

The most well-known *power indices* in the literature are undoubtedly the Shapley-Shubik power index (Shapley and Shubik, 1954) and the Banzhaf power index (Banzhaf, 1965). Essential to the construction of these power indices is the concept of pivotal agent. Given a voting game $(N, \mathcal{W}) \in V$, agent $i \in N$ is *pivotal* in coalition $S \ni i$ if leaving this coalition turns it from a winning coalition ($S \in \mathcal{W}$) into a losing coalition ($S \setminus i \in 2^N \setminus \mathcal{W}$).

The *Banzhaf power index* $\mathrm{Bz}_i(N, \mathcal{W})$ of an agent $i \in N$ in a voting game $(N, \mathcal{W}) \in V$ is defined as the fraction of coalitions for which i is pivotal under the assumption that each coalition of agents has the same probability to be formed. Formally,

$$\forall i \in N, \quad \mathrm{Bz}_i(N, \mathcal{W}) = \frac{1}{2^{n-1}} \big| \big\{ S \in \mathcal{W} : S \setminus i \in 2^N \setminus \mathcal{W} \big\} \big|.$$

The Shapley-Shubik index is based on a different probabilistic model of coalition formation. The agents are totally ordered according to a bijection $\pi : N \longrightarrow N$. Let Π_N be the set of bijections on N. For N, $i \in N$ and $\pi \in \Pi_N$, coalition

$$P_i(\pi) = \big\{ j \in N : \pi(j) \leq \pi(i) \big\} \subseteq N,$$

contains agent i together with the agents that appear before i according to π. Assuming that the probability of drawing each $\pi \in \Pi_N$ is $1/n!$, the *Shapley-Shubik power index* for voter i in $(N, \mathcal{W}) \in V$ is defined as the fraction of orderings π in which i is pivotal in $P_i(\pi)$:

$$\forall i \in N, \quad \mathrm{Sh}_i(N, \mathcal{W}) = \frac{1}{n!} \big| \big\{ \pi \in \Pi : P_i(\pi) \in \mathcal{W}, P_i(\pi) \setminus i \in 2^N \setminus \mathcal{W} \big\} \big|.$$

Another well-known power index is the *Equal division power index* ED which distributes the same fraction of power to each agent, i.e,

$$\forall i \in N, \quad \mathrm{ED}_i(N, \mathcal{W}) = n^{-1}.$$

Note that the Harsanyi power solution obtained for the Equal division power index ,Φ^{ED} on GV, is the *Myerson value for union stable systems* (Algaba et al. 2001), i.e.,

$$\forall i \in N, \quad \Phi_i^{ED}(N, v, \mathcal{W}) = \sum_{S \subseteq \mathcal{W}, S \ni i} \frac{1}{s} \Delta_{v^{\mathcal{W}}}(S),. \tag{7}$$

In this paper, we consider mainly the subclass of games on union stable systems $(N, v, \mathcal{W})$ where $(N, v) \in G$ and $(N, \mathcal{W}) \in V$. Denote by $GV \subseteq GUS$ this class of games. In particular, we provide an axiomatic characterization of the Harsanyi power solutions, as defined in (4), on GV.

It should be noted that such a restriction is not as innocent as it seems. To characterize the Harsanyi power solutions on GUS, Theorem 5 in Algaba et al. (2015) invokes the *inessential support property*. This axiom says that given a unanimity game (N, u_S) on a nonempty feasible coalition $S \in \mathcal{F}$, the allocation rule does not depend on those supports which contain at least one agent outside of coalition S.

Inessential support property An allocation rule Φ on GUS satisfies the inessential support property if, for each union stable system $(N, \mathcal{F}) \in US$, $c \in \mathbb{R}$, $S \in \mathcal{F}$, $S \neq \emptyset$, and $T \in B(\mathcal{F})$ such that $T \not\subseteq S$, it holds that

$$\Phi(N, cu_S, \mathcal{F}) = \Phi(N, cu_S, \overline{B(\mathcal{F}) \setminus \{T\}}).$$

We cannot use this axiom on GV because $\overline{B(\mathcal{F}) \setminus \{T\}}$ does not necessarily belongs to V as illustrated in the following example.

Example 1 Let $(N, \mathcal{W})$ be a voting game, with $N = \{1, 2, 3\}$ and $\mathcal{W} = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. The basis $B(\mathcal{W})$ or the set of minimal winning coalitions is given by $M(\mathcal{W}) = \{\{1, 2\}, \{1, 3\}\}$. If the support $\{1, 3\}$ is removed of the structure then $\overline{B(\mathcal{W}) \setminus \{1, 3\}} = \{\{1, 2\}\}$, but $(N, \{\{1, 2\}\})$ does not belong to V since N is not a winning coalition. $\square$

Theorem 6 in Algaba et al. (2015) considers the *superfluous agent property* to provide an alternative characterization of the Harsanyi power solutions on GUS. Agent i is superfluous in $(N, v, \mathcal{F}) \in GUS$ if i is null in the $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$. The superfluous agent property indicates that the allocation rule is insensible to the removal of feasible coalitions containing agent i.

Superfluous agent property An allocation rule Φ on GUS satisfies the superfluous agent property if for each $(N, v, \mathcal{F}) \in GUS$ and each superfluous agent i, it holds that

$$\Phi(N, v, \mathcal{F}) = \Phi(N, v, \mathcal{F}_{N\setminus i}),$$

where $\mathcal{F}_{-i} = \{S \in \mathcal{F} : i \notin S\}$ is given by all those feasible coalitions in $\mathcal{F}$ which do not contain agent i. When deleting coalitions containing a particular agent of a union stable system, the remaining collection of coalitions is still a union stable system. However, this property cannot be used on GV since the set of feasible coalitions defined by $\mathcal{W}_{N\setminus i}$ does not contain N. Consequently, the set system $(N, \mathcal{W}_{N\setminus i})$ does not belong to the set V of voting games on N.

