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▶ To cite this version:

Ling Ma, Vincent Andrieu, Daniele Astolfi, Cheng-Zhong Xu, Xuyang Lou. Boundary control with integral action for a class of gantry crane systems. 62nd IEEE Conference on Decision and Control (CDC 2023), IEEE, Dec 2023, Singapour, Singapore. 10.1109/CDC49753.2023.10383266 . hal-04195996

HAL Id: hal-04195996
https://hal.science/hal-04195996v1

Submitted on 5 Sep 2023

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Boundary control with integral action for a class of gantry crane systems

Ling Ma, Vincent Andrieu, Daniele Astolfi, Cheng-Zhong Xu, and Xuyang Lou

Abstract—**In this paper, we focus on the output regulation problem of a class of gantry crane systems governed by partial differential equations (PDEs) in the presence of unknown constant disturbances. We first design a preliminary state-feedback controller for the system without perturbations and establish the existence of a strict Lyapunov function under the preliminary control law. By employing the forwarding method, we then add an integral action to the preliminary controller and analyze the well-posedness of the resulting closed-loop system, demonstrating its exponential stability. Furthermore, we extend our analysis to consider the system with perturbations, solving the output regulation problem. Finally, we provide numerical simulation results to demonstrate the effectiveness of our proposed strategy.**

Index Terms—**Integral action, boundary control, output regulation, forwarding method, gantry crane system.**

I. Introduction

In this article, our main focus is on the output regulation problem of a class of gantry crane systems that are widely used in various industry settings. Due to the complexity of the structure, gantry crane systems can be more accurately described by partial differential equations (PDEs), which has drawn the attention of many researchers, e.g. [1]–[3]. Controlling such systems has therefore been an important research topic. For example, in [4] and [5], the authors proposed feedback controllers to achieve asymptotic and exponential stability of a PDE interconnected with an ordinary differential equation (ODE) system. More recently, [6] introduced a nonlinear feedback law for a PDE-ODE system that achieves finite-time stabilization of the closed-loop system. However, to the best of the authors' knowledge, the output regulation problem and the use of integral actions for this class of systems have not been studied yet.

With output regulation we refer to the problem of designing a controller to ensure that the output converges to a desired behavior despite external disturbances or model uncertainties. This problem can be solved by following the so-called internal model principle [7], [8] and it has attracted a lot of attention in the context of PDE control, see, e.g. [9]–[13]. In the context of constant references and perturbations, this implies adding an integral action in the control feedback. Such an approach has been used in the context of PDE control in many works. For instance, Trinh et al. design an integral controller for nonlinear systems described by scalar hyperbolic PDEs to tackle the output regulation problem and achieve local exponential stability in [14]. Similarly, in [15], a PI controller is suggested to regulate the bottom velocity of a drill pipe, leading to exponential stability. Based on these prior work, we aim to design a feedback controller with integral action for regulating the trolley's position of the system with perturbations. This control design ensures accurate control of the trolley's position, which is crucial for safe and efficient operation of the system. Other examples of use of integral-based controllers in the context of PDEs and linear operators can be found in [16]–[23].

The forwarding method has been shown to be an effective tool for obtaining an integral-based controller to solve both regulation and stabilization problems [23]–[25]. This approach involves constructing a strict Lyapunov function, as demonstrated in [19], [26], and applying the Lyapunov direct method to derive the controller and the feedback gains. Besides, the forwarding method can be applied both in finite-dimensional [27] and infinite-dimensional systems [22], [23]. Overall, the forwarding method offers a promising approach to controller design in a variety of systems. Inspired by this method, we prove that the regulation and stabilization control goals can be achieved in this paper. The article makes two main contributions:

1) the well-posedness of the closed-loop system under the designed boundary controller with integral action is proven by using semigroup theory;
2) the proposed boundary controller with integral action achieves both exponential stability and output regulation for the gantry crane system with perturbations.

The paper is structured as follows. Section II introduces the problem to be addressed in this work, and Section III presents the boundary controller with integral action. The main results of this paper are outlined in Section IV. The simulation results are illustrated in Section V, and concluded remarks are given in Section VI.

Notation: In this article, the non-negative real numbers are denoted by $\mathbb{R}_+ := [0, \infty)$. Given two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we will use the compact notation $(x, y) := (x^\top, y^\top)^\top$. Given a function $y : (x, t) \mapsto y(x, t)$ defined on the domain $[0, L] \times \mathbb{R}_+$, we use the notation $y_t = \frac{\partial y}{\partial t}$, $y_x = \frac{\partial y}{\partial x}$, $y_{tt} = \frac{\partial^2 y}{\partial t^2}$, $y_{xx} = \frac{\partial^2 y}{\partial x^2}$, and $y_{xt} = \frac{\partial^2 y}{\partial x \partial t}$.

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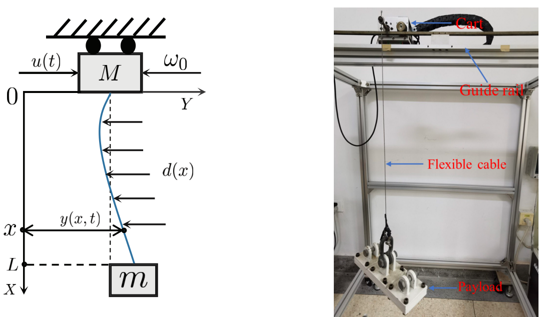
This work was supported by the China Scholarship Council (No. 202206790086) and by the French Grants ODISSE (ANR-19-CE48-0004-01) and Alligator (ANR-22-CE48-0009-01).

If y is a single-variable function, then y' and $\dot{y}$ represent its space and time derivatives, respectively. $L^2(0, L)$ represents the space of the square-integral functions y in $[0, L]$ such that $\int_0^L |y(x)|^2 \mathrm{d}x < +\infty$ with associated norm $\|y(x)\|_{L^2}^2 = \int_0^L |y(x)|^2 \mathrm{d}x$. $H^p(0, L)$ denotes the Sobolev space of order p. Given a Hilbert space $\mathbb{X}_1$, the notation $\mathfrak{L}(\mathbb{X}_1, \mathbb{X}_1)$ represents the set of linear bounded operators that map from $\mathbb{X}_1$ to $\mathbb{X}_1$. Let $T > 0$, the function space $C^1([0, T])$ consists of continuous functions defined on the interval $[0, T]$ that have a continuous first-order derivative on this interval.

II. Problem statement



(a) Schematic diagram. (b) Physical system.

Fig. 1: The gantry crane system: schematic diagram and physical system.

