INTUITIONISTIC LOGIC WITH A GALOIS CONNECTION HAS THE FINITE MODEL PROPERTY*

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ABSTRACT. We show that the intuitionistic propositional logic with a Galois connection (IntGC), introduced by the authors, has the finite model property.

1. Introduction

In [1], we introduced the intuitionistic propositional logic with a Galois connection (IntGC). In addition to the intuitionistic logic axioms and inference rule of modus ponens, IntGC contains just two rules of inference mimicking the condition defining Galois connections. A *Galois connection* between partially ordered sets P and Q consists of two maps $f\colon P \to Q$ and $g\colon Q \to P$ such that for all $a \in P$ and $b \in Q$, we have $f(a) \leq b$ if and only if $a \leq g(b)$. Note that in the literature can be found two ways to define Galois connections – the one adopted here, in which the maps are order-preserving, and the other, in which they are reversing the order.

We proved in [1] that IntGC is complete with respect to both Kripke style and algebraic semantics. Our intention was also to show that IntGC has the *finite model property* (FMP), that is, for every formula which is not provable, there exists a finite counter Kripke model. Together with the other results presented in the paper, this would imply that the following assertions are equivalent for every IntGC-formula A:

(i) A is provable.
(ii) A is valid in any finite distributive lattice with an additive and normal operator f;
(iii) A is valid in any finite distributive lattice with a multiplicative and co-normal operator g;
(iv) A is valid in any finite Kripke model for IntGC.

Unfortunately, our proof of FMP presented in [1] is incomplete and has some faults. For instance, we did not show that the frame on which the filtration is defined really forms a required Kripke frame. Therefore, here we present a more complete proof based on improved filtration model.

The paper is organised as follows. In Section 2, we recall the syntax, Kripke semantics and Kripke completeness of IntGC. Section 3 is devoted to proving the finite model property of IntGC.

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2. Logic IntGC

The language $\mathcal{L}$ of IntGC is constructed from a countable set of propositional variables P and the connectives $\neg$, $\to$, $\vee$, $\wedge$, $\blacktriangle$, $\triangledown$. The constant *true* is defined by $\top := p \to p$ for some fixed propositional variable $p \in P$, and the constant *false* is defined by $\bot := \neg\top$.

The logic IntGC is the smallest logic in $\mathcal{L}$ that contains the intuitionistic propositional logic Int, and is closed under the rules of *substitution*, *modus ponens*, and the rules:

(GC1) If $A \to \triangledown B$ is provable, then $\blacktriangle A \to B$ is provable.
(GC2) If $\blacktriangle A \to B$ if provable, then $A \to \triangledown B$ is provable.

It is known that the following rules are admissible in IntGC:

(r1) If A is provable, then $\triangledown A$ is provable.
(r2) If $A \to B$ is provable, then $\triangledown A \to \triangledown B$ and $\blacktriangle A \to \blacktriangle B$ are provable.

In addition, the following formulas are provable:

(f1) $A \to \triangledown\blacktriangle A$ and $\blacktriangle\triangledown A \to A$.
(f2) $\blacktriangle A \leftrightarrow \blacktriangle\triangledown\blacktriangle A$ and $\triangledown A \leftrightarrow \triangledown\blacktriangle\triangledown A$.
(f3) $\triangledown\top$ and $\neg\blacktriangle\bot$.
(f4) $\triangledown(A \wedge B) \leftrightarrow \triangledown A \wedge \triangledown B$ and $\blacktriangle(A \vee B) \leftrightarrow \blacktriangle A \vee \blacktriangle B$.
(f5) $\triangledown(A \to B) \to (\triangledown A \to \triangledown B)$.

