# A conditional, a fuzzy and a probabilistic interpretation of self-organising maps

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**Abstract.** In this paper we establish a link between fuzzy and preferential semantics for description logics and Self-Organising Maps, which have been proposed as possible candidates to explain the psychological mechanisms underlying category generalisation. In particular, we show that the input/output behavior of a Self-Organising Map after training can be described by a fuzzy description logic interpretation as well as by a preferential interpretation, based on a concept-wise multipreference semantics, which takes into account preferences with respect to different concepts and has been recently proposed for ranked and for weighted defeasible description logics. Properties of the network can be proven by model checking on the fuzzy or on the preferential interpretation. Starting from the fuzzy interpretation, we also provide a probabilistic account for this neural network model.

#### 1 Introduction

Conditional logics have their roots in philosophical logic. They have been studied first by Lewis [70,79] to formalize hypothetical and counterfactual reasoning (if A were the case then B) that cannot be captured by classical logic with its material implication. From the 80's they have been considered in computer science and artificial intelligence and they have provided an axiomatic foundation of non-monotonic and common sense reasoning [31,75,81,65,82,68,6].

KLM preferential approaches [65,68,69] to common sense reasoning have been more recently extended to description logics, to deal with inheritance with exceptions in ontologies, allowing for non-strict forms of inclusions, called *typicality or defeasible inclusions*, corresponding to conditional implications, based on different preferential semantics [45,17,46] and closure constructions [20,19,48,47,83,23,16,42].

Fuzzy description logics have been widely studied in the literature for representing vagueness in DLs [88,87,74,11,8], based on the idea that concepts and roles can be interpreted as fuzzy sets and fuzzy binary relations.

In this paper we aim at developing a logical interpretation of self-organising maps (SOMs) [62], which have been proposed as possible candidates to explain the psychological mechanisms underlying category generalisation. They are psychologically and biologically plausible neural network models that can also learn after limited exposure to positive category examples, without any need of contrastive information. We consider a "concept-wise" multi-preferential semantics [40], which has been recently introduced for a lightweight description logic of the  $\mathcal{EL}^{\perp}$  family [3] and takes into account preferences with respect to different concepts. We show that both the fuzzy semantics and the multi-preferential semantics can be used to provide a logical interpretation of SOMs, and allow for the verification of properties of the trained SOM by model checking.

Both interpretations are based on the idea of associating each learned category to a concept in the language of the simple description logic  $\mathcal{LC}$ , which does not allow for roles and role restrictions, but allows for the boolean combination of concepts (as we will see, restricting to a language without roles is enough for the purpose of developing a logical interpretation of SOMs). We show that the learning process in self-organising maps produces, as a result, either a *fuzzy model*, in which each concept (or learned category) is interpreted as a fuzzy set over the domain of input stimuli, or a *multipreference model* by associating a preference relation to each concept (each learned category). Both models can be exploited to extract or validate knowledge from the empirical data used in the learning process and the evaluation of such knowledge can be done by model checking, using the information recorded in the SOM. The verification of logical properties of a neural network can be useful for post-hoc explanation, in view of a trustworthy, reliable and explainable AI [1,52,2].

Concerning the preferential semantics, based on the assumption that the abstraction process in the SOM is able to identify the most typical exemplars for a given category, in the semantic representation of a category, we identify some specific stimuli as the *typical exemplars* of the category, and define a preference relation among the exemplars of a category. To this purpose, we use the notion of distance of an input stimulus from a category representation. We then exploit a notion of relative distance, introduced by Gliozzi and Plunkett in their similarity-based account of category generalization

based on self-organising maps [51], for developing another semantic interpretation of SOMs based on *fuzzy DL interpretations*. This is done by interpreting each category (concept) as a function mapping each input stimulus to a value in [0,1], based on the map's generalization degree of category membership to the stimulus used in [51]. Our fuzzy interpretation of SOMs sticks to this specific use for category generalization.

The multipreference model of the SOM will be defined as a multipreference  $\mathcal{LC}$  interpretation, while the fuzzy model of the SOM will be defined as a fuzzy  $\mathcal{LC}$  interpretation. In both cases, model checking can be used for the verification of inclusions (either defeasible inclusions or fuzzy inclusion axioms) over the respective models of the SOM. Starting from the fuzzy interpretation of the SOM we also provide a probabilistic interpretation of this neural network model based on Zadeh's probability of fuzzy events [94]. The paper extends the results in [43] which only discusses a preferential interpretation of SOMs.

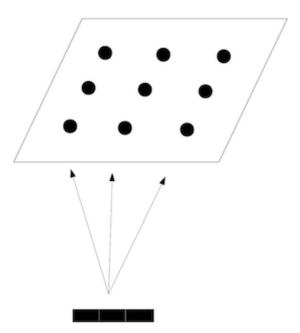
The paper is organized as follows. Section 2 shortly describes self-organising maps. Section 3 and 4 contain preliminaries about the description logic  $\mathcal{LC}$  and fuzzy  $\mathcal{LC}$  interpretations. A concept-wise multipreference semantics for  $\mathcal{LC}$  with typicality inclusions is described in Section 5. Section 6 and Section 7, respectively, relate self-organising maps with multipreference and a fuzzy DL interpretations, while Section 8 provides a probabilistic interpretation of SOMs. Section 9 hints at a possible relation between the process of updating the category representation in the SOM with change operators in knowledge representation. Section 10 concludes the paper and discusses related work.

# 2 Self-organising maps

Self-organising maps (SOMs, introduced by Kohonen [62]) are particularly plausible neural network models that learn in a human-like manner. In particular: SOMs learn to organize stimuli into categories in an *unsupervised* way, without the need of a teacher providing a feedback; they can learn with just a few positive stimuli, without the need for negative examples or contrastive information; they reflect basic constraints of a plausible brain implementation in different areas of the cortex [77], and are therefore biologically plausible models of category formation; they have proven to be capable of explaining experimental results.

In this section we shortly describe the architecture of SOMs and report Gliozzi and Plunkett's similarity-based account of category generalization based on SOMs [51]. In brief, in [51] the authors judge a new stimulus as belonging to a category by comparing the distance of the stimulus from the category representation to the precision of the category representation.

SOMs consist of a set of neurons, or units, spatially organized in a grid [62], as in Figure 1. Each map unit u is associated with a weight vector  $w_u$  of the same dimensionality as the input vectors. At the beginning of training, all weight vectors are initialized to random values, outside the range of values of the input stimuli. During training, the input elements (a set X) are sequentially presented to all neurons of the map. After each presentation of an input x, the best-matching unit (BMU $_x$ ) is selected: this is a unit i whose weight vector  $w_i$  is closest to the stimulus x (i.e. it minimizes the distance  $||x-w_j||$ ).



**Fig. 1.** An example of SOM. The set of rectangles stands for the input presented to the SOM (in the example the input is three-dimensional). This is presented to *all* neurons of the SOM (these are the neurons in the upper grid) in order to find the BMU.

The weights of the best matching unit and of its surrounding units are updated in order to maximize the chances that the same unit (or its surrounding units) will be selected as the best matching unit for the same stimulus or for similar stimuli on subsequent presentations. In particular, the weight update reduces the distance between the best matching unit's weights (and its surrounding neurons' weights) and the incoming input. Furthermore, it organizes the map topologically so that weights of close-by neurons are updated in a similar direction, and come to react to similar inputs.

The learning process is incremental: after the presentation of each input, the map's representation of the input (and in particular the representation of its best-matching unit) is updated in order to take into account the new incoming stimulus. At the end of the whole process, the SOM has learned to organize the stimuli in a topologically significant way: similar inputs (with respect to Euclidean distance) are mapped to close by areas in the map, whereas inputs which are far apart from each other are mapped to distant areas of the map.

Once the SOM has learned to categorize, in order to assess category generalization, Gliozzi and Plunkett [51] define the map's disposition to consider a new stimulus y as a member of a known category C. This disposition is defined as a function of the *distance* of y from the *map's representation* of C. They take a minimalist notion of what is the map's category representation: this is the ensemble of best-matching units corresponding to known instances of the category. They use  $BMU_C = \{BMU_x \mid x \in X \text{ and } x \in C_i\}$  to refer to the map's representation of category C and define category generalization as depending on two elements:

- the distance of the new stimulus y ( $y \notin X$ ) with respect to the category representation:  $min\|y BMU_C\|$  (in the following, also denoted by  $d(y, C_i)$ )
- compared to the maximal distance from that representation of all known instances of the category

This captured by the following notion of *relative distance* (rd for short) [51]:

$$rd(y,C) = \frac{\min \|y - BMU_C\|}{\max_{x \in C} \|x - BMU_x\|}$$
 (1)

where  $min\|y - BMU_C\|$  is the (minimal) Euclidean distance between y and C's category representation, and  $max_{x \in C}\|x - BMU_x\|$  expresses the *precision* of category representation, and is the (maximal) Euclidean distance between any known member of the category and the category representation.

With this definition, a given Euclidean distance from y to C's category representation will give rise to a higher *relative distance rd* if category representation is precise (and the maximal distance between C and its known examples is low) than if category representation is coarse (and the maximal distance between C and its known examples is high). As a function of the relative distance above, Gliozzi and Plunkett then define the *map's Generalization Degree* of category C membership to a new stimulus y. This is a function of the relative distance of Equation (1). The map's Generalization Degree exponentially decreases with the increase of the relative distance as follows:

Generalization Degree= 
$$e^{-rd(y,C)}$$
 (2)

It was observed that the above notion of relative distance (Equation 1) requires there to be a memory of some of the known instances of the category being used (this is needed to calculate the denominator in the equation). This gives rise to a sort of hybrid model in which category representation and some exemplars coexist. An alternative way of formulating the same notion of relative distance would be to calculate *online* the distance between known category instance currently examined and the representation of the category being formed.

By judging a new stimulus as belonging to a category by comparing the distance of the stimulus from the category representation to the precision of the category representation, Gliozzi and Plunkett demonstrate [51] that the Numerosity and Variability effects of category generalization, described by Griffiths and Tenenbaum [90], and usually explained with Bayesian tools, can be accommodated within a simple and psychologically plausible similarity-based account. In the next sections, we show that the notions of distance and relative distance can be used as a basis for a logical semantics for SOMs.

#### 3 The description logic $\mathcal{LC}$

In this section and in Section 4 we will consider the boolean fragment of the description logic  $\mathcal{ALC}$  [4] as well as its fuzzy extension [74]. As we will see, the boolean fragment does not allow for roles but it is expressive enough for the purpose of providing a logical interpretation of a SOM, in which categories are interpreted as atomic concepts and their combination through union, intersection and complement allows for the formulation of properties to be validated over a SOM model.

Description logics are based on a simple set-theoretic semantics and, in the following, we will only focus on the semantics of boolean concepts. In the two-valued case, concepts are interpreted as sets over a domain  $\Delta$  of elements, that is, each concept (e.g., Elephant) is interpreted as a set of elements (the set of all elephants in  $\Delta$ ). In the fuzzy case, concepts are interpreted as fuzzy sets. For the two-valued case, the interpretation of union, intersection and complement is the usual one in set theory, while in the fuzzy case it depends on the underlying fuzzy logic combination functions. Individual names (dumbo, garfield, etc.) represent specific domain elements in  $\Delta$ .

