53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Event-Triggered Cooperative Control with General Linear Dynamics and Communication Delays

Eloy Garcia*, Yongcan Cao, and David W. Casbeer

***Abstract*— The consensus problem with general linear dynamics and communication delays is investigated. An event-triggered consensus protocol is proposed, where each agent implements a model of the decoupled dynamics of its neighbors. This approach not only avoids the need for continuous communication between agents but also provides a decentralized and asynchronous method for transmission of information in the presence of time-varying communication delays. This method gives more flexibility for scheduling information broadcasting compared to periodic and sampled-data implementations. Lower-bounds on the inter-event times are computed and shown to be positive in order to avoid Zeno behavior.**

I. Introduction

Cooperative control of multi-agent systems is an active research area with broad and relevant applications in commercial, academic, and military areas [1]. Design of decentralized and scalable control algorithms provides the necessary coordination for a group of agents to outperform a single or a number of systems operating independently. In scenarios where communication among agents is limited, decentralized computation of the time instants that each agent needs to transmit relevant information is also necessary.

Consensus with limited communication has been studied using the sampled-data (periodic) approach [2], and [3]. An important drawback of periodic transmission is that it requires synchronization between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations.

In the present paper, in lieu of periodic approaches, we use an asynchronous communication scheme based on event-triggered control strategies and we consider agents that are described by general linear models and subject to communication delays. In the context of event-triggered control, measurements are not transmitted periodically in time but they are triggered by the occurrence of certain events. In event-triggered broadcasting [4], [5], [6], [7], and [8], a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control strategies have been used for stabilization of multiple coupled subsystems as in [9], and [10] and in other multi-agent applications [11]. Consensus problems have also been studied using these techniques [12], [13], [14], [15], [16]. Event-triggered control provides a more robust and efficient use of network bandwidth. Its implementation in multi-agent systems also provides a highly decentralized way to schedule transmission instants which does not require synchronization compared to periodic sampled-data approaches.

Consensus problems where all agents are described by general linear models have been considered by different authors [17], [18], [19], [20], [21], and [22]. In these papers it is assumed that continuous communication between agents is possible.

Event-triggered consensus of agents with linear dynamics and limited communication was recently explored in [23] and [24]. In our previous work [25] and [26] we proposed a novel approach in which each agent implements models of the decoupled dynamics of each one of its neighbors and uses the model states to compute the local control input. This approach offered better performance than Zero-Order-Hold (ZOH) approach used in [23] and [24] where the updates from neighbors are kept constant by the local agent. A similar model-based framework was proposed in [27] where only constant thresholds were used. In this paper we follow a similar approach as in [25], [26]. One difference with respect to those papers is that the local control input is based on the local state and model states of neighbors rather than on model states of all, local agent and neighbors.

The main contribution of the present paper with respect to previous work [25], [26] is the design of events in the presence of communication delays while in those papers delays were assumed to be negligible. Decentralized event thresholds that guarantee practical consensus and strictly positive inter-event times are designed in this paper. The lower-bounds on the inter-event time intervals are independent of the particular system trajectories, therefore they hold for any two consecutive local events.

The remainder of the paper is organized as follows. Section II provides a brief background on graph theory and describes the problem. Section III gives a result assuming continuous communication which will be used throughout the document. Design of decentralized event thresholds which guarantee positive inter-event times is addressed in Section IV for the case of no communication delay. Section V extends this approach in order to consider time-varying, in a range, communication delays. Section VI presents an example and Section VII concludes the paper.

This work has been supported in part by AFOSR LRIR No. 12RB07COR.

Eloy Garcia is a contractor (Infoscitex Corp.) with the Control Science Center of Excellence, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433.

Yongcan Cao and David Casbeer are with the Control Science Center of Excellence, Air Force Research Laboratory, Wright-Patterson AFB, OH 45433.

* Corresponding Author: elgarcia@infoscitex.com

II. Preliminaries

A. Graph Theory

Consider a graph $\mathcal{G}=\{\mathcal{V},\mathcal{E}\}$ consisting of a set of vertices or nodes $\mathcal{V}=\{1,...,N\}$ and a set of edges $\mathcal{E}$. An edge between nodes i and j is represented by the pair $(i,j)\in\mathcal{E}$. A graph $\mathcal{G}$ is called undirected if $(i,j)\in\mathcal{E}\Leftrightarrow(j,i)\in\mathcal{E}$ and the nodes are called adjacent. The adjacency matrix $\mathcal{A}$ is defined by $a_{ij}=1$ if the nodes i and j are adjacent and $a_{ij}=0$ otherwise. If $(j,i)\in\mathcal{E}$, then j is said to be a neighbor of i. The set $\mathcal{N}_i$ is called the set of neighbors of node i, and N_i is its cardinality. A node j is an element of $\mathcal{N}_i$ if $(j,i)\in\mathcal{E}$. A path from node i to node j is a sequence of distinct nodes that starts at i and ends at j, such that every pair of consecutive nodes is adjacent. An undirected graph is connected if there is a path between every pair of distinct nodes. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined as $\mathcal{L}=\mathcal{D}-\mathcal{A}$ where $\mathcal{D}$ represents the degree matrix which is a diagonal matrix with entries $d_{ii}=\sum_{j\in\mathcal{N}_i}a_{ij}$. For undirected graphs, $\mathcal{L}$ is symmetric and positive semi-definite. $\mathcal{L}$ has zero row sums and, therefore, zero is an eigenvalue of $\mathcal{L}$ with associated eigenvector 1_N (a vector with all entries equal to one), that is, $\mathcal{L}1_N=0$. If an undirected graph is connected then $\mathcal{L}$ has exactly one eigenvalue equal to zero and all its non-zero eigenvalues are positive; they can be set in increasing order $\lambda_1(\mathcal{L})<\lambda_2(\mathcal{L})\leq\lambda_3(\mathcal{L})\leq...\leq\lambda_N(\mathcal{L})$, with $\lambda_1(\mathcal{L})=0$.