A structure $\mathcal{F} = (X, \leq, R)$ is called a *Kripke frame* of IntGC, if X is a non-empty set, $\leq$ is a preorder on X, and R is a relation on X such that

$$(\star) \qquad (\geq \circ R \circ \geq) \subseteq R.$$

Let v be a function $v \colon P \to \wp(X)$ assigning to each propositional variable p a subset $v(p)$ of X. Such functions are called *valuations* and the pair $\mathcal{M} = (\mathcal{F}, v)$ is called an IntGC-*model*. For any $x \in X$ and $A \in \Phi$, we define a *satisfiability relation* in $\mathcal{M}$ inductively by the following way:

$$\begin{aligned}
x \models p &\iff x \in v(p),\\
x \models A \wedge B &\iff x \models A \text{ and } x \models A,\\
x \models A \vee B &\iff x \models A \text{ or } x \models A,\\
x \models A \to B &\iff \text{ for all } y \geq x,\ y \models A \text{ implies } y \models B,\\
x \models \neg A &\iff \text{ for no } y \geq x \text{ does } y \models A,\\
x \models \blacktriangle A &\iff \text{ exists } y \text{ such that } x\,R\,y \text{ and } y \models A, \text{ and}\\
x \models \triangledown A &\iff \text{ for all } y, y\,R\,x \text{ implies } y \models A.
\end{aligned}$$

Let $x \leq y$. If $x \models \blacktriangle A$, there exists z such that $x\,R\,z$ and $z \models A$. Now $y \geq x$, $x\,R\,z$, and $z \geq z$ imply $y\,R\,z$ by $(\star)$. Thus, $y \models \blacktriangle A$. Similarly, if $y \not\models \triangledown A$, then there exists z such that $z\,R\,y$ and $z \not\models A$. Now $z \geq z$, $z\,R\,y$, and $y \geq x$ imply $z\,R\,x$. This means $x \not\models \triangledown A$. Hence, the frame is *persistent*.

An IntGC-formula A is *valid in a Kripke model* $\mathcal{M}$, if $x \models A$ for all $x \in X$. The formula is *valid in a Kripke frame* $\mathcal{F}$, if A is valid in every model based on $\mathcal{F}$. The formula A is *Kripke valid* if A is valid in every frame.

We proved in [1] that every formula is Kripke valid if and only if it is provable.

3. IntGC has FMP

Let A be a formula that is not provable. Then, there exists a Kripke model $\mathcal{M} = (X, \leq, R)$ such that A is not valid in $\mathcal{M}$. We construct a counter model for A on a finite frame.

Let $\mathrm{Sub}(A)$ be the set of subformulas of A. We define the set

$$\Gamma = \mathrm{Sub}(A) \cup \{\triangledown\blacktriangle B \mid \blacktriangle B \in \mathrm{Sub}(A)\} \cup \{\blacktriangle\triangledown B \mid \triangledown B \in \mathrm{Sub}(A)\}.$$

From this set, we can now define the set

$$\begin{aligned}\Sigma = \mathrm{Sub}(A) &\cup \{(\triangledown\blacktriangle)^n \triangledown B \mid n \geq 0 \text{ and } \triangledown B \in \Gamma\} \\ &\cup \{\blacktriangle(\triangledown\blacktriangle)^n \triangledown B \mid n \geq 0 \text{ and } \triangledown B \in \Gamma\} \\ &\cup \{(\blacktriangle\triangledown)^n \blacktriangle B \mid n \geq 0 \text{ and } \blacktriangle B \in \Gamma\} \\ &\cup \{\triangledown(\blacktriangle\triangledown)^n \blacktriangle B \mid n \geq 0 \text{ and } \blacktriangle B \in \Gamma\}.\end{aligned}$$

Obviously, $\mathrm{Sub}(A) \subseteq \Gamma \subseteq \Sigma$.

Lemma 3.1. (a) *If $\triangledown B \in \Sigma$, then $\blacktriangle\triangledown B \in \Sigma$.*
(b) *If $\blacktriangle B \in \Sigma$, then $\triangledown\blacktriangle B \in \Sigma$.*

Proof. (a) Suppose that $\triangledown B \in \Sigma$. If $\triangledown B$ is of the form $(\triangledown\blacktriangle)^n \triangledown C$ for some $n \geq 0$, where $\triangledown C \in \Gamma$, then $\blacktriangle\triangledown B = \blacktriangle(\triangledown\blacktriangle)^n \triangledown C$ belongs to Σ by definition. If $\triangledown B$ has the form of $\triangledown(\blacktriangle\triangledown)^m \blacktriangle C$ for some $m \geq 0$ where $\blacktriangle C \in \Gamma$, then by the definition, $\blacktriangle\triangledown B = \blacktriangle\triangledown(\blacktriangle\triangledown)^m \blacktriangle C = (\blacktriangle\triangledown)^{m+1} \blacktriangle C$ is in Σ.