Let  $\mathcal{LC}$  be the fragment of the description logic  $\mathcal{ALC}$  which does not admit roles and universal and existential restrictions, namely, the fragment only containing union, intersection, and complement as concept constructors. While, as mentioned above, we restrict our consideration to the fragment of  $\mathcal{ALC}$  without roles, both the fuzzy and the preferential description logics considered in the following have been first introduced for description logics including roles.

Let  $N_C$  be a set of concept names and  $N_I$  a set of individual names. The set of  $\mathcal{LC}$  concepts (or, simply, concepts) can be defined inductively as follows:

- $A \in N_C$ ,  $\top$  and  $\bot$  are concepts;
- if C and D are concepts, then  $C \sqcap D, C \sqcup D, \neg C$  are concepts.

A knowledge base (KB) K is a pair  $(\mathcal{T}, \mathcal{A})$ , where  $\mathcal{T}$  is a TBox – i.e., a set of *concept inclusions* (or subsumptions)  $C \sqsubseteq D$ , where C, D are concepts – and  $\mathcal{A}$  is an ABox –

i.e., a set of assertions of the form C(a) where C is a concept and a an individual name in  $N_I$ .

As an example, given the concept names Elephant,  $White\_Animal$ ,  $African\_Animal$  and Young, the complex concepts  $Elephant \sqcap White\_Animal$  and  $Elephant \sqcap White\_Animal \sqcup African\_Animal$ ) represent, respectively, the set of white elephants and the set of elephants which are white or live in Africa. For an individual name, dumbo, the assertion ( $Elephant \sqcap White\_Animal$ )(dumbo) means that Dumbo is a white elephant. The concept inclusion  $Young \sqcap White\_Animal \sqsubseteq African\_Animal$  means that all young white elephants live in Africa (the set of young white elephants is a subset of the set of African animals).

An  $\mathcal{LC}$  interpretation is defined as for  $\mathcal{ALC}$  as a pair  $I=\langle \Delta, \cdot^I \rangle$  where:  $\Delta$  is a domain—a set whose elements are denoted by  $x,y,z,\ldots$ —and  $\cdot^I$  is an extension function that maps each concept name  $C \in N_C$  to a set  $C^I \subseteq \Delta$ , and each individual name  $a \in N_I$  to an element  $a^I \in \Delta$ . It is extended to complex concepts as follows:

The notion of satisfiability of a KB in an interpretation and the notion of entailment are defined as follows:

**Definition 1** (Satisfiability and entailment). Given an  $\mathcal{LC}$  interpretation  $I = \langle \Delta, \cdot^I \rangle$ :

- I satisfies an inclusion  $C \sqsubseteq D$  if  $C^I \subseteq D^I$ ;
- I satisfies an assertion C(a) if  $a^I \in C^I$ ;

Given a KB  $K = (\mathcal{T}, \mathcal{A})$ , an interpretation I satisfies  $\mathcal{T}$  (resp.,  $\mathcal{A}$ ) if I satisfies all inclusions in  $\mathcal{T}$  (resp., all assertions in  $\mathcal{A}$ ). I is an  $\mathcal{LC}$  model of K if I satisfies  $\mathcal{T}$  and  $\mathcal{A}$ .

Letting a query F to be either an inclusion  $C \sqsubseteq D$  (where C and D are concepts) or an assertion C(a), F is entailed by K, written  $K \models_{\mathcal{LC}} F$ , if for all  $\mathcal{LC}$  models  $I = \langle \Delta, \cdot^I \rangle$  of K, I satisfies F.

Given a knowledge base K, the *subsumption* problem is the problem of deciding whether an inclusion  $C \sqsubseteq D$  is entailed by K. The *instance checking* problem is the problem of deciding whether an assertion C(a) is entailed by K.

#### 4 Fuzzy $\mathcal{LC}$ interpretations

Fuzzy description logics allow to represent vagueness in DLs [88,87,74,11,8], by interpreting concepts and roles as fuzzy sets. As in Mathematical Fuzzy Logic [25] a formula has a degree of truth in an interpretation, rather than being either true or false, in a fuzzy DL, axioms are associated to a degree of truth (typically in the interval [0,1]). In the

following we shortly recall the semantics of a fuzzy extension of  $\mathcal{LC}$ , as a fragment of fuzzy  $\mathcal{ALC}$ , referring to the survey by Lukasiewicz and Straccia [74]. We limit our consideration to  $\mathcal{LC}$  constructs and only to the few features of a fuzzy DLs which are relevant for defining an interpretation of SOMs and, in particular, we omit considering datatypes.

A fuzzy interpretation for  $\mathcal{LC}$  is a pair  $I=\langle \Delta, \cdot^I \rangle$  where:  $\Delta$  is a non-empty domain and  $\cdot^I$  is fuzzy interpretation function that assigns to each concept name  $A \in N_C$  a function  $A^I: \Delta \to [0,1]$ , and to each individual name  $a \in N_I$  an element  $a^I \in \Delta$ . A domain element  $x \in \Delta$  belongs to the extension of A to some degree in [0,1], i.e.,  $A^I$  is a fuzzy set.

The interpretation function  $\cdot^{I}$  is extended to complex concepts as follows:

$$\begin{aligned} & \top^I(x) = 1, \perp^I(x) = 0, \\ & (\neg C)^I(x) = \Delta \ominus C^I(x), \\ & (C \sqcap D)^I(x) = C^I(x) \otimes D^I(x) \\ & (C \sqcup D)^I(x) = C^I(x) \oplus D^I(x) \end{aligned}$$

(for  $x \in \Delta$ ), and to non-fuzzy axioms (i.e., to strict inclusions and assertions of an  $\mathcal{LC}$  knowledge base) as follows:

$$(C \sqsubseteq D)^I = inf_{x \in \Delta} \ C^I(x) \triangleright D^I(x)$$
  
 $(C(a))^I = C^I(a^I)$ 

where  $\otimes$ ,  $\oplus$ ,  $\triangleright$  and  $\ominus$  are so-called "combination functions, namely, triangular norms (or t-norms), triangular co-norms (or s-norms), implication functions, and negation functions, respectively, which extend the classical Boolean conjunction, disjunction, implication, and negation, respectively, to the many-valued case" [74]. They have been studied for multi-valued logics [53] and for fuzzy DLs. For instance, in both Zadeh and Gödel logics  $a \otimes b = min\{a,b\}$ ,  $a \oplus b = max\{a,b\}$ . In Zadeh logic  $a \triangleright b = max\{1-a,b\}$  and  $\ominus a = 1-a$ . In Gödel logic  $a \triangleright b = 1$  if  $a \le b$  and b otherwise;  $\ominus a = 1$  if a = 0 and b = 0 otherwise. Following [74], we will not commit to a specific fuzzy logic.

A fuzzy  $\mathcal{LC}$  knowledge base K is a pair  $(\mathcal{T}, \mathcal{A})$  where  $\mathcal{T}$  is a fuzzy TBox and  $\mathcal{A}$  a fuzzy ABox. A fuzzy TBox is a set of fuzzy concept inclusions of the form  $C \sqsubseteq D \theta n$ , where  $C \sqsubseteq D$  is an  $\mathcal{LC}$  concept inclusion axiom,  $\theta \in \{\geq, \leq, >, <\}$  and  $n \in [0, 1]$ . A fuzzy ABox  $\mathcal{A}$  is a set of fuzzy assertions of the form  $C(a) \theta n$ , where C is an  $\mathcal{LC}$  concept,  $a \in N_I$ ,  $\theta \in \{\geq, \leq, >, <\}$  and  $n \in [0, 1]$ . Following Bobillo and Straccia [8], we assume that fuzzy interpretations are witnessed, i.e., the inf (as well as the sup) is attained at some point of the involved domain.

In a fuzzy DL interpretation, an assertion like  $White\_Animal(dumbo)$  is interpreted as having a degree in [0,1], rather than having a value true or false. For instance, in a given interpretation I, one might have that dumbo is white with degree 0.6, i.e.,  $White\_Animal^I(dumbo^I) = 0.6$ , and that Dumbo is an elephant with degree 0.9,  $Elephant^I(dumbo^I) = 0.9$ . In Zadeh logic, one would get  $((Elephant \sqcap White\_Animal)(dumbo))^I = (Elephant \sqcap White\_Animal)^I(dumbo^I) = min\{Ele-phant^I(dumbo^I), White\_Animal^I(dumbo^I)\} = min(0.9, 0.6) = 0.6$ , i.e., Dumbo belongs to the set of white elephants with degree 0.6. Hence, a fuzzy assertion  $(Elephant \sqcap White\_Animal)(dumbo) \geq 0.5$  would be satisfied in the interpretation I, according to the definition of satisfiability given below.

The notions of satisfiability of a KB in a fuzzy interpretation and of entailment are defined in the natural way.

**Definition 2** (Satisfiability and entailment for fuzzy knowledge bases). A fuzzy interpretation I satisfies a fuzzy  $\mathcal{LC}$  axiom E (denoted  $I \models E$ ), as follows. For  $\theta \in \{\geq, \leq, >, <\}$ :

- I satisfies a fuzzy  $\mathcal{LC}$  inclusion axiom  $C \sqsubseteq D \theta \ n \ \text{if} \ (C \sqsubseteq D)^I \theta \ n$ ;
- I satisfies a fuzzy  $\mathcal{LC}$  assertion C(a)  $\theta$  n if  $C^{I}(a^{I})\theta$  n.

Given a fuzzy KB  $K = (\mathcal{T}, \mathcal{A})$ , a fuzzy interpretation I satisfies  $\mathcal{T}$  (resp.  $\mathcal{A}$ ) if I satisfies all fuzzy inclusions in  $\mathcal{T}$  (resp. all fuzzy assertions in  $\mathcal{A}$ ). A fuzzy interpretation I is a model of K if I satisfies  $\mathcal{T}$  and  $\mathcal{A}$ . A fuzzy axiom E is entailed by a fuzzy knowledge base K, written  $K \models E$ , if for all models  $I = \langle \mathcal{\Delta}, \cdot^I \rangle$  of K, I satisfies E.

# 5 A concept-wise multipreference semantics for $\mathcal{LC}$ plus typicality

In this section we describe an extension of  $\mathcal{LC}$  with typicality inclusions, defined along the lines of the extension of description logics with typicality [45,47], but under a different semantics with multiple preferences [40]. This concept-wise multi-preference semantics has been originally introduced for the description logic  $\mathcal{EL}_{\perp}^+$  of the  $\mathcal{EL}$  family [3], which is at the basis of OWL2 EL Profile. The semantics is a variant of the multipreferential semantics developed by Gliozzi [50] to define a refinement of the Rational Closure semantics for description logics. In the following we reformulate concept-wise multipreference semantics for  $\mathcal{LC}$ .