Lemma 1: Let $\mathcal{L}$ be the symmetric Laplacian of an undirected and connected graph. Then, consensus is achieved if and only if

$$V=\xi^T\hat{\mathcal{L}}\xi=0, \tag{1}$$

where $\hat{\mathcal{L}}=\mathcal{L}\otimes Q$, $Q\in\mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\xi(t)=\left[\xi_1(t)^T\xi_2(t)^T\ldots\xi_N(t)^T\right]^T$, and $\xi_i\in\mathbb{R}^n$.

B. Problem Statement

Consider a group of N agents with fixed communication graphs and fixed weights. Each agent can be described by

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t),\quad i=1...N, \tag{2}$$

with

$$u_i(t)=cF\sum_{j\in\mathcal{N}_i}(x_i(t)-y_j(t)),\quad i=1...N, \tag{3}$$

where $x_i\in\mathbb{R}^n$ is the state of agent i, $u_i\in\mathbb{R}^m$ is its control input. The matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$. F and c are design parameters that are defined below. The variables $y_j\in\mathbb{R}^n$ represent a model (or estimate) of the j^{th} agent's state using the decoupled dynamics:

$$\dot{y}_j(t)=Ay_j(t),\quad j=1...N. \tag{4}$$

Every agent in the network implements a model of itself $y_i(t)$ and also models of its neighbors $y_j(t)$. The model state $y_i(t)$ is not used by agent i for control since the real state is locally available but it is used to trigger local events. Local events for agent i are defined as follows. When an event is triggered, agent i will transmit its current state x_i to its neighbors. For negligible communication delays, agent i and its neighbors will all update their local models $y_i(t)$ at the same time instant. Since agent i and its neighbors use the same measurements to update the models, say, $x_i(t_{k_i})$ and the model dynamics (4) represent the decoupled dynamics where all agents use the same state matrix, then the model states $y_i(t)$ implemented by agent i and by its neighbors are the same. In the presence of communication delays the previous statement will not hold and we will differentiate between $y_{ii}(t)$, the model state of agent i as seen by agent i, and $y_{ij}(t)$, the model state of agent i as seen by agents j, $j\in\mathcal{N}_i$. More details concerning communication delays are presented in Section V.

The model update process is similar for all agents $i=1,...,N$. The local control input (3) is decentralized since it only depends on local information, that is, on the state of the local agent and on the model states of its neighbors. Continuous access to the states of neighbors is not needed. This approach can be seen as a generalization of the sample-data approach where Zero-Order-Hold (ZOH) models are used. However, because we consider unstable trajectories in general, the choice of ZOH models does not provide a good performance when considering general linear dynamics as it was in the case of single integrators [12], [13].

Note that the difference between the agent dynamics (2) and our proposed models (4) is given by the input term in (2) and this input decreases as the agents approach a consensus state. It can also be seen that in the particular case when systems (2) represent single integrator dynamics, then our models degenerate to ZOH models as in [12], [13].

III. Consensus with continuous measurements

Let us assume in this section that continuous communication between agents is possible, then (3) is given by:

$$u_i(t)=cF\sum_{j\in\mathcal{N}_i}(x_i(t)-x_j(t)),\quad i=1...N. \tag{5}$$

Assume that the pair (A,B) is controllable. Then, for $\alpha>0$ there exists a symmetric and positive definite solution P to

$$PA+A^TP-2PBB^TP+2\alpha P<0. \tag{6}$$

Let

$$F=-B^TP \tag{7}$$

$$c\geq 1/\lambda_2. \tag{8}$$

Theorem 1: Assume the pair (A,B) is controllable and the communication graph is connected and undirected. Define F and c as in (7) and (8). Then the following symmetric matrix

$$\bar{\mathcal{L}}=\hat{\mathcal{L}}A_c+A_c^T\hat{\mathcal{L}} \tag{9}$$

has only n eigenvalues equal to zero and the rest of its eigenvalues are negative. In addition, the eigenvectors associated with its n zero eigenvalues belong to the subspace spanned by the eigenvectors associated with the n zero eigenvalues of $\hat{\mathcal{L}}$, where $\hat{\mathcal{L}}=\mathcal{L}\otimes P$, $A_c=\bar{A}+\bar{B}$, $\bar{A}=I_N\otimes A$, $\bar{B}=c\mathcal{L}\otimes BF$.

Proof. See [25] for proof.