Assertion (b) can be proved analogously. □

A set of IntGC-formulas Σ is said to be *closed under subformulas* if $B \in \Sigma$ and $C \in \mathrm{Sub}(B)$ imply $C \in \Sigma$.

Lemma 3.2. *The set Σ is closed under subformulas.*

Proof. Let $B \in \Sigma$. If B is of the form $C \vee D$, $C \wedge D$, $C \to D$, or $\neg C$, then B must be in $\mathrm{Sub}(A)$ by the definition of Σ. Thus, $C, D \in \mathrm{Sub}(A) \subseteq \Sigma$.

If $\blacktriangle B \in \Sigma$ is of the form $\blacktriangle(\triangledown\blacktriangle)^n \triangledown C$ for some $\triangledown C \in \Gamma$ and $n \geq 0$, then $B = (\triangledown\blacktriangle)^n \triangledown C \in \Sigma$.

If $\blacktriangle B \in \Sigma$ has the form $(\blacktriangle\triangledown)^n \blacktriangle C$ for some $n \geq 0$ and $\blacktriangle C \in \Gamma$, then $\blacktriangle B = \blacktriangle(\triangledown\blacktriangle)^n \triangledown D$, since $\blacktriangle C \in \Gamma$ means that $C = \triangledown D \in \mathrm{Sub}(A)$. Then, $B = (\triangledown\blacktriangle)^n \triangledown D \in \Sigma$.

The remaining two cases are proved analogously. □

We now define for every formula $B \in \Sigma$, a unique formula $B^* \in \Gamma$ as follows:

(i) If $B \in \mathrm{Sub}(A)$ and B is not of the form $\triangledown C$ nor $\blacktriangle C$, then $B^* = B$.
(ii) If B is of the form $(\triangledown\blacktriangle)^n \triangledown C$, where $\triangledown C \in \Gamma$, then $B^* = \triangledown C$.
(iii) If B is of the form $\blacktriangle(\triangledown\blacktriangle)^n \triangledown C$, where $\triangledown C = \triangledown\blacktriangle D \in \Gamma$ for some $\blacktriangle D \in \mathrm{Sub}(A)$, then $B^* = \blacktriangle D$.
(iv) If B is of the form $(\blacktriangle\triangledown)^n \blacktriangle C$, where $\blacktriangle C \in \Gamma$, then $B^* = \blacktriangle C$.
(v) If B is of the form $\triangledown(\blacktriangle\triangledown)^n \blacktriangle C$, where $\blacktriangle C = \blacktriangle\triangledown D \in \Gamma$ for some $\triangledown D \in \mathrm{Sub}(A)$, then $B^* = \triangledown D$.

Related to the above definitions, we can write the following lemma.

Lemma 3.3. *For every $B \in \Sigma$, there exists a unique $B^* \in \Gamma$ such that the formula $B \leftrightarrow B^*$ is provable in* IntGC.

Proof. We consider cases (ii) and (iii) only.

(ii) $B = (\triangledown\blacktriangle)^n \triangledown C = (\triangledown\blacktriangle)^{n-1} \triangledown\blacktriangle\triangledown C \leftrightarrow (\triangledown\blacktriangle)^{n-1} \triangledown C \leftrightarrow \cdots \leftrightarrow \triangledown C = B^*$, because $\triangledown C \leftrightarrow \triangledown\blacktriangle\triangledown C$ by (f2).

(iii) $B = \blacktriangle(\triangledown\blacktriangle)^n \triangledown C = \blacktriangle\triangledown\blacktriangle(\triangledown\blacktriangle)^{n-1} \triangledown C \leftrightarrow \blacktriangle(\triangledown\blacktriangle)^{n-1}\triangledown C \leftrightarrow \cdots \leftrightarrow \blacktriangle\triangledown C = \blacktriangle\triangledown\blacktriangle D = \blacktriangle D = B^*$, since $\blacktriangle A \leftrightarrow \blacktriangle\triangledown\blacktriangle A$ for any A. □

Lemma 3.3 says that since Γ is finite, also the set Σ can be considered "finitary", because it can be divided into classes of provably equivalent formulas such that each class corresponds to one formula of Γ.

