THE COMPUTATIONAL HARDNESS OF COUNTING IN TWO-SPIN MODELS ON d-REGULAR GRAPHS

*ALLAN SLY AND †NIKE SUN

ABSTRACT. The class of two-spin systems contains several important models, including random independent sets and the Ising model of statistical physics. We show that for both the hard-core (independent set) model and the anti-ferromagnetic Ising model with arbitrary external field, it is NP-hard to approximate the partition function or approximately sample from the model on d-regular graphs when the model has non-uniqueness on the d-regular tree. Together with results of Jerrum-Sinclair, Weitz, and Sinclair-Srivastava-Thurley giving FPRAS's for all other two-spin systems except at the uniqueness threshold, this gives an almost complete classification of the computational complexity of two-spin systems on bounded-degree graphs.

Our proof establishes that the normalized log-partition function of any two-spin system on bipartite locally tree-like graphs converges to a limiting "free energy density" which coincides with the (non-rigorous) Bethe prediction of statistical physics. We use this result to characterize the local structure of two-spin systems on locally tree-like bipartite expander graphs, which then become the basic gadgets in a randomized reduction to approximate MAX-CUT. Our approach is novel in that it makes no use of the second moment method employed in previous works on these questions.

1. Introduction

Spin systems are stochastic models defined by local interactions on networks. The class of spin systems includes well-known combinatorial counting and constraint satisfaction problems. In this paper we classify the complexity of approximating the partition function for all homogeneous two-spin systems on bounded-degree graphs.

When interactions favor agreement of adjacent spins, the model is said to be ferromagnetic. Jerrum and Sinclair [11] gave a fully polynomial-time randomized approximation scheme (FPRAS) for approximating the partition function (the normalizing constant in the probability distribution) of the ferromagnetic Ising model, which covers all ferromagnetic two-spin systems. For anti-ferromagnetic systems such as the hard-core and anti-ferromagnetic Ising models, the complexity of approximating the partition function depends on the model parameters, and is known to be NP-hard when the interactions are sufficiently strong. Our first main result establishes that the computational transition for such models on d-regular graphs is located precisely at the uniqueness threshold (see Defn. 1.6) for the corresponding model on the d-regular tree.

Theorem 1. For $d \geq 3$ and $\lambda > \lambda_c(d) = \frac{(d-1)^{d-1}}{(d-2)^d}$, unless NP = RP there exists no FPRAS for the partition function of the hard-core model with fugacity λ on d-regular graphs.

Date: June 19, 2018.

^{*}Research partially supported by Alfred P. Sloan Research Fellowship.

[†]Research partially supported by Department of Defense NDSEG Fellowship.

The transition point $\lambda_c(d)$ is the uniqueness threshold for the hard-core model on the d-regular tree: it marks the point above which distant boundary conditions have a non-vanishing influence on the spin at the root. In a seminal paper [18], Weitz used computational tree methods to provide a FPTAS for the partition function of the hard-core model on graphs of maximum degree d at any $\lambda < \lambda_c(d)$. Together with Weitz's result, Thm. 1 completes the classification of the complexity of the hard-core model except at the threshold λ_c .

Previously it was shown that there is no FPRAS for the hard-core model at $\lambda d \geq 10000$ [13]. In the case of $\lambda = 1$ this was improved to $d \geq 25$ [6, 7], using random regular bipartite graphs as basic gadgets in a hardness reduction. Mossel et al. [15] showed that local MCMC algorithms are exponentially slow for $\lambda > \lambda_c(d)$, and conjectured that λ_c is in fact the threshold for existence of an FPRAS.

The first rigorous result establishing a computational transition at the uniqueness threshold appeared in [17], where hardness was shown for $\lambda_c(d) < \lambda < \lambda_c(d) + \epsilon(d)$ for some $\epsilon(d) > 0$. The proof relies on a detailed analysis of the hard-core model on random bipartite graphs, which are then used in a randomized reduction to MAX-CUT. More precisely the result of [17] gives hardness subject to a technical condition which was an artifact of a difficult second moment calculation from [15], and which could only be verified for $\lambda < \lambda_c(d) + \epsilon(d)$. Hardness was subsequently shown by Galanis et al. [8] for all $\lambda > \lambda_c(d)$ when $d \neq 4, 5$ by verifying the technical condition of [17].

In this paper we follow a different approach which is more conceptual and completely circumvents second moment method calculations. Moreover the same method of proof gives the analogous result for anti-ferromagnetic Ising models with arbitrary external field:

Theorem 2. For $d \geq 3$, $B \in \mathbb{R}$ and $\beta < \beta_{c,af}(B,d) < 0$, unless NP = RP there does not exist an FPRAS for the partition function of the anti-ferromagnetic Ising model with inverse temperature β and external field B on d-regular graphs.

Here $\beta_{c,af}(B,d)$ denotes the uniqueness threshold for the anti-ferromagnetic Ising model with external field B on the d-regular tree. Extending the methods of Weitz [18], Sinclair et al. [16] (see also [12]) gave a FPTAS for the anti-ferromagnetic Ising model on d-regular graphs at inverse temperature $\beta > \beta_{c,af}(B,d)$, so together with Thm. 2 this again establishes that the computational transition coincides with the tree uniqueness threshold.

The hard-core and anti-ferromagnetic Ising models together encompass all (non-degenerate) homogeneous two-spin systems on d-regular graphs (see §2.2). Thus, the results of [18, 11, 16] combined with Thms. 1 and 2 give a full classification of the computational complexity of approximating the partition function for (homogeneous) two-spin systems on d-regular graphs, except at the uniqueness thresholds $\lambda_c(d)$ and $\beta_{c,af}(B,d)$.

In fact, we will show inapproximability in non-uniqueness regimes in a strong sense: not only does there not exist an FPRAS, but for any fixed choice of model parameters and d there exists c > 0 such that it is NP-hard even to approximate the partition function within a factor of e^{cn} on the class of d-regular graphs.

Independent results of Galanis-Štefankovič-Vigoda. In a simultaneous and independent work, Galanis, Štefankovič and Vigoda [9] established the result of Thm. 1, and Thm. 2 in the case of zero external field (B=0). Their methods differ from ours: they analyze the

second moment of the partition function on random bipartite d-regular graphs, and establish the condition necessary to apply the approach of [17]. Their proof analyzes a difficult optimization of a real function in several variables by relating the problem to certain tree recursions.

1.1. Reduction to max-cut via bipartite graphs. Our proof is based on a detailed characterization (Thm. 5) of the local structure of anti-ferromagnetic two-spin systems on symmetric bipartite d-regular locally tree-like graphs. Specifically, we show that the joint distribution of all the spins in a large neighborhood of a typical vertex in the graph converges to a known (Gibbs) measure on the d-regular tree. Under the additional assumption that that the graph is an edge expander, when the model has non-uniqueness on the d-regular tree the spin distribution on the graph is divided into + and - phases where one or the other side of the graph has a linear number more vertices with + spin.

Our main results Thms. 1 and 2 are then proved by a variation on the construction of [17], using the bipartite graphs in a randomized reduction approximate MAX-CUT on 3-regular graphs, which is known to be NP-hard [1]. First, we use Thm. 5 to construct a symmetric bipartite d-regular locally tree-like graph G of large constant size such that, conditioned on the phase of the global configuration, spins at distant vertices are asymptotically independent with known marginals depending only on the side of the graph (Propn. 4.2).

Given a 3-regular graph H on which we wish to approximate MAX-CUT, first we take a disjoint copy G_v of G for each vertex $v \in H$. After removing 3k edges from each G_v , for each edge $(u, v) \in H$ we add k edges joining each side of G_u to the corresponding side of G_v in such a way that the resulting graph H^G is d-regular.

The connections between gadgets do not substantially change the spin distributions inside them, and in particular the \pm phases remain. The anti-ferromagnetic nature of the interation, however, results in neighboring copies of G in H^G preferring to be in opposing phases. Using the asymptotic conditional independence result Propn. 4.2 we can estimate the partition function for the model on H^G restricted to configurations of given phase on each copy of G within a factor of $e^{\epsilon|H|}$ (Lem. 4.3). We find that the distribution is concentrated on configurations where the vector of phases gives a good cut of H, and the effect is strengthened as k is increased. Thus, for any $\epsilon > 0$, by taking k (hence G) to be sufficiently large a $(1+\epsilon)$ -approximation of MAX-CUT(H) can be determined from the partition function of the model on H^G , thereby completing the reduction.