Lemma 2: Assume the pair (A,B) is controllable and the communication graph is connected and undirected. Then, protocol (5), with F and c defined in (7) and (8), solves the consensus problem for agents described by (2). Furthermore, the Lyapunov function defined by $V = x^T\hat{\mathcal{L}}x$ has a time derivative along the trajectories of (2) with inputs (5) given by $\dot{V} = x^T\bar{\mathcal{L}}x$.

From Theorem 1 it can be seen that $\dot{V}$ is negative when the overall system is in disagreement and is equal to zero only when the corresponding states are in total agreement. In the latter case we also have $V = 0$, see Lemma 1. Different from consensus with single integrators, where the agents converge to a constant value, here it is only required that the difference between states of agents tends to zero, regardless of the particular response of the systems.

IV. Decentralized Event Triggered Consensus

In this section we consider the case when agents use event-triggered communication strategies in order to reduce the frequency of transmissions. It is assumed in this section that communication delays are negligible. We derive decentralized thresholds that depend only on local information and can be measured and applied in a decentralized way. Lower-bounds on the inter-event times for each agent are also determined which exclude Zeno phenomena. Zeno phenomena in event-triggered control refers to the occurrence of an infinite number of triggering events in a finite time interval. Instead of c, the new coupling factor c_1 is now used in the inputs (3).

Define $e_i(t) = y_i(t) - x_i(t)$, $x = \left[x_1^T \dots x_N^T\right]^T$, $y = \left[y_1^T \dots y_N^T\right]^T$, $e = \left[e_1^T \dots e_N^T\right]^T$.

Theorem 2: Assume the pair (A, B) is controllable and the communication graph is connected and undirected. Define F in (7) and $c_1 = c + c_2$ where $c \geq 1/\lambda_2$ and $c_2 > 0$. Then agents (2) with inputs (3) achieve a bounded consensus error where the difference between any two states is bounded by

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|^2 \leq \frac{N\eta}{\beta\lambda_{\min}(P)} \tag{10}$$

for $i, j = 1, ..., N$, if the events are triggered when

$$k_e e_i^T PBB^T Pe_i > \sigma k_z z_i^T PBB^T Pz_i + \eta, \tag{11}$$

where $0 < \sigma < 1$, $\eta > 0$, $\beta = \frac{\lambda_{\min\neq 0}(-\bar{\mathcal{L}})}{\lambda_{\max}(\hat{\mathcal{L}})} > 0$,

$$\begin{aligned} k_e &= N_i\left(\tfrac{c-c_2}{b_i} + c(N-1)(b_i + \tfrac{1}{b_i})\right) \\ k_z &= 2c_2 - (c - c_2)b_i N_i \end{aligned} \tag{12}$$

and

$$z_i = \sum_{j\in\mathcal{N}_i} (x_i - y_j)\,. \tag{13}$$

The parameter b_i is given by $0 < b_i < \frac{2c_2}{N_i(c-c_2)}$ if $c > c_2$, and $b_i > 0$ otherwise. Furthermore, the agents do not exhibit Zeno behavior and the inter-event times $t_{k_i+1} - t_{k_i}$ for every agent $i = 1, ..., N$ are bounded by the *positive* times τ_i, that is

$$\tau_i \leq t_{k_i+1} - t_{k_i} \tag{14}$$

where

$$\tau_i = \frac{\ln\left[(\frac{\eta}{k_\tau})^{1/2} + 1\right]}{\|A\|}, \tag{15}$$

$$k_\tau = k_e \left\|PBB^T P\right\| \left(\frac{z_{i,\max}\|cBF\|}{\|A\|}\right)^2, \tag{16}$$

and $z_{i,\max}$ represents a bound on $z_i(t)$, that is, $\|z_i(t)\| \leq z_{i,\max}$.

Sketch of Proof. By implementing the coupling factor $c_1 = c+c_2$ in (3), we can write (2)-(3) in compact form as follows:

$$\dot{x} = (\bar{A} + \bar{B}_{\mathcal{D}})x + \bar{B}_{\mathcal{A}}y = (A_c + \bar{B}_2)x + \bar{B}_{\mathcal{A}}e \tag{17}$$

where $\bar{B}_{\mathcal{D}} = c_1\mathcal{D}\otimes BF$ and $\bar{B}_{\mathcal{A}} = -c_1\mathcal{A}\otimes BF$, $\bar{B}_1 = c_1\mathcal{L}\otimes BF$, $\bar{B}_2 = c_2\mathcal{L}\otimes BF$.

Consider the candidate Lyapunov function $V = x^T\hat{\mathcal{L}}x$ and evaluate the derivative along the trajectories of systems (2) with inputs (3). Assume that $b_i = b$ for simplicity of notation (note that we can always select $b = \min(b_i)$). We can express $\dot{V}$ as follows:

$$\begin{aligned} \dot{V} &= x^T\hat{\mathcal{L}}\left((A_c + \bar{B}_2)x + \bar{B}_{\mathcal{A}}e\right) \\ &\quad + \left((A_c + \bar{B}_2)x + \bar{B}_{\mathcal{A}}e\right)^T\hat{\mathcal{L}}x \\ &= x^T\bar{\mathcal{L}}x + 2\textstyle\sum_{i=1}^N \Big[-c_2\sum_{k\in\mathcal{N}_i}(x_i - x_k)^T \\ &\quad PBB^TP\textstyle\sum_{j\in\mathcal{N}_i}(x_i - x_j) \\ &\quad + c_1\textstyle\sum_{k\in\mathcal{N}_i}(x_i - x_k)^T PBB^TP\sum_{j\in\mathcal{N}_i} e_j\Big] \end{aligned} \tag{18}$$