Remark 2 For each $(N, v, \mathcal{W}) \in GV$, the expression of the $\mathcal{F}$-restricted game $(N, v^{\mathcal{W}}) \in G_{\mathcal{W}}$ as defined in (3) takes the following form

$$v^{\mathcal{W}}(S) = \begin{cases} v(S), & \text{if } S \in \mathcal{W}, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, note that

$$G^N_{\mathcal{W}} = \{v \in G^N : \forall S \in 2^N \setminus \mathcal{W},\, v(S) = 0\}. \tag{8}$$

Indeed, from the above expression of $v^{\mathcal{W}}$, it is obvious that

$$G^N_{\mathcal{W}} \subseteq \{v \in G^N : \forall S \in 2^N \setminus \mathcal{W},\, v(S) = 0\}.$$

Conversely, consider v such that, for each $S \in 2^N \setminus \mathcal{W}$, $v(S) = 0$. First, if $S \in 2^N \setminus \mathcal{W}$, then, by definition of $(N, \mathcal{W})$, $C_{\mathcal{W}}(S) = \emptyset$ and so $v^{\mathcal{W}}(S) = 0 = v(S)$. Second, if $S \in \mathcal{W}$ we also have, by definition of $v^{\mathcal{W}}$, $v(S) = v^{\mathcal{W}}(S)$. Therefore,

$$G^N_{\mathcal{W}} \supseteq \{v \in G^N : \forall S \in 2^N \setminus \mathcal{W},\, v(S) = 0\},$$

showing that equality (8) holds. □

By Lemma 1, for each power index σ on GV, the corresponding Harsanyi power solution Φ^σ on GV can be written as

$$\forall i \in N, \quad \Phi^\sigma_i(N, v, V) = \sum_{S \in \mathcal{W}, S \ni i} \frac{\sigma_i(S, \mathcal{W}_S)}{\sum_{j \in S} \sigma_j(S, \mathcal{W}_S)} \Delta_{v^{\mathcal{W}}}(S) \tag{9}$$

The above expression is well-defined since, by assumption, we do not consider the null power index (i.e., the situation $\sum_{j \in S} \sigma_j(S, \mathcal{W}_S) = 0$ is excluded).

3. Axiomatic characterizations on *GV*

In this section, we propose a characterization of the Harsanyi power solutions on GV. First, we recall two standard axioms for allocation rules Φ on GV.

Efficiency For each $(N, v, \mathcal{W}) \in GV$, it holds that

$$\sum_{i \in N} \Phi_i(N, v, \mathcal{W}) = v(N).$$

Additivity For every $(N, v, \mathcal{W}) \in GV$ and $(N, u, \mathcal{W}) \in GV$, it holds that

$$\Phi(N, v, \mathcal{W}) + \Phi(N, u, \mathcal{W}) = \Phi(N, v + u, \mathcal{W}),$$

where the TU-game $(N, v + u) \in GV$ is defined as $(v + u)(S) = v(S) + u(S)$ for each $S \subseteq N$.

Next, we introduce two new axioms. The first one indicates that if the game (N, v) vanishes on the subset W of winning coalitions, then all agents are treated equally. If we interpret (N, W) as a political game and (N, v) as an economic game, this axiom indicates that differences in payoffs between agents can only appear if coalitions with political power also have economic power.

Equality For each $(N, v, \mathcal{W}) \in GV$ such that $v \in G^N_{\mathcal{W}}$, it holds that for each $i, j \in N$,

$$\Phi_i(N, v, \mathcal{W}) = \Phi_j(N, v, \mathcal{W}).$$

Proposition 1 *If Φ is an allocation rule on GV that satisfies efficiency, additivity, and equality, then*

$$\Phi(N, v, \mathcal{W}) = \Phi(N, v^{\mathcal{W}}, \mathcal{W}). \tag{10}$$

Proof. Let Φ be an allocation rule as hypothesized, and take any $(N, v, \mathcal{W}) \in GV$. We have $(N, v, \mathcal{W}) = (N, v - v^{\mathcal{W}} + v^{\mathcal{W}}, \mathcal{W})$. By additivity,

$$\Phi(N, v - v^{\mathcal{W}} + v^{\mathcal{W}}, \mathcal{W}) = \Phi(N, v - v^{\mathcal{W}}, \mathcal{W}) + \Phi(N, v^{\mathcal{W}}, \mathcal{W}).$$

The coalition function $v - v^{\mathcal{W}}$ belongs to $G^N_{\mathcal{W}}$ and $(v - v^{\mathcal{W}})(N) = 0$ since $N \in \mathcal{W}$. By equality and efficiency, we obtain

$$\Phi(N, v - v^{\mathcal{W}}, \mathcal{W}) = (0, \dots, 0),$$

and, hence,

$$\Phi(N, v - v^{\mathcal{W}} + v^{\mathcal{W}}, \mathcal{W}) = \Phi(N, v, \mathcal{W}) = \Phi(N, v^{\mathcal{W}}, \mathcal{W}).$$

■

The second axiom is a weak version of the null agent out axiom introduced by Derks and Haller (1999) for allocation rules on G. This axiom indicates that deleting a null agent from the game does not affect the payoff of the other agents. An agent is a *null agent* in $(N, v) \in G$ if, for each $S \ni i$, $v(S) = v(S \setminus i)$.

Weak null agent out For each $(N, v, \mathcal{W}) \in GV$ such that $v \in G^N_{\mathcal{W}}$ and each null agent $i \in N$ in (N, v), it holds that for each $j \in N \setminus i$,

$$\Phi_j(N, v, \mathcal{W}) = \Phi_j(N \setminus i, v_{N \setminus i}, \mathcal{W}_{N \setminus i}).$$

Remark 3 Weak Null agent out is equivalent to the fact that, for each $(N, v, \mathcal{W}) \in GV$ such that $v \in G^N_{\mathcal{W}}$ and each $S \neq N$ of null agents in (N, v), it holds that for each $j \in N \setminus S$,

$$\Phi_j(N, v, \mathcal{W}) = \Phi_j(N \setminus S, v_{N \setminus S}, \mathcal{W}_{N \setminus S}).$$

□

Remark 4 The combination of efficiency and weak null agent out implies that for each $(N, v, \mathcal{W}) \in GV$ such that $(N, v) \in G^N_{\mathcal{W}}$ and each null agent in (N, v), the latter receives a null payoff. To see this, note that by weak null agent out and efficiency we have

$$\sum_{j \in N \setminus i} \Phi_j(N, v, \mathcal{W}) = \sum_{j \in N \setminus i} \Phi_j(N \setminus i, v_{N \setminus i}, \mathcal{W}_{N \setminus i}) = v(N \setminus i) = v(N),$$

where the last equality comes from the fact that i is null in (N, v). Therefore,

$$\sum_{j \in N \setminus i} \Phi_j(N, v, \mathcal{W}) = v(N),$$

implies $\Phi_i(N, v, \mathcal{W}) = 0$. In particular, for each $(N, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}) \in GV$ such that $S \in \mathcal{W}$, we have $\Delta_{v^{\mathcal{W}}}(S) u_S \in G^N_{\mathcal{W}}$ and agent $i \in N \setminus S$ is null in $(N, \Delta_{v^{\mathcal{W}}}(S) u_S)$. So, $\Phi_i(N, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}) = 0$.
□

Proposition 2 *If Φ is an allocation rule on GV that satisfies efficiency, additivity, weak null agent out, and equality, then for each $i \in N$,*

$$\Phi_i(N, v, \mathcal{W}) = \sum_{S \in \mathcal{W}: S \ni i} \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}_S). \tag{11}$$