In addition to standard concept inclusions  $C \sqsubseteq D$  (called *strict* inclusions in the following), typicality inclusions are allowed having the form  $\mathbf{T}(C) \sqsubseteq D$ , where C and D are  $\mathcal{LC}$  concepts. A typicality inclusion  $\mathbf{T}(C) \sqsubseteq D$  means that "typical C's are D's" or "normally C's are D's" and corresponds to a conditional implication  $C \vdash D$  in Kraus, Lehmann and Magidor's (KLM) preferential approach [65,68]. Such inclusions are defeasible, i.e., admit exceptions, while strict inclusions must be satisfied by all domain elements.

Let  $\mathcal{C}=\{C_1,\ldots,C_k\}$  be a set of (general)  $\mathcal{LC}$  concepts, called distinguished concepts. For each concept  $C_i\in\mathcal{C}$ , we introduce a modular preference relation  $<_{C_i}$  which describes the preference among domain elements with respect to  $C_i$ . Each preference relation  $<_{C_i}$  has the same properties of preference relations in KLM-style ranked interpretations [68], it is a modular and well-founded strict partial order (i.e., an irreflexive and transitive relation). In particular, a preference relation  $<_{C_i}$  is well-founded if, for all  $S\subseteq \Delta$ , if  $S\neq\emptyset$ , then  $min_{<_{C_i}}(S)\neq\emptyset$ ; relation  $<_{C_i}$  is modular if, for all  $x,y,z\in\Delta$ , if  $x<_{C_i}y$  then  $x<_{C_i}z$  or  $z<_{C_i}y$ .

Observe that, as usual, the strict preference relation  $<_{C_i}$  can be defined from a total preorder  $\leq_{C_i}$  by letting  $x <_{C_i} y$  iff  $x \leq_{C_i} y$  and not  $y \leq_{C_i} x$ . An equivalence relation  $\sim_{C_i}$  can be defined as:  $x \sim_{C_i} y$  iff  $x \leq_{C_i} y$  and  $y \leq_{C_i} x$ .

**Definition 3** (Multipreference interpretation). A multipreference interpretation is a tuple  $\mathcal{M} = \langle \Delta, <_{C_1}, \dots, <_{C_k}, \cdot^I \rangle$ , where:

(a)  $\Delta$  is a non-empty domain;

- (b)  $<_{C_i}$  is an irreflexive, transitive, well-founded and modular relation over  $\Delta$ ;
- (d)  $\cdot^{I}$  is an interpretation function, defined as in  $\mathcal{LC}$  interpretations (see Section 3).

Observe that, given a multipreference interpretation, a triple  $\mathcal{M}_{C_i} = \langle \Delta, <_{C_i}, \cdot^I \rangle$ , which can be associated to each concept  $C_i$ , is a ranked interpretation as the ones considered for  $\mathcal{ALC}$  plus typicality in [47]. The preference relation  $<_{C_i}$  allows the set of *prototypical*  $C_i$ -elements to be defined as the set of  $C_i$ -elements which are minimal with respect to  $<_{C_i}$ , i.e., the set  $min_{<_{C_i}}(C_i^I)$ . As a consequence, the multipreference interpretation above is able to single out the typical  $C_i$ -elements, for all distinguished concepts  $C_i \in \mathcal{C}$ .

The multipreference interpretations have been first introduced in [40], to develop a semantics for ranked knowledge bases in a lightweight description logic which is at the basis of OWL2 EL Profile [80], based on an approach inspired by Brewka's framework of basic preference descriptions [13]. In the following we will shortly reformulate for  $\mathcal{LC}$  the notion of ranked knowledge base, and we will just recall the main ideas and results concerning concept-wise multipreference entailment without providing a reformulation for  $\mathcal{LC}$ . In fact, in the following sections, we will not focus on entailment but we will construct a multipreference model of a SOM and use it for validation of strict and preferential properties.

A ranked  $\mathcal{LC}$  knowledge base is a tuple  $\langle \mathcal{T}_{strict}, \mathcal{T}_{C_1}, \dots, \mathcal{T}_{C_k}, \mathcal{A} \rangle$ , where  $\mathcal{T}_{strict}$  is a set of standard concept and role inclusions,  $\mathcal{A}$  is an ABox and, for each  $C_j \in \mathcal{C}$ ,  $\mathcal{T}_{C_j}$  is a ranked TBox containing all the defeasible inclusions,  $\mathbf{T}(C_i) \sqsubseteq D$ , specifying the typical properties of  $C_i$ -elements, with their ranks (non-negative integers). In the following, we will denote  $\mathcal{T}_{C_j}$  as the set  $\{(d_i^j, r_i^j)\}$ , where each  $d_i^j$  is a typicality inclusion of the form  $\mathbf{T}(C_j) \sqsubseteq D_i^j$ , and  $r_i^j$  its rank. The defeasible inclusions with higher ranks are considered to be more plausible (and hence more important) than the ones with lower ranks. Let us consider the following example:

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Example 1. Consider the ranked KB K = \langle T_{strict}, T_{Horse}, T_{Zebra}, T_{Bird}, T_{Penguin}, \mathcal{A} \rangle (with \mathcal{A} = \emptyset), where \mathcal{T}_{strict} contains the strict inclusions:

Horse \sqsubseteq Mammal   Mammal \sqsubseteq Animal   Zebra \sqsubseteq Mammal   Horse \sqcap Zebra \sqsubseteq \bot the ranked TBox \mathcal{T}_{Horse} = \{(d_1,0),(d_2,1),(d_3,1),(d_4,2)\} contains the defeasible inclusions:

(d_1) \mathbf{T}(Horse) \sqsubseteq Has\_Long\_Mane   (d_2) \mathbf{T}(Horse) \sqsubseteq Tall   (d_3) \mathbf{T}(Horse) \sqsubseteq RunFast   (d_4) \mathbf{T}(Horse) \sqsubseteq Has\_Tail the ranked TBox \mathcal{T}_{Zebra} = \{(d_5,0),(d_6,1),(d_7,2),(d_8,2)\} contains the defeasible
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- $(d_5) \mathbf{T}(Zebra) \sqsubseteq RunFast$
- $(d_6) \mathbf{T}(Zebra) \sqsubseteq \neg Tall$

inclusions:

- $(d_7) \mathbf{T}(Zebra) \sqsubseteq Striped$
- $(d_8) \mathbf{T}(Zebra) \sqsubseteq Has\_Tail$

The ranked Tbox  $\mathcal{T}_{Horse}$  can be used to define an ordering among domain elements comparing their typicality as horses. For instance, given two horses *Spirit* and *Buddy*, if

Spirit has long mane, is tall, has a tail, but does not run fast, it is intended to be more typical than Buddy, a horse running fast, tall, with long mane, but without tail, as having a tail (rank 2) is a more important property for horses wrt running fast (rank 1). We expect that the ranked knowledge base K above gives rise to multipreference models, where the two preference relations  $<_{Horse}$  and  $<_{Zebra}$  represent the preference among the elements of the domain  $\Delta$  according to concepts Horse and Zebra, respectively. We omit the specification of defeasible inclusions for the other distinguished concepts.

Given a DL interpretation  $I = \langle \Delta, \cdot^I \rangle$ , a modular partial order  $<_{C_i}$  can be defined over  $\Delta$  for each concept  $C_i \in \mathcal{C}$ , where  $x <_{C_i} y$  means that x is more typical than y wrt  $C_i$  (in the example,  $Spirit <_{Horse} Buddy$ ). The definition of the preference relations  $<_{C_1}, \ldots, <_{C_k}$  starting from a ranked knowledge base, can exploit different closure constructions. A lexicographic strategy has been considered in [40], which is one of the strategies considered in Brewka's framework of basic preference descriptions [13] and relates to Lehmann's lexicographic closure construction. However, the concept-wise multipreference semantics can better be regarded as a framework in which alternative notions of preference can be combined [39], starting from a modular knowledge base, and alternative preference constructions can been considered for each module, including rational closure, lexicographic closure (as in lexicographic modules [39]), Kern-Isberner's c-representations [59,61] or the closure construction introduced for weighted defeasible  $\mathcal{EL}^{\perp}$  knowledge bases [41]. An algebraic framework for preference combination in Multi-Relational Contextual Hierarchies has been recently developed by Bozzato et al. [12].

In the following, we will assume that the preferences with respect to concepts are given and we reformulate for  $\mathcal{LC}$  the notion of concept-wise multi-preference interpretation by *combining* the preference relations  $<_{C_i}$  into a global preference relation <. This is needed for reasoning about the typicality of arbitrary  $\mathcal{LC}$  concepts which do not belong to the set of distinguished concepts C. For instance, we may want to verify whether the typicality inclusion  $\mathbf{T}(Horse \sqcup Zebra) \sqsubseteq RunFast$  is satisfied in some multipreference interpretation, i.e., whether the typical instances of concept  $Horse \sqcup Zebra$  have the property of running fast. To check inclusions of this kind more than one preference relation may be relevant. In this example, both preference relations  $<_{Horse}$  and  $<_{Zebra}$ are relevant, and they might as well be conflicting for pairs of domain elements. For instance, if Buddy has all typical properties of a horse and Marty has all typical properties of a zebra, Buddy can be more typical than Marty as a horse ( $Buddy <_{Horse} Marty$ ), but more exceptional as a zebra (  $Marty <_{Zebra} Spirit$ ). By combining the preference relations  $<_{C_i}$  into a single global preference <, one can interpret the concept  $\mathbf{T}(C)$ (the set of typical instances of C) as the set of the minimal C-elements with respect to the global preference <. In the case above, neither Marty would be globally preferred to Spirit, nor vice-versa, but both of them should be regarded as typical among  $(Horse \sqcup Zebra)$ -elements.

In order to define a global preference relation, we take into account the specificity relation among concepts, such as, for instance, the fact that a concept like  $Baby\_Horse$  is more specific than concept Horse. The idea is that, in case of conflicts, the properties of a more specific class (such as that baby horses normally are not tall) should override the properties of a less specific class (such as that horses normally are tall).

**Definition 4 (Specificity).** A specificity relation *among concepts in C is a binary relation*  $\succ \subseteq C \times C$  *which is irreflexive and transitive.* 

For  $C_h, C_j \in \mathcal{C}$ ,  $C_h \succ C_j$  means that  $C_h$  is more specific than  $C_j$ . The simplest notion of specificity among concepts with respect to a knowledge base K (an ontology) is based on the subsumption hierarchy:  $C_h \succ C_j$  holds iff  $C_h \sqsubseteq C_j$  is subsumed from the ontology, while  $C_j \sqsubseteq C_h$  is not. The notion of specificity based on the subsumption hierarchy has been considered in many approaches to non-monotonic reasoning in DLs, including prioritized defaults [5], prioritized circumscription [10], and  $\mathcal{DL}^N$  [9]. Another notion of specificity considered in the literature is the one given by the ranking of concepts in the rational closure [68] of the knowledge base.