As illustrated in Fig. 1, a typical gantry crane system contains three parts: the top cart of mass M, which can move along the horizontal guide rail, a flexible cable of length L and mass per unit length ρ, and a payload of mass m. In this article we model the gantry crane system as in [28], with the additional assumption (iii) in [5] that the acceleration of the load mass is negligible with respect to the gravitational acceleration g. As a consequence, the governing equation is given by

$$\rho y_{tt}(x,t) = (S(x) y_x(x,t))_x + d(x) \tag{1a}$$

for all $(x, t) \in (0, L) \times \mathbb{R}_+$, with two boundary conditions given by

$$M y_{tt}(0,t) = u(t) + S(0) y_x(0,t) + \omega_0, \tag{1b}$$
$$y_x(L,t) = 0, \tag{1c}$$

where $y(x, t)$ describes the transversal displacement of the cable at space variable x and time variable t, u is a control force which is applied at the trolley and $S(x) = mg + \rho g(L - x)$ is the tension of the flexible cable for $x \in [0, L]$. Note that $S(x) \geq S^0 > 0$ for some constant S^0. The parameter $\omega_0 \in \mathbb{R}$ is an unknown constant disturbance and d is a constant unknown distributed perturbation acting on the flexible cable, such as wind. These disturbances represent model uncertainties that may impact the behavior of the system.

Our control objective is to obtain both an asymptotic stability property of an equilibrium point and moreover to achieve output regulation for the output $y(0, t)$ at a desired reference. Specifically, for any constant perturbation (ω_0, d), the asymptotic stability of the equilibrium of the resulting closed-loop system should be guaranteed (even in the presence of small model uncertainties, see, e.g. [11]), and moreover, at that equilibrium the output $y(0, t)$ has to be regulated to a given constant reference y_{ref}, i.e., the regulation objective is

$$\lim_{t \to \infty} |y(0,t) - y_{\mathrm{ref}}| = 0. \tag{2}$$

III. Preliminary state-feedback design

A. Change of coordinates and state space definition

To begin with, we consider the case in which the perturbations d and ω_0 and the reference y_{ref} are set to be zero. In this case, the considering system can be rewritten as follows:

$$\begin{cases} \rho y_{tt}(x,t) = (S(x) y_x(x,t))_x, \\ \dot{\chi}(t) = \dfrac{1}{M}\big(u(t) + S(0) y_x(0,t)\big), \end{cases} \tag{3a}$$

with the boundary conditions

$$\begin{aligned} y_t(0,t) &= \chi(t), \\ y_x(L,t) &= 0, \end{aligned} \tag{3b}$$

for all $(x, t) \in (0, L) \times \mathbb{R}_+$. We introduce now a linear operator $\mathcal{T}$, which is used to map the state variables (y, y_t, χ) into a set of new state variables (y, y_t, ξ) and expressed as:

$$\Theta = \begin{pmatrix} y \\ y_t \\ \chi \end{pmatrix} \mapsto \zeta = \begin{pmatrix} y \\ y_t \\ \xi \end{pmatrix} := \mathcal{T} \begin{pmatrix} y \\ y_t \\ \chi \end{pmatrix}, \tag{4}$$

where $\xi = -k_a y'(0) + k_p y(0) + \chi$ with $k_a > 0$ and $k_p > 0$ are adjustable parameters. In the new coordinates, the boundary conditions (3b) read

$$\begin{aligned} y_t(0,t) &= \xi(t) + k_a y_x(0,t) - k_p y(0,t), \\ y_x(L,t) &= 0. \end{aligned}$$

We define now the following Hilbert spaces

$$\begin{aligned} \mathbb{X} &= H^1(0,L) \times L^2(0,L) \times \mathbb{R}, \\ \mathbb{X}_1 &= H^2(0,L) \times H^1(0,L) \times \mathbb{R}, \end{aligned}$$

which are respectively equipped with the following inner products

$$\begin{aligned} \langle \zeta_1, \zeta_2 \rangle_{\mathbb{X}} :=& \xi_1 \xi_2 + \int_0^L y_{t1}(x) y_{t2}(x) \mathrm{d}x \\ &+ \int_0^L \big(y_1'(x) y_2'(x) + y_1(x) y_2(x)\big) \mathrm{d}x, \end{aligned} \tag{5}$$

$$\begin{aligned} \langle \zeta_1, \zeta_2 \rangle_{\mathbb{X}_1} :=& \int_0^L \left(y_{t1}'(x) y_{t2}'(x) + y_{t1}(x) y_{t2}(x)\right) \mathrm{d}x + \xi_1 \xi_2 \\ &+ \int_0^L \big(y_1''(x) y_2''(x) + y_1'(x) y_2'(x) + y_1(x) y_2(x)\big) \mathrm{d}x, \end{aligned} \tag{6}$$

where $\zeta_i = \big(y_i, y_{ti}, \xi_i\big) \in \mathbb{X}$ or $\zeta_i = \big(y_i, y_{ti}, \xi_i\big) \in \mathbb{X}_1$ $i = 1, 2$. The following lemma shows that $\mathcal{T}$ is a bounded linear operator in $\mathbb{X}_1$ which defines a valid change of coordinates.

Lemma 1. *$\mathcal{T}$ is a bounded linear operator from $\mathbb{X}_1$ to $\mathbb{X}_1$, i.e., $\mathcal{T} \in \mathfrak{L}(\mathbb{X}_1, \mathbb{X}_1)$, and there exist two positive real numbers $\overline{C}$ and $\underline{C}$ such that $\underline{C}\|\Theta\|_{\mathbb{X}_1} \leq \|\mathcal{T}\Theta\|_{\mathbb{X}_1} \leq \overline{C}\|\Theta\|_{\mathbb{X}_1}$ for all $\Theta \in \mathbb{X}_1$.*

The proof is given in Appendix A.

Remark 1. *In the rest of the article, we will state our main results only for solutions in $\mathbb{X}_1$. This is motivated by two reasons. First because the operator $\mathcal{T}$ is invertible only in $\mathbb{X}_1$, and second because for output regulation more regular solutions are typically required, see, e.g. [15].*

B. Preliminary feedback and ISS Lyapunov functional

In the new coordinates system defined in (4), we introduce a preliminary state-feedback control law, which employs signals obtained from sensors located at the trolley end.

More specifically, we design a preliminary boundary controller as follows

$$\begin{aligned} u_p(t) = &- k_1\xi(t) - S(0)y_x(0,t) + Mk_a y_{xt}(0,t) \\ &- Mk_p y_t(0,t) + \nu, \end{aligned} \tag{7}$$

for all t on the time domain of existence of the solution and where $k_1 > 0$ is a controller gain and ν can be considered either as an external perturbation taking values in $C^1(\mathbb{R}_+, \mathbb{R})$, see, e.g., ω_0 in (1b), or as extra control input (this will be the case later in Section IV in order to incorporate the integral action).