Our reduction depends crucially on the detailed picture of the spin distribution developed in Thm. 5 and Propn. 4.2. Using methods developed in [14], these results in turn are obtained as consequences of precise asymptotics for the partition function of two-spin models on bipartite d-regular graphs: we show that the log-partition function, normalized by the number of vertices in the graph, has an asymptotic value, the "free energy density," which is easily computed from the (non-rigorous) "Bethe prediction" of statistical physics (see §2.1). This is a result of independent interest, since lower bounds for partition functions on graphs have proved to be in general challenging. Asymptotics for the partition function on general tree-like graphs were established for the ferromagnetic Ising model in [3, 5, 4], and for more general spin systems in uniqueness regimes in [4]. Our result for anti-ferromagnetic models is stated somewhat informally as follows; for the precise statement see Thm. 4.

Theorem 3. For any non-degenerate homogeneous two-spin model on bipartite d-regular locally tree-like graphs, the log-partition function normalized by the number of vertices has an asymptotic value which coincides with the Bethe free energy prediction.

In the remainder of this introductory section we formally introduce the models which we consider. We then define the notion of local (weak) convergence of graphs and give precise statements of our results on the partition function (Thm. 4) and local structure (Thm. 5) of these models on bipartite graphs.

1.2. **Definition of spin systems.** Let G = (V, E) be a finite undirected graph, and \mathscr{X} a finite alphabet of *spins*. A *spin system* or *spin model* on G is a probability measure on the space of (spin) configurations $\underline{\sigma} \in \mathscr{X}^V$ of form

$$\nu_{G}^{\underline{\psi}}(\underline{\sigma}) = \frac{1}{Z_{G}(\underline{\psi})} \prod_{(ij) \in E} \psi(\sigma_{i}, \sigma_{j}) \prod_{i \in V} \bar{\psi}(\sigma_{i}), \tag{1.1}$$

where ψ is a symmetric function $\mathscr{X}^2 \to \mathbb{R}_{\geq 0}$, $\bar{\psi}$ is a positive function $\mathscr{X} \to \mathbb{R}_{\geq 0}$, and $Z_G(\underline{\psi})$ is the normalizing constant, called the *partition function*. The pair $\underline{\psi} \equiv (\psi, \bar{\psi})$ is called a specification for the spin system (1.1).

In this paper we consider spin systems with an alphabet of size two; without loss $\mathscr{X} \equiv \{\pm 1\}$. The *Ising* model on G at inverse temperature β and external field B is given by

$$\nu_G^{\beta,B}(\underline{\sigma}) = \frac{1}{Z_G(\beta,B)} \prod_{(ij)\in E} e^{\beta\sigma_i\sigma_j} \prod_{i\in V} e^{B\sigma_i}.$$
 (1.2)

The hard-core (or independent set) model on G at activity or fugacity λ is given by

$$\nu_G^{\lambda}(\underline{\sigma}) = \frac{1}{Z_G(\lambda)} \prod_{(ij) \in E} \mathbf{1} \{ \bar{\sigma}_i \bar{\sigma}_j \neq 1 \} \prod_{i \in V} \lambda^{\bar{\sigma}_i}$$
 (1.3)

where $\bar{\sigma} \equiv \mathbf{1}\{\sigma = +1\} = (1+\sigma)/2$. The edge interaction has no temperature parameter and includes a hard constraint. Our definition (1.3) is trivially equivalent to the standard definition of the hard-core model which has spin 0 in place of -1, but we take $\mathscr{X} = \{\pm 1\}$ throughout to unify the notation.

1.3. Local convergence and the Bethe prediction. If G is any graph and v a vertex in G, write $B_t(v)$ for the subgraph induced by the vertices of G at graph distance at most t from v, and $\partial v \equiv B_1(v) \setminus \{v\}$ for the neighbors of v. We let $T \equiv (T, o)$ denote a general tree with root o, with $T^t \equiv B_t(o) \subseteq T$ the subtree of depth t. We also fix d throughout and write $\mathbb{T} \equiv (\mathbb{T}, o)$ for the rooted d-regular tree.

Definition 1.1. Let $G_n = (V_n = [n], E_n)$ be a sequence of (random) finite undirected graphs, and let I_n denote a uniformly random vertex in V_n . The sequence G_n is said to *converge locally* to the d-regular tree \mathbb{T} if for all $t \geq 0$, $B_t(I_n)$ converges to \mathbb{T}^t in distribution with respect to the joint law \mathbb{P}_n of (G_n, I_n) : that is, $\lim_{n\to\infty} \mathbb{P}_n(B_t(I_n) \cong \mathbb{T}^t) = 1$ (where \cong denotes graph isomorphism).

We write \mathbb{E}_n for expectation with respect to \mathbb{P}_n and impose the following integrability condition on the degree of I_n :

Definition 1.2. The sequence G_n is uniformly sparse if the random variables $|\partial I_n|$ are uniformly integrable, that is, if

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{E}_n[|\partial I_n| \mathbf{1}\{|\partial I_n| \ge L\}] = 0.$$

We assume throughout that G_n $(n \ge 1)$ is a uniformly sparse graph sequence converging locally to the d-regular tree \mathbb{T} ; this setting is hereafter denoted $G_n \to_{loc} \mathbb{T}$. The free energy density for a specification ψ on G_n is defined by

$$\phi \equiv \lim_{n \to \infty} \phi_n \equiv \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n[\log Z_n], \quad Z_n \equiv Z_{G_n}(\underline{\psi}), \tag{1.4}$$

provided the limit exists. For ferromagnetic spin systems on a broad class of locally tree-like graphs, heuristic methods from statistical physics yield an explicit (conjectural) formula for the value of ϕ , the so-called "Bethe prediction" Φ whose definition we recall in §2.1. For antiferromagnetic two-spin models, the Bethe prediction is well-defined only on graph sequences G_n which are nearly bipartite, in the following sense: let \mathbb{T}_+ denote the d-regular tree \mathbb{T} with vertices colored +1 (black) or -1 (white) according to whether they are at even or odd distance from the root o; let \mathbb{T}_- be \mathbb{T}_+ with the colors reversed. Let \mathbf{T} be the random tree which equals \mathbb{T}_+ or \mathbb{T}_- with equal probability; write \mathbf{P} for the law of \mathbf{T} and \mathbf{E} for expectation with respect to \mathbf{P} .

Definition 1.3. For $G_n \to_{loc} \mathbb{T}$, we say the G_n are nearly bipartite, and write $G_n \to_{loc} \mathbf{T}$ (equivalently $G_n \to_{loc} \mathbf{P}$), if there exists a (not necessarily proper) black-white coloring of G_n such that for all $t \geq 0$, $B_t(I_n) \to \mathbf{T}^t$ in distribution.

The precise statement of Thm. 3 is then as follows:

Theorem 4. Let ψ specify a non-degenerate homogeneous two-spin system.

- (a) If $\underline{\psi}$ is ferromagnetic, then ϕ exists for any $G_n \to_{loc} \mathbb{T}$ and equals $\Phi_{\{\mathbb{T}\}}$ as defined by $(2.\overline{2})$ (and given more explicitly by (2.4)).
- (b) If $\underline{\psi}$ is anti-ferromagnetic, then ϕ exists for any $G_n \to_{loc} \mathbf{T}$ and equals $\Phi_{\{\mathbb{T}_{\pm}\}}$ as defined in (2.2) (and given more explicitly by (2.3)).

Remark 1.4. Hereafter we treat $G_n \to_{loc} \mathbb{T}$ and $G_n \to_{loc} \mathbf{T}$ in a unified manner when possible by writing $G_n \to_{loc} \mathbb{P}_{\mathcal{T}}$ for $\mathbb{P}_{\mathcal{T}}$ the uniform measure on \mathcal{T} , which always denotes either $\{\mathbb{T}\}$ or $\{\mathbb{T}_{\pm}\}$. We write $\mathbb{E}_{\mathcal{T}}$ for expectation with respect to $\mathbb{P}_{\mathcal{T}}$.