Proof. By Proposition 1, it holds that

$$\Phi(N, v, \mathcal{W}) = \Phi(N, v^{\mathcal{W}}, \mathcal{W}).$$

By additivity of Φ and Lemma 1, we have

$$\Phi(N, v^{\mathcal{W}}, \mathcal{W}) = \sum_{S \in \mathcal{W}} \Phi(N, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}).$$

Next, take any $S \in \mathcal{W}$. As mentioned in Remark 4, each agent $i \in N \setminus S$ is null in $(N, \Delta_{v^{\mathcal{W}}}(S)u_S)$ and $\Delta_{v^{\mathcal{W}}}(S)u_S \in G^N_{\mathcal{W}}$. Again by Remark 4, we get that for each $S \in \mathcal{W}, i \in N \setminus S$,

$$\Phi_i(N, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}) = 0.$$

Therefore, we conclude that for each $i \in N$,

$$\Phi_i(N, v, \mathcal{W}) = \sum_{S \in \mathcal{W} : S \ni i} \Phi_i(N, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}).$$

On the other hand, since each agent of $N \setminus S$ is null in $(N, \Delta_{v^{\mathcal{W}}}(S)u_S)$ and $\Delta_{v^{\mathcal{W}}}(S)u_S \in G^N_{\mathcal{W}}$, an application of Remark 3 yields that for each $S \in \mathcal{W}$, $i \in S$,

$$\Phi_i(N, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}) = \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S).$$

Thus, we finally get for each $i \in N$,

$$\Phi_i(N, v, \mathcal{W}) = \sum_{S \in \mathcal{W} : S \ni i} \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S),$$

that is (11) holds, as desired. ■

The next axiom plays the same role as the axiom of σ-point unanimity used in Algaba et al. (2015). The latter axiom generalizes the communication ability axiom for communication graph TU-games introduced by Borm et al. (1992).

σ-unanimity Let σ be a power index. For each $(N, au_N, \mathcal{W}) \in GV$, where $a \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that

$$\Phi(N, au_N, \mathcal{W}) = b\sigma(N, \mathcal{W}).$$

Theorem 1 *Let σ a power index. The Harsanyi power solution Φ^σ on GV is the unique allocation rule that satisfies efficiency, additivity, weak null agent out, equality and σ-unanimity.*

Proof. (*Uniqueness*). Let Φ be an allocation rule on GV satisfying efficiency, additivity, null agent out, and equality and σ-unanimity. Let $(N, v, \mathcal{W}) \in GV$. By Proposition 2, for each $i \in N$,

$$\Phi_i(N, v, \mathcal{W}) = \sum_{S \in \mathcal{W} : S \ni i} \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S). \tag{12}$$

Let $S \in \mathcal{W}$ be a winning coalition. We can apply σ-unanimity on $(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S)$ to obtain

$$\Phi(S, \Delta_{v^{\mathcal{W}}}(N)u_S, \mathcal{W}_S) = b\sigma(S, \mathcal{W}_S),$$

for some $b \in \mathbb{R}$. By efficiency,

$$\sum_{i\in N} \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S) = \Delta_{v^{\mathcal{W}}}(S) = b\sum_{i\in S}\sigma_i(S, \mathcal{W}_S),$$

so that

$$b = \frac{\Delta_{v^{\mathcal{W}}}(S)}{\sum_{i\in S}\sigma_i(S, \mathcal{W}_S)}.$$

It follows that $\Phi(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S)$ is uniquely determined and for each $i \in N$,

$$\Phi_i(S, \Delta_{v^{\mathcal{W}}}(S)u_S, \mathcal{W}_S) = \frac{\sigma_i(S, \mathcal{W}_S)}{\displaystyle\sum_{j\in S}\sigma_j(S, \mathcal{W}_S)}\Delta_{v^{\mathcal{W}}}(S).$$

This complete the proof of the uniqueness part.

(*Existence*) We show that the Harsanyi power solution Φ^σ satisfies the five axioms.
Efficiency. For each $(N, v, \mathcal{W}) \in GV$, we have

$$\begin{aligned}
\sum_{i\in N}\Phi^\sigma(N, v, \mathcal{W}) &= \sum_{i\in N}\sum_{S\subseteq\mathcal{W}:S\ni i}\frac{\sigma_i(S, \mathcal{W}_S)}{\displaystyle\sum_{j\in S}\sigma_j(S, \mathcal{W}_S)}\Delta_{v^{\mathcal{W}}}(S)\\
&= \sum_{S\in\mathcal{W}}\sum_{i\in S}\frac{\sigma_i(S, \mathcal{W}_S)}{\displaystyle\sum_{j\in S}\sigma_j(S, \mathcal{W}_S)}\Delta_{v^{\mathcal{W}}}(S)\\
&= \sum_{S\in\mathcal{W}}\Delta_{v^{\mathcal{W}}}(S)\\
&= v^{\mathcal{W}}(N)\\
&= v(N),
\end{aligned}$$

where the fourth equality is a consequence of Lemma 1. This shows that Φ^σ satisfies efficiency.
Additivity. The proof is similar to the one given by Algaba et al. (2015, Theorem 1), and so it is omitted.
Weak null agent out. Consider any $(N, v, \mathcal{W}) \in GV$ such that $v \in G^N_{\mathcal{W}}$ and some null agent $i \in N$ in (N, v). Because $v \in G^N_{\mathcal{W}}$, $v = v^{\mathcal{W}}$. Therefore, for each nonempty $S \subseteq N$, $\Delta_v(S) = \Delta_{v^{\mathcal{W}}}(S)$. In particular, by point 1 of Remark 1, for each $S \ni i$, $\Delta_v(S) = 0$. By point 2 of Remark 1, we also have, for each $S \subseteq N \setminus i$, $\Delta_v(S) = \Delta_{v_{N\setminus i}}(S)$. It follows that for each $j \in N \setminus i$,

$$\begin{aligned}
\Phi^\sigma_j(N, v, \mathcal{W}) &= \sum_{S\in\mathcal{W}:S\ni j}\frac{\sigma_j(S, \mathcal{W}_S)}{\displaystyle\sum_{k\in S}\sigma_k(S, \mathcal{W}_S)}\Delta_{v^{\mathcal{W}}}(S)\\
&= \sum_{S\in\mathcal{W}_{N\setminus i}:S\ni j}\frac{\sigma_j(S, \mathcal{W}_S)}{\displaystyle\sum_{k\in S}\sigma_k(S, \mathcal{W}_S)}\Delta_{v^{\mathcal{W}}}(S)\\
&= \sum_{S\in\mathcal{W}_{N\setminus i}:S\ni j}\frac{\sigma_j(S, \mathcal{W}_S)}{\displaystyle\sum_{k\in S}\sigma_k(S, \mathcal{W}_S)}\Delta_{((v_{N\setminus i})^{\mathcal{W}})}(S)\\
&= \Phi^\sigma_j(N \setminus i, v_{N\setminus i}, \mathcal{W}_{N\setminus i}),
\end{aligned}$$

showing that Φ^σ satisfies weak null agent out.