After several manipulations and using the properties of undirected graphs it is possible to write

$$\begin{aligned} \dot{V} &\leq x^T\bar{\mathcal{L}}x + \textstyle\sum_{i=1}^N \left[-k_z z_i^T PBB^TPz_i\right. \\ &\quad \left. + k_e e_i^T PBB^T Pe_i\right] \end{aligned} \tag{19}$$

When threshold (11) holds, say, at time t_{k_i} then the error resets to zero, that is, $e_i(t_{k_i}) = 0$, since $y_i(t_{k_i}) = x_i(t_{k_i})$. This means that $k_e e_i^T(t_{k_i})PBB^TPe_i(t_{k_i}) = 0$; then the expression $k_e e_i^T PBB^T Pe_i \leq \sigma k_z z_i^T PBB^T Pz_i + \eta$ holds. Consequently,

$$\begin{aligned} \dot{V} &\leq x^T\bar{\mathcal{L}}x + \textstyle\sum_{i=1}^N \left[(\sigma - 1)k_z z_i^T PBB^TPz_i + \eta\right] \\ &\leq x^T\bar{\mathcal{L}}x + N\eta \end{aligned} \tag{20}$$

Furthermore, it can be shown that

$$V(t) \leq \left(V(0) - \tfrac{N\eta}{\beta}\right)e^{-\beta t} + \tfrac{N\eta}{\beta}. \tag{21}$$

and

$$\lambda_{\min}(P)\|x_i - x_j\|^2 \leq \left(V(0) - \tfrac{N\eta}{\beta}\right)e^{-\beta t} + \tfrac{N\eta}{\beta}. \tag{22}$$

Finally, the difference between any two states can be bounded as in (10).

In order to prove that the inter-event times are lower bounded by a positive constant we consider the following

$$\|e_i(t)\| \leq \frac{z_{i,\max}\|cBF\|}{\|A\|}\left(e^{\|A\|t} - 1\right). \tag{23}$$

for $t \in [t_{k_i}, t_{k_i+1})$. We use (23) to analyze the growth of the term $k_e e_i^T PBB^T Pe_i$ in the time interval $t \in [t_{k_i}, t_{k_i+1})$

$$k_e e_i(t)^T PBB^T Pe_i(t) \leq k_\tau \left(e^{\|A\|(t-t_{k_i})} - 1\right)^2 \tag{24}$$

where k_τ was defined in (16). We can see that the time it takes $k_e e_i^T PBB^T Pe_i$ to grow from zero to $\sigma k_z z_i^T PBB^T Pz_i + \eta$ after the last event at time t_{k_i} is no smaller than the time it takes the last expression in (24) to reach η. Then we solve for the time τ_i in the following equation, where τ_i represents a lower bound on the inter-event-times, i.e. $\tau_i \leq t_{k_i+1} - t_{k_i}$,

$$\left(e^{\|A\|\tau_i} - 1\right)^2 = \frac{\eta}{k_\tau}. \tag{25}$$

Since the right hand side of (25) is positive we obtain:

$$e^{\|A\|\tau_i} = \left(\frac{\eta}{k_\tau}\right)^{1/2} + 1 \tag{26}$$

Note that the right hand side of (26) is strictly greater than one. Solving for the time $\tau_i > 0$ we obtain (15) which shows that the inter-event times are strictly positive and we can guarantee that Zeno behavior does not occur at any node. •

Remark 1. The design parameter η provides a tradeoff between performance as measured by the consensus error (10) and the bound on the inter-event intervals (15). Also note that the variables used to compute the threshold (11), which define the events at node i, are available locally.

Remark 2. Note that the triggering of a local event by agent i, i.e. the transmission of a measurement $x_i(t_{k_i})$, does not change the local control input u_i since u_i is not a function of y_i. The transmission of a new measurement updates the control inputs of neighbor agents and for the local agent, it only resets its local state error e_i.

V. Decentralized Event Triggered Consensus With Communication Delays

In this section we consider the existence of time-varying but bounded communication delays. Since the measurement updates will be delayed, the neighbors of an agent i will have a version of agent i's model state that it is different than agent i's version. It is necessary to distinguish between the model state as seen by the local agent and as seen by its neighbors. Define the dynamics and update law of the model state of agent i as seen by agent i as

$$\dot{y}_{ii}(t) = Ay_{ii}(t),\ y_{ii}(t_{k_i}) = x_i(t_{k_i}). \tag{27}$$

The measurement $x_i(t_{k_i})$ is transmitted by agent i at time t_{k_i} and will arrive to agents j, $j \in \mathcal{N}_i$, at time $t_{k_i} + d_i(t_{k_i})$. Define the dynamics and update law of the model state of agent i as seen by agent j, $j \in \mathcal{N}_i$, as

$$\begin{aligned}&\dot{y}_{ij}(t) = Ay_{ij}(t),\\ &y_{ij}(t_{k_i} + d_i(t_{k_i})) = f_d(x_i(t_{k_i}), d_i(t_{k_i}))\end{aligned} \tag{28}$$

where t_{k_i} represents the update instants triggered by agent i and $d_i(t_{k_i})$ represents the communication delay associated to the triggering instant t_{k_i}.