Exploiting the specificity relation among concepts, the global preference < can be defined by a modified Pareto combination of the relations  $<_{C_1}, \ldots, <_{C_k}$ , which takes into account specificity, as follows:

$$x < y \text{ iff} \quad (i) \ x <_{C_i} y, \text{ for some } C_i \in \mathcal{C}, \text{ and}$$

$$(ii) \text{ for all } C_j \in \mathcal{C}, \ x \leq_{C_j} y \text{ or } \exists C_h(C_h \succ C_j \text{ and } x <_{C_h} y)$$

$$(3)$$

The intuition is that x < y holds if there is at least a concept  $C_i \in \mathcal{C}$  such that  $x <_{C_i} y$  and, for all other concepts  $C_j \in \mathcal{C}$ , either  $x \leq_{C_j} y$  holds or, in case it does not, there is some  $C_h$  more specific than  $C_j$  such that  $x <_{C_h} y$  (i.e., preference  $<_{C_h}$  overrides  $<_{C_j}$ ). In the example above, for two baby horses (who are also horses) x and y, if  $x <_{Baby\_Horse} y$  and  $y <_{Horse} x$ , we will have x < y, that is, x is regarded as being globally more typical than y as it satisfies more properties of typical baby horses wrt y, although y may satisfy more properties of typical horses wrt x.

We can now formulate the notion of concept-wise multipreference interpretation [40] for  $\mathcal{LC}$ .

**Definition 5** (concept-wise multipreference interpretation). A concept-wise multipreference interpretation (or *cwm*-interpretation) is a tuple  $\mathcal{M} = \langle \Delta, <_{C_1}, \dots, <_{C_k}, <, \cdot^I \rangle$  such that:

- (a)  $\Delta$  is a non-empty domain;
- (b) for all i = 1, ..., k,  $<_{C_i}$  is an irreflexive, transitive, well-founded and modular relation over  $\Delta$ ;
- (c) < is the (global) preference relation over  $\Delta$  defined from  $<_{C_1}, \ldots, <_{C_k}$  in (3);
- (d)  $\cdot^{I}$  is an interpretation function, as defined for  $\mathcal{LC}$  interpretations (see Section 3), with the addition that, for typicality concepts, we let:

$$(\mathbf{T}(C))^I = min_{<}(C^I)$$

where 
$$Min_{\leq}(S) = \{u : u \in S \text{ and } \nexists z \in S \text{ s.t. } z < u\}.$$

It has been proven that the relation < is an irreflexive, transitive and well-founded relation [40]. As a consequence, the triple  $\langle \Delta, <, \cdot^I \rangle$  is a KLM-style preferential interpretation, as those introduced for  $\mathcal{ALC}$  with typicality [46], while it is not necessarily a modular interpretation.

The notion of cwm-model of a ranked KB and the notion of cwm-entailment can be defined in a natural way for ranked  $\mathcal{LC}$  knowledge bases, as it has been done for

the lightweight description logic  $\mathcal{EL}^+_{\perp}$  [40]. Let us mention that concept-wise multipreference entailment has been proven to satisfy the KLM postulates of a preferential consequence relation [68], and to have good properties such as avoiding the blocking inheritance problem, a well known problem of the rational closure and System Z [82,6], roughly speaking the problem that, if a subclass of C is exceptional for a given aspect, it is exceptional tout court and does not inherit any of the typical properties of C. Proof methods for reasoning with ranked and weighted knowledge bases in description logics under the concept-wise multipreference semantics, have been investigated for lightweight description logics of the  $\mathcal{EL}$ -family [3] (namely, for ranked  $\mathcal{EL}^+_{\perp}$  KBs [40] and for weighted  $\mathcal{EL}^+$  KBs [49]), by exploiting Answer Set Programming [36] and asprin [14] for defeasible inference.

In the following, we will address the problem of defining an  $\mathcal{LC}$  multipreference interpretation as a semantic model of a self-organising map. Given such a model, the verification of the logical properties that hold in the SOM can be done by model checking, i.e., by verifying the satisfiability of strict and defeasible inclusions in the model.

### 6 Relating self-organising maps and multi-preference models

We aim at showing that, once the SOM has learned to categorize, we can regard the result of the categorization as a multipreference interpretation. Let X be the set of input stimuli from different categories  $C_1, \ldots, C_k$ , which have been considered during the learning process.

For each category  $C_i$ , we let  $BMU_{C_i}$  be the set of best-matching units corresponding to input stimuli of category  $C_i$ , as in Section 2. We regard the learned categories  $C_1,\ldots,C_k$  as being concept names (atomic concepts) in the description logic (i.e., concept names in  $N_C$ ) and we let them constitute our set of distinguished concepts  $C=\{C_1,\ldots,C_k\}$ . We introduce an individual name  $a_x\in N_I$  for each possible stimulus x in the space of the possible stimuli.

In order to construct a multi-preference interpretation we proceed as follows: first, we fix the *domain*  $\Delta^s$  to be the space of possible stimuli, that we will assume to be finite and to include X, the set of all the input stimuli considered during training; then, for each category (concept)  $C_i$ , we define a preference relation  $<_{C_i}$  over  $\Delta^s$  by exploiting a notion of distance of a stimulus y from the map's representation of  $C_i$ . Finally, we define the interpretation of concepts.

Let  $\Delta^s$  be a finite set of possible stimuli, including all input stimuli considered during training  $(X \subseteq \Delta^s)$  as well as the best matching units of input stimuli (i.e.,  $\{BMU_x \mid x \in X\} \subseteq \Delta^s$ ). We therefore build a hybrid model, in which input exemplars and category representations coexist. Notice that we can consider these elements together because they have the same dimensionality. As we will see, category representations will be useful in reasoning about typicality.

Once the SOM has learned to categorize, the notion of distance  $d(x, C_i)$  of a stimulus x from a category  $C_i$  introduced above can be used to build a binary preference relation  $<_{C_i}$  among the stimuli in  $\Delta^s$  w.r.t. category  $C_i$  as follows: for all  $x, x' \in \Delta^s$ ,

$$x <_{C_i} x' \text{ iff } d(x, C_i) < d(x', C_i)$$

$$\tag{4}$$

Each preference relation  $<_{C_i}$  is a strict partial order relation on  $\Delta^s$ . The relation  $<_{C_i}$  is also well-founded as we have assumed  $\Delta^s$  to be finite  $^1$ .

We exploit this notion of preference to define a concept-wise multipreference interpretation associated to the SOM, and we call it a *cwm*-model of the SOM.

**Definition 6 (Concept-wise multipreference-model of a SOM).** The concept-wise multipreference model (or *cwm*-model) of the SOM is a *cwm-interpretation*  $\mathcal{M}^{som} = \langle \Delta^s, <_{C_1}, \dots, <_{C_k}, <, \cdot^I \rangle$  such that:

- (i)  $\Delta^s$  is the set of the possible stimuli, as introduced above;
- (ii) for each  $C_i \in \mathcal{C}$ ,  $<_{C_i}$  is the preference relation defined by equivalence (4);
- (iii) < is the global preference relation defined from  $<_{C_1}, \ldots, <_{C_k}$ , as in equation (3);
- (iv) the interpretation function  $\cdot^I$  is defined for individual names  $a_x \in N_I$  as  $a_x^I = x$ , and for concept names (i.e. categories)  $C_i$  as follows:

$$C_i^I = \{ y \in \Delta^s \mid d(y, C_i) \le d_{max, C_i} \}$$

where  $d_{max,C_i}$  is the maximal distance of an input stimulus  $x \in C_i$  from category  $C_i$ , that is,  $d_{max,C_i} = max_{x \in C_i} \{d(x,C_i)\}$ . The interpretation function  $\cdot^I$  is extended to complex concepts according to Definition 5, point (d).

Informally, we interpret as  $C_i$ -elements the stimuli whose distance from category  $C_i$  is not larger than the distance of any input exemplar belonging to category  $C_i^2$ .

Given  $<_{C_i}$ , we can identify the most typical  $C_i$ -elements (wrt  $<_{C_i}$ ) as the  $C_i$ -elements in  $min_{<_{C_i}}(C_i^I)$ , whose distance from category  $C_i$  is minimal. Observe that the best matching unit  $BMU_x$  of an input stimulus  $x \in C_i$  is an element of  $\Delta^s$ . Hence, for  $y = BMU_x$ , the distance  $d(y, C_i)$  of y from category  $C_i$  is 0, as  $min\|y - BMU_{C_i}\| = 0$ . Therefore,  $min_{<_{C_i}}(C_i^I) = \{y \in \Delta^s \mid d(y, C_i) = 0\}$  and  $BMU_{C_i} \subseteq min_{<_{C_i}}(C_i^I)$ . The converse inclusion  $BMU_{C_i} \supseteq min_{<_{C_i}}(C_i^I)$  might not hold. In fact, in case there is some input exemplar  $x \in C$  whose weight vector exactly coincides with the weight vector of its best matching unit  $BMU_x$ , then both x and  $BMU_x$  belong to  $min_{<_{C_i}}(C_i^I)$ . Notice that x and  $BMU_x$  are different elements of  $\Delta^s$ , even when they are associated to the same vector.

In  $\mathcal{M}^{som}$ , as in all  $\mathit{cwm}$ -interpretations (see Definition 5), the interpretation of typicality concepts  $\mathbf{T}(C)$  is defined based on the global preference relation < as  $(\mathbf{T}(C))^I = min_<(C^I)$ , for all concepts C. The model  $\mathcal{M}^{som}$  can be considered a sort of canonical model representing what holds in the SOM after the learning phase. Note that, for all domain elements  $y \in \Delta^s$  which have not been considered during training and do not correspond to some best matching unit, whether  $y \in C_i^I$  for a concept

Observe that we could have equivalently used the notion of relative distance  $rd(x, C_i)$  of a stimulus x from a category  $C_i$  to define the preference relation, as done in [43], rather than using the notion of distance  $d(x, C_i)$ .

<sup>&</sup>lt;sup>2</sup> As we can see, roles are not necessary to define a preferential model of a SOM. This is the reason why we have not introduced them in the language, although they were present (but not used) in the preliminary version of the paper [43], exploiting the description logic  $\mathcal{EL}^{\perp}$ . On the other hand, here, we have preferred to considered a logic with union and complement operators, which are not present in  $\mathcal{EL}^{\perp}$  but useful for the specification of the properties to be verified.

 $C_i$  and whether  $y <_{C_i} x$ , for  $x \in \Delta^s$ , is determined based on the distance  $d(y, C_i)$  of y from category  $C_i$ , as for the input stimuli  $x \in X$  (see points (iii), (iv) in Definition 6).

As  $\mathcal{M}^{som}$  is a *cwm*-interpretation, the triple  $\langle \Delta^s, <, \cdot^I \rangle$  is a KLM style preferential interpretation [65,68]. It follows that  $\mathcal{M}^{som}$  provides a preferential semantics of the SOM. The preferential interpretation  $\mathcal{M}^{som}$  determines a preferential consequence relation (the set of all conditionals  $\mathbf{T}(C) \sqsubseteq D$  true in  $\mathcal{M}^{som}$ ), which satisfies all KLM postulates of a preferential consequence relation [65].