Equality. Let $(N, v, \mathcal{W}) \in GV$ such that $v \in G^N_{\mathcal{W}}$. In such a case, the $\mathcal{W}$-restricted-game $(N, v^{\mathcal{W}})$ is the null game, which implies that $\Delta_{v^{\mathcal{W}}}(S) = 0$, for each nonempty coalition $S \subseteq N$. By definition of Φ^σ, we have

$$\Phi^\sigma(N, v, \mathcal{W}) = (0, \ldots, 0).$$

Thus, Φ^σ satisfies equality.

σ-unanimity. Let $(N, au_N, \mathcal{W}) \in GV$ for some $a \in \mathbb{R}$. We have $\Delta_{au_N}(N) = a$, and, by definition, Φ^σ distributes the following payoffs

$$\forall i \in N, \quad \Phi^\sigma_i(N, au_N, \mathcal{W}) = \frac{\sigma_i(N, \mathcal{W})}{\sum_{j \in N} \sigma_j(N, \mathcal{W})} \Delta_{v^{\mathcal{W}}}(N).$$

By setting $b = \Delta_{v^{\mathcal{W}}}(N)/(\sum_{j \in N} \sigma_j(N, \mathcal{W}))$, we have

$$\forall i \in N, \quad \Phi^\sigma_i(N, au_N, \mathcal{W}) = b\sigma_i(N, \mathcal{W}),$$

showing that Φ^σ satisfies σ-unanimity. ■

Note that Theorem 4 (Theorem 6, respectively), in Algaba et al. (2015), characterizes the Harsanyi power solution Φ^σ with respect to σ on the full domain GUS by using *component efficiency*, *component-dummy*, *additivity*, *σ-point unanimity*, the *inessential support property* (the *superfluous agent property* respectively) and *connectedness*. Component-dummy indicates that agents who do not belong to any support receive a null payoff. Component efficiency, already used in Myerson (1977), indicates that the worth of each component of the grand coalition is distributed among these members. Connectedness indicates that the worth of non-feasible coalitions are irrelevant to compute the payoffs. We have already mentioned why it is not possible to use the inessential support property or the superfluous agent property on the restricted domain GW. Instead, we use a weak version of the null agent out axiom. The main difference between these two axioms is that the former indicates that the allocation rules is indifferent to the removal of some supports in specific situations, whereas the second indicates that the allocation rule is insensible to the removal of some agents – in this case, these are null agents – in specific situations. When some supports are removed, some agents may become isolated. To determine the payoff of these isolated agents, the axiom of component-dummy is used. Conversely, when null agents are removed, the combination of weak null agent and efficiency implies that these agents receive a null payoff (see Remark 4). That is the reason why we don't need to use component-dummy. We use efficiency instead of component-efficiency because in each voting game the grand coalition N is feasible so that each Harsanyi power solution on GW distributes the worth obtained by the members of N when they cooperate. The axioms of σ-point unanimity and σ-unanimity are similar. The requirement contained in connectedness is partially encapsulated in equality which indicates that in situations where the worth of feasible coalitions is null, the worth of non-feasible coalitions are of no use in differentiating agents' payoffs.

4. The Harsanyi-Shapley-Shubik solution, the Harsanyi-Banzhaf solution, and the Myerson value

In this section, we focus our attention on three Harsanyi power solutions on GV. The Myerson value, given by (7), is the Harsanyi power solution Φ^{ED}, where the Equal division power index

ED is used to distribute the Harsanyi dividends of the $\mathcal{W}$-restricted game. We introduce two other Harsanyi power solutions. The first one, defined as Φ^{Sh}, is such that the Harsanyi dividend of each coalition in the $\mathcal{W}$-restricted game is distributed to its members proportionally to their Shapley-Shubik power index with respect to this coalition. For this reason, it is referred to as the *Harsanyi-Shapley-Shubik solution*. The third Harsanyi power solution is the *Harsanyi-Banzhaf solution* defined as Φ^{Bz}. It is such that the Banzhaf power index is used to distribute the Harsanyi dividends of the $\mathcal{W}$-restricted game. Formally, we have

$$\forall i \in N \quad \Phi_i^{Sh}(N, v, \mathcal{W}) = \sum_{S \subseteq \mathcal{W}: S \ni i} \frac{Sh_i(S, \mathcal{W}_S)}{\sum_{j \in S} Sh_j(S, \mathcal{W}_S)} \Delta_{v^{\mathcal{W}}}(S),$$

and

$$\forall i \in N, \quad \Phi_i^{Bz}(N, v, \mathcal{W}) = \sum_{S \subseteq \mathcal{W}: S \ni i} \frac{Bz_i(S, \mathcal{W}_S)}{\sum_{j \in S} Bz_j(S, \mathcal{W}_S)} \Delta_{v^{\mathcal{W}}}(S).$$

We provide a comparable axiomatization of these three Harsanyi power solutions on GV. To this end, we need to introduce other axioms for an allocation rule Φ on GV.

Strong equal treatment of equal agents For every $(N, v, \mathcal{W}) \in GV$ and pair of agents $\{i, j\} \subset N$ which are equal in (N, v) and $(N, \mathcal{W})$, it holds that

$$\Phi_i(N, v, \mathcal{W}) = \Phi_j(N, v, \mathcal{W}).$$

Null voting power For every $(N, v, \mathcal{W}) \in GV$ and null agent $i \in N$ in $(N, \mathcal{W})$, it holds that

$$\Phi_i(N, v, \mathcal{W}) = 0.$$

Strong equal treatment of equal agents indicates that two equal agents both in the economic game (N, v) and the voting game $(N, \mathcal{W})$ are treated equally. Null voting power is the null agent principle applied to the voting game: if an agent is null in the voting game, his or her payoff is null whatever his or her role in the economic game $(N, v) \in G$ is. This axiom reveals that although an agent is productive in an economic game, if he or she is never pivotal in the voting game, then he or she will still get a null payoff. The three other axioms concern situations $(N, au_N, \mathcal{W})$ where agents are involved in (a multiple of) a unanimity game on the grand coalition and the voting structure is arbitrary.

Weak modularity For every $(N, au_N, \mathcal{W})$ and $(N, au_N, \mathcal{W}')$, where $a \in \mathbb{R}$, it holds that

$$\Phi(N, au_N, \mathcal{W}) + \Phi(N, au_N, \mathcal{W}') = \Phi(N, au_N, \mathcal{W} \cap \mathcal{W}') + \Phi(N, au_N, \mathcal{W} \cup \mathcal{W}').$$

Laruelle and Valenciano (2001) reformulate "modularity" in an equivalent but more transparent version stating that the effect (gain or loss) on any agent's payoff of eliminating a single minimal winning coalition from the set of winning ones is the same in any game in which this coalition is minimal winning. Formally, for any $(N, \mathcal{W})$ different from $(N, \mathcal{W}_N^*)$, and any minimal winning coalition $S \in M(\mathcal{W}) \cap M(\mathcal{W}')$, weak modularity can be rewritten as

$$\Phi(N, au_N, \mathcal{W}) - \Phi(N, au_N, \mathcal{W} \setminus S) = \Phi(N, au_N, \mathcal{W}') - \Phi(N, au_N, \mathcal{W}' \setminus S).$$

The principle of modularity, also called transfer, has been extensively used in the cooperative game literature to characterize the Shapley-Shubik index and the Banzhaf index (Dubey, 1975; Dubey and Shapley, 1979; Feltkamp, 1995; Laruelle and Valenciano, 2001, among others).