The closed-loop system under the preliminary controller (7) is rewritten in the operator form as follows,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\zeta &= \mathcal{A}\zeta + \mathcal{B}\nu \\ &:= \begin{pmatrix} y_t \\ \frac{1}{\rho}\left(Sy'\right)' \\ -\frac{k_1}{M}\xi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\nu, \end{aligned} \tag{8}$$

with the domain of operator $\mathcal{A} : D(\mathcal{A}) \to \mathbb{X}$ given by

$$\begin{aligned} D(\mathcal{A}) = \{(y, y_t, \xi) \in \mathbb{X}_1 : y'(L) = 0, \\ \xi = -k_a y'(0) + k_p y(0) + y_t(0)\}. \end{aligned}$$

In order to achieve the output regulation objective, we will use the following strategy: 1) add an integral action and 2) add a stabilizing feedback. For the design of the stabilizing feedback we will rely on the forwarding method (see, e.g. [22]–[24]). As a first step, we need to prove that an input-to-state stable (ISS) Lyapunov functional (see [29] for more details) exists for the dynamics given by equation (8).

Next, we introduce the following inner product $\langle\cdot,\cdot\rangle_{\mathbb{Z}}$ which is a symmetric bilinear form defined as

$$\begin{aligned} \langle\zeta_1,\zeta_2\rangle_{\mathbb{Z}} &:= \frac{\beta}{2}\int_0^L S(x)y_1'(x)y_2'(x)\mathrm{d}x \\ &+ \frac{\beta S(0)k_p}{2k_a}y_1(0)y_2(0) + \frac{\alpha M}{2}\xi_1\xi_2 + \frac{\beta\rho}{2}\int_0^L y_{t1}(x)y_{t2}(x)\mathrm{d}x \\ &+ \frac{\gamma\sigma\rho}{2}\int_0^L \big(y_1(x)y_{t2}(x) + y_2(x)y_{t1}(x)\big)\mathrm{d}x \\ &+ \frac{\gamma\rho}{2}\int_0^L (x-L)S(x)\big(y_1'(x)y_{t2}(x) + y_2'(x)y_{t1}(x)\big)\mathrm{d}x \end{aligned} \tag{9}$$

for any $\zeta_i = (y_i, y_{ti}, \xi_i) \in \mathbb{X}, i = 1, 2$, and where α, β, γ and σ are positive constants to be chosen.

The linearity in both of its arguments and symmetry of the above inner product (9) are readily seen. Furthermore, the following lemma shows that if β is sufficiently large, $\langle\cdot,\cdot\rangle_{\mathbb{Z}}$ is an inner product on $\mathbb{X}$ which satisfies the positive definiteness condition and the conditions for equivalence between the induced norm on $\mathbb{X}$ and the inner product $\langle\cdot,\cdot\rangle_{\mathbb{Z}}$.

Lemma 2. *Select*

$$\alpha_1 = \tfrac{\gamma\rho(L+\sigma)}{2}, \quad \alpha_2 = \tfrac{\gamma\rho\left(LS(0)S(L) + 4\sigma L^2\right)}{2S(L)}, \quad \alpha_3 = \sigma\gamma\rho L,$$

and let β be a positive constant β satisfying

$$\beta > \max\left\{\tfrac{2\alpha_1}{\rho}, 2\alpha_2, \tfrac{2k_a\alpha_3}{S(0)k_p}\right\}. \tag{10}$$

Then, the following inequality holds

$$\underline{a}\|\zeta\|_{\mathbb{X}}^2 \leq \langle\zeta,\zeta\rangle_{\mathbb{Z}} \leq \bar{a}\|\zeta\|_{\mathbb{X}}^2, \qquad \forall\,\zeta \in \mathbb{X}, \tag{11}$$

with the real numbers $\bar{a} > \underline{a} > 0$ given by

$$\underline{a} \leq \min\left\{\tfrac{\beta\rho - 2\alpha_1}{2}, \tfrac{(\beta-2\alpha_2)S(L)}{2(1+4L^2)}, \tfrac{\beta S(0)k_p - 2k_a\alpha_3}{4Lk_a}, \tfrac{\alpha M}{2}\right\}, \tag{12}$$

$$\bar{a} \geq \max\left\{\tfrac{\beta\rho + 2\alpha_1}{2}, \tfrac{(\beta+2\alpha_2)S(L)}{2(1+4L^2)}, \tfrac{\beta S(0)k_p + 2k_a\alpha_3}{4Lk_a}, \tfrac{\alpha M}{2}\right\}. \tag{13}$$

The proof can be found in Appendix B.

Based on the inner product (9), we can now show the existence of an ISS-Lyapunov functional for the system (8).

Theorem 1. *For any choice of constant $\alpha, \beta, \gamma, \sigma > 0$ and controller gains $k_1, k_a, k_p > 0$, satisfying*

$$k_1 > \tfrac{\beta S(0)}{2\alpha k_a}, \quad \beta > \max\{\tfrac{\gamma\sigma}{k_p}, k_a\gamma\rho L\}, \quad \tfrac{LS(0)k_p}{k_a} < \sigma < \tfrac{S(L)}{2}, \tag{14}$$

then, the Lyapunov functional $V : \mathbb{X} \to \mathbb{R}_+$, $V(\zeta) = \langle\zeta,\zeta\rangle_{\mathbb{Z}}$, is an exponentially input-to-state stable (ISS) Lyapunov functional for the system (8), i.e., there exist constants $\lambda, \mu > 0$ such that

$$\langle\zeta, \mathcal{A}\zeta + \mathcal{B}\nu\rangle_{\mathbb{Z}} + \langle\mathcal{A}\zeta + \mathcal{B}\nu, \zeta\rangle_{\mathbb{Z}} \leq -\lambda\langle\zeta,\zeta\rangle_{\mathbb{Z}} + \mu|\nu|^2 \tag{15}$$

for all $\zeta \in \mathcal{D}(\mathcal{A})$ and $\nu \in \mathbb{R}$.

The proof can be found in the in Appendix C.

C. Well-posedness and exponential stability via state feedback

With Theorem 1 in hand, the well-posedness and exponential stability of the origin of the closed-loop system (8) under the preliminary controller (7) can be proved.