Define a positive and constant upper bound on the communication delays by $d \leq \tau$, that is, $d_i(t_{k_i}) < d$ for any triggering instant t_{k_i} and for $i = 1, ..., N$, where τ represents a lower-bound on the inter-event times of any agent and it will be computed later in this section. Assume that the current delay is known to the receiving agents, for instance, by applying time-stamping techniques.

Since both, y_{ii} and y_{ij}, use the same state matrix for their continuous evolution between their corresponding update instants, then we define

$$f_d(x_i(t_{k_i}), d_i(t_{k_i})) = e^{Ad_i(t_{k_i})}x_i(t_{k_i}). \tag{29}$$

By definition, we have that the following *local* triggering event will occur at time $t_{k_i+1} \geq t_{k_i} + d$, this means that $y_{ii}(t_{k_i} + d_i(t_{k_i})) = e^{Ad_i(t_{k_i})}x_i(t_{k_i})$ because no other local event has been triggered since time instant t_{k_i}. Therefore,

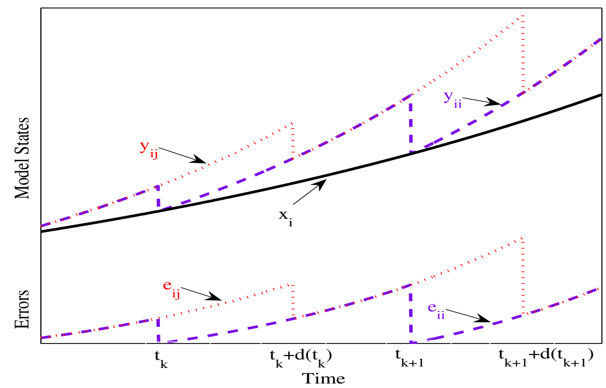

Fig. 1. Relation between state x_i, model states y_{ii}, y_{ij}, and corresponding errors e_{ii}, e_{ij}.

we have that $y_{ii}(t) \neq y_{ij}(t)$ for $t \in [t_{k_i}, t_{k_i} + d_i(t_{k_i}))$ and $y_{ii}(t) = y_{ij}(t)$ for $t \in [t_{k_i} + d_i(t_{k_i}), t_{k_i+1})$.

Define the state errors

$$e_{ii}(t) = y_{ii}(t) - x_i(t), \tag{30}$$

$$e_{ij}(t) = y_{ij}(t) - x_i(t). \tag{31}$$

Note that $e_{ii}(t_{k_i}) = 0$ and $e_{ij}(t) = e_{ii}(t)$, for $t \in [t_{k_i} + d_i(t_{k_i}), t_{k_i+1})$. These relations are pictured in Fig. 1.

Also define $\zeta = [e_{11}^T \ldots e_{NN}^T]^T \in \mathbb{R}^{nN}$ and $\zeta_d = [e_{1j_1}^T \ldots e_{Nj_N}^T]^T \in \mathbb{R}^{nN}$, where the components e_{ij_i} in ζ_d represent the errors defined in (31), that is, the error of agent i as seen by agents j_i, $j_i \in \mathcal{N}_i$.

The dynamics of every agent in (2) with communication delays captured by the new input definitions (which are functions of delayed model states y_{ji}):

$$u_i(t) = c_1 F \sum_{j \in \mathcal{N}_i} (x_i(t) - y_{ji}(t)), \quad i = 1...N, \tag{32}$$

can be written in compact form as follows:

$$\dot{x} = (A_c + \bar{B})x + \bar{B}_{\mathcal{A}}\zeta_d \tag{33}$$

where the coupling strength $c_1 = 2c$ has been used.

Note that if we follow the same analysis as in Theorem 2 we can arrive at a similar expression to (19) involving the local control inputs z_i and the delayed errors e_{ij}. This will create a major difficulty in designing the local events since the local agent i does not have access to the errors e_{ij} as seen by its neighbors. More importantly, the local agent is not able to reset the error e_{ij} but only the local error e_{ii}.

The following theorem provides a method to design local events in the presence of communication delays and using the local state errors e_{ii}.

Theorem 3: Assume the pair (A, B) is controllable and the communication graph is connected and undirected. Define F in (7) and $c_1 = 2c$ where $c \geq 1/\lambda_2$. Then agents (2) with inputs (32) achieve, in the presence of communication delays $d_i < d$, a bounded consensus error where the difference between any two states is bounded by (10) for $i, j = 1, ..., N$, if the events are triggered when

$$k_{\mathcal{A}}\delta_i > \sigma z_i^T PBB^T Pz_i + \eta \tag{34}$$

where $0 < \sigma < 1$,

$$z_i = \sum_{j \in \mathcal{N}_i} (x_i - y_{ji}), \tag{35}$$

$$\delta_i = e_{ii}^T (e^{Ad})^T e^{Ad} e_{ii} + 2k_m \left\| e^{Ad} e_{ii} \right\| (e^{\|A\|d} - 1) + k_m^2 (e^{\|A\|d} - 1)^2, \tag{36}$$

$$\eta = k_{\mathcal{A}} k_m^2 (\left\| e^{Ad} \right\| + 1)^2 (e^{\|A\|d} - 1)^2, \tag{37}$$

$k_m = \frac{z_{i,\max} \|cBF\|}{\|A\|}$, and $k_{\mathcal{A}} \geq \left\| (\mathcal{A} \otimes P) \bar{B}_{\mathcal{A}} \right\|$. Furthermore, the agents do not exhibit Zeno behavior and the inter-event times $t_{k_i+1} - t_{k_i}$ for every agent $i = 1, ..., N$ are bounded by the *positive* times $\tau = d$, that is

$$d \leq t_{k_i+1} - t_{k_i}. \tag{38}$$

Proof. Since the only condition on the design parameter c_2 is $c_2 > 0$ we made the choice $c_2 = c$ (which makes $c_1 = 2c$) in order to simplify the following analysis which addresses the case of communication delays.