The next axiom is a slight modification of the previous one which takes into account the role of a power index.

Weak σ-modularity Let σ be a power index on V. For every $(N, au_N, \mathcal{W})$ and $(N, au_N, \mathcal{W}')$, where $a \in \mathbb{R}$, it holds that

$$\begin{aligned}\sigma_N(N,\mathcal{W})\Phi_i(N, au_N, \mathcal{W}) + \sigma_N(N,\mathcal{W}')\Phi_i(N, au_N, \mathcal{W}') &= \\ \sigma_N(N,\mathcal{W}\cap\mathcal{W}')\Phi_i(N, au_N, \mathcal{W}\cap\mathcal{W}') + \sigma_N(N,\mathcal{W}\cup\mathcal{W}')\Phi_i(N, au_N, \mathcal{W}\cup\mathcal{W}'),\end{aligned}$$

where $\sigma_N(N,\mathcal{W})$ stands for the sum of the $\sigma_i(N,\mathcal{W})$ over N.

The last axiom relies on an invariance principle. It says that if a set of agents are involved in (a multiple of) a unanimity game on the grand coalition, then their payoffs are indifferent to a change of the voting game.

Invariance to the voting structure for unanimity situations For every $(N, au_N, \mathcal{W})$ and $(N, au_N, \mathcal{W}')$ in GV, it holds that

$$\Phi(N, au_N, \mathcal{W}) = \Phi(N, au_N, \mathcal{W}').$$

Using these axioms, we can determine the distribution of the payoffs for unanimity situations $(N, au_N, \mathcal{W}) \in GV$. Formally, we have the following result.

Proposition 3 *Let Φ be an allocation rule on GV, satisfying efficiency and strong equal treatment of equal agents. Then, for each $(N, au_N, \mathcal{W}) \in GV$, it holds that $\Phi(N, au_N, \mathcal{W})$ is uniquely determined, in the two following cases.*

1. *The allocation rule Φ satisfies null voting power and either weak modularity or weak σ-modularity.*
2. *The allocation rule Φ satisfies invariance to the voting structure for unanimity situations.*

Proof.
POINT 1. Let Φ be any allocation rule on GV satisfying efficiency, strong equal treatment of equal agents, null power agent and weak modularity. Take $(N, au_N, \mathcal{W}) \in GV$ where $a \in \mathbb{R}$. The proof is by induction on the number of elements in the set $M(\mathcal{W})$ of minimal winning coalitions.

INITIALIZATION. Suppose that $M(\mathcal{W})$ contains a unique coalition, say S, so that $\mathcal{W} = \mathcal{W}^*_S$. On the one hand, each agent in $N \setminus S$ is null in $(N, \mathcal{W}^*_S)$. By null voting power, $\Phi_i(N, v, \mathcal{W}^*_S) = 0$ for each $i \in N \setminus S$. On the other hand, each pair of distinct agents in S are equal agents in (N, au_N) and $(N, \mathcal{W}^*_S)$. By strong equal treatment of equal agents, for each pair $\{i, j\} \subseteq S$, we have $\Phi_i(N, au_N, \mathcal{W}^*_S) = \Phi_j(N, au_N, \mathcal{W}^*_S)$. Applying efficiency, we get, for each $i \in S$, $\Phi_i(N, au_N, \mathcal{W}^*_S) = as^{-1}$. Thus, $\Phi(N, v, \mathcal{W}^*_S)$ is uniquely determined.

INDUCTION HYPOTHESIS. Assume that $\Phi(N, v, \mathcal{W})$ is uniquely determined for situations where $M(\mathcal{W})$ contains at most q elements for some $q \geq 1$.

Induction step. Suppose that $M(\mathcal{W})$ contains $q+1$ minimal winning coalitions, $S_1, S_2, \dots, S_{q+1}$. By (6), $\mathcal{W}$ admits the following decomposition

$$\mathcal{W} = (\mathcal{W}^*_{S_1}) \bigcup \Big(\bigcup_{k=2}^{q+1} \mathcal{W}^*_{S_k} \Big).$$

Denote by $\mathcal{W}_{-S_1}$ the voting structure $\mathcal{W}_{S_2} \cup \dots \cup \mathcal{W}_{S_{q+1}}$. Note that $M(\mathcal{W}_{-S_1})$ contains q elements. By weak modularity, we obtain

$$\Phi(N, au_N, \mathcal{W}) = -\Phi(N, au_N, \mathcal{W}^*_{S_1} \cap \mathcal{W}_{-S_1}) + \Phi(N, au_N, \mathcal{W}^*_{S_1}) + \Phi(N, au_N, \mathcal{W}_{-S_1})$$

By the induction hypothesis, $\Phi(N, au_N, \mathcal{W}^*_{S_1})$ and $\Phi(N, au_N, \mathcal{W}_{-S_1})$ are uniquely determined. In addition, we have

$$\begin{aligned} \mathcal{W}^*_{S_1} \cap \mathcal{W}_{-S_1} &= \bigcup_{k=2}^{q+1} \Big(\mathcal{W}^*_{S_1} \cap \mathcal{W}^*_{S_k} \Big) \\ &= \bigcup_{k=2}^{q+1} \mathcal{W}^*_{S_1 \cup S_k}, \end{aligned}$$

so that $M(\mathcal{W}^*_{S_1} \cap \mathcal{W}_{-S_1})$ contains exactly q elements. By the induction hypothesis, the payoff vector $\Phi(N, au_N, \mathcal{W}^*_{S_1} \cap \mathcal{W}_{-S_1})$ is uniquely determined. Thus, $\Phi(N, au_N, \mathcal{W})$ is uniquely determined, as desired.

The proof for the case where Φ satisfies weak σ-modularity is similar, so it is omitted.