Corollary 1. *Let $\alpha, \beta, \gamma, \sigma > 0$ and the controller gains $k_1, k_a, k_p > 0$, be chosen according to Theorem 1. Then, the operator $\mathcal{A}$ defined in (8) generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathbb{X}$. Moreover, for any initial condition $\zeta_0 \in \mathbb{X}$ (resp. $D(\mathcal{A})$), there exists a unique weak solution ζ in $C^0(\mathbb{R}_+;\mathbb{X})$(resp. strong solution $\zeta \in C^1(\mathbb{R}_+;\mathbb{X}) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$) to system (8) satisfying*

$$\|\zeta(t)\|_{\mathbb{X}} \leq ke^{-\lambda t}\|\zeta(0)\|_{\mathbb{X}}, \qquad \forall\, t \geq 0$$

for some $k, \lambda > 0$.

The proof of Corollary 1 is omitted for space reasons.

Remark 2. *Note that the convergence is obtained in $\mathbb{X}$. Going back to the coordinates $\Theta = (y, y_t, \chi)$, we obtain exponential stability of the origin of system* (3) *with the feedback law only in $\mathbb{X}_1$ since the bounds given in Lemma 1 holds only in $\mathbb{X}_1$.*

IV. Main Results

Consider now the following control law with integral action, which is given as

$$\begin{aligned} u(t) &= u_p(t) - k_i\eta(t), \\ \dot{\eta}(t) &= y(0,t), \end{aligned} \tag{16}$$

where $k_i > 0$ is a positive constant and the integral action $\dot{\eta}(t)$ is used to update the output error.

We study now the well-posedness of the new closed-loop system after adding the integral action. To this end, we define the following Hilbert space

$$\begin{aligned} \mathbb{H} :=& \mathbb{X} \times \mathbb{R} \\ =& H^1(0,L) \times L^2(0,L) \times \mathbb{R}^2 \end{aligned}$$

equipped with the inner product given by

$$\langle \phi_1, \phi_2 \rangle_{\mathbb{H}} := \langle \zeta_1, \zeta_2 \rangle_{\mathbb{Z}} + (\eta_1 - \mathcal{N}\zeta_1)(\eta_2 - \mathcal{N}\zeta_2), \tag{17}$$

where $\phi_i = \big(y_i, y_{ti}, \xi_i, \eta_i\big) \in \mathbb{H}$, $i = 1, 2$ and $\mathcal{N} : H^1(0,L) \times L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ is a linear operator defined as solution to the equation

$$\mathcal{N}\mathcal{A}\zeta = y(0)\,, \tag{18}$$

with ζ defined in (4).

It can be verified that it takes the form

$$\begin{aligned} \mathcal{N}\zeta = &\int_0^L \big(n_1(x)y(x) + n_2(x)y_t(x) + n_3(x)y'(x)\big)\mathrm{d}x \\ &- n_4\big(y(0) + \xi\big), \end{aligned} \tag{19}$$

for some choice of $n_i \in \mathbb{R}$, $i = 1, 2, 4$ and $n_3 \in L^2(0,L)$ to be chosen. Under the designed control law (16) with integral action, the closed-loop system can be expressed as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\phi = \mathcal{F}\phi := \begin{pmatrix} y_t \\ \frac{1}{\rho}\big(Sy'\big)' \\ -\frac{k_1}{M}\xi - \frac{k_i}{M}\eta \\ y(0) \end{pmatrix}, \quad \phi = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}, \tag{20}$$

with the domain of operator $\mathcal{F} : D(\mathcal{F}) \to \mathbb{H}$ given as

$$\begin{aligned} D(\mathcal{F}) = \big\{&(y, y_t, \xi, \eta) \in \mathbb{H}_1 : y'(L) = 0, \\ &\xi = -k_a y'(0) + k_p y(0) + y_t(0)\big\}, \end{aligned}$$

with $\mathbb{H}_1 := H^2(0,L) \times H^1(0,L) \times \mathbb{R}^2$. Based on the above abstract form (20), we can obtain the following result.

Theorem 2. *Consider system* (20). *There exists $k_i^* > 0$ such that, for any $k_i \in (0, k_i^*)$, the operator $\mathcal{F}$ generates a C_0-semigroup $e^{\mathcal{F}t}$ on $\mathbb{H}$. Furthermore, for any initial value $\phi_0 \in \mathbb{H}$ (resp. $D(\mathcal{F})$), system* (20) *admits a unique weak solution $\phi \in C^0(\mathbb{R}_+; \mathbb{H})$ (resp. strong solution in $C^1(\mathbb{R}_+; \mathbb{H}) \cap C^0(\mathbb{R}_+; D(\mathcal{F}))$) satisfying*

$$\|\phi(t)\|_{\mathbb{H}} \leq k e^{-\lambda t}\|\phi_0\|_{\mathbb{H}}, \qquad \forall\, t \geq 0$$

for some $k, \lambda > 0$.

Proof. First, let's consider the operator $\mathcal{N}$ introduced in (19). We select the parameters n_i, $i = 1 \ldots, 4$ as

$$\begin{aligned} n_1 &= -\frac{k_1 - M}{k_1 k_p L}, \quad n_2 = -\frac{\rho k_a}{k_p S(0)}, \quad n_4 = \frac{M}{k_1 k_p}, \\ n_3(x) &= \frac{k_1 - M}{k_1 k_p}\left(\frac{L - x}{L}\right). \end{aligned} \tag{21}$$

It can be verified that with such a choice, the identity (18) is verified. Details are omitted for space reasons.

Subsequently, we prove that under the designed boundary controller (16), the closed-loop system (20) is well-posed. To this end, we remark that the operator $\mathcal{F}$ can be alternatively defined as

$$\mathcal{F}\phi = \begin{pmatrix} \mathcal{A}\zeta - \mathcal{B}\frac{k_i}{M}\eta \\ y(0) \end{pmatrix}$$

with $\mathcal{B}$ defined as in (8), and therefore, in view of the definition of $\mathcal{N}$ given via the Sylvester equation (18), we have

$$y(0) - \mathcal{N}(\mathcal{A}\zeta - \mathcal{B}\tfrac{k_i}{M}\eta) = -n_4\frac{k_i}{M}\eta\,.$$

As a consequence, according to the definition of the inner product in (17), we obtain

$$\begin{aligned} \langle \mathcal{F}\phi, \phi \rangle_{\mathbb{H}} + \langle \phi, \mathcal{F}\phi \rangle_{\mathbb{H}} &= \langle \mathcal{A}\zeta - \mathcal{B}\tfrac{k_i}{M}\eta, \zeta \rangle_{\mathbb{Z}} \\ &+ \langle \zeta, \mathcal{A}\zeta - \mathcal{B}\tfrac{k_i}{M}\eta \rangle_{\mathbb{Z}} - 2n_4\frac{k_i}{M}\eta(\eta - \mathcal{N}\zeta). \end{aligned}$$