1.4. Local structure of measures. Under some additional assumptions on G_n , Thm. 4, together with the arguments of [14], characterizes the asymptotic local structure of the spin systems $\nu_n \equiv \nu_{G_n}$. For $G_n \to_{loc} \mathbf{T}$, let $\tau : V_n \to \{\pm\}$ denote the given black-white coloring of the vertices of G_n (hereafter writing \pm as shorthand for ± 1). We say that G_n is symmetric if it is isomorphism-invariant to reversing the black-white coloring. For a spin configuration $\underline{\sigma} \in G_n$ we define the phase of $\underline{\sigma}$ to be

$$Y(\underline{\sigma}) \equiv \operatorname{sgn} \sum_{i} \tau_{i} \sigma_{i}, \text{ where } \operatorname{sgn} x \equiv \mathbf{1}\{x \geq 0\} - \mathbf{1}\{x < 0\}.$$

Let ν_n^{\pm} denote the measure ν_n conditioned on the configurations of \pm phase: that is,

$$\nu_n^{\pm}(\underline{\sigma}) \equiv \frac{1}{Z_n^{\pm}} \mathbf{1} \{ Y(\underline{\sigma}) = \pm \} \prod_{(ij) \in E_n} \psi(\sigma_i, \sigma_j) \prod_{i \in V_n} \bar{\psi}(\sigma_i),$$

where Z_n^{\pm} is the partition function restricted to the \pm configurations. We will characterize the local structure of the measures ν_n^{\pm} on graph sequences satisfying an edge-expansion assumption, as follows:

Definition 1.5. A graph G = (V, E) is a $(\delta, \gamma, \lambda)$ -edge expander if, for any set of vertices $S \subseteq V$ with $\delta|V| \leq |S| \leq \gamma|V|$, there are at least $\lambda|S|$ edges joining S to $V \setminus S$.

The measures ν_n^{\pm} will be related to Gibbs measures on the infinite tree. In particular, recall the definition of (Gibbs) uniqueness:

Definition 1.6. For a rooted tree T, let \mathscr{G}_T denote the set of Gibbs measures for the specification ψ on T. The specification is said to have (Gibbs) uniqueness (on T) if $|\mathscr{G}_T| = 1$.

Recalling Rmk. 1.4, let $\mathscr{G}_{\mathcal{T}}$ denote the space of mappings $\nu: T \mapsto \nu(T), T \in \mathcal{T}$ (with $\mathscr{G}_{\{\mathbb{T}\}} \hookrightarrow \mathscr{G}_{\{\mathbb{T}_{\pm}\}}$ in the obvious manner). When $\mathcal{T} = \{\mathbb{T}_{\pm}\}$ we write ν_{\pm} as shorthand for $\nu(\mathbb{T}_{\pm})$.

Definition 1.7. An element $\nu \in \mathscr{G}_{\mathcal{T}}$ is translation-invariant if for $(T, o) \in \mathcal{T}$ and any vertex $x \in T$, the law on spin configurations of (T, x) induced by $\nu(T, o)$ coincides with $\nu(T, x)$.

For a two-spin model, let ν^+ (resp. ν^-) be the elements of $\mathscr{G}_{\mathcal{T}}$ defined by conditioning on all spins identically equal to 1 on the t-th level of black (resp. white) vertices and taking the weak limit as $t \to \infty$; the ν^{\pm} are translation-invariant. The projections $\mu^+ \equiv \nu^+_+ \equiv \nu^+(\mathbb{T}_+)$ and $\mu^- \equiv \nu^-_+ \equiv \nu^-(\mathbb{T}_+)$, disregarding the black-white coloring on \mathbb{T}_+ , are the extremal semi-translation-invariant Gibbs measures for the model on \mathbb{T} , and by symmetry

$$\mu^+ = \nu_-^- \equiv \nu^-(\mathbb{T}_-), \quad \mu^- = \nu_-^+ \equiv \nu^+(\mathbb{T}_-).$$

The model has uniqueness if and only if $\mu^+ = \mu^-$.

Definition 1.8. For $G_n \sim \mathbb{P}_n$ a random graph sequence and ν_n any law on spin configurations $\underline{\sigma}_n$ of G_n , we say that $\mathbb{P}_n \otimes \nu_n$ converges locally (weakly) to $\mathbb{P}_{\mathcal{T}} \otimes \nu$ (for $\nu \in \mathscr{G}_{\mathcal{T}}$), and write $\mathbb{P}_n \otimes \nu_n \to_{loc} \mathbb{P}_{\mathcal{T}} \otimes \nu$, if it holds for all $t \geq 0$ that $(B_t(I_n), \underline{\sigma}_{B_t(I_n)})$ converges in distribution to $(T^t, \underline{\sigma}_t)$ where $T \sim \mathbb{P}_{\mathcal{T}}$ and $\underline{\sigma}_t$ is the restriction to T^t of $\underline{\sigma} \sim \nu(T)$.

Remark 1.9. In [14, Defn. 2.3] three forms A, B, C of local convergence of measures are distinguished, with $C \Rightarrow B \Rightarrow A$. Our Defn. 1.8 corresponds to the weakest form A: however, as explained in the proof of [14, Thm. 2.4 (II)], if the $(\nu(T))_{T \in \mathcal{T}}$ are extremal Gibbs measures then A, B, C are easily seen to be equivalent, so convergence in the sense of Defn. 1.8 implies convergence in the a priori stronger sense of

$$\|\mathbb{P}_n[(B_t(I_n),\underline{\sigma}_{B_t(I_n)}) = \cdot] - \mathbb{P}_{\mathcal{T}}[(T^t,\underline{\sigma}_{T^t}) = \cdot]\|_{TV} \to 0.$$

Theorem 5. For any anti-ferromagnetic two-spin system on $G_n \to_{loc} \mathbf{T}$, the following hold: (a) If the G_n are symmetric, then $\mathbb{P}_n \otimes \nu_n \to_{loc} \mathbf{P} \otimes [(\nu^+ + \nu^-)/2]$.

¹If $\mathcal{T} = \{\mathbb{T}\}$ this agrees with the usual definition of translation-invariance, whereas if $\mathcal{T} = \{\mathbb{T}_{\pm}\}$ then the projections $\nu(\mathbb{T}^{\pm})$ are *semi*-translation-invariant.

(b) If for all $\delta > 0$ the G_n are $(\delta, 1/2, \lambda_{\delta})$ -edge expanders for some $\lambda_{\delta} > 0$, then

$$\mathbb{P}_n \otimes \nu_n^{\pm} \to_{loc} \mathbf{P} \otimes \nu^{\pm}. \tag{1.5}$$

Further, with $\langle \rangle_{\mu}$ denoting expectation with respect to the Gibbs measure μ ,

$$\frac{1}{n}Y(\underline{\sigma})\sum_{i\in V}\tau_{i}\sigma_{i} \to \frac{1}{2}[\langle\sigma_{o}\rangle_{\mu^{+}} - \langle\sigma_{o}\rangle_{\mu^{-}}] \quad \text{in probability.}$$
(1.6)

Outline of the paper. In §2 we review the Bethe prediction in the *d*-regular setting and prove Thm. 3 (in its form Thm. 4). In §3 we show how to deduce Thm. 5 from Thm. 4 by the methods of [14]. In §4 we prove the approximate conditional independence statement (Propn. 4.2) and demonstrate the randomized reduction to MAX-CUT to prove our main results Thms. 1 and 2.

2. Partition function for two-spin models

In this section we prove Thm. 4, establishing the free energy density ϕ (and verifying the Bethe prediction) for two-spin models on graph sequences $G_n \to_{loc} \mathbf{T}$. We refer to [2, 4] for more general background and references on the Bethe prediction, and in §2.1 describe only its specialization to the d-regular setting. In §2.2 we show that for purposes of computing ϕ on d-regular locally tree-like graph sequences, all non-degenerate two-spin systems reduce to Ising or hard-core. In §2.3 we compute the free energy density for these models by applying an interpolation scheme described in [4], thereby completing the proof of Thm. 4.

2.1. **The Bethe prediction.** Recalling the notation of Rmk. 1.4, we now review the Bethe prediction for $G_n \to_{loc} \mathbb{P}_{\mathcal{T}}$. Given \mathcal{T} , let \mathcal{T}_e denote the set of trees T rooted not at a vertex but at an oriented edge $x \to y$, obtained by distinguishing an oriented edge in $T \in \mathcal{T}$ and forgetting the root. Elements of \mathcal{T} , \mathcal{T}_e are regarded modulo isomorphism: thus if $\mathcal{T} = \{\mathbb{T}\}$ then $\mathcal{T}_e = \{(\mathbb{T}, o \to j)\}$, and if $\mathcal{T} = \{\mathbb{T}_{\pm}\}$ then $\mathcal{T}_e = \{(\mathbb{T}_{\pm}, o \to j)\}$.