#### 6.1 Evaluation of strict and defeasible concept inclusions by model checking

In the previous section, we have defined a multipreference interpretation  $\mathcal{M}^{som}$  from a SOM where we are able to identify, on the domain  $\Delta^s$  of the possible stimuli, the set of  $C_i$ -elements as well as the set of most typical  $C_i$ -elements wrt  $<_{C_i}$ , for each category  $C_i$ . Provided the necessary information are recorded during training, we are then able to evaluate strict inclusions  $C \sqsubseteq D$  and defeasible inclusions  $\mathbf{T}(C) \sqsubseteq D$ , by model checking, i.e., by verifying their satisfiability in the model  $\mathcal{M}^{som}$ .

For instance, we may want to check whether elephants are African or Asiatic animals (i.e., if  $Elephant \sqsubseteq African\_Animal \sqcup Asian\_Animal$  holds in the model), or whether typical elephants are African or Asiatic animals (i.e.,  $T(Elephant) \sqsubseteq African\_Animal \sqcup Asian\_Animal$ ), or whether typical big elephants are African (i.e.,  $T(Elephant \sqcup Big\_Animal) \sqsubseteq African\_Animal$ ), provided the concepts Elephant,  $African\_Animal$ ,  $Asiatic\_Animal$  and  $Big\_Animal$  correspond to learned categories. The latter inclusions represent weaker properties with respect to the first one as they are only concerned with the typical instances of concepts. A strict inclusion  $Elephant \sqsubseteq Big\_Animal$ , for instance, cannot be expected to be true in the model of the SOM, if input stimuli also include small elephants such as baby elephants, while a defeasible inclusion  $T(Elephant) \sqsubseteq Big\_Animal$  might hold.

The verification of general concept inclusions, where C and D are  $\mathcal{LC}$  concepts, i.e., boolean combinations of concepts in  $\mathcal{C}$ , may require to record the extensions of concepts  $C_i$ 's in the model  $\mathcal{M}^{som}$  and to perform set-theoretic operation to compute the interpretation of concepts. For instance, concept  $(C_1 \sqcap C_2) \sqcup \neg C_3$  is interpreted as  $(C_1^I \cap C_2^I) \cup (\Delta^s \backslash C_3^I)$ . Finding typical instances of this concept would further require to compute the global preference relation < among its instances and, in the worst case, to verify for all pairs of input stimuli  $x,y \in \Delta^s$  whether x < y or y < x or x and y are incomparable. This, in turn, requires to check, for all categories  $C_i$ , whether  $x <_{C_i} y$  or  $y <_{C_i} x$  or  $x \sim_{C_i} y$ , based on the distances  $d(x,C_i)$  and  $d(y,C_i)$ . This might be challenging in practice, depending on the size of the domain, although the verification requires a polynomial number of checks. In the following of this section, we will let b to be the largest number of best matching units for a category. Note that b cannot be larger than the size of the set X of all input stimuli.

**Proposition 1.** The verification that a general typicality inclusion is satisified in the multipreference interpretation  $\mathcal{M}^{som}$  is  $O(n^3 \times k)$ , where n is the size of  $\Delta^s$  and k the number of categories.

*Proof.* Consider a typicality inclusion  $\mathbf{T}(C) \sqsubseteq D$ , where C and D are general  $\mathcal{LC}$  concepts.

Observe that computing  $d(x, C_i)$  for some  $x \in \Delta^s$  and concept  $C_i \in \mathcal{C}$  requires to compute the distance of x from all the best matching units in  $BMU_{C_i}$ . This requires a number of steps equal to the size of  $BMU_{C_i}$ . Hence, computing  $d(x, C_i)$  is O(b).

Verifying whether  $x \in C_i$ , for  $x \in \Delta^s$  and concept  $C_i \in \mathcal{C}$ , amounts to check whether  $d(x,C_i) \leq d_{max,C_i}$ , and is O(b), if  $d_{max,C_i}$  has been computed during training. Verifying whether  $x <_{C_i} y$  for  $x,y \in \Delta^s$  and  $i=1,\ldots,k$ , requires to check whether  $d(x,C_i) < d(y,C_i)$ , and is O(b) (and, similarly, verifying  $x \sim_{C_i} y$ ). Checking x < y, requires to verify whether  $x <_{C_i} y$  and whether  $x \sim_{C_i} y$  for all  $C_i \in \mathcal{C}$ . Hence, checking x < y for some  $x,y \in \Delta^s$  is  $O(b \times k)$ .

Identifying the instances of a given boolean concept C requires to verify, for each domain element  $x \in \Delta^s$ , if  $x \in C^I$ . It requires to check whether x belongs to  $C_i$ , for all concepts  $C_i$  occurring in C. As checking whether x belongs to  $C_i$  is O(b), verifying this for all  $C_i$  occurring in C is  $O(b \times k)$ .

Once we know whether  $x \in C_i^I$  is known for all i's, determining whether  $x \in C^I$  requires to evaluate a boolean expression, which is linear in the size of the expression C (that we assume to be O(k)). Overall, verifying whether  $x \in C^I$  is  $O(b \times k^2)$ . Thus, identifying all C-elements in the domain  $\Delta^s$  is  $O(n \times b \times k^2)$  and, as b cannot be larger than n, it is  $O(n^2 \times k^2)$ .

Let S be the set of all C-elements. Identifying the <-minimal C-elements, among all C-elements, can be done in  $O(n^2)$  steps, starting from a set S (whose size may be comparable to the size of  $\Delta$ ), initialized to contain all C-elements, by iterating on all C-elements y and by removing from S all elements  $y \in S$  such that y' < y for some  $y' \in \Delta$  (using a two nested loop). Verifying y' < y takes constant time only if the distances  $d(x, C_i)$ , for all  $x \in \Delta$  and all  $C_i$ , are computed and stored in advance, which is unlikely when the size of  $\Delta^s$  is large. Otherwise, verifying y' < y is  $O(b \times k)$ , as seen above. In this second case, identifying the <-minimal C-elements, among all C-elements is  $O(b \times k \times n^2)$ , and hence  $O(n^3 \times k)$ .

For each  $x \in min_{<}(C^I)$ , verifying that  $x \in D^I$  (as for C above) is  $O(b \times k^2)$ . The size of the set  $min_{<}(C^I)$  is smaller than the size of  $\Delta^s$ , hence verifying, for all  $x \in min_{<}(C^I)$ , that  $x \in D^I$  is  $O(n \times b \times k^2)$  and, hence,  $O(n^2 \times k^2)$ .

We then have a sequence of steps, where each step is either  $O(n^2 \times k^2)$  or  $O(n^3 \times k)$ . As  $n \gg k$ , verifying that a general typicality inclusion is satisified in  $\mathcal{M}^{som}$  is  $O(n^3 \times k)$ 

We have seen that, in the general case, verifying the satisfiability of an inclusion on the model of the SOM may be non trivial, depending on the number of input stimuli that are considered in the learning phase (the size of the set X of input exemplars and their best matching units) and on the size of  $\Delta^s$ . Gliozzi and Plunkett have considered self-organising maps that are able to learn from a limited number of input stimuli, although this is not generally true for all self-organising maps [51]. In the following we will see that, however, verifying the satisfiability of strict inclusions  $C_i \sqsubseteq C_j$  and defeasible inclusions  $\mathbf{T}(C_i) \sqsubseteq C_j$  where  $C_i$  and  $C_j$  are categories (i.e. distinguished concepts in C), requires a smaller number of steps, as it does not require to compute the distance of each stimulus from each concept representation.

In order to verify that a typicality inclusion  $\mathbf{T}(C_i) \sqsubseteq C_j$  is satisfied in  $\mathcal{M}^{som}$ , we have to check that the most typical  $C_i$  elements wrt  $<_{C_i}$  are  $C_j$  elements, that is,

 $min_{<_{C_i}}(C_i^I) \subseteq C_j^I$ . Note that, besides the elements in  $BMU_{C_i}$ ,  $min_{<_{C_i}}(C_i^I)$  might contain other elements of  $\Delta^s$  having distance 0 from  $C_i$ , namely those elements y from the domain having distance 0 from their best matching unit (elements whose weight vector exactly coincides with the weight vector of a best matching unit  $BMU_x$  for  $C_i$ ). In such a case, as  $BMU_x$  is already in  $BMU_{C_i}$ , it is enough to verify that all elements in  $BMU_{C_i}$  are  $C_j$ -elements, that is:

for all input stimuli 
$$x \in C_i$$
,  $d(BMU_x, C_i) \le d_{max,C_i}$  (5)

that is, the distance of all best matching units of  $C_i$  from  $C_j$  is smaller than the distance from  $C_j$  of some  $C_j$ -element in the input set. Let the distance of  $BMC_{C_i}$  from  $C_j$  be defined as follows:

$$d(BMC_{C_i}, C_j) = \max_{x \in C_i} \{d(BMU_x, C_j)\}\$$

as the maximal distance of any  $BMU_x$ , for  $x \in C_i$ , from  $C_j$ . Then we can rewrite condition (5) simply as

$$d(BMC_{C_i}, C_j) \le d_{max, C_j}. (6)$$

Observe that the distance  $d(BMC_{C_i}, C_j)$  provides a measure of plausibility of the defeasible inclusion  $\mathbf{T}(C_i) \sqsubseteq C_j$ : the higher is the distance of  $BMU_{C_i}$  from  $C_j$ , the more implausible is the defeasible inclusion  $\mathbf{T}(C_i) \sqsubseteq C_j$ . The lower is  $d(BMC_{C_i}, C_j)$ , the more plausible is the defeasible inclusion  $\mathbf{T}(C_i) \sqsubseteq C_j$ . This is relevant if we aim at extracting knowledge in the form of a set of conditionals from a SOM. In particular, using the relative distance and the generalization degree in [51], the value  $e^{-rd(BMC_{C_i}, C_j)}$  provides a degree of plausibility of a defeasible inclusion  $\mathbf{T}(C_i) \sqsubseteq C_j$  in the interval (0,1].

Let us now consider the case of a strict inclusion  $C_i \sqsubseteq C_j$ , with  $C_i, C_j \in \mathcal{C}$ . Verifying that  $C_i \sqsubseteq C_j$  is satisfied by  $\mathcal{M}^s$ , requires to check that  $C_i^I$  is included in  $C_j^I$ . Exploiting the fact that the map is organized topologically, and using the distance  $d(BMC_{C_i}, C_j)$  of  $BMC_{C_i}$  from  $C_j$  defined above, we can verify that the distance of  $BMC_{C_i}$  from  $C_j$  plus the maximal distance of a  $C_i$ -element from  $C_i$  is not greater than the maximal distance of a  $C_j$ -element from  $C_j$ :

$$d(BMC_{C_i}, C_i) + d_{max, C_i} \le d_{max, C_i}, \tag{7}$$

that is, the most distant  $C_i$ -element from  $C_j$  is nearer to  $C_j$  than the most distant  $C_j$ -element.