Using inequality (15) one immediately gets

$$\begin{aligned} &\langle \mathcal{F}\phi, \phi \rangle_{\mathbb{H}} + \langle \phi, \mathcal{F}\phi \rangle_{\mathbb{H}} \leq \\ &\quad -\lambda\langle \zeta, \zeta \rangle_{\mathbb{Z}} + \mu\big|\tfrac{k_i}{M}\eta\big|^2 - 2n_4\frac{k_i}{M}\eta^2 + 2n_4\frac{k_i}{M}\eta\mathcal{N}\zeta. \end{aligned}$$

Then, since $\mathcal{N}$ is a bounded operator, there exists $\overline{N} > 0$ such that $\|\mathcal{N}\zeta\|_{\mathbb{X}} \leq \overline{N}\|\zeta\|_{\mathbb{X}}$. This gives

$$\begin{aligned} \langle \mathcal{F}\phi, \phi \rangle_{\mathbb{H}} + \langle \phi, \mathcal{F}\phi \rangle_{\mathbb{H}} &\leq -\lambda\|\zeta\|_{\mathbb{Z}}^2 + \lambda\underline{a}c\|\zeta\|_{\mathbb{X}}^2 \\ &- \left[n_4 - \left(\mu + \frac{n_4^2\overline{N}^2}{\lambda\underline{a}c}\right)\left(\frac{k_i}{M}\right)\right]\left(\frac{k_i}{M}\right)\eta^2 \end{aligned}$$

for any $c > 0$. Recalling inequality (11) and selecting k_i^* as

$$k_i^* = \frac{M n_4 \lambda\underline{a}c}{\lambda\underline{a}c\mu + n_4^2\overline{N}^2}$$

it can be shown that for any $k_i \in (0, k_i^*)$ we can select $c < 1$ and therefore there exists a $\varepsilon > 0$ such that

$$\langle \mathcal{F}\phi, \phi \rangle_{\mathbb{H}} + \langle \phi, \mathcal{F}\phi \rangle_{\mathbb{H}} \leq -\varepsilon\|\zeta\|_{\mathbb{Z}}^2 - \varepsilon|\eta|^2 \leq -\varepsilon\|\phi\|_{\mathbb{H}}^2$$

showing the dissipativity of the operator $\mathcal{F}$. Next, we show that the operator $\mathcal{F}$ is maximal. To this end, we need to show that there exists a $\tilde{\lambda}_0 > 0$ such that the range $\mathrm{Ran}(\tilde{\lambda}_0 I - \mathcal{F})$ is equal to $\mathbb{H}$. For this purpose, we consider a function vector $\bar{\phi} = (\bar{\psi}(x), \bar{\varphi}(x), \bar{h}, \bar{b}) \in \mathbb{H}$. By using the definition of $\mathcal{F}$ given in (20), we can show that there exists a unique $\phi \in D(\mathcal{F})$ satisfying the equation $\mathcal{F}\phi = \bar{\phi}$. Explicitly, the solution $\phi = (y, y_t, \xi, \eta)$ can be expressed in the following form

$$\begin{cases} y(x) = \bar{b} - \rho \int_0^x \frac{1}{S(s)} \int_s^L \bar{\varphi}(\mu) \mathrm{d}\mu \mathrm{d}s, \\ y_t(x) = \bar{\psi}(x), \\ \xi = \frac{k_1 \rho}{S(0)} \int_0^L \bar{\varphi}(\mu) \mathrm{d}\mu + k_p \bar{b} + \bar{\psi}(0), \\ \eta = \frac{k_1 k_a \rho}{k_i S(0)} \int_0^L \bar{\varphi}(\mu) \mathrm{d}\mu + \frac{k_1 k_p}{k_i} \bar{b} + \frac{k_1}{k_i} \bar{\psi}(0) - \frac{M}{k_i} \bar{h}. \end{cases}$$

This shows that the operator $\mathcal{F}$ is m-dissipative. The statement of the theorem follows from Lumer-Phillips theorem [30, Theorem 4.3]. □

Note that a precise estimation of the bound k_i^* is given by

$$k_i^* = \frac{2Mk_1k_p}{2M + k_1^2 k_p^2}. \tag{22}$$

Computations are omitted for space reasons and the bound given in the proof of Theorem 2 is more conservative.

The second main result concerns the output regulation objective and stability of the gantry crane system (1a)-(1c) with unknown disturbances. Here, we consider that the gantry's position $y(0,t)$ should be regulated at a given constant reference y_{ref} as illustrated in (2). To this end, we propose the state-feedback controller with integral action $\dot{\eta}_n$ as follows:

$$\begin{aligned} u(t) =& u_{pn}(t) - k_i \eta_n(t), \\ \dot{\eta}_n(t) =& y(0,t) - y_{\mathrm{ref}}, \\ u_{pn}(t) =& - k_1 \xi_n(t) - S(0) y_x(0,t) + M k_a y_{xt}(0,t) \\ & - M k_p y_t(0,t), \\ \xi_n(t) =& - k_a y_x(0,t) + k_p (y(0,t) - y_{\mathrm{ref}}) + y_t(0,t), \end{aligned} \tag{23}$$

where $k_1 > 0, k_p > 0$, $k_a > 0$, and $k_i > 0$ are the controller gains. We leverage on the stability result of Theorem 2 in order to study the overall closed-loop dynamics (20). In particular, we will show the existence of an equilibrium (in general, different from the origin) which is asymptotically stable on which the regulation objective $\lim_{t\to\infty} |y(0,t) - y_{\mathrm{ref}}| = 0$ is achieved.