Let Δ denote the $(|\mathcal{X}|-1)$ -dimensional simplex of probability measures on \mathcal{X} . A message is a mapping $h: \mathcal{T}_e \to \Delta$; we write $\mathcal{H} \equiv \mathcal{H}(\mathcal{T})$ for the space of messages on \mathcal{T}_e . For $T \in \mathcal{T}$, $x \to y$ in T, and $h \in \mathcal{H}$, write $h_{x \to y}$ for the image of $(T, x \to y) \in \mathcal{T}_e$ under h, and define

$$\Phi_T(h) \equiv \Phi_T^{\text{vx}}(h) - \Phi_T^{\text{e}}(h)$$

where

$$\Phi_T^{\text{vx}}(h) \equiv \log \left\{ \sum_{\sigma_o} \bar{\psi}(\sigma_o) \prod_{j \in \partial o} \left(\sum_{\sigma_j} \psi(\sigma_o, \sigma_j) h_{j \to o}(\sigma_j) \right) \right\},
\Phi_T^{\text{e}}(h) \equiv \frac{1}{2} \sum_{j \in \partial o} \log \left\{ \sum_{\sigma_o} \psi(\sigma_o, \sigma_j) h_{o \to j}(\sigma_o) h_{j \to o}(\sigma_j) \right\}.$$

The Bethe free energy functional on $\mathcal{H}(\mathcal{T})$ is defined by $\Phi_{\mathcal{T}}(h) \equiv \mathbb{E}_{\mathcal{T}}[\Phi_{\mathcal{T}}(h)]$. The Bethe or belief propagation (BP) recursion is the map

$$\mathsf{BP} \equiv \mathsf{BP}_{\mathcal{T}} : \mathcal{H}(\mathcal{T}) \to \mathcal{H}(\mathcal{T}), \quad (\mathsf{BP}h)_{x \to y}(\sigma) \equiv \bar{\mathsf{F}}[(h_{v \to x})_{v \in \partial x \setminus y}]$$

for $\bar{\mathsf{F}}:\Delta^{d-1}\to\Delta$ defined by

$$[\bar{\mathsf{F}}(\underline{h})](\sigma) \cong \bar{\psi}(\sigma) \prod_{j=1}^{d-1} \left\{ \sum_{\sigma_j} \psi(\sigma, \sigma_j) h_j(\sigma_j) \right\}, \quad \underline{h} \equiv (h_1, \dots, h_{d-1}) \in \Delta^{d-1}$$
 (2.1)

(where \cong denotes equivalence up to a positive normalizing factor).

Definition 2.1. For any homogeneous spin system on $G_n \to_{loc} \mathbb{P}_{\mathcal{T}}$, the *Bethe prediction* is that the free energy density ϕ of (1.4) exists and equals

$$\Phi \equiv \Phi_{\mathcal{T}} \equiv \sup_{h \in \mathcal{H}_{\star}} \Phi_{\mathcal{T}}(h) \tag{2.2}$$

with $\mathcal{H}_{\star} \equiv \mathcal{H}_{\star}(\mathcal{T}) \subseteq \mathcal{H}(\mathcal{T})$ the set of all fixed points of $\mathsf{BP}_{\mathcal{T}}$.

For $h \in \Delta$ write $F(h) \equiv \bar{F}(h, ..., h)$: then $\mathcal{H}_{\star}(\{\mathbb{T}\})$ corresponds simply to the fixed points of F in simplex. For $h \in \mathcal{H}(\{\mathbb{T}_{\pm}\})$ we write $h_{\pm} \equiv h(\mathbb{T}_{\pm}, o \to j) \in \Delta$: then any $h \in \mathcal{H}_{\star}(\{\mathbb{T}_{\pm}\})$ must satisfy $h_{\pm} = F(h_{\mp})$, so $\mathcal{H}_{\star}(\{\mathbb{T}_{\pm}\})$ corresponds to the fixed points of the double recursion $F^{(2)} \equiv F \circ F$.

In verifying the Bethe prediction we will identify the fixed points attaining the supremum in (2.2). In the anti-ferromagnetic case, with h^+ (resp. h^-) denoting the elements $h \in \mathcal{H}_{\star}(\{\mathbb{T}_{\pm}\})$ maximizing $h_+(+)$ (resp. $h_-(+)$), we will see that

$$\Phi_{\{\mathbb{T}_{\pm}\}} = \Phi_{\{\mathbb{T}_{\pm}\}}(h^{+}) = \Phi_{\{\mathbb{T}_{\pm}\}}(h^{-}). \tag{2.3}$$

Explicitly, $h_+^+ = h_-^-$ (resp. $h_-^+ = h_+^-$) will be the fixed points of $\mathsf{F}^{(2)}$ giving maximal (resp. minimal) probability to spin +. The ferromagnetic case reduces to the Ising model: here, with h^{\pm} denoting the elements of $\mathcal{H}_{\star}(\{\mathbb{T}\})$ maximizing $h_{o\to j}(\pm)$ on \mathbb{T} , we will see that

$$\Phi_{\{\mathbb{T}\}} = \Phi_{\{\mathbb{T}\}}(h^{\operatorname{sgn} B}). \tag{2.4}$$

The remainder of this section is devoted to the proof of Thm. 4.

2.2. Reduction to Ising and hard-core on d-regular graphs. We first show that for the computation of the free energy density, all (non-degenerate) homogeneous two-spin models on graph sequences $G_n \to_{loc} \mathbf{T}$ reduce to either the Ising or hard-core model. Indeed, let $\underline{\psi} \equiv (\psi, \overline{\psi})$ be a specification for a two-spin system with alphabet $\mathscr{X} = \{\pm\}$. If we define $\underline{\psi}'$ by $\psi'(\sigma, \sigma') \equiv \psi(\sigma, \sigma') \overline{\psi}(\sigma)^{1/d} \overline{\psi}(\sigma')^{1/d}$, and $\overline{\psi}'(\sigma) \equiv 1$, then

$$\frac{1}{n}\log Z_G(\underline{\psi}) - \frac{1}{n}\log Z_G(\underline{\psi}') = O(\mathbb{E}_n[|\partial I_n|\mathbf{1}\{|\partial I_n| \neq d\}]),$$

which for $G_n \to_{loc} \mathbb{T}$ tends to zero as $n \to \infty$ by uniform sparsity. Therefore we assume without loss $\bar{\psi} \equiv 1$, and consider the possibilities for ψ :

(1) If $\psi > 0$, then $\psi(\sigma, \sigma') = e^{B_0} e^{\beta \sigma \sigma'} e^{B\sigma/d} e^{B\sigma'/d}$ for β, B, B_0 defined by

$$\frac{\psi(+,+)}{\psi(-,-)} = e^{4B/d}, \quad \frac{\psi(+,+)\psi(-,-)}{\psi(+,-)^2} = e^{4\beta}, \quad \psi(+,+)\psi(+,-)^2\psi(-,-) = e^{4B_0},$$

so $\phi_n - (d/2)B_0$ is asymptotically equal to the free energy density for the Ising model on G_n with parameters (β, B) .

(2) If $\psi(+,-) = \psi(-,+) > 0$ and $\psi(-,-) > \psi(+,+) = 0$, then, recalling $\bar{\sigma} \equiv \mathbf{1}\{\sigma = +\}$, we have $\psi(\sigma,\sigma') = e^{B_0}\mathbf{1}\{\bar{\sigma}\bar{\sigma}' \neq 1\}\lambda^{\bar{\sigma}/d}\lambda^{\bar{\sigma}'/d}$ for B_0,λ defined by

$$\psi(-,-) \equiv e^{B_0}, \quad \frac{\psi(+,-)}{\psi(-,-)} \equiv \lambda^{1/d}.$$

Therefore $\phi_n - (d/2)B_0$ is asymptotically equal to the free energy density for the independent set model on G_n at fugacity λ .

The remaining two-spin models are degenerate, with free energy density which is easy to calculate:

(3) Suppose $\psi(+,-) = \psi(-,+) = 0$, so that $\psi(\sigma,\sigma')$ may be written as $\mathbf{1}\{\sigma = \sigma'\}e^{B_0}e^{B\sigma'/d}e^{B\sigma'/d}$.

$$\phi_n = B_0 \frac{\mathbb{E}_n[|E_n|]}{n} + B + \frac{1}{n} \mathbb{E}_n \left[\sum_{j=1}^{k(G_n)} \log(1 + e^{-2B|C_j|}) \right]$$

where the sum is taken over the connected components $C_1, \ldots, C_{k(G_n)}$ of G_n . We claim $\phi_n \to \phi = (d/2)B_0 + B$: we have $\liminf_{n\to\infty} (\phi_n - \phi) \ge 0$ (using uniform sparsity), and

$$\limsup_{n \to \infty} (\phi_n - \phi) \le \limsup_{n \to \infty} \log 2 \frac{\mathbb{E}_n[k(G_n)]}{n},$$

so it suffices to show $\mathbb{E}_n[k(G_n)]/n \to 0$. Indeed, if this fails then there exists $\epsilon > 0$ such that for infinitely many n, the event $\{k(G_n) \ge \epsilon n\}$ occurs with \mathbb{P}_n -probability at least ϵ . On this event, G_n has at least $\epsilon n/2$ components of size $\le 2/\epsilon$, so for $t > \log_k(2/\epsilon)$, $\limsup_{n\to\infty} \mathbb{P}_n(B_t(I_n) \not\cong \mathbb{T}^t) \ge \epsilon^2/2 > 0$, in contradiction of $G_n \to_{loc} \mathbb{T}$.