Now we define an equivalence $\sim$ on the set X by setting

$$x \sim y \iff (\forall B \in \Sigma)\, x \models B \text{ iff } y \models B.$$

This means that points x and y are equivalent if they satisfy exactly the same formulas of Σ. We denote by $[x]$ the $\sim$-class of x, and $X/{\sim}$ is the set of all $\sim$-classes.

Lemma 3.4. *The quotient set $X/{\sim}$ is finite.*

Proof. Let $x \in X$. For all $y \in X$, $[x] \neq [y]$ means that there exists a formula $B \in \Sigma$ that "separates" x and y, that is, either (i) $x \models B$ and $y \not\models B$, or (ii) $y \models B$ and $x \not\models B$. For instance, in case (i) this means by Lemma 3.3 that $x \models B^*$, $y \not\models B^*$, and $B^* \in \Gamma$. Because the set Γ is finite, only a finite number of classes can be "separated" from $[x]$. Hence, also the quotient set $X/{\sim}$ must be finite. □

We denote $X/{\sim}$ simply by X^f. We define in X^f the relations $\leq^f$ and R^f by setting:

$$[x] \leq^f [y] \iff (\forall B \in \Sigma)\, x \models B \text{ implies } y \models B;$$
$$[x]\, R^f\, [y] \iff (\forall B \in \Sigma)\, \triangledown B \in \Sigma \text{ and } y \models \triangledown B \text{ imply } x \models B.$$

We can now write the following lemma.

Lemma 3.5. (a) *If $x \leq y$, then $[x] \leq^f [y]$.*
(b) *If $x\, R\, y$, then $[x]\, R^f\, [y]$.*

Proof. Claim (a) is obvious, because our Kripke frames are persistent.

(b) Assume $x\, R\, y$, $B \in \Sigma$, and $\triangledown B \in \Sigma$. By Lemma 3.1, also $\blacktriangle\triangledown B \in \Sigma$. If $y \models \triangledown B$, then $x\, R\, y$ gives $x \models \blacktriangle\triangledown B$. We have $x \models B$, because $\blacktriangle\triangledown B \to B$ is a valid formula. Hence, $|x|\, R^f\, |y|$. □

Lemma 3.6. *The structure $\mathcal{F}^f = (X^f, \leq^f, R^f)$ is a Kripke frame.*

Proof. It is clear that $\leq^f$ is a preorder. Therefore, it is enough to show that

$$(\geq^f \circ R^f \circ \geq^f) \subseteq R^f.$$

Suppose that $[x] \geq^f [y]$, $[y]\, R^f [z]$, and $[z] \geq^f [w]$. For all $B \in \Sigma$, if $\triangledown B \in \Sigma$ and $w \models \triangledown B$, then $z \models \triangledown B$ because $[z] \geq^f [w]$. Now $y \models B$ by $[y]\, R^f [z]$. Finally, $[x] \geq^f [y]$ implies $x \models B$. Thus, $[x]\, R^f [w]$, □

Our next lemma gives another condition for R^f.

Lemma 3.7. $[x]\, R^f [y] \iff (\forall B \in \Sigma)\, \blacktriangle B \in \Sigma$ *and* $y \models B$ *imply* $x \models \blacktriangle B$.

Proof. Let $B \in \Sigma$. Assume $[x]\, R^f [y]$, $\blacktriangle B \in \Sigma$ and $y \models B$. Since $B \to \triangledown\blacktriangle B$ is a provable formula, we have $y \models B \to \triangledown\blacktriangle B$ and so $y \models \triangledown\blacktriangle B$. Because $\blacktriangle B \in \Sigma$, Lemma 3.1 gives $\triangledown\blacktriangle \in \Sigma$. Since $[x]\, R^f [y]$, we get $x \models \blacktriangle B$.

Conversely, assume that the right-side of the condition holds. If $\triangledown B \in \Sigma$ and $y \models \triangledown B$, then by Lemma 3.1, $\blacktriangle\triangledown B \in \Sigma$, from which we get $x \models \blacktriangle\triangledown B$ by the assumption. Because $\blacktriangle\triangledown B \to B$ is a provable formula, we have $x \models B$. Thus, $[x]\, R^f [y]$. □

We define the valuation v^f in such a way that for all proposition variables $p \in \Sigma$:

$$v^f(p) = \{[x] \mid x \models p\}.$$

Then, $\mathcal{M}^f = (X^f, \leq^f, R^f, v^f)$ is called *filtration of* $\mathcal{M}$ *through* Σ.