Theorem 3. *Consider system* (1a)-(1c) *in closed-loop with the feedback* (23)*. Let the parameters of the feedback* (23) *be chosen according to Theorems 1 and 2. Then, for any set of perturbations/references* $(d, y_{\mathrm{ref}}, \omega_0) \in H^1(0,L) \times \mathbb{R}^2$ *and for any initial condition* $(\tilde{y}(\cdot,0), \tilde{y}_t(\cdot,0), \tilde{\xi}(0), \tilde{\eta}(0))$ *in* $H^2(0,L) \times H^1(0,L) \times \mathbb{R}^2$ *the trolley's position* $y(0,t)$ *is asymptotically regulated to the given reference* y_{ref}*, namely* (2) *holds.*

Proof. With the integral controller (23), the closed-loop system is

$$\begin{cases} \rho y_{tt}(x,t) = (S(x) y_x(x,t))_x + d(x), \\ y_t(0,t) = \xi_n(t) + k_a y_x(0,t) - k_p (y(0,t) - y_{\mathrm{ref}}), \\ \dot{\xi}_n(t) = -\frac{k_1}{M} \xi_n(t) - \frac{k_i}{M} \eta_n(t) + \frac{1}{M} \omega_0, \\ \dot{\eta}_n(t) = y(0,t) - y_{\mathrm{ref}}, \\ y_x(L,t) = 0. \end{cases} \tag{24}$$

It is worth mentioning that due to the presence of ω_0, d and y_{ref}, the equilibrium $(y_{ne}, \eta_{ne}, \xi_{ne})$ of the closed-loop system is not at $(0,0,0)$. Instead, according to (24), it admits an equilibrium $(y_{ne}(x), \eta_{ne}, \xi_{ne})$ given by

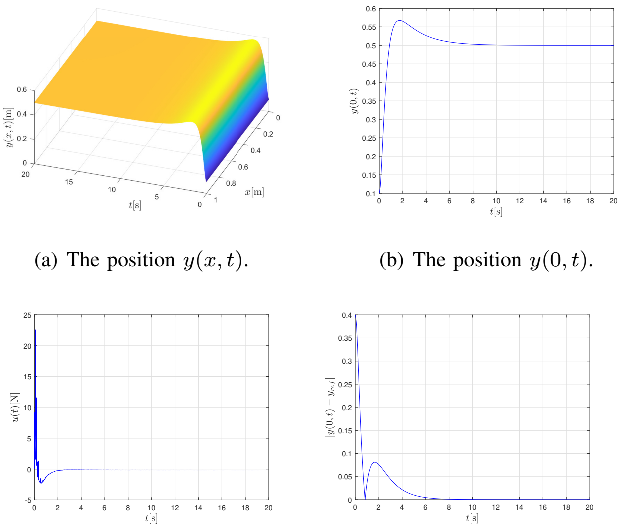
$$\begin{aligned} y_{ne}(x) &= y_e(0) + \int_0^x \tfrac{1}{S(\mu)} \int_\mu^L d(s) \mathrm{d}s \mathrm{d}\mu, \\ \eta_{ne} &= -\tfrac{k_1}{k_i} \xi_{ne} + \tfrac{1}{k_i} \omega_0, \\ \xi_{ne} &= -\tfrac{k_a}{S(0)} \int_0^L d(s) \mathrm{d}s, \end{aligned}$$

on which $y_e(0) = y_{\mathrm{ref}}$. Changing the coordinates according to such an equilibrium, one obtains an unperturbed system of the form for which the origin is the equilibrium. Furthermore, applying a second change of coordinates defined by $\mathcal{T}$ one obtains a system of the form (20). As a consequence, applying Theorem 3 one can show that the origin is exponentially stable. By restricting the analysis on strong solutions, that is, on initial conditions on $\mathbb{H}_1$, one can show that these solution satisfies also $\|\mathcal{F}(\tilde{\zeta}(t), \tilde{\eta}(t))\|_{\mathbb{H}} \leq k e^{-\lambda t} \|\mathcal{F}(\tilde{\zeta}(0), \tilde{\eta}(0)\|_{\mathbb{H}}$ for all $t \geq 0$, which implies $\lim_{t\to\infty} \|\dot{\tilde{\eta}}(t)\| = 0$. Hence, the regulation is obtained. □

V. Simulations

In order to verify the efficiency of the proposed boundary integral controller (23), we present simulation results in this section. The parameters of the gantry crane system (1a)-(1c) are given as $M = 2.1\,\mathrm{kg}$, $L = 1\,\mathrm{m}$, $m = 10\,\mathrm{kg}$, $\rho = 0.2\,\mathrm{kg/m}$, and $g = 9.8\,\mathrm{m/s^2}$, in which the parameters are consistent with those of the physical system made in the lab, as shown in Fig.1(b). The distributed disturbance is chosen as $d(x) = 0.5x$, the constant disturbance is $\omega_0 = 0.1$, the constant reference is $y_{\mathrm{ref}} = 0.5\,\mathrm{m}$. The numerical simulations are performed using a finite difference method with a space step $0.01\,\mathrm{m}$ and a time step $0.1\,\mathrm{ms}$. The initial conditions are $y(x,0) = 0.1\,\mathrm{m}$ and $y_t(x,0) = 0\,\mathrm{m/s}$. The control gains are set $k_1 = 100, k_a = 60$, and $k_p = 2$. Based on the Lyapunov control design, we obtain $k_i^* \approx 0.02$, which is very restrictive. Hence, in simulations, larger k_i is used instead, we choose $k_i = 100$.

The evolution of the states $y(x,t)$ and $y(0,t)$ are depicted in Figs. 2(a) and 2(b), respectively. Fig. 2(a) displays that the system can converge well to the given reference. In particular, Fig. 2(b) demonstrates the successful regulation of the trolley's position $y(0,t)$ to the given reference value of $y_{\mathrm{ref}} = 0.5\,\mathrm{m}$. The evolution of the control law (23), which is bounded, is shown in Fig. 2(c). Fig. 2(d) displays the evolution of the error $|y(0,t) - y_{\mathrm{ref}}|$, which also indicates that the proposed controller (23) drives the output towards the given reference.



(a) The position $y(x, t)$. (b) The position $y(0, t)$.

(c) Evolution of controller $u(t)$. (d) Evolution of $|y(0, t) - y_{\mathrm{ref}}|$.

Fig. 2: The transient dynamics of the gantry crane system with unknown perturbations.

VI. Conclusions

In this paper, we present a boundary controller with integral action to address the output regulation problem for a class of gantry crane systems with unknown perturbations. The proposed controller design is based on the construction of a strict Lyapunov function and the application of the forwarding method to achieve the desired regulation objective. Moreover, we establish the well-posedness of the closed-loop system using semigroup theory. The effectiveness of the proposed controller is demonstrated through simulation results, which clearly illustrate the efficiency of output regulation in the presence of unknown disturbances.