(4) Suppose instead $\psi(+,+) = \psi(-,-) = 0$ while $\psi(+,-) = \psi(-,+) > 0$. If the G_n are not exactly bipartite then $\phi_n = -\infty$. If they are exactly bipartite then

$$\phi_n = \log \psi(+, -) \frac{\mathbb{E}_n[|E_n|]}{n} + \log 2 \frac{\mathbb{E}_n[k(G_n)]}{n},$$

and by the observation of (3) this converges to $\phi = (d/2) \log \psi(+, -)$.

2.3. **Bethe interpolation.** We now evaluate the hard-core and Ising free energy densities by interpolating in the model parameters. Write $\xi \equiv \log \psi$, $\bar{\xi} \equiv \log \bar{\psi}$, and for the hard-core model take $B \equiv \log \lambda$. Let $\langle \, \rangle_n^{\beta,B}$ denote expectation with respect to $\nu_n^{\beta,B} \equiv \nu_{G_n}^{\beta,B}$, and define

$$a_n^{\text{vx}}(\beta, B) \equiv \partial_B \phi_n(\beta, B) = \mathbb{E}_n[\langle \partial_B \bar{\xi}(\sigma_{I_n}) \rangle_n^{\beta, B}],$$

$$a_n^{\text{e}}(\beta, B) \equiv \partial_\beta \phi_n(\beta, B) = \frac{1}{2} \mathbb{E}_n \Big[\sum_{j \in \partial I_n} \langle \partial_\beta \xi(\sigma_{I_n}, \sigma_j) \rangle_n^{\beta, B} \Big]$$

(with $a_n^{\rm e}(\beta, B) \equiv 0$ for hard-core). We also define analogous quantities on the limiting tree $T \sim \mathbb{P}_{\mathcal{T}}$: for $h \in \mathcal{H}$ let

$$a^{\text{vx}}(\beta, B, h) \equiv a_{\mathcal{T}}^{\text{vx}}(\beta, B, h) \equiv \mathbb{E}_{\mathcal{T}}[[\![\partial_{B}\bar{\xi}(\sigma_{o})]\!]_{h}^{\beta, B}],$$

$$a^{\text{e}}(\beta, B, h) \equiv a_{\mathcal{T}}^{\text{e}}(\beta, B, h) \equiv \frac{1}{2}\mathbb{E}_{\mathcal{T}}\Big[\sum_{j \in \partial o} [\![\partial_{\beta}\xi(\sigma_{o}, \sigma_{j})]\!]_{h}^{\beta, B}\Big]$$
(2.5)

where $[\![\cdot]\!]_h$ denotes expectation with respect to the measure $\nu_{T^1}^h$ on spin configurations on T^1 defined by

$$\nu_{T^1}^h(\underline{\sigma}_{T^1} = \cdot) \cong \bar{\psi}(\sigma_o) \prod_{j \in \partial o} \psi(\sigma_o, \sigma_j) h_{j \to o}(\sigma_j).$$

The following lemma, describing our interpolation scheme, may be verified directly or obtained as a consequence of [4, Propn. 2.4]. We always interpolate in one parameter at a time, keeping the other fixed and suppressing it from the notation.

Lemma 2.2. If for $B \in [B_0, B_1]$ we have $h \equiv h(B) \in \mathcal{H}^B_{\star}$ which is continuous and of bounded total variation in B, then

$$\Phi_{\mathcal{T}}(B_1) - \Phi_{\mathcal{T}}(B_0) = \int_{B_0}^{B_1} a_{\mathcal{T}}^{vx}(B, h) \ dB.$$

The same result holds for B, a_T^{vx} replaced with β, a_T^e .

We now make explicit the connection between BP fixed points and Gibbs measures; for a discussion in a more general setting and further references see [4, Rmk. 2.6]. Recall that for $T \in \mathcal{T}$, \mathscr{G}_T denotes the set of Gibbs measures for the specification $\underline{\psi}$ on T, and $\mathscr{G}_{\mathcal{T}}$ denotes the space of mappings $T \mapsto \nu(T) \in \mathscr{G}_T$ with $T \in \mathcal{T}$. An element $\mu = \nu(T) \in \mathscr{G}_T$ is a Markov chain or splitting Gibbs measure (see [19]) if there exists a collection $h^{\mu} \equiv (h^{\mu}_{x \to y})$ of elements of Δ indexed by the oriented edges of T such that for any finite connected induced subgraph $U = (V_U, E_U)$ of T,

$$\mu(\underline{\sigma}_{U}) = \frac{1}{z} \prod_{i \in V_{U}} \bar{\psi}(\sigma_{i}) \prod_{(ij) \in E_{U}} \psi(\sigma_{i}, \sigma_{j}) \prod_{j \in \partial U} \left\{ \sum_{\sigma_{j}} \psi(\sigma_{p(j)}, \sigma_{j}) h_{j \to p(j)}^{\mu}(\sigma_{j}) \right\}, \tag{2.6}$$

where p(j) denotes the unique neighbor of j inside U for j belonging to the external boundary ∂U of U. Extremal Gibbs measures are Markov chains but the converse is false. We say that an element $\nu \in \mathscr{G}_{\mathcal{T}}$ is Markovian if $\nu(T)$ is a Markov chain for each $T \in \mathcal{T}$: the associated collection $h^{\nu} \equiv (h^{\nu(T)})_{T \in \mathcal{T}}$ is called an $entrance\ law$: it satisfies consistency conditions imposed by (2.6) (which closely resemble the BP equation), and the correspondence between Markovian $\nu \in \mathscr{G}_{\mathcal{T}}$ and entrance laws h^{ν} is bijective. If ν is also translation-invariant (in the sense of Defn. 1.7), then each $h^{\nu(T)}_{x\to y}$ depends only on the isomorphism class of $(T, x \to y)$ in \mathcal{T}_e , so $h \in \mathcal{H}(\mathcal{T})$, and in fact by the consistency conditions $h \in \mathcal{H}_{\star}(\mathcal{T})$. Thus there is a bijection between BP fixed points $h \in \mathcal{H}_{\star}(\mathcal{T})$ and translation-invariant Markovian $\nu^h \in \mathscr{G}_{\mathcal{T}}$. In particular, for two-spin models, the $\nu^{\pm} \in \mathscr{G}_{\mathcal{T}}$ of §1.4 and the $h^{\pm} \in \mathcal{H}_{\star}$ of §2.1 are related by this correspondence and so may be regarded as essentially equivalent.

The main implication of Lem. 2.2 is the following (which may also be obtained as a special case of [4, Thm. 1.13]): if for $B \in [B_0, B_1]$ we have $h \equiv h(B) \in \mathcal{H}^B_{\star}$ which is continuous and of bounded total variation in B, then

$$\limsup_{n \to \infty} a_n^{\text{vx}}(B) \le a_{\mathcal{T}}^{\text{vx}}(B, h), \tag{2.7}$$

implies $\limsup_{n\to\infty} [\phi_n(B_1) - \phi_n(B_0)] \le \Phi_{\mathcal{T}}(B_1) - \Phi_{\mathcal{T}}(B_0)$. (The statement also holds with B, a^{vx} replaced by β, a^{e} .) From the above discussion we can re-express

$$a_{\mathcal{T}}^{\text{vx}}(B,h) \equiv \mathbb{E}_{\mathcal{T}}[\langle \partial_B \bar{\xi}(\sigma_o) \rangle_{\nu^h}^B], \quad a_{\mathcal{T}}^{\text{e}}(\beta,h) \equiv \frac{1}{2} \mathbb{E}_{\mathcal{T}}\Big[\sum_{i \in \partial o} \langle \partial_{\beta} \xi(\sigma_o,\sigma_j) \rangle_{\nu^h}^{\beta}\Big],$$

so showing (2.7) amounts to proving a relation between the expectation of a local observable, in this case $\partial_B \bar{\xi}(\sigma_i)$, in the finite graph to the expectation of the analogous observable $\partial_B \bar{\xi}(\sigma_o)$ under translation-invariant Markov chains on the limiting tree. In the remainder of the section we carry out this scheme. Note that for the Ising model

$$a_n^{\text{vx}}(B) = \mathbb{E}_n[\langle \sigma_{I_n} \rangle_n^B], \quad a_n^{\text{e}}(\beta) = \frac{1}{2} \mathbb{E}_n \Big[\sum_{j \in \partial I_n} \langle \sigma_{I_n} \sigma_j \rangle_n^\beta \Big].$$

For the hard-core model $a_n^{\text{vx}}(B) = \mathbb{E}_n[\langle \bar{\sigma}_{I_n} \rangle_n^B].$

2.3.1. Interpolation for hard-core.

Lemma 2.3. For the hard-core model at fugacity λ , the supremum of $\langle \bar{\sigma}_o + d^{-1} \sum_{j \in \partial_o} \bar{\sigma}_j \rangle_{\mu}$ over $\mu \in \mathscr{G}_{\mathbb{T}}$ is achieved precisely by the measures μ^{\pm} .