Note that the verification of conditions (6) and (7) does not require to record all the instances of the concepts  $C_i$  and  $C_j$  in  $\Delta^s$ , but to compute and record some measures, namely  $d_{max,C_j}$  and  $d(BMC_{C_i},C_j)$ , for all concepts (categories)  $C_i,C_j\in\mathcal{C}$ . While computing  $d_{max,C_j}$  during training only requires a constant overhead (by updating the current value of  $d_{max,C_j}$  for each input stimulus in  $C_j$ ), computing  $d(BMC_{C_i},C_j)$  requires to compute the maximum distance between the best matching units for  $C_i$  and the best matching units for  $C_j$ , requiring  $O(b^2)$  steps, where b is the number of best matching units for the input stimuli. Once such measures have been computed, the verification of conditions (6) and (7) requires constant time. The next proposition follows:

**Proposition 2.** Checking the satisfiability of a strict (resp., defeasible) inclusion of the form  $C_i \sqsubseteq C_j$  (resp.,  $\mathbf{T}(C_i) \sqsubseteq C_j$ ) in the interpretation  $\mathcal{M}^{som}$  is  $O(b^2)$ , when  $C_i$  and  $C_j$  are distinguished concepts in C corresponding to learned categories.

Notice that the number of the best matching units for  $C_i$  is smaller than the size of X and of the domain  $\Delta^s$ . Furthermore, as observed by Gliozzi and Plunkett, the number of best matching units may sometimes be significantly smaller than the set of the input stimuli. Even when the domain  $\Delta^s$  only contains the input stimuli in X and their best matching units, the verification of general typicality formulas over the SOM model is  $O(n_X^3 \times k)$ , where  $n_X$  is the size of set of input stimuli X considered during training. It may be challenging in practice, due to the large number of input stimuli, and more challenging than validating inclusions of the form  $C_i \sqsubseteq C_j$  and  $\mathbf{T}(C_i) \sqsubseteq C_j$ , as b may be significantly lower than  $n_X$ . An alternative approach for reasoning about the knowledge learned by the SOM might be: first identify the set K of strict and defeasible inclusions of the form  $C_i \sqsubseteq C_j$  and  $\mathbf{T}(C_i) \sqsubseteq C_j$  satisfied in  $\mathcal{M}^{som}$  (where  $C_1, \ldots, C_k$  are the learned categories); then exploit the defeasible knowledge base K extracted from the SOM for symbolic reasoning in a conditional logic formalism.

For instance, as we have already mentioned, an approach which exploits Answer Set Programming and *asprin* [14] to achieve defeasible reasoning under the concept-wise multipreference semantics has been proposed for reasoning with ranked  $\mathcal{EL}_{\perp}^{+}$  knowledge bases [40]. The approach has been extended to deal with weighted knowledge bases with real valued weights [49]. As we have mentioned above, a measure of plausibility can be determined for the defeasible inclusions satisfied by the SOM, and it can be exploited for the definition of a weighted knowledge base. Other proposals, which are not based on multiple preferences, have been developed in the literature of non-monotonic and conditional reasoning for dealing with weighted knowledge bases. Let us mention Weydert's System JLZ [92], an approach which allows rational and real-valued ranking measures, and the work by Kern-Isberner and Eichhorn [61], based on c-representations which also allow for plausibility weights.

#### 6.2 Dealing with specificity

Let us conclude this section by commenting on the notion of specificity, that we have left aside up to this point. Reasoning about specificity is an important feature of a non-monotonic formalism, when it is intended to deal with exceptions among classes. This was recognized from the beginning by Baader and Hollunder [5] who, in their work on prioritized defaults in description logics, observe that

"the question of how to prefer more specific defaults over more general ones [...] is of general interest for default reasoning but is even more important in the terminological case where the emphasis lies on the hierarchical organization of concepts".

Many non-monotonic extensions of description logics, including the ones based on circumscription [10], on the rational closure [20,47], on the lexicographic closure [21] and their refinements, conform to the principle that the specificity relation among concepts is to be taken into account.

To see that specificity is also relevant in this context, let us suppose that, among the categories  $C_1, \ldots, C_k$ , both the category Bird and the category Penguin are present. Let us further suppose that, as a result of the categorization, the inclusion  $Penguin \sqsubseteq Bird$  is satisfied in the model  $\mathcal{M}^{som}$ . Clearly, the preference relations  $<_{Penguin}$  and  $<_{Bird}$  might not agree as typical penguins do not fly and are atypical as birds. In this case, the class Penguin is more specific than the class Bird. As we expect that typical  $Penguin \sqcap Bird$ -elements are penguins and hence do not fly, we would expect the typical instances of concept  $Penguin \sqcap Bird$ , i.e.,  $min_{<}(Penguin \sqcap Bird)$ , should correspond to the minimal elements with respect to  $<_{Penguin}$ , i.e., to  $min_{<_{Penguin}}(Penguin \sqcap Bird)$ .

In Section 5 we have defined the global preference relation in such a way that the specificity relation among concepts is taken into account. The idea is that, in case of conflicts, the properties of a more specific class, Penguin, should override the properties of a less specific class, Bird, as  $Penguin \succ Bird$  (concept Penguin is more specific than Bird).

The definition of global preference allows us to deal with the specificity issues when they emerge, that is, when a category  $C_i$  is subsumed by another  $C_j$ , as a result of categorization. In such a case, we expect the subsumption  $C_i \sqsubseteq C_j$  to hold in the model of the SOM  $\mathcal{M}^{som}$ , whereas we do not expect the converse inclusion to hold.

In other cases, the SOM might not recognize that concept  $C_i$  is more specific than  $C_j$ , but we may know that it is the case from some alternative knowledge source (e.g., an ontology) rather than from the analysis of empirical data in the SOM model. In such a case, we can nevertheless exploit the specificity information Penguin > Bird in the construction of the global preference relation > from the preferences  $<_{C_j}$  built from the SOM, or we can add the specificity inclusions to the defeasible knowledge base containing the strict and defeasible inclusions extracted from the SOM. This is a possible way of exploiting symbolic knowledge in combination with the knowledge extracted from empirical data, i.e., from the preferential model of SOM.

In this section, we have used the relative distance of a stimulus x from a category  $C_i$  to define a preference relation  $<_{C_i}$  on the set of input stimuli. This preference relation determines the typical  $C_i$ -elements, the domain elements with the lowest relative distance from category  $C_i$ . In the next section, we will exploit the same notion of relative distance of a stimulus x from a category  $C_i$  to define a degree of membership of x to category  $C_i$ . We will do this by associating a fuzzy  $\mathcal{LC}$  interpretation to a SOM.

#### 7 A fuzzy interpretation of a Self-Organising Maps

In this section, we consider an alternative interpretation of Self-Organising Maps based on fuzzy description logic interpretations. As mentioned in Section 4, fuzzy description logics allows for representing vagueness in DLs and have been widely studied in the literature [88,87,74,11,8]. In fuzzy description logics, a concept C is interpreted as a function  $C^I$  mapping each domain element x to a value in the interval [0,1], a degree of membership in C.

We again follow Gliozzi and Plunkett's similarity-based account of category generalization [51] (see Section 2) where, from the notion of relative distance  $rd(x, C_i)$  of an input stimulus x from category  $C_i$ , they define the map's Generalization De-

gree of category C membership to a new stimulus y (here denoted by  $gd_C(y)$ ), as:  $gd_C(y) = e^{-rd(y,C)}$ . The value of  $gd_C(y,C)$  is in (0,1] and can be regarded as the degree of membership of y in  $C_i$ .

As in the definition of the multipreference model  $\mathcal{M}^{som}$  in Section 6, we let the domain  $\Delta^s$  be the set of all the possible stimuli, including the input stimuli considered in the training phase and their best matching units in the map. We also consider the learned categories  $C_1,\ldots,C_k$  as the concept names (atomic concepts) in the description logic, i.e.,  $N_C = \{C_1,\ldots,C_k\}$ . We let  $N_I$  be the set of individual names as in previous section.

**Definition 7** (Fuzzy model of a SOM). The fuzzy model of the SOM is a fuzzy interpretation  $I^s = \langle \Delta^s, \cdot^I \rangle$  such that:

- $\Delta^s$  is the set of possible stimuli, as introduced above;
- the interpretation function  $\cdot^I$  is defined for individual names  $a_x \in N_I$  as  $a_x^I = x$ , and for all named concepts  $C_i$  as:  $C_i^I(x) = e^{-rd(x,C_i)}$ .

With this definition, the fuzzy interpretation  $I^s$  interprets each concept  $C_i$  corresponding to a learned category as a fuzzy set. As recalled in Section 4, the fuzzy interpretation of complex concepts is defined inductively, given a choice of t-norm, s-norm, implication function, and negation function.

Observe that, in the fuzzy model of the SOM, the (normalized) relative distance  $rd(x,C_i)$  of x from  $C_i$  is used to define the value of  $C_i^I(x)$ , rather than the distance  $d(x,C_i)$ . As  $rd(x,C_i)\leq 1$  for all instances x of  $C_i$ , for all instances x of a category  $C_i$ , it holds that  $C_i^I(x)\leq e^{-1}$ . For  $y\in\Delta^s$ , the value of  $C_i^I(y)$  is 1 when  $rd(y,C_i)=0$  and approaches 0 when  $rd(y,C_i)$  approaches  $\infty$ .

As we have seen in section 4, in the fuzzy interpretation  $I^s$ , a concept inclusion axiom  $C_i \sqsubseteq C_j$  is interpreted as  $(C_i \sqsubseteq C_j)^{I^s} = inf_{x \in \Delta} \ C_i^{I^s}(x) \rhd C_j^{I^s}(x)$ , which describes the degree of subsumption between the two fuzzy sets  $C_i^{I^s}$  and  $C_j^{I^s}$ . Computing  $(C_i \sqsubseteq C_j)^{I^s}$  requires the values  $C_i^{I^s}(x)$  and  $C_j^{I^s}(x)$  to be computed or recorded for all domain elements x, that is, for all the stimuli considered in  $\Delta^s$ . The satisfiability of a fuzzy inclusion axiom such as  $C_i \sqsubseteq C_j \geq n$  can then be evaluated in the fuzzy model  $I^s$ , by verifying that  $(C_i \sqsubseteq C_j)^{I^s} \geq n$ . Although conceptually this is just model checking, it might be challenging in practice, depending on the size of the domain  $\Delta^s$ .

The possibility of verifying the satisfiability of fuzzy axioms over the model  $I^{som}$  of the SOM allows for symbolic knowledge to be extracted from the SOM in the form of fuzzy inclusion axioms. This knowledge can then be used for symbolic reasoning, based on the proof methods that have been developed for both expressive and lightweight fragments of fuzzy OWL [7,8]. The problem of learning fuzzy rules has been widely investigated in the context of fuzzy description logics based on other machine learning approaches [72,89].