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APPENDIX

A. Proof of Lemma 1

A norm on $\mathbb{X}_1$ induced by the inner product (6) is expressed as follows:

$$\begin{aligned}\|\mathcal{T}\Theta\|_{\mathbb{X}_1}^2 &= \|y\|_{H^2(0,L)}^2 + \|y_t\|_{H^1(0,L)}^2 + |\xi|^2 \\ &\leq \|y\|_{H^2(0,L)}^2 + 3|\chi|^2 + 3k_a^2|y'(0)|^2 \\ &\quad + 3k_p^2|y(0)|^2 + \|y_t\|_{H^1(0,L)}^2. \end{aligned} \tag{25}$$

Using the Sobolev embedding,

$$\sup_{x\in[0,L]} |y(x,t)|^2 \leq C_1\|y(\cdot,t)\|_{H^1(0,L)}^2,$$
$$\sup_{x\in[0,L]} |y_x(x,t)|^2 \leq C_2\|y(\cdot,t)\|_{H^2(0,L)}^2.$$

the equation (25) can be rewritten as:

$$\begin{aligned}\|\mathcal{T}\Theta\|_{\mathbb{X}_1}^2 &\leq (1+3C_2k_a^2)\|y\|_{H^2(0,L)}^2 + \|y_t\|_{H^1(0,L)}^2 \\ &\quad + 3C_1k_p^2\|y\|_{H^1(0,L)}^2 + 3|\chi|^2 \\ &\leq (1+3C_1k_p^2+3C_2k_a^2)\|y\|_{H^2(0,L)}^2 \\ &\quad + \|y_t\|_{H^1(0,L)}^2 + 3|\chi|^2 \\ &\leq \overline{C}\|\Theta\|_{\mathbb{X}_1}^2,\end{aligned}$$

where $\overline{C} \geq \max\{1 + C_1 3k_p^2 + 3C_2k_a^2, 3\}$, C_1 and C_2 are positive real numbers. Thus, we have shown that the operator $\mathcal{T}$ is linear and bounded. Similarly, a lower bound can be found.

B. Proof of Lemma 2

We can express the norm with respect to the inner product defined in (9) as follows:

$$\begin{aligned}\|\zeta\|_{\mathbb{Z}}^2 &= \langle\zeta,\zeta\rangle_{\mathbb{Z}} \\ &= \frac{\beta\rho}{2}\int_0^L y_t^2(x)\mathrm{d}x + \frac{\beta}{2}\int_0^L S(x)y'^2(x)\mathrm{d}x \\ &\quad + \frac{\beta S(0)k_p}{2k_a}y^2(0) + \frac{\alpha M}{2}\xi^2 + A_1,\end{aligned} \tag{26}$$

where we denote

$$\begin{aligned}A_1 &= \gamma\rho\int_0^L (x-L)S(x)y'(x)y_t(x)\mathrm{d}x \\ &\quad + \gamma\sigma\rho\int_0^L y(x)y_t(x)\mathrm{d}x.\end{aligned}$$

By using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, Poincaré Inequality (as shown in [31, Lemma 2.6]), and the fact that $S(0) \geq S(x) \geq S(L) > 0$, it gives that

$$\begin{aligned}A_1 \leq |A_1| &\leq \frac{\gamma\rho}{2}\int_0^L |x-L|\big(S^2(x)y'^2(x) + y_t^2(x)\big)\mathrm{d}x \\ &\quad + \frac{\gamma\sigma\rho}{2}\int_0^L \big(y^2(x) + y_t^2(x)\big)\mathrm{d}x \\ &\leq \frac{\gamma\rho(L+\sigma)}{2}\int_0^L y_t^2(x)\mathrm{d}x \\ &\quad + \frac{\gamma\rho\big(LS(0)S(L) + 4\sigma L^2\big)}{2S(L)}\int_0^L S(x)y'^2(x)\mathrm{d}x \\ &\quad + \sigma\gamma\rho L y^2(0).\end{aligned}$$

Furthermore, we get

$$\begin{aligned}\|\zeta\|_{\mathbb{Z}}^2 &\leq \big(\frac{\beta\rho}{2} + \alpha_1\big)\int_0^L y_t^2(x)\mathrm{d}x + \big(\frac{\beta}{2} + \alpha_2\big)\int_0^L S(x)y'^2(x)\mathrm{d}x \\ &\quad + \big(\frac{\beta S(0)k_p}{2k_a} + \alpha_3\big)y^2(0) + \frac{\alpha M}{2}\xi^2,\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\big(\frac{\beta\rho}{2} - \alpha_1\big)\int_0^L y_t^2(x)\mathrm{d}x + \big(\frac{\beta}{2} - \alpha_2\big)\int_0^L S(x)y'^2(x)\mathrm{d}x \\ &\quad + \big(\frac{\beta S(0)k_p}{2k_a} - \alpha_3\big)y^2(0) + \frac{\alpha M}{2}\xi^2 \leq \|\zeta\|_{\mathbb{Z}}^2,\end{aligned} \tag{28}$$

where the positive constants $\alpha_1 = \frac{\gamma\rho(L+\sigma)}{2}, \alpha_2 = \frac{\gamma\rho\big(LS(0)S(L)+4\sigma L^2\big)}{2S(L)}$, and $\alpha_3 = \sigma\gamma\rho L$. If the condition in (10) is satisfied, then $\frac{\beta\rho}{2} - \alpha_1$, $\frac{\beta}{2} - \alpha_2$, and $\frac{\beta S(0)k_p}{2k_a} - \alpha_3$ are positive.

Next, we need to show that there exist two positive constants $\bar{a}$ and $\underline{a}$ such that $\underline{a}\|\zeta\|_{\mathbb{X}}^2 \leq \|\zeta\|_{\mathbb{Z}}^2 \leq \bar{a}\|\zeta\|_{\mathbb{X}}^2$ holds.

Based on the inner product defined in (5) on $\mathbb{X}$ and using the Poincaré inequality (as shown in [31, Lemma 2.6]), we obtain the following expression:

$$\begin{aligned}\bar{a}\|\zeta\|_{\mathbb{X}}^2 &= \bar{a}\langle\zeta,\zeta\rangle_{\mathbb{X}} \\ &= \bar{a}\int_0^L \big(y'^2(x) + y^2(x)\big)\mathrm{d}x + \bar{a}\xi^2 + \bar{a}\int_0^L y_t^2(x)\mathrm{d}x \\ &\leq \frac{\bar{a}(1+4L^2)}{S(L)}\int_0^L S(x)y'^2(x)\mathrm{d}x + \bar{a}\xi^2 \\ &\quad + \bar{a}\int_0^L y_t^2(x)\mathrm{d}x + 2\bar{a}Ly^2(0).\end{aligned} \tag{29}$$

Firstly, to verify the inequality $\|\zeta\|_{\mathbb{Z}}^2 \leq \bar{a}\|\zeta\|_{\mathbb{X}}^2$, it means to show the existence of a positive constant $\bar{a}$. From (27) and (29), it is obvious that if we let $\bar{a} \geq \max\big\{\frac{\beta\rho+2\alpha_1}{2}, \frac{(\beta+2\alpha_2)S(L)}{2(1+4L^2)}, \frac{\beta S(0)k_p+2k_a\alpha_3}{4Lk_a}, \frac{\alpha M}{2}\big\}$, then $\|\zeta\|_{\mathbb{Z}}^2 \leq \bar{a}\|\zeta\|_{\mathbb{X}}^2$.