Consequently, the supremum of $\mathbf{E}[\langle \bar{\sigma}_o \rangle_{\nu}]$ over translation-invariant $\nu \in \mathscr{G}_{\mathcal{T}}$ is achieved precisely by the ν^{\pm} .

Proof. By extremal decomposition, assume without loss that μ is itself extremal, with $h^{\mu} \equiv (h^{\mu}_{x \to y})$ as defined above. For $j \in \partial o$ write $q_j \equiv h^{\mu}_{j \to o}(-)$: then

$$\left\langle \bar{\sigma}_o + \frac{1}{d} \sum_{j \in \partial o} \bar{\sigma}_j \right\rangle_{\mu} = \frac{\lambda \prod_{j \in \partial o} q_j + d^{-1} \sum_{j \in \partial o} (1 - q_j)}{1 + \lambda \prod_{j \in \partial o} q_j}$$
$$= 1 - \frac{1}{d} \frac{\sum_{j \in \partial o} q_j}{1 + \lambda \prod_{j \in \partial o} q_j}.$$

For fixed $\sum_{j} q_{j}$ this is (strictly) maximized by taking all $q_{j} \equiv q$, so the above is

$$\leq 1 - \frac{1}{\max_{q^- \leq q \leq q^+} [q^{-1} + \lambda q^{d-1}]},$$

where q^- and q^+ are the minimal and maximal values for q, corresponding to μ^- and μ^+ respectively. Since $q^{-1} + \lambda q^{d-1}$ is convex, the maximum can only be attained at the endpoints, and in fact it is attained at both endpoints with value $(1/q^- + 1/q^+ - 1)^{-1}$.

Lemma 2.4. For the hard-core model, $\mathcal{H}^{\lambda}_{\star}(\{\mathbb{T}\})$ consists of a single message $h^{\star} \equiv h^{\star}(\lambda)$. $\mathcal{H}^{\lambda}_{\star}(\{\mathbb{T}_{\pm}\})$ consists of the messages h^{\star}, h^{\pm} which coincide for $\lambda \leq \lambda_c$ and are distinct for $\lambda > \lambda_c$. The messages are continuous in λ , smooth except possibly at $\lambda = \lambda_c$.

Proof. For the hard-core model, the function F(h) $(h \in \Delta)$ of (2.1) is expressed in terms of $q \equiv h(-)$ as $F(q) = (1 + \lambda q^{d-1})^{-1}$. As q increases from 0 to 1, F decreases from 1 to $(1 + \lambda)^{-1}$, so F has a unique fixed point q_{\star} which is smoothly decreasing in λ . We compute

$$\mathsf{F}' = -\frac{(d-1)}{q}\mathsf{F}[1-\mathsf{F}], \quad \mathsf{F}'' = -\frac{d-1}{q^2}\mathsf{F}(1-\mathsf{F})[(d-1)2\mathsf{F}-d],$$

so $G \equiv F^{(2)}$ has second derivative

$$G'' = (F' \circ F)F'' + (F'' \circ F)(F')^2 = \frac{(d-1)^2}{q^2} (1-F)(1-G)G \cdot Q,$$

$$Q \equiv -2(d-1)^2 (1-F)G + (d-2)[d(1-F)+F].$$

Setting this to zero gives

$$\mathsf{G} = \frac{(d-2)}{2(d-1)^2} \left(d + \frac{\mathsf{F}}{1-\mathsf{F}} \right).$$

The left-hand side is decreasing in F while the right-hand side is increasing, so G has at most one inflection point, hence at most three fixed points. If G has a fixed point which is not equal to q_{\star} , then necessarily it has exactly three fixed points $F(q_{\circ}) < q_{\star} < q_{\circ}$, so $G'(q_{\star}) > 1$. But

$$\mathsf{G}'(q_{\star}) = \mathsf{F}'(q_{\star})^2 = (d-1)^2 (1-q_{\star})^2$$

is smoothly increasing in λ with $\mathsf{G}'(q_{\star})=1$ precisely at $\lambda=\lambda_c(d)$, so we see G has a unique fixed point h_{\star} when $\lambda\leq\lambda_c$, and when $\lambda>\lambda_c$ it has three fixed points $\mathsf{F}(q_{\circ})< q_{\star}< q_{\circ}$ which are smooth on the open interval (λ_c,∞) .

It remains to verify that $q_{\circ} \to q_{\star}$ as $\lambda \downarrow \lambda_c$. Suppose otherwise, so that

$$\limsup_{\lambda \downarrow \lambda_c} \mathsf{F}(q_\circ) + 2\epsilon < q_\star < \liminf_{\lambda \downarrow \lambda_c} q_\circ - 2\epsilon$$

for some $\epsilon > 0$. It is possible to take a sequence $\lambda \downarrow \lambda_c$ along which the inflection point of G always lies on the same side of q_{\star} : assume it is $\leq q_{\star}$ (the argument for the $\geq q_{\star}$ case is symmetric), so that G' is decreasing on $q \geq q_{\star}$. By the mean value theorem applied to the interval $[q_{\star} + \epsilon, q_{\star} + 2\epsilon]$,

$$\mathsf{G}'(q_{\star}+\epsilon) \geq \frac{\mathsf{G}(q_{\star}+2\epsilon) - \mathsf{G}(q_{\star}+\epsilon)}{\epsilon} \geq \frac{q_{\star}+2\epsilon - [q_{\star}+\mathsf{G}'(q_{\star})\epsilon]}{\epsilon} = 2 - \mathsf{G}'(q_{\star}).$$

Therefore $2 - \mathsf{G}'(q_{\star}) \leq \mathsf{G}'(q) \leq \mathsf{G}'(q_{\star})$ for all $q \in [q_{\star}, q_{\star} + \epsilon]$, and consequently $\mathsf{F}'(q) = 1$ for all $q \in [q_{\star}, q_{\star} + \epsilon]$ at λ_c . This gives the desired contradiction and the lemma follows.

Proposition 2.5. For the hard-core model,

- (a) The definitions (2.2) and (2.3) of $\Phi \equiv \Phi_{\{\mathbb{T}_{\pm}\}}$ coincide. If $G_n \to_{loc} \mathbb{T}$ then $\phi = \Phi$ for $\lambda \leq \lambda_c$ and $\limsup_n \phi_n \leq \Phi$ for $\lambda > \lambda_c$.
- (b) If $G_n \to_{loc} \mathbf{T}$ then $\phi = \Phi$ for all $\lambda > 0$.

Proof. (a) Since any subsequential local weak limit of the measures ν_n must be translation-invariant, the second part of Lem. 2.3 implies

$$\limsup_{n \to \infty} a_n^{\text{vx}}(B) = \limsup_{n \to \infty} \mathbb{E}_n[\langle \bar{\sigma}_{I_n} \rangle_n^B] \le \mathbf{E}[\langle \bar{\sigma}_o \rangle_{\nu^{\pm}}] \equiv a^{\text{vx}}(B, h^{\pm})$$

(the inequality can alternatively be obtained by expressing $a_n^{\text{vx}}(B)$ as $o(1) + \mathbb{E}_n[\langle \bar{\sigma}_{I_n} + d^{-1} \sum_{j \in \partial I_n} \bar{\sigma}_j \rangle_n^B]/2$ and directly applying the first part of Lem. 2.3). Also, $a^{\text{vx}}(B, h^*) \leq a^{\text{vx}}(B, h^+) = a^{\text{vx}}(B, h^-)$, with equality for $\lambda \leq \lambda_c$ and with strict inequality for $\lambda > \lambda_c$. It then follows from Lem. 2.2 (using Lem. 2.4) that for $\lambda_c < \lambda$,

$$\limsup [\phi_n(\lambda) - \phi_n(\lambda_c)] \le \Phi(\lambda, h^{\pm}) - \Phi(\lambda_c, h^{\pm}) > \Phi(\lambda, h^{\star}) - \Phi(\lambda_c, h^{\star}).$$

It was shown in [4, Thm. 1.11] that $\phi = \Phi(h^*) = \Phi(h^{\pm})$ for $\lambda \leq \lambda_c$ so the claim follows (again making use of Lem. 2.4).