Consider the candidate Lyapunov function $V = x^T \hat{\mathcal{L}} x$ and evaluate the derivative along the trajectories of systems (2) with inputs (32).

$$\begin{aligned} \dot{V} &= x^T \hat{\mathcal{L}} \left((A_c + \bar{B}) x + \bar{B}_{\mathcal{A}} \zeta_d \right) \\ &\quad + \left((A_c + \bar{B}) x + \bar{B}_{\mathcal{A}} \zeta_d \right)^T \hat{\mathcal{L}} x. \end{aligned} \tag{39}$$

It can be shown that (39) can be written as

$$\begin{aligned} \dot{V} &= x^T \bar{\mathcal{L}} x + 2c \textstyle\sum_{i=1}^N \left[-z_i^T P B B^T P z_i \right. \\ &\quad \left. + \textstyle\sum_{k \in \mathcal{N}_i} e_{ki}^T P B B^T P \textstyle\sum_{j \in \mathcal{N}_i} e_{ji} \right] \end{aligned} \tag{40}$$

where e_{ji} represents the state error of agent j as seen by agent i for $j \in \mathcal{N}_i$. The last term in (40) satisfies

$$\begin{aligned} &2c \textstyle\sum_{i=1}^N \sum_{k \in \mathcal{N}_i} e_{ki}^T P B B^T P \sum_{j \in \mathcal{N}_i} e_{ji} \\ &= \zeta_d^T (\mathcal{A} \otimes P) \bar{B}_{\mathcal{A}} \zeta_d \leq k_{\mathcal{A}} \zeta_d^T \zeta_d. \end{aligned} \tag{41}$$

Define $\nu_i = y_{ij} - y_{ii}$, then we have

$$e_{ij} = y_{ij} - x_i = y_{ij} - (y_{ii} - e_{ii}) = \nu_i + e_{ii}. \tag{42}$$

The term $\nu_i(t)$ is piece-wise continuous defined as follows:

$$\nu_i(t) = \begin{cases} e^{A(t - t_{k_i})} e_{ii}(t_{k_i}^-), & t \in [t_{k_i}, t_{k_i} + d_i(t_{k_i})) \\ 0, & t \in [t_{k_i} + d_i(t_{k_i}), t_{k_i+1}) \end{cases} \tag{43}$$

The initial condition in (43) is obtained by simply realizing that $\nu_i(t_{k_i}) = y_{ij}(t_{k_i}) - y_{ii}(t_{k_i}) = y_{ii}(t_{k_i}^-) - x_i(t_{k_i}^-) = e_{ii}(t_{k_i}^-)$, (the local error just before it resets to zero), since the local model y_{ii} is updated using $x_i(t_{k_i})$ and $y_{ij}(t_{k_i}) = y_{ij}(t_{k_i}^-) = y_{ii}(t_{k_i}^-)$. Define $\nu = [\nu_1^T \ldots \nu_N^T]^T$ and we have that

$$\zeta_d = \nu + \zeta. \tag{44}$$

Consider the worst case scenario (greatest difference between ζ_d and ζ) given when all agents transmit at the same instant t_k and the greatest possible delay $d^- = d - \epsilon$, for a small $\epsilon > 0$, is present. Then, we have the following

$$\begin{aligned} &\zeta_d^T (t_k + d^-) \zeta_d (t_k + d^-) \\ &= \zeta^T (t_k^-) (e^{\bar{A}d})^T e^{\bar{A}d} \zeta(t_k^-) + 2\zeta^T (t_k + d^-) e^{\bar{A}d} \zeta(t_k^-) \\ &\quad + \zeta^T (t_k + d^-) \zeta(t_k + d^-) \\ &= \textstyle\sum_{i=1}^N \left[e_{ii}^T (t_k^-) (e^{Ad})^T e^{Ad} e_{ii}(t_k^-) \right. \\ &\quad \left. + 2e_{ii}^T (t_k + d^-) e^{Ad} e_{ii}(t_k^-) + e_{ii}^T (t_k + d^-) e_{ii}(t_k + d^-) \right] \\ &\leq \textstyle\sum_{i=1}^N \left[e_{ii}^T (t_{k_i}^-) (e^{Ad})^T e^{Ad} e_{ii}(t_{k_i}^-) \right. \\ &\quad \left. + 2k_m \left\| e^{Ad} e_{ii}(t_{k_i}^-) \right\| (e^{\|A\|d} - 1) + k_m^2 (e^{\|A\|d} - 1)^2 \right] \end{aligned} \tag{45}$$

where $e_{ii}^T(t_{k_i}^-)$ represents the error just before the update instant, i.e., before it is reset to zero because of the update at time t_{k_i}. On the other hand, $e_{ii}(t_{k_i} + d^-)$ represents the error after the update at time t_{k_i} and it can only be estimated using (23).