Lemma 3.8. *For any* $B \in \Sigma$ *and* $x \in X$, $x \models B$ *iff* $[x] \models B$.

Proof. By induction on B. This can be done, because Σ is closed under subformulas. The base case follows immediately from the definition of v^f, and with respect to $\vee$ and $\wedge$ the proof is obvious.

(i) Let B of the form $\neg C \in \Gamma$. Assume $[x] \models \neg C$. If $x \not\models \neg C$, then there exists $y \geq x$ such that $y \models C$. Since Γ is closed under subformulas, also $C \in \Gamma$ and $[y] \models C$ by the induction hypothesis. Because $y \geq x$, we have $[y] \geq^f [x]$ by Lemma 3.5. This gives that $[x] \not\models \neg C$, a contradiction. So, $x \models \neg C$.

Conversely, suppose that $x \models \neg C$. Because $C \in \Gamma$, then by the definition, $[y] \geq^f [x]$ implies $y \models \neg C$ and $y \not\models C$. By the induction hypothesis, we have that $[y] \geq^f [x]$ implies $[y] \not\models C$, that is, $[x] \models \neg C$.

(ii) Let B of the form $C \to D \in \Gamma$. Assume $x \models C \to D$ and $[x] \not\models C \to D$. Then, there exists $[y] \geq^f [x]$ such that $[y] \models C$, but $[y] \not\models D$. By induction hypothesis, $y \models C$ and $y \not\models D$. Therefore, $y \not\models C \to D$, which is impossible because $[y] \geq^f [x]$. Thus, $[x] \models C \to D$.

On the other hand, if $[x] \models C \to D$, then for all $[y] \geq^f [x]$, $[y] \models C$ implies $[y] \models D$. If $x \not\models C \to D$, there exists $y \geq x$ such that $y \models C$ and $y \not\models D$. Now $y \geq x$ gives $[y] \geq^f [x]$, and $[y] \models C$ and $[y] \not\models D$ by the induction hypothesis. But this is impossible. So, $x \models C \to D$.

(iii) Let B be of the form $\triangledown C \in \Gamma$. Assume that $x \models \triangledown C$. If $[y]\, R^f [x]$, then $y \models C$, and $[y] \models C$ follows from the induction hypothesis. Hence, $[x] \models \triangledown C$.

Conversely, assume that $[x] \models \triangledown C$ and $y\, R\, x$. Then, $[y]\, R^f [x]$ by Lemma 3.5, which gives $[y] \models C$. We obtain $y \models C$ by the induction hypothesis, and so $x \models \triangledown C$.

(iv) Let B be of the form $\blacktriangle C \in \Gamma$. If $x \models \blacktriangle C$, then there exists y such that $x\, Ry$ and $y \models C$. By the induction hypothesis, $[y] \models C$. Since $x\, Ry$, we have $[x]R^f[y]$ and $[x] \models \blacktriangle C$.

On the other hand, if $[x] \models \blacktriangle C$, then there exists y such that $[x]\, R^f [y]$ and $[y] \models C$. This implies $y \models C$ by the induction hypothesis. By Lemma 3.7, we get $x \models \blacktriangle C$. □

Finally, we may write the following proposition.

Proposition 3.9. IntGC *has the finite model property and is decidable.*

Proof. Suppose that a formula A is not provable. Then, there exists a model $\mathcal{M} = (X, \leq, R)$ such that A is not valid in $\mathcal{M}$. This means that there exists $x \in X$ such that $x \not\models A$. We may define the set Σ and the filtration of $\mathcal{M}$ through Σ as above. Because $A \in \Sigma$, then $[x] \not\models A$ by Lemma 3.8, and hence A is not valid in the finite model $\mathcal{M}^f$.

In addition, it is well known that if a logic is finitely axiomatised with the finite model property, then the logic is decidable. □

References

[1] Wojciech Dzik, Jouni Järvinen, and Michiro Kondo, *Intuitionistic propositional logic with Galois connections*, Logic Journal of the IGPL **18** (2010), 837–858.

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