(b) It suffices to show that for $G_n \to_{loc} \mathbf{T}$,

$$\lim_{\lambda \to \infty} \liminf_{n \to \infty} (\phi_n - \Phi) \ge 0. \tag{2.8}$$

Indeed,

$$\Phi^{\text{vx}}(\lambda) = \frac{1}{2} \log[\lambda(q^+)^d + 1] + \frac{1}{2} \log[\lambda(q^-)^d + 1],$$

$$\Phi^{\text{e}}(\lambda) = \frac{d}{2} \log(1 - (1 - q^-)(1 - q^+)),$$

and $\lim_{\lambda\to\infty} q^+ = \lim_{\lambda\to\infty} (1-q^-) = 1$, so $\lim_{\lambda\to\infty} \Phi(\lambda) - [\log(\lambda+1)]/2 = 0$. But

$$\liminf_{n \to \infty} \phi_n \ge \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}_n \left[\log \sum_{j=0}^{\alpha_n} \binom{n}{j} \lambda^j \right] = \log(1+\lambda) \frac{\mathbb{E}_n[\alpha_n]}{n}$$

for α_n the independence number of G_n . But α_n is at least the number of black vertices with no black neighbors, so $G_n \to_{loc} \mathbf{T}$ implies $\limsup_n \mathbb{E}_n[\alpha_n]/n \leq 1/2$. This proves (2.8) from which the result follows.

2.3.2. Interpolation for Ising.

Lemma 2.6. For the Ising model with parameters $\beta < 0$ and $B \in \mathbb{R}$, the supremum of $\langle \sigma_o + d^{-1} \sum_{j \in \partial o} \sigma_j \rangle_{\mu}$ over $\mu \in \mathscr{G}_{\mathbb{T}}$ is achieved precisely by the μ^{\pm} .

Consequently, the supremum of $\mathbf{E}[\langle \sigma_o \rangle_{\nu}]$ over translation-invariant $\nu \in \mathcal{G}_{\mathcal{T}}$ is achieved precisely by the ν^{\pm} .

Proof. We argue as in the proof of Lem. 2.3: assume μ is extremal, and write $(h_j, 1 - h_j) \equiv (h_{j\to o}^{\mu}(+), h_{j\to o}^{\mu}(-))$: then

$$\left\langle \sigma_o + \frac{1}{d} \sum_{j \in \partial o} \sigma_j \right\rangle_{\mu} = \frac{\left[e^B - e^{-B} \prod_{k \in \partial o} R_k \right] + d^{-1} \sum_{j \in \partial o} \left[e^B A_j + \left(e^{-B} \prod_{k \in \partial o} R_k \right) B_j \right]}{e^B + e^{-B} \prod_{k \in \partial o} R_k}$$

where
$$R_j \equiv [e^{-\beta}h_j + e^{\beta}(1 - h_j)]/[e^{\beta}h_j + e^{-\beta}(1 - h_j)]$$
, and
$$A_j \equiv \frac{e^{\beta}h_j - e^{-\beta}(1 - h_j)}{e^{\beta}h_j + e^{-\beta}(1 - h_j)} = \frac{2R_j - (e^{2\beta} + e^{-2\beta})}{e^{-2\beta} - e^{2\beta}},$$

$$B_j \equiv \frac{e^{-\beta}h_j - e^{\beta}(1 - h_j)}{e^{-\beta}h_j + e^{\beta}(1 - h_j)} = \frac{(e^{2\beta} + e^{-2\beta}) - 2R_j^{-1}}{e^{-2\beta} - e^{2\beta}}.$$

If $(h_k)_{k\neq j}$ are fixed, the expression is maximized by taking h_j as large or as small as possible. Since $\beta < 0$, for any fixed $\prod_{k \in \partial o} R_k$, both $\sum_{j \in \partial o} A_j$ and $\sum_{j \in \partial o} B_j$ are maximized by taking all the R_j equal, i.e. with $h_j \equiv h$. The overall maximum is then attained for h equal to h^+ or h^- , the minimal and maximal values for q corresponding to the μ^{\pm} . In fact it is attained at both endpoints with value

$$\frac{2(h^- + h^+ - 1)}{1 + (e^{-2\beta} - 1)(h^- + h^+) - 2(e^{-2\beta} - 1)h^-h^+},$$

which concludes the proof.

Lemma 2.7. For the Ising model with parameters $\beta < 0$ and B = 0, the supremum of $-\langle \sum_{j \in \partial o} \sigma_o \sigma_j \rangle_{\mu}$ over $\mu \in \mathscr{G}_{\mathbb{T}}$ is achieved precisely by the μ^{\pm} .

Proof. As in the proof of Lem. 2.6, assume μ is extremal and write $(h_{x\to y}, 1-h_{x\to y}) \equiv (h^{\mu}_{x\to y}(+), h^{\mu}_{x\to y}(-))$. Then

$$-\langle \sigma_o \sigma_j \rangle_{\mu} = \frac{e^{-\beta} - (e^{\beta} + e^{-\beta})[h_{o \to j}h_{j \to o} + (1 - h_{o \to j})(1 - h_{j \to o})]}{e^{-\beta} - (e^{-\beta} - e^{\beta})[h_{o \to j}h_{j \to o} + (1 - h_{o \to j})(1 - h_{j \to o})]}.$$
 (2.9)

The partial derivative with respect to $h_{j\to o}$ has the same sign as $1/2 - h_{o\to j}$, so $-\langle \sigma_o \sigma_j \rangle_{\mu}$ is maximized with $(h_{o\to j}, h_{o\to j})$ equal to (h^-, h^+) and (h^+, h^-) , corresponding to the measures μ^{\pm} .

Lemma 2.8. For the anti-ferromagnetic Ising model, $\mathcal{H}_{\star}^{\beta,B}(\{\mathbb{T}\})$ consists of a single message $h^{\star} \equiv h^{\star}(\beta,B)$. $\mathcal{H}_{\star}^{\lambda}(\{\mathbb{T}_{\pm}\})$ consists of the messages h^{\star},h^{\pm} which coincide for $\beta_{c,af}(B,d) \leq \beta \leq 0$ and are distinct for $\beta < \beta_{c,af}(B,d)$. The messages are continuous in β,B , smooth except possibly where $\beta = \beta_{c,af}(B,d)$.

Proof. For the Ising model, the function F(h) $(h \in \Delta)$ of (2.1) is expressed in terms of $t \equiv \log[h(+)/h(-)]$ as

$$\mathsf{F}(t) = 2B + (d-1)\log\left\{\frac{e^t + \theta}{\theta e^t + 1}\right\}, \quad \theta \equiv e^{-2\beta}. \tag{2.10}$$

F – 2B is an odd function of $t \in \mathbb{R}$, identically zero when $\beta = 0$ and strictly monotone otherwise, going from $(d-1)\log\theta$ to $-(d-1)\log\theta$ as t increases from $-\infty$ to ∞ . Suppose from now on that $\beta < 0$. Then F has a unique fixed point t_{\star} of the same sign as B, smoothly increasing in B, and smooth in β with absolute value increasing as β becomes more negative.