Since the worst case is given by the maximum delay d^- we can use (45) and the current local error $e_{ii}(t)$ to bound the delayed error $e_{ij}(t + d)$ for any time $t > 0$ and the following holds:

$$\dot{V} \leq x^T \bar{\mathcal{L}} x + \sum_{i=1}^N \left[-2c z_i^T P B B^T P z_i + k_{\mathcal{A}} \delta_i \right]. \tag{46}$$

Then, the local thresholds can be defined based on the local errors $e_{ii}(t)$ as in (34) with δ_i given by (36). When an event is triggered the error e_{ii} is reset to zero and the following holds

$$\begin{aligned} \dot{V} &\leq x^T \bar{\mathcal{L}} x + 2c \textstyle\sum_{i=1}^N (\sigma - 1) z_i^T P B B^T P z_i + N\eta \\ &\leq x^T \bar{\mathcal{L}} x + N\eta \end{aligned} \tag{47}$$

and the bound (10) on the difference between any two states follows.

The final task is to determine η such that the inter-event times satisfy (38) for a given d. At time t_{k_i} we have $e_{ii}(t_{k_i}) = 0$ and $\delta_i(t_{k_i}) = k_m^2 (e^{\|A\|d} - 1)^2$. The error response $e_{ii}(t)$ can be bounded using (23) for $t \in [t_{k_i}, t_{k_i+1})$. During the same time interval we can use (23) to obtain the following

$$\begin{aligned} k_{\mathcal{A}} \|\delta_i\| \leq k_{\mathcal{A}} &\left[k_m^2 \left\| e^{Ad} \right\|^2 (e^{\|A\|(t - t_{k_i})} - 1)^2 \right. \\ &+ 2k_m^2 \left\| e^{Ad} \right\| (e^{\|A\|(t - t_{k_i})} - 1)(e^{\|A\|d} - 1) \\ &\left. + k_m^2 (e^{\|A\|d} - 1)^2 \right]. \end{aligned} \tag{48}$$

Similarly, it can be seen that the time it takes $k_{\mathcal{A}} \delta_i$ to grow from $k_{\mathcal{A}} k_m^2 (e^{\|A\|d} - 1)^2$ to $\sigma z_i^T P B B^T P z_i + \eta$ is no smaller than the time it takes the right hand side of (48) to grow from $k_{\mathcal{A}} k_m^2 (e^{\|A\|d} - 1)^2$ to η. Then, in order to guarantee that the inter-event times are lower-bounded by $\tau = d > d_i$ we find the term η using (48) when $t - t_{k_i} = d$ and we obtain (37) which guarantees that Zeno behavior does not occur at any node since the inter-event times are strictly positive and lower-bounded by d as in (38). Furthermore, by design of $\tau = d$ we also guarantee that no additional event is triggered by the local agent before the current transmitted measurement is received by the neighbor agents, i.e., $t_{k_i+1} > t_{k_i} + d_i(t_{k_i})$, for $i = 1, ..., N$. •

VI. EXAMPLE

Consider a decentralized model-based implementation of four second order agents ($N = 4$, $n = 2$) with unstable linear dynamics given by:

$$A = \begin{bmatrix} 0.38 & -2 \\ 1.62 & 0.08 \end{bmatrix}, B = \begin{bmatrix} 0.45 \\ -1.87 \end{bmatrix}, P = \begin{bmatrix} 0.5352 & -0.1804 \\ -0.1804 & 0.4009 \end{bmatrix}$$

where the matrix P is obtained by solving (6). The upper-bound on the communication delays is $d = 0.01$ seconds; The nonzero elements of the undirected adjacency matrix are $a_{12} = a_{23} = a_{34} = 1$ (the corresponding symmetric elements are also equal to one). Fig. 2 shows the response of the agents and the models. Fig. 3 shows that the disagreement between agents converge to a bounded region; it also shows the communication time instants for every agent.

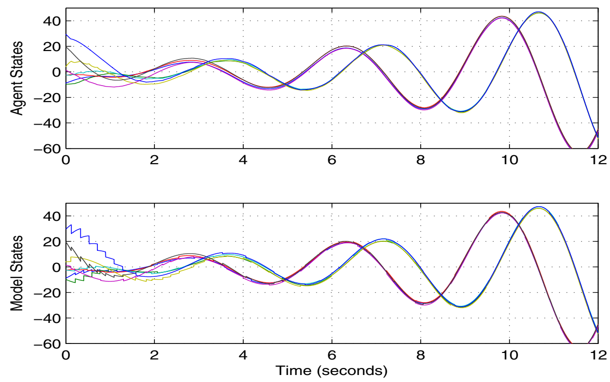

Fig. 2. Top: States of four agents in Example 1. Bottom: States of corresponding models y_{ij}.