Point 2. Assume that Φ on GV satisfies efficiency, strong equal treatment of equal agents, and invariance to the voting structure for unanimity situations. Take $(N, au_N, \mathcal{W}) \in GV$ where $a \in \mathbb{R}$. By invariance to the voting structure for unanimity situations,

$$\Phi(N, (N, au_N, \mathcal{W}) = \Phi(N, au_N, \mathcal{W}^*_N).$$

Each pair of distinct agents are equal agents in (N, au_N) and $(N, \mathcal{W}^*_N)$. By strong equal treatment of equal agents and efficiency, $\Phi(N, au_N, \mathcal{W}^*_N) = (a/n \dots, a/n)$, and so is uniquely determined. Thus, $\Phi(N, (N, au_N, \mathcal{W})$ is uniquely determined. This completes the proof of Proposition 3. ■

Next, we obtain comparable axiomatizations for the three specific Harsanyi power solutions above mentioned.

Theorem 2 *Among all those allocation rules in GV satisfying efficiency, additivity, weak null agent out, equality, and strong equal treatment of equal agents,*

1. *The Myerson value Φ^{ED} is the only one which satisfies invariance to the voting structure for unanimity situations.*
2. *The Harsanyi-Shapley-Shubik solution Φ^{Sh} is the only one which satisfies null voting power and weak modularity.*
3. *The Harsanyi-Banzhaf solution Φ^{Bz} is the only one which satisfies null voting power and weak σ-modularity.*

Proof. (*Uniqueness*) Let Φ be an allocation rule on GV satisfying efficiency, additivity, weak null agent out, equality, and strong equal treatment of equal agents. By Proposition 2, we have

$$\forall i \in N, \quad \Phi_i(N, v, \mathcal{W}) = \Phi_i(N, v^{\mathcal{W}}, \mathcal{W}) = \sum_{S \in \mathcal{W}: S \ni i} \Phi_i(S, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}_S) \tag{13}$$

Take $i \in N$ and $S \in \mathcal{W}$ such that $S \ni i$.

Point 1. Suppose that Φ satisfies invariance to the voting structure for unanimity situations. By point 2 of Proposition 3, the payoff $\Phi_i(S, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}_S)$ is uniquely determined. Thus, $\Phi(N, v, \mathcal{W})$ is uniquely determined.

Point 2. Assume that Φ satisfies null voting power and weak modularity. By point 1 of Proposition 3, the payoff $\Phi_i(S, \Delta_{v^{\mathcal{W}}}(S) u_S, \mathcal{W}_S)$ is uniquely determined. Thus, $\Phi(N, v, \mathcal{W})$ is uniquely determined.

Point 3. Similar to point 2.

(*Existence*) The allocation rules Φ^{ED}, Φ^{Sh} and Φ^{Bz} belong to the family of Harsanyi power solutions. By Theorem 1, they satisfy efficiency, additivity, weak null agent out and equality.

Strong equal treatment of equal agents. Let $(N, v, \mathcal{W}) \in GV$ such that there exists a pair of (distinct) equal agents $\{i, j\} \subseteq N$ in (N, v) and $(N, \mathcal{W})$. Take $S \not\ni i, j$. If $S \in \mathcal{W}$, then $S \cup i$ and $S \cup j$ belong to $\mathcal{W}$ and so

$$v^{\mathcal{W}}(S \cup i) = v(S \cup i) = v(S \cup j) = v^{\mathcal{W}}(S \cup j).$$

Next, assume that $S \in 2^N \setminus \mathcal{W}$. As i and j are equal agents in $\mathcal{W}$, either $S \cup i$ and $S \cup j$ belong to $\mathcal{W}$ or $S \cup i$ and $S \cup j$ belong to $2^N \setminus \mathcal{W}$. In both cases, $v^{\mathcal{W}}(S \cup i) = v^{\mathcal{W}}(S \cup i)$. It follows, that i and j are equal agents in $v^{\mathcal{W}}$ so that, for any $S \not\ni i, j$, $\Delta_{v^{\mathcal{W}}}(S \cup i) = \Delta_{v^{\mathcal{W}}}(S \cup j)$. It is well-known that the power indices ED, Sh and Bz treat equally equal agents. From this fact, we can easily deduce that, for two equal agents i and j in $(N, \mathcal{W})$ and for each $S \in \mathcal{W}$ containing neither i nor j, we have $\sigma_i(S \cup i, \mathcal{W}_{S \cup i}) = \sigma_j(S \cup j, \mathcal{W}_{S \cup j})$ for $\sigma \in \{ED, Sh, Bz\}$. Therefore, for Φ^σ, where $\sigma \in \{ED, Sh, Bz\}$, we obtain

$$\begin{aligned}
\Phi_i^\sigma(N, v, \mathcal{W}) &= \sum_{S \in \mathcal{W}: S \ni i} \frac{\sigma_i(S, \mathcal{W}_S)}{\sum_{j \in S} \sigma_k(S, \mathcal{W}_S)} \Delta_{v^{\mathcal{W}}}(S) \\
&= \sum_{S \cup i \in \mathcal{W}} \frac{\sigma_i(S \cup i, \mathcal{W}_{S \cup i})}{\sum_{k \in S} \sigma_k(S \cup i, \mathcal{W}_{S \cup i})} \Delta_{v^{\mathcal{W}}}(S \cup i) \\
&= \sum_{S \cup j \in \mathcal{W}} \frac{\sigma_j(S \cup j, \mathcal{W}_{S \cup j})}{\sum_{k \in S} \sigma_k(S \cup j, \mathcal{W}_{S \cup j})} \Delta_{v^{\mathcal{W}}}(S \cup j) \\
&= \Phi_j^\sigma(N, v, \mathcal{W}).
\end{aligned}$$

Invariance to the voting structure for unanimity situations. This follows directly from the definition of the Myerson value Φ^{ED}.

Null voting power. If $i \in N$ is null in $(N, \mathcal{W})$, then, i remains a null agent in each voting game $(S, \mathcal{W}_S) \in V$ such that $S \in \mathcal{W}$ and $S \ni i$. If $\sigma \in \{Bz, Sh\}$, then $\sigma_i(S, \mathcal{W}_S) = 0$ by definition of Bz and Sh, and so $\Phi_i^{Bz}(N, v, \mathcal{W}) = \Phi_i^{Sh}(N, v, \mathcal{W}) = 0$.

Weak modularity For every $(N, au_N, \mathcal{W})$ and $(N, v, \mathcal{W}')$ in GV. By definition of the $\mathcal{W}$-restricted TU-game and by efficiency of the Shapley-Shubik power index, we obtain $(au_N)^{\mathcal{W}} = au_N$ and $\sum_{i\in N} Sh_i(N, \mathcal{W}) = 1$. Therefore,

$$\forall i \in N, \quad \Phi_i^{Sh}(N, au_N, \mathcal{W}) = aSh_i(N, \mathcal{W}).$$

From Dubey (1975), the Shapley-Shubik power index satisfies modularity on V, from which we deduce that Φ_i^{Sh} satisfies weak modularity.
Weak σ-modularity The proof is similar to the previous one, except that weak σ-modularity takes into account that the Banzhaf power index is not efficient on V. Then, it suffices to use the fact that the Banzhaf power index satisfies modularity (Dubey and Shapley, 1979).