For $G \equiv F^{(2)}$ we compute

$$\mathsf{G}''(t) = \frac{(d-1)^2 \theta(\theta^2 - 1)^2 e^{\mathsf{F}+t} (A_+ e^{2\mathsf{F}} + A e^{\mathsf{F}} + A_-)}{(e^{\mathsf{F}} + \theta)^2 (\theta e^{\mathsf{F}} + 1)^2 (e^t + \theta)^2 (\theta e^t + 1)^2},$$

$$A_{\pm} \equiv -\theta(e^{2t} - 1) \pm (d - 1)(\theta^2 - 1) e^t,$$

$$A \equiv -(\theta^2 + 1)(e^{2t} - 1).$$

If $A_+ = 0$ then clearly G can have at most one inflection point, so suppose $A_+ > 0$: then setting G" to zero results in $e^{\mathsf{F}} = r_{\pm}$ where

$$r_{\pm} \equiv (-A \pm \sqrt{D})/(2A_{+}),$$

 $D \equiv A^{2} - 4A_{+}A_{-} = (\theta^{2} - 1)^{2}[1 + 4((d - 1)^{2} - 1/2)e^{2t} + e^{4t}] > 0.$

If $A/A_+ \ge 0$ then at most one of the r_\pm can be positive. If $A \ge 0$ then $t \le 0$ which implies $A_+ > 0$, so it remains only to consider the case $A < 0 < A_+$: in this case, A < 0 implies t > 0 and so $A_- < 0$, therefore

$$|D| - A^2 = -4A_+A_- > 0$$

which implies $r_{-} < 0 < r_{+}$. Thus G has at most one inflection point for any $\beta < 0$.

By implicit differentiation we find

$$\partial_{\theta}[\mathsf{F}'(t_{\star})] = \left[\partial_{\theta}\mathsf{F}'(t)\right]|_{t=t_{\star}} + \mathsf{F}''(t_{\star})\partial_{\theta}t_{\star} = \left[\partial_{\theta}\mathsf{F}'(t)\right]|_{t=t_{\star}} + \mathsf{F}''(t_{\star})\frac{\left[\partial_{\theta}\mathsf{F}(t)\right]|_{t=t_{\star}}}{1 - \mathsf{F}'(t_{\star})}$$

$$= -\frac{(d-1)e^{t}}{(e^{t}+\theta)(\theta e^{t}+1)} \frac{\left[d(\theta^{2}-1)+2\right](e^{2t}+1)+4\theta e^{t}}{\left[d(\theta^{2}-1)+2\right]e^{t}+\theta(e^{2t}+1)}\Big|_{t=t_{\star}} < 0,$$

so $\mathsf{G}'(t_\star) = \mathsf{F}'(t_\star)^2$ increases smoothly as β decreases. The result then follows by repeating the argument of Lem. 2.4.

Proposition 2.9. For the anti-ferromagnetic Ising model,

(a) The definitions (2.2) and (2.3) of $\Phi \equiv \Phi_{\{\mathbb{T}_{\pm}\}}$ coincide. If $G_n \to_{loc} \mathbb{T}$ then $\phi = \Phi$ for $\beta \geq \beta_{c,af}(B)$ and $\limsup_n \phi_n \leq \Phi$ for $\beta < \beta_{c,af}(B)$.

(b) If $G_n \to_{loc} \mathbf{T}$ then $\phi = \Phi$ for all β, B .

Proof. (a) First fix B = 0: since any subsequential local weak limit of the measures ν_n must be translation-invariant, Lem. 2.7 gives

$$\lim_{n \to \infty} \inf a_n^{e}(\beta) = \frac{1}{2} \lim_{n \to \infty} \inf \mathbb{E}_n \left[\sum_{j \in \partial I_n} \langle \sigma_{I_n} \sigma_j \rangle_n^{\beta} \right]
\geq \frac{1}{2} \mathbf{E} \left[\sum_{j \in \partial o} \langle \sigma_o \sigma_j \rangle_{\nu^{\pm}} \right] \geq a^{e}(\beta, h^{\pm}).$$

Also, $a^{\rm e}(\beta, h^{\star}) \geq a^{\rm e}(\beta, h^{\pm})$ with equality for $\beta \geq \beta_{c,\rm af}$ and with strict inequality for $\beta < \beta_{c,\rm af}$. It then follows from Lem. 2.2 (together with Lem. 2.8 and the previous result for $\beta \geq 0$) that for B = 0 and $\beta \leq 0$ we have

$$\lim \sup_{n \to \infty} \phi_n \le \Phi(h^{\pm}) > \Phi(h^{\star}). \tag{2.11}$$

Using Lem. 2.6 to interpolate in B (as in the proof of Propn. 2.5) then gives (2.11) for all $\beta \leq 0, B \in \mathbb{R}$.

(b) Consider the limits $\beta \to -\infty$ and $B \to \infty$:

$$\lim_{\beta \to -\infty} [\Phi(\beta, 0) + \beta d/2] = 0, \quad \lim_{B \to \infty} [\Phi(\beta, B) - B - \beta d/2] = 0.$$

If $G_n \to_{loc} \mathbf{T}$, then

$$\lim_{n \to \infty} \inf \phi_n \ge \liminf_{n \to \infty} -\beta \frac{\mathbb{E}_n[|E_n|]}{n} = -\beta d/2,
\lim_{n \to \infty} \inf \phi_n \ge B + \liminf_{n \to \infty} \beta \frac{\mathbb{E}_n[|E_n|]}{n} = B + \beta d/2,$$

so in fact $\phi = \Phi$ for all $\beta, B \in \mathbb{R}$.

For completeness we review what is known for the ferromagnetic Ising model:

Proposition 2.10. For the ferromagnetic Ising model on $G_n \to_{loc} \mathbb{T}$, ϕ exists and equals $\Phi_{\{\mathbb{T}\}}$ as defined by (2.2) (and given more explicitly by (2.4)).

Proof. In this setting \mathcal{H}_{\star} corresponds simply to the fixed points of a single iteration of the map F of (2.10), which is analyzed for example in [4, Lem. 4.6]. F always has between one and three fixed points, and we write t^+ and t^- for the maximal and minimal fixed points respectively, corresponding to h^+ and h^- as in (2.4). By symmetry we may always suppose $B \geq 0$.

At B=0, a fixed point is always given by $t^{\circ}=0$, unique provided $\mathsf{F}'(0)\leq 1$. $\mathsf{F}'(0)$ increases monotonically in β and reaches 1 at $\beta_{c,\mathsf{f}}(B=0,d)$, and for $\beta>\beta_{c,\mathsf{f}}(0,d)$ there are three distinct fixed points $t^-<0=t^{\circ}< t^+$, with $t^+=-t^-\downarrow 0=t^{\circ}$ as $\beta\downarrow\beta_{c,\mathsf{f}}(0,d)$. Since adding B simply shifts the map F of (2.10) by the constant 2B, it is easy to deduce the behavior for general $B\geq 0$: if $\mathsf{F}'(t)|_{t=0}\leq 1$ then F has a unique fixed point $t^+=t^-$ which is zero when B=0 and increases smoothly in B. If $\mathsf{F}'(t)|_{t=0}>1$, then at B=0 the map F has three fixed points $t^-< t^{\circ}=0< t^+$. As B increases, t^\pm increase smoothly while t° decreases smoothly. The fixed points t^- and t° merge at the threshold $B=B_{c,\mathsf{f}}(\beta,d)$, and for B above this threshold we again have $t^-=t^+$.

It follows from [3, Thm. 2.4] (see also [4, Thm. 1.8]) that for $G_n \to_{loc} \mathbb{T}$, ϕ exists and equals Φ as defined by (2.4). Therefore it remains to verify that

$$\Phi(h^+) \ge \Phi(h^-) \lor \Phi(h^\circ) \quad \text{for all } B \ge 0. \tag{2.12}$$

At B = 0, it follows from the above that $\Phi(h^{\circ})$ and $\Phi(h^{+}) = \Phi(h^{-})$ are continuous in β . Writing $h \equiv h(+)$ and recalling (2.9), we compute

$$\partial_h \langle \sigma_o \sigma_j \rangle = \frac{4e^{2\beta}(2h-1)}{[e^{2\beta} - 2(e^{2\beta} - 1)h(1-h)]^2},$$

so $\langle \sigma_o \sigma_j \rangle_{h^+} \geq \langle \sigma_o \sigma_j \rangle_{h^\circ}$, hence $a^{\rm e}(\beta, h^{\pm}) \geq a^{\rm e}(\beta, h^\circ)$, and then Lem. 2.2 gives $\Phi(h^{\pm}) \geq \Phi(h^\circ)$ at B = 0. Next, for all $B \geq 0$, clearly $a^{\rm e}(\beta, h^+) \geq a^{\rm e}(\beta, h^-) \vee a^{\rm e}(\beta, h^\circ)$, so another application of Lem. 2.2 gives (2.12) from which the proposition follows.

Proof of Thms. 3 and 4. Follows by combining the reduction of $\S 2.2$ with the results of Propns. 2.5, 2.9, and 2.10.

3. Local structure of measures

In this section we show how Thm. 4 can be used to deduce Thm