Similarly, to verify the inequality $\underline{a}\|\zeta\|_{\mathbb{X}}^2 \leq \|\zeta\|_{\mathbb{Z}}^2$, it suffices to show the existence of a positive constant $\underline{a}$. As discussed above, we can choose $\underline{a}$ to be the maximum of the following four quantities: $\underline{a} \leq \min\big\{\frac{\beta\rho-2\alpha_1}{2}, \frac{(\beta-2\alpha_2)S(L)}{2(1+4L^2)}, \frac{\beta S(0)k_p-2k_a\alpha_3}{4Lk_a}, \frac{\alpha M}{2}\big\}$. This completes the proof.

C. Proof of Theorem 1

For each (ζ,ν) in $\mathcal{D}(\mathcal{A})\times\mathbb{R}$, let us denote $\dot{V}(\zeta,\nu) = \langle\zeta, \mathcal{A}\zeta + \mathcal{B}\nu\rangle_{\mathbb{Z}} + \langle\mathcal{A}\zeta + \mathcal{B}\nu, \zeta\rangle_{\mathbb{Z}}$. Note that for each (ζ,ν) in $\mathcal{D}(\mathcal{A})\times\mathbb{R}$, the following decomposition can be made

$$\dot{V}(\zeta,\nu) = \dot{V}_1(\zeta) + \dot{V}_2(\zeta,\nu) + \dot{V}_3(\zeta), \tag{30}$$

with

$$\dot{V}_1 = \beta\int_0^L S(x)y'(x)y_t'(x)\mathrm{d}x + \frac{\beta S(0)k_p}{k_a}y(0)y_t(0) + \beta\rho\int_0^L y_t(x)\frac{1}{\rho}\big(S(x)y'(x)\big)'\mathrm{d}x$$

$$\dot{V}_2 = \alpha M\xi(-\frac{k_1}{M}\xi+\nu)$$

$$\dot{V}_3 = \gamma\sigma\rho\int_0^L \big(y(x)\frac{1}{\rho}\big(S(x)y'(x)\big)' + y_t(x)y_t(x)\big)\mathrm{d}x + \gamma\rho\int_0^L (x-L)S(x)\big(y'(x)\frac{1}{\rho}\big(S(x)y'(x)\big)' + y_t'(x)y_t(x)\big)\mathrm{d}x$$

Hence, it implies with integration by parts and completion of the square

$$\dot{V}_1 = \frac{\beta S(0)}{2k_a}\xi^2 - \frac{\beta S(0)}{2k_a}k_p^2y^2(0) - \frac{\beta S(0)}{2k_a}y_t^2(0) - \frac{\beta k_a S(0)}{2}y'^2(0) + \beta S(L)y_t(L)y'(L) + \beta S(0)k_p y(0)y'(0),$$

$$\dot{V}_2 \le -k_1\alpha\xi^2 + \frac{\alpha M}{2}\xi^2 + \frac{\alpha M}{2}\nu^2,$$

$$\dot{V}_3 \le \frac{\gamma\rho L S(0)}{2}y_t^2(0) + \frac{\gamma L}{2}S^2(0)y'^2(0) - \gamma\big(\frac{S(L)}{2}+\sigma\big)\int_0^L S(x)y'^2(x)\mathrm{d}x - \gamma\rho\big(\frac{S(L)}{2}-\sigma\big)\int_0^L y_t^2(x)\mathrm{d}x + \gamma\sigma y(L)S(L)y'(L) - \gamma\sigma y(0)S(0)y'(0).$$

From this, we obtain

$$\dot{V}(\zeta,\nu) \le -2\gamma_1\int_0^L y_t^2(x)\mathrm{d}x - 2\gamma_2\int_0^L S(x)y'^2(x)\mathrm{d}x - 2\gamma_3 y^2(0) - 2(\gamma_4 - \frac{\alpha M}{4})\xi^2 - 2\gamma_6 y'^2(0) - 2\gamma_5\big(k_a y'(0) - k_p y(0,t)\big)^2 - 2\gamma_7 y_t^2(0) + \frac{\alpha M}{2}\nu^2, \tag{31}$$

where $\gamma_1 = \frac{\gamma\rho}{2}(\frac{S(L)}{2}-\sigma)$, $\gamma_2 = \frac{\gamma}{2}(\frac{S(L)}{2}+\sigma)$, $\gamma_3 = \frac{\gamma\sigma S(0)k_p}{4k_a}$, $\gamma_4 = \frac{1}{2}(k_1\alpha - \frac{\beta S(0)}{2k_a})$, $\gamma_5 = \frac{S(0)}{4k_a}(\beta - \frac{\gamma\sigma}{k_p})$, $\gamma_6 = \frac{\gamma S(0)}{4}(\frac{\sigma k_a}{k_p} - S(0)L)$, $\gamma_7 = \frac{S(0)}{4k_a}(\beta - k_a\gamma\rho L)$.

Using the conditions in (14) with $\gamma_4 > \frac{\alpha M}{4}$, it gives the following inequality:

$$\begin{aligned}\dot{V}(\zeta,\nu) &\le -2\gamma_1\int_0^L y_t^2(x)\mathrm{d}x - 2\gamma_2\int_0^L S(x)y'^2(x)\mathrm{d}x - 2\gamma_3 y^2(0) - 2(\gamma_4 - \frac{\alpha M}{4})\xi^2 + \frac{\alpha M}{2}\nu^2\\ &\le -\gamma_0(V_1+V_2) + \frac{\alpha M}{2}\nu^2\\ &\le -\frac{\gamma_0}{\bar{a}}V(\zeta) + \frac{\alpha M}{2}\nu^2,\end{aligned} \tag{32}$$

where $\gamma_0 = \min\{\frac{4\gamma_1}{\beta\rho}, \frac{4\gamma_2}{\beta}, \frac{4k_a\gamma_3}{\beta S(0)k_p}, \frac{4\gamma_4-\alpha M}{\alpha M}\}$. The above inequality shows (15) with $\lambda = \frac{\gamma_0}{\bar{a}}$ and $\mu = \frac{\alpha M}{2}$.