This completes the proof of Theorem 2. ■

5. Extension to GUS

Another advantage of Theorem 1 over Theorem 4 and Theorem 6 in Algaba et al. (2015) is that it is still valid on the full domain GUS, provided that the efficiency axiom is slightly adapted. Indeed, on GUS, the grand coalition is not necessarily feasible so that a Harsanyi power solution might not satisfy efficiency.

Restricted efficiency For each $(N, v, \mathcal{F}) \in GUS$, it holds that

$$\sum_{i\in N} \Phi_i(N, v, \mathcal{F}) = v^{\mathcal{F}}(N).$$

Note that restricted efficiency is a weaker requirement that the combination of component efficiency and component-dummy: if an allocation rule satisfies component efficiency and component-dummy, then it satisfies restricted efficiency, but the converse is not true.

All other axioms in the statement of Theorem 1 are extended to the domain GUS in a natural way. In particular, weak null agent out and σ-unanimity can be written as follows.

Weak null agent out For each $(N, v, \mathcal{F}) \in GUS$ such that $v \in G^N_{\mathcal{F}}$ and each null agent $i \in N$ in (N, v), it holds that

$$\forall j \in N \setminus i, \quad \Phi_j(N, v, \mathcal{F}) = \Phi_j(N \setminus i, v_{N\setminus i}, \mathcal{F}_{N\setminus i}).$$

σ-unanimity Let σ be a power measure. For each $(N, au_N, \mathcal{F}) \in GUS$, such that $N \in \mathcal{F}$ and $a \in \mathbb{R}$, there exists $b \in \mathbb{R}$ such that

$$\Phi(N, au_N, \mathcal{F}) = b\sigma(N, \mathcal{F}).$$

Theorem 3 *Let σ be a power index. The Harsanyi power solution Φ^σ on GUS is the unique allocation rule that satisfies restricted efficiency, additivity, weak null agent out, equality and σ-unanimity.*

Before turning to the proof of this result, we verify that an analog of Proposition 1 and Proposition 2 hold when efficiency is replaced by restricted efficiency.

Proposition 4 *Let Φ be an allocation rule on GUS satisfying restricted efficiency, additivity, and equality, for each $(N, v, \mathcal{F}) \in GUS$, it holds that*

1. $\Phi(N, v, \mathcal{W}) = \Phi(N, v^{\mathcal{F}}, \mathcal{F})$.
2. *If, moreover, Φ satisfies weak null agent out, then*

$$\forall i \in N, \quad \Phi_i(N, v, \mathcal{F}) = \sum_{S \in \mathcal{F}: S \ni i} \Phi_i(S, \Delta_{v^{\mathcal{F}}}(S) u_S, \mathcal{F}_S).$$

Proof. POINT 1. consider Φ satisfying restricted efficiency, additivity, and equality, and $(N, v, \mathcal{W}) \in GUS$. We have $(N, v, \mathcal{F}) = (N, v - v^{\mathcal{F}} + v^{\mathcal{F}}, \mathcal{F})$. By additivity,

$$\Phi(N, v - v^{\mathcal{F}} + v^{\mathcal{F}}, \mathcal{F}) = \Phi(N, v - v^{\mathcal{F}}, \mathcal{F}) + \Phi(N, v^{\mathcal{F}}, \mathcal{F}). \tag{14}$$

For each $S \in \mathcal{F}$, $(v - v^{\mathcal{F}})(S) = 0$, which implies that the coalition function $v - v^{\mathcal{F}}$ belongs to $G^N_{\overline{\mathcal{F}}}$. On the one hand, by equality, we have

$$\Phi_i(N, v - v^{\mathcal{F}}, \mathcal{F}) = \Phi_j(N, v - v^{\mathcal{F}}, \mathcal{F}), \quad \forall i, j \in N.$$

On the other hand, by restricted efficiency, we have

$$\sum_{i \in N} \Phi_i(N, v - v^{\mathcal{F}}, \mathcal{F}) = (v - v^{\mathcal{F}})^{\mathcal{F}}(N) = v^{\mathcal{F}}(N) - v^{\mathcal{F}}(N) = 0.$$

Thus,

$$\Phi_i(N, v - v^{\mathcal{F}}, \mathcal{F}) = 0, \quad \forall i \in N.$$

From (14) and $\Phi(N, v - v^{\mathcal{F}}, \mathcal{F}) = (0, \ldots, 0)$, we get

$$\Phi(N, v - v^{\mathcal{W}} + v^{\mathcal{W}}, \mathcal{W}) = \Phi(N, v, \mathcal{W}) = \Phi(N, v^{\mathcal{W}}, \mathcal{W}),$$

showing point 1.
POINT 2. The proof is similar to the one of Proposition 2, and so it is omitted. In particular, it is important to note that Remark 4 still holds on GUS. ■

Remark 5 Point 2. of Proposition 4 indicates that agents who do not belong to any support receive a null payoff. Thus, the combination of restricted efficiency, additivity, equality and weak null agent out imply component dummy. □

Proof. (of *Theorem 3*) *(Existence)*. Let Φ^σ be a Harsanyi power solution on GUS. Algaba et al. (2015) have already shown that Φ^σ satisfies additivity. It is easy to verify Φ^σ continues to satisfy equality and σ-unanimity on GUS. It remains to verify that Φ^σ satisfies restricted efficiency and weak null agent out on GUS.
Restricted efficiency. If $(N, v, \mathcal{F}) \in GUS$. Then,

$$\begin{aligned}
\sum_{i\in N}\Phi_i^{\sigma}(N,v,\mathcal{F}) &= \sum_{i\in N}\sum_{S\in\mathcal{F}:S\ni i}\frac{\sigma_i(S,\mathcal{F}_S)}{\sum_{j\in S}\sigma_j(S,\mathcal{F}_S)}\,\Delta_{v^{\mathcal{F}}}(S)\\
&= \sum_{S\in\mathcal{F}}\sum_{i\in S}\frac{\sigma_i(S,\mathcal{F}_S)}{\sum_{j\in S}\sigma_j(S,\mathcal{F}_S)}\,\Delta_{v^{\mathcal{F}}}(S)\\
&= \sum_{S\in\mathcal{F}}\Delta_{v^{\mathcal{F}}}(S)\\
&= v^{\mathcal{F}}(N),
\end{aligned}$$

showing that Φ^{σ} satisfied restricted efficiency.
Weak null agent out. Consider $(N,v,\mathcal{F})\in GUS$ such that $v\in G^N_{\mathcal{F}}$ and assume that $i\in N$ is null in (N,v). By definition of Φ^{σ}, we have

$$\forall j\in N\setminus i,\quad \Phi_j^{\sigma}(N,v,\mathcal{F})=\sum_{S\in\mathcal{F}:S\ni j}\frac{\sigma_j(S,\mathcal{F}_S)}{\sum_{k\in S}\sigma_k(S,\mathcal{F}_S)}\Delta_{v^{\mathcal{F}}}(S).$$