For instance, referring to the running example, we may want to check whether elephants are big animals with a degree  $\geq 0.6$ , i.e., whether the fuzzy axiom  $Elephant \sqsubseteq Big\_Animal \geq 0.7$  is satisfied in the interpretation  $I^s$ , and whether the fuzzy axioms  $Elephant \sqcup Hyppos \sqsubseteq Big\_Animal \geq 0.7$  and  $Elephant \sqcap White\_Animal \sqsubseteq African\_Animal \sqcup Asian\_Animal \geq 0.8$  hold in  $I^s$ .

Notice that, while the verification of fuzzy axioms on the fuzzy model  $I^s$  depends on all domain elements in  $\Delta^s$ , the verification on the preferential model  $\mathcal{M}^s$  of the prototypical properties of a category  $C_i$  by an inclusion  $\mathbf{T}(C_i) \subseteq C_i$  only requires to consider the typical  $C_i$ -elements, i.e., the elements in  $BMU_{C_i}$ , rather than all  $C_i$ elements. Indeed, although the preferential and the fuzzy semantics of the SOM are constructed starting from the measures of distance of the input stimuli from the categories, the two semantics may provide quite different information about the SOM, which are expressed, respectively in the form of conditional assertions and of fuzzy concept inclusions. For instance, in the presence of high variability in the examples of a category, typical elements do not necessarily provide a precise representation of the category and of the properties of all its elements (we will return to the variability effect in the next section). Although a typicality inclusion  $T(C) \subseteq D$  may be satisfied, as the typical C-elementas are D-elements, it might not be the case that the value of  $(C \subseteq D)^I$  is high (i.e., that it approaches 1), as there might be elements in C which are very exceptional with respect to property D. The presence in the domain  $\Delta^s$  of additional stimuli with respect to the input stimuli in X (and their best matching units), may affect the valuation  $(C \subseteq D)^I$  of an inclusion  $C \subseteq D$  in the fuzzy interpretation  $I^s$ . This is the same for general typicality inclusions in the multipreference interpretation  $\mathcal{M}^{som}$  but, as mentioned above, this is not the case for typicality inclusion  $\mathbf{T}(C_i) \subseteq C_i$ , with  $C_i$ and  $C_i$  distinguished concepts in  $\mathcal{C}$ , whose evaluation only depends on the prototypical elements in the category  $C_i$ , i.e., on the best matching units for category  $C_i$ .

# 8 Towards a probabilistic interpretation of Self-Organising maps

In previous section, we have defined a fuzzy DL interpretation  $I^s = \langle \Delta^s, \cdot^I \rangle$  as the semantics of a SOM after the training phase. Each concept C in  $I^s$  is interpreted as a fuzzy set on  $\Delta$  with membership function  $C^I$ . The relationships between fuzzy sets and probability theory have been widely investigated in the literature [94,63,33], and we will follow the proposal by Zadeh, who has shown that "the notions of an event and its probability can be extended in a natural fashion to fuzzy events" [94]. In particular, we refer to a recent characterization of the continuous t-norms compatible with Zadeh's probability of fuzzy events ( $P_Z$ -compatible t-norms) by Montes et al. [78], to provide a probabilistic interpretation starting from the fuzzy interpretation  $I^s$ . We will see that, following this approach, we can regard  $C^I(x)$  as the conditional probability of the (fuzzy) concept C, given element x. This is in agreement with Kosko's account of "fuzziness in a probabilistic world" [63].

Zadeh has proven that the set of fuzzy events forms a  $\sigma$ -field with respect to the operations of complement, union and intersection in (Zadeh's) fuzzy logic. Montes et al. [78] have studied the problem of determining which t-norms make Zadeh's probability of fuzzy events fulfill Kolmogorov's axioms (called  $P_Z$ -compatible t-norms). For a given t-norm T, they consider a generalization of the notion of algebra to the fuzzy framework by means of T-clans.

For a given universe  $\Omega$ , they let  $\mathcal{F}(\Omega)$  denote the set of all fuzzy subsets on  $\Omega$ , and a T-clan  $\mathcal{A}$  be a subset of  $\mathcal{F}(\Omega)$  satisfying the properties that:

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-\emptyset \in \mathcal{A}, where \emptyset(x) = 0 for all x \in \Omega;
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- if  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ , where  $A^c(w) = 1 A(w)$ ; - if  $A, B \in \mathcal{A}$ , then  $A \cap_T B \in \mathcal{A}$ .
- Given the universe  $\Omega$ , the set  $\mathcal{F}(\Omega)$  of all fuzzy subsets on  $\Omega$ , and a probability measure P over  $\Omega$ , they define the probability of a measurable fuzzy event A in a T-clan A on  $\mathcal{F}(\Omega)$  as

$$P(A) = \int_{\Omega} A(w)dP(w). \tag{8}$$

Montes et al. [78] extend the result by Zadeh by identifying a class of  $P_Z$ -compatible t-norms, for which the probability P defined by equation (8) satisfies Kolmogorov's axioms (formulated in the fuzzy framework). For a t-norm T, they consider the t-conorm  $S_T(x,y) = 1 - T(1-x,1-y)$ .

Let us restrict to a  $P_Z$ -compatible t-norm T, with associated t-conorm  $S_T$  and the negation function  $\ominus x = 1 - x$ . For instance, we can assume the combination functions in Zadeh logic or in Lukasiewicz logic. Observe that in the interpretation  $I^s = \langle \Delta^s, \cdot^I \rangle$  constructed from the SOM, the domain  $\Delta^s$  is a finite set of input stimuli and, based on the chosen t-norm T, each concept C is interpreted as a fuzzy set on  $\Delta^s$  with membership function  $C^I: \Delta^s \to [0,1]$  (a measurable function, being  $\Delta$  finite). Let  $\mathcal{F}(\Delta^s)$  denote the set of all fuzzy subsets on  $\Delta^s$ . Given  $I^s = \langle \Delta^s, I \rangle$ , the collection  $\mathcal{A}$  of the fuzzy sets  $C^I$ , for all DL concepts C, forms a T-clan on  $\mathcal{F}(\Delta^s)$ .

By assuming a discrete probability distribution p over  $\Delta^s$ , we can define the probability of the fuzzy set  $C^I$ , for each DL concepts C as:

$$P(C^{I}) = \sum_{d \in \Delta^{s}} C^{I}(d) \ p(d) \tag{9}$$

As the interpretation  $I^s$  is the unique fuzzy interpretation associated to the SOM, we can simply write P(C), rather than  $P(C^I)$  and read P(C) as the probability of a fuzzy concept C, given a probability distribution p over the domain  $\Delta^s$  of the input stimuli.

For a given input stimulus  $x \in \Delta^s$ , the probability the x is an instance of a fuzzy concept C, can then be interpreted as the conditional probability  $P(C \mid x)$  of C given x (where x stands for the crisp set  $\{x\}$ ). Following Smets [86], we let the conditional probability of a fuzzy event C given the fuzzy event D be

$$P(C \mid D) = \frac{P(D \sqcap C)}{P(D)}$$

(provided P(D) > 0) and depart from Zadeh's definition of conditional probability, which exploits the product of the fuzzy sets rather than the intersection. As observed by Dubois and Prade [33], letting  $P(G \mid F) = P(F \cap G)/P(F)$  (for the fuzzy events F and G), generalizes both conditional probability and the fuzzy inclusion index advocated by Kosko [64].

From the definition above of conditional probability and from condition (9), it easily follows that  $P(C \mid x) = C^{I}(x)$ . In fact:

$$P(C \mid x) = \frac{P((C \sqcap \{x\})^I)}{P(\{x\}^I)} = \frac{\sum_{d \in \Delta} (C \sqcap \{x\})^I(d) \ P(d)}{\sum_{d \in \Delta} \{x\}^I(d) \ P(d)}$$

As  $(C \cap \{x\})^I(d) = 0$  for  $d \neq x$ , and  $\{x\}^I(d) = 0$  for  $d \neq x$ , this simplifies to:

$$\frac{(C \sqcap \{x\})^I(x) \ P(x)}{\{x\}^I(x) \ P(x)}$$

and, observing that  $(C \cap \{x\})^I(x) = C^I(x)$  and  $\{x\}^I(x) = 1$ :

$$P(C \mid x) = C^{I}(x)$$

Furthermore, if we assume a uniform probability distribution P (i.e., P(x)=1/n, for all  $x\in \Delta^s$ , where  $n=|\Delta^s|$ ), it holds that

$$P(x|C) = C^{I}(x)/M(C)$$

where  $M(C) = \sum_{x \in \Delta^s} C^I(x)$  is called the *size* of the fuzzy concept  $C^I$ . In fact,

$$P(x \mid C) = \frac{P((C \sqcap \{x\})^I)}{P(C^I)} = \frac{\sum_{d \in \Delta} (C \sqcap \{x\})^I(d) \ P(d)}{\sum_{d \in \Delta} C^I(d) \ P(d)}$$

As  $(C \cap \{x\})^I(d) = 0$  for  $d \neq x$  and  $(C \cap \{x\})^I(x) = C^I(x)$ , this simplifies to:

$$\frac{C^{I}(x) P(x)}{\sum_{d \in \Lambda} C^{I}(d) P(d)}$$

and, assuming a uniform probability distribution over  $\Delta$ :

$$P(x \mid C) = \frac{C^{I}(x) (1/n)}{(1/n) \sum_{d \in A} C^{I}(d)} = \frac{C^{I}(x)}{\sum_{d \in A} C^{I}(d)} = \frac{C^{I}(x)}{M(C^{I})}$$

Observe that the likelihood  $P(x|C^I)$  decreases when the size of  $C^I$  increases, an effect that recalls about the *size principle* by Tenenbaum and Griffiths [90]); this principle is at the basis of their explanation of the numerosity and variability effects in category generalization.

Gliozzi and Plunkett [51] have shown that in self-organising maps the numerosity and the variability of the known instances of a category affect the quality of a category representation: the numerosity of known examples of a category improves the precision of the category representation whereas the variability of these examples diminishes this precision. The probabilistic interpretation of self-organising maps given above is in agreement with their experimental results.

For variability, if the variability increases,  $d_{max,C_i} = max_{x \in C} \|x - BMU_x\|$  increases as well. Then,  $rd(y,C_i)$  decreases and consequently  $C_i^I(y) = gd_{C_i}(y)$  increases. As observed by Gliozzi and Plunkett [51], an increase in variability of the known category examples generates an increase in generalization; the higher is variability, the higher the probability of generalization outside the range.

For numerosity, let us consider an increase in the number of input stimuli of category  $C_i$ , all other things being equal (including the range of values of input stimuli). Under these conditions, a larger number of input stimuli allows to form a more precise representation of the category (i.e., a representation with a lower  $d_{max,C_i}$ ). In fact, repeatedly

updating the weights of the best matching units of  $C_i$  (and their neighbours), makes their distance from the input stimuli smaller.