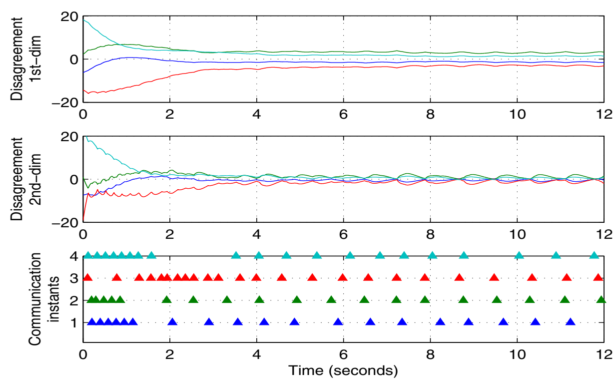

Fig. 3. First two plots show the disagreement for the dimensions of the states of the agents. The last plot shows the transmission instants for each one of the four agents.

VII. Conclusions

Event-based consensus protocols for linear systems with limited communication and transmission delays have been studied. Decentralized events have been designed in order for each agent to broadcast measurement updates only when it is necessary, that is, when a function of discrepancy between real and model states is greater than a specified threshold. The decentralized event triggered technique allows each agent to transmit information based on its own decisions and synchronization of updates is not required as in periodic approaches. The use of models and event-based techniques provides a formal framework that reduces communication and provides freedom to each agent in order to determine its own broadcasting instants.

References

[1] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[2] Y. Cao and W. Ren, "Multi-vehicle coordination for double integrator dynamics under fixed undirected/directed interaction in a sampled-data setting," *International Journal of Robust and Nonlinear Control*, vol. 20, pp. 987–1000, 2010.

[3] J. Qin and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double integrator dynamics with nonuniform and time-varying communication delays," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2417–2422, 2012.

[4] K. J. Astrom, "Event based control," in *A. Astolfi and L. Marconi, (eds.), Analysis and Design of Nonlinear Control Systems*, Springer-Verlag, Berlin, 2008, pp. 127–147.

[5] M. C. F. Donkers and W. P. M. H. Heemels, "Output-based event-triggered control with guaranteed linfty-gain and improved event-triggering," in *49th IEEE Conference on Decision and Control*, 2010, pp. 3246–3251.

[6] C. D. Persis, R. Sailer, and F. Wirth, "Parsimonious event-triggered distributed control: a zeno free approach," *Automatica*, vol. 49, no. 7, pp. 2116–2124, 2013.

[7] E. Garcia and P. J. Antsaklis, "Model-based event-triggered control for systems with quantization and time-varying network delays," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 422–434, 2013.

[8] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, 2007.

[9] E. Garcia and P. J. Antsaklis, "Decentralized model-based event-triggered control of networked systems," in *American Control Conference*, 2012, pp. 6485–6490.

[10] M. Guinaldo, D. V. Dimarogonas, K. H. Johansson, J. Sanchez, and S. Dormido, "Distributed event-based control for interconnected linear systems," in *2011 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*. IEEE, 2011, pp. 2553–2558.

[11] C. Nowzari and J. Cortés, "Self-triggered coordination of robotic networks for optimal deployment," *Automatica*, vol. 48, no. 6, pp. 1077–1087, 2012.

[12] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[13] E. Garcia, Y. Cao, H. Yu, P. J. Antsaklis, and D. W. Casbeer, "Decentralized event-triggered cooperative control with limited communication," *International Journal of Control*, vol. 86, no. 9, pp. 1479–1488, 2013.

[14] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[15] X. Chen and F. Hao, "Event-triggered average consensus control for discrete-time multi-agent systems," *IET Control Theory and Applications*, vol. 6, no. 16, pp. 2493–2498, 2012.

[16] M. Guo and D. V. Dimarogonas, "Nonlinear consensus via continuous, sampled, and aperiodic updates," *International Journal of Control*, vol. 86, no. 4, pp. 567–578, 2013.

[17] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint," *IEEE Transactions on Circuits and Systems-I: Regular papers*, vol. 57, no. 1, pp. 213–224, 2010.

[18] C. Q. Ma and J. F. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1263–1268, 2010.

[19] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.

[20] Y. Su and J. Huang, "Stability of a class of linear switching systems with applications to two consensus problems," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1420–1430, 2012.

[21] S. E. Tuna, "Synchronizing linear systems via partial-state coupling," *Automatica*, vol. 44, no. 8, pp. 2179–2184, 2008.

[22] S. E. Tuna, "Conditions for synchronizability in arrays of coupled linear systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2416–2420, 2009.

[23] T. Liu, D. J. Hill, and B. Liu, "Synchronization of dynamical networks with distributed event-based communication," in *51st IEEE Conference on Decision and Control*, 2012, pp. 7199–7204.

[24] W. Zhu, Z.-P. Jiang, and G. Feng, "Event-based consensus of multi-agent systems with general linear models," *Automatica*, vol. 50, no. 2, pp. 552–558, 2014.

[25] E. Garcia, Y. Cao, and D. W. Casbeer, "Cooperative control with general linear dynamics and limited communication: centralized and decentralized event-triggered control strategies," in *American Control Conference*, 2014.

[26] E. Garcia and Y. Cao and D. W. Casbeer, "Decentralized event-triggered consensus with general linear dynamics," *Automatica*, 2014.

[27] O. Demir and J. Lunze, "Event-based synchronisation of multi-agent systems," in *IFAC Conference on Analysis and Design of Hybrid Systems*, 2012, pp. 1–6.