Since $v\in G^N_{\mathcal{F}}$, $v=v^{\mathcal{F}}$. Therefore, for each nonempty $S\subseteq N$, $\Delta_v(S)=\Delta_{v^{\mathcal{F}}}(S)$. In particular, by point 1 of Remark 1, for each $S\ni i$, $\Delta_v(S)=0$. By point 2 of Remark 1, we also have, for each $S\subseteq N\setminus i$, $\Delta_v(S)=\Delta_{v_{N\setminus i}}(S)$. Note also that for such a coalition S, we have

$$\forall k\in S,\quad \sigma_k(S,\mathcal{F}_S)=\sigma_k(S,(\mathcal{F}_{N\setminus i})_S).$$

It follows that

$$\begin{aligned}
\forall j\in N\setminus i,\quad \Phi_j^{\sigma}(N,v,\mathcal{F}) &= \sum_{S\in\mathcal{F}_{N\setminus i}:S\ni j}\frac{\sigma_j(S,(\mathcal{F}_{N\setminus i})_S)}{\sum_{k\in S}\sigma_k(S,(\mathcal{F}_{N\setminus i})_S)}\,\Delta_{v_{N\setminus i}}(S)\\
&= \Phi_j^{\sigma}(N\setminus i,v_{N\setminus i},\mathcal{F}_{N\setminus i}).
\end{aligned}$$

showing that Φ^{σ} satisfies weak null agent out.
(Uniqueness). Let Φ be an allocation rule satisfying restricted efficiency, additivity, weak null agent out, equality and σ-unanimity. By point 2 of Proposition 3, we have

$$\forall i\in N,\quad \Phi_i(N,v,\mathcal{F})=\Phi_i(N,v^{\mathcal{F}},\mathcal{F})=\sum_{S\in\mathcal{F}:S\ni i}\Phi_i(S,\Delta_{v^{\mathcal{F}}}(S)u_S,\mathcal{F}_S).$$

Then, it suffices to apply σ-unanimity on $(S,\Delta_{v^{\mathcal{F}}}(S)u_S,\mathcal{F}_S)$, $S\in\mathcal{F}$ in a similar way as in the proof of Theorem 1. Notice that, for each $S\in\mathcal{F}$, we have $S\in\mathcal{F}_S$, which ensures that restricted σ-unanimity is sufficient to conclude.

This completes the proof of Theorem 3. ■

It is also possible to extend the characterization of the Myerson value in Theorem 2 to the domain GUS. This can be done by keeping restricted efficiency, additivity, weak null agent out, equality and strong equal treatment of equal agents, and by the following minor adaptation of invariance to the voting structure for unanimity situations: the same principle is applied but only when N is feasible.

Invariance to the voting structure for unanimity situations (for GUS) For every $(N, au_N, \mathcal{W})$ and $(N, au_N, \mathcal{W}')$ in GUS, such that $N \in \mathcal{W} \cap \mathcal{W}'$, it holds that

$$\Phi(N, au_N, \mathcal{W}) = \Phi(N, au_N, \mathcal{W}').$$

Theorem 4 *The Myerson value Φ^{ED} on GUS is the unique allocation rule that satisfies restricted efficiency, additivity, weak null agent out, equality and invariance to the voting structure for unanimity situations.*

On the contrary, we cannot extend the characterization of the Harsanyi-Shapley-Shubik solution and the Harsanyi-Banzhaf solution for GUS in a simple way. The reason is that the principle of modularity is not well-defined on GUS. Indeed, if $(N, \mathcal{F})$ and $(N, \mathcal{F}')$ are in US, the set system $(N, \mathcal{F} \cup \mathcal{F}')$ does not necessarily belong to GUS.

6. Concluding remarks

The union stable system model assumes that if two feasible coalitions have common elements, they will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the union of these coalitions. In this article, the system of winning coalitions of a voting game is introduced as a specific subclass of union stable systems. For this special subclass of union stable systems the characteristic of power is contained in the implicit interpretation of the winning coalitions in a voting game. Moreover, the interpretation of power in these systems is quite different from the one given by accessible union stable systems (Algaba et al., 2018) which combines communication and hierarchical properties. Namely, these set systems could be also studied taking into account only the feature of power which is in contrast with the system of feasible coalitions derived from an acyclic directed graph by the conjunctive approach (Gilles et al., 1992) or under precedence constraints (Faigle and Kern, 1992). Under this setting, Harsanyi power solutions when restrictions to cooperation coming from this special subclass of union stable systems are introduced. This class of solutions constitutes a very interesting subclass of the Harsanyi power solutions on union stable systems (Algaba et al., 2015) since it is possible to take into consideration not only the economic influence of each coalition but also its political power established in the voting game. These solutions allocate the Harsanyi dividends of the restricted game proportional to the power values of the agents according to any positive power measure for the given winning coalition system. Such a power measure assigns a non-negative real number to every agent, which measures the power or strength of this agent in the winning coalition system. It is obtained through power indices, and leads to an axiomatic characterization of the class of Harsanyi power solutions when the union stable system is the system of winning coalitions of a voting game by efficiency, additivity, equality, weak null agent out and σ-unanimity. In the property of equality is revealed the strong connection between economic influence and political power. It is also worth mentioning that the weak null agent out evokes the irrelevant agent independence which is introduced for games under precedence constraints to characterize the hierarchical solution (Algaba et al., 2017). The main distinction between the latter two axioms can be found in the structural differences of the system of feasible coalitions of each approach. The axiom of σ–unanimity has also been used to characterize the hierarchical solution. In fact, considering only the unanimity game of the grand coalition, all agents are identical in the game, and therefore, the only difference, in the setting of precedence constraints, is with respect to their position in the digraph, whereas in the case of the set of feasible coalitions arising from the winning coalitions in a voting game, the payoff allocation in the unanimity game is fully determined by the chosen power index. Additionally, special solutions called the Harsanyi Shapley-Shubik solution, the Harsanyi Banzhaf solution and the Myerson value which are obtained when the influence of agents is defined by the Shapley-Shubik index, the Banzhaf index and the Equal division power index, respectively. Comparable axiomatizations highlight the essential differences among these specific Harsanyi power solutions. Finally, it is important to emphasize that the axiomatization given for the class of Harsanyi power solutions on the subclass of winning coalitions can be extended to the whole class of union stable systems with a light modification of the efficiency axiom. As a result, a new characterization of the Myerson value on the whole class of union stable systems is provided when considering the Equal division power measure.

Further research will include more studies on the structure given by the feasible coalitions systems coming from the winning coalitions of a voting game as well as axiomatizations of solutions introduced for union stable systems in this framework. The analysis of how the additional feature of power in the structure influences the properties of the solutions is of interest in the literature.

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