While Gliozzi and Plunkett [51] have demonstrated that the Numerosity and Variability effects can be accommodated within a simple and psychologically plausible similarity based account, in this section we have shown that their generalization degree can indeed be considered as a probability measure.

# 9 The training process as a belief change process

We have seen that one can give an interpretation of a self-organising map after the learning phase, as a preferential model. However, the state of the SOM during the learning phase can as well be represented as a multipreference model, precisely in the same way. During training, the current state of the SOM corresponds to a model representing the beliefs about the input stimuli considered so far (beliefs concerning the category of the stimuli).

The learning process can then be regarded as a model building process and, in a way, as a belief change process. Initially we do not know the category of the stimuli in the domain  $\Delta^s$ . In the initial model, call it  $\mathcal{M}_0^{som}$  (over the domain  $\Delta^s$ ) the interpretation of each concept  $C_i$  is empty.  $\mathcal{M}_0^{som}$  can be regarded as the model of a knowledge base  $K_0$  containing a strict inclusion  $C_i \sqsubseteq \bot$ , for all  $C_i$ .

Each time a new input stimulus  $(x \in C_i)$  is considered, the model is revised adding the stimulus x (and its best matching unit  $BMU_x$ ) into the proper category  $(C_i)$ . Not only the category interpretation is revised by the addition of x and  $BMU_x$  in  $C_i^I$  (so that  $C_i \sqsubseteq \bot$  does not hold any more), but also the associated preference relation  $<_{C_i}$  is revised as the addition of  $BMU_x$  modifies the set of best matching units  $BMU_{C_i}$  for category  $C_i$ , as well as the distance  $d(y,C_i)$  of a stimulus y from  $C_i$ . That is, a revision/update step may change the set of conditionals which are satisfied by the model.

At the end of the training process, the final state of the SOM is captured by the model  $\mathcal{M}^{som}$  obtained by a sequence of revision steps which, starting from  $\mathcal{M}^{som}_{i}$ , gives rise to a sequence of models  $\mathcal{M}^{som}_{0}, \mathcal{M}^{som}_{i_1}, \ldots, \mathcal{M}^{som}_{i_r}$  (with  $\mathcal{M}^{som} = \mathcal{M}^{som}_{i_r}$ ). At each step the knowledge base is not represented explicitly, but the model  $\mathcal{M}^{som}_{i_j}$  of the knowledge base at step j is used to determine the model at step j+1 as a result of the revision/update step  $(\mathcal{M}^{som}_{i_j+1} = \mathcal{M}^{som}_{i_j} \star C_{i_j}(x_{i_j}))$ . The knowledge base K (the set of all the strict and defeasible inclusions and assertions satisfied in  $\mathcal{M}^{som}$ ) can then be regarded as the knowledge base obtained from  $K_0$  through a sequence of revision/update steps. For future work, it would be interesting to study the properties of this notion of change and compare its properties with the properties of the notions of belief revision [34,35,57,58] and iterated belief revision [26,44,60,24] studied in the literature.

#### 10 Conclusions

The concept-wise multipreference semantics has recently been introduced for dealing with typicality in description logics [40], based on the idea that reasoning about exceptions in ontologies requires taking into account preferences with respect to different

concepts and integrating them into a single global preference providing a preferential semantics which allows a standard, KLM style, interpretation of defeasible inclusions.

In this paper, we have explored the relationships between the concept-wise multipreference semantics for the description logic  $\mathcal{LC}$  and self-organising maps. On the one hand, we have seen that self-organising maps can be given a logical semantics in terms of KLM-style preferential interpretations. The model can be used to learn or to validate conditional knowledge from the empirical data used for training and generalization, based on model checking. The learning process in the self-organising map can be regarded as an iterated belief change process. On the other hand, the plausibility of concept-wise multipreference semantics is supported by the fact that self-organising maps are considered as psychologically and biologically plausible neural network models.

The concept-wise multipreference semantics is related to the multipreference semantics for  $\mathcal{ALC}$  developed by Gliozzi [50], which is based on the idea of refining the rational closure construction considering the preference relations  $<_{A_i}$  associated to different aspects, but the concept-wise multipreference semantics follows a different route concerning both the definition of the preference relations associated with concepts, and the way of combining them in a single preference relation. The idea of having different preference relations, associated to different typicality operators, has been studied by Gil [37] to define a multipreference formulation of the description logic  $\mathcal{ALC} + \mathbf{T}_{min}$ , a typicality DL with a minimal model preferential semantics. In this proposal, we associate preferences with concepts, and we combine such preferences into a single global one. An extension of DLs with multiple preferences has also been developed by Britz and Varzinczak [18,15] to define defeasible role quantifiers and defeasible role inclusions, by associating multiple preference relations with roles. A related semantics with multiple preferences has also been proposed in the first-order logic setting by Delgrande and Rantsaudis [32].

Let us observe that alternative notions of preference combinations can be devised beyond the definition of a global preference relation <. In some cases, one may want to define preferences with respect to specified criteria of preference combination, as in Brewka's framework of basic preference descriptions [13]. In this direction, an algebraic framework for preference combination in Multi-Relational Contextual Hierarchies has been recently developed by Bozzato et al. [12]. A related problem of commonsense concept combination has been addressed in a probabilistic extension of the typicality description logic  $\mathcal{ALC} + \mathbf{T}_{B}$  [71]. Another simple approach to concept combination exploits the fuzzy interpretation of concepts, as introduced in Section 7, for associating preferences to complex concepts [41]. In fact, a fuzzy interpretation induces a preference ordering on the domain for each concept. This approach has been exploited for developing a fuzzy-multipreference semantics for Multilayer Perceptrons, which allows a deep neural network to be regarded as a weighted conditional knowledge base. The approach exploits a combination of the fuzzy and the multipreference semantics considered in this paper. A fuzzy extension of preferential logics has been previously studied by Casini and Straccia [22] for Gödel logic, based on the Rational closure construction.

The logical interpretation of a self-organising map through a fuzzy DL interpretation allows properties of the SOM, expressed as fuzzy concept inclusions, to be verified by model checking. The correspondence between neural network models and fuzzy systems

has been first investigated by Kosko in his seminal work [64]. In his view, "at each instant the n-vector of neuronal outputs defines a fuzzy unit or a fit vector. Each fit value indicates the degree to which the neuron or element belongs to the n-dimentional fuzzy set." In our approach, in a fuzzy interpretation of a SOM, each concept (representing a learned category) is regarded as a fuzzy set over a domain (here, a set of input stimuli) which is the usual way of viewing concepts in fuzzy description logics [88,73,7]. This allows the validation of fuzzy inclusion axioms over the model of the SOM, as usual in fuzzy DLs. In this direction, one could also consider using fuzzy modifiers (*very, slightly*, etc.), which have also been introduced in fuzzy DLs [73], for the formulation of properties to be checked over the fuzzy model of the SOM (e.g., are very big animals either elephants or hippos with a degree  $\geq 0.8$ ?).

The problem of learning fuzzy rules has been widely investigated in the context of fuzzy description logics [72,89] based on other machine learning approaches. The objective of our work is to show that a fuzzy interpretation of self-organising maps is possible and natural, and that, it can be used for the validation of fuzzy inclusion axioms over the SOM by model checking.

Much work has been devoted, in recent years, to the combination of neural networks and symbolic reasoning [27,29,28], leading to the definition of new computational models [66,85,56,67] and to extensions of logic programming languages with neural predicates [76,93]. Among the earliest systems combining logical reasoning and neural learning are the Knowledge-Based Artificial Neural Network (KBANN) [91] and the Connectionist Inductive Learning and Logic Programming (CILP) [30] systems. Penalty Logic [84], a non-monotonic reasoning formalism, was proposed as a mechanism to represent weighted formulas in energy-based symmetric connectionist networks (SCNs), where the search performed by the SCN for a global minimum may be viewed as a search for a model minimizing penalty. The relationships between normal logic programs and connectionist network have been investigated by Garcez and Gabbay [30,27] and by Hitzler et al. [55]. None on these approaches addresses the problem of developing a logical interpretation of SOMs.

The logical interpretation of self-organising maps in terms of multipreference and fuzzy interpretations, besides providing a logical interpretation to SOMs, which may be of interest from the side of explainable AI [1,52,2], can potentially be exploited, as described above, as a basis for an integrated use of self-organising maps and defeasible knowledge bases (resp., fuzzy knoweldge bases). While a neural network, once trained, is able and fast in classifying the new stimuli (that is, it is able to do instance checking), all other reasoning services such as satisfiability, entailment and model-checking are missing. These capabilities are needed for dealing with tasks combining empirical and symbolic knowledge, e.g., proving whether the network validates some (strict or conditional) properties; learning the weights of a conditional KB from empirical data; combining the defeasible inclusions extracted from a neural network with other defeasible or strict inclusions for inference. We have seen in the paper that some of these tasks (including property validation and the extraction of defeasible inclusions with a weight) can be achieved over the SOM model.

The idea of constructing a semantic interpretation of a neural network based on a fuzzy and/or a multipreference semantics has also been pursued for Multilayer Percep-

trons (MLPs) in [41]. A deep network is considered after the training phase, when the synaptic weights have been learned, to show that it can be associated with a preferential DL interpretation with multiple preferences, as well as with a semantics based on fuzzy DL interpretations and another one combining fuzzy interpretations with multiple preferences. The three semantics allow the input-output behavior of the network to be captured by interpretations built over a set of input stimuli through a simple construction, which exploits the activity level of neurons for the stimuli, those units whose meaning we want to reason about, including hidden units (see, for instance, the discussion in [41] of the well known Hinton's family example [54]). Logical properties can be verified over such models by model checking. Due to the diversity of the two neural models (SOMs and MLPs) we expect that this approach might be extended to other neural network models and learning approaches, by exploiting units activations or notions of distance of a stimulus from a category (as done for SOMs).

The relationships between the logics of common sense reasoning and Multilayer Perceptrons are even deeper, as a deep neural network can be regarded as a conditional knowledge base with weighted conditionals. This has been achieved by developing a concept-wise fuzzy multipreference semantics for a DL with weighted defeasible inclusions [41], as a combination of the fuzzy and the multipreference semantics considered in this paper.

Whether conditional description logics under a multi-preferential and/or fuzzy semantics can be regarded as possible candidates for neuro-symbolic integration is a subject for future investigation. An issue is the development of proof methods for such logics. An open problem is whether the notion of fuzzy-multipreference entailment is decidable, for which DLs fragments and under which choice of fuzzy logic combination functions. In the two-valued case multipreference entailment is decidable for weighted  $\mathcal{EL}^{\perp}$  KBs and can be computed based on ASP encodings [49], by exploiting preferential reasoning in *asprin* [14]. This is a first step towards the definition of proof methods for multi-valued extensions of our concept-wise preferential semantics based on a notion of faithful interpretations [38]. Another issue is whether the mapping of multilayer networks to weighted conditional knowledge bases can be extended to more complex neural network models, such as Graph neural networks [66], or whether different logical formalisms and semantics would be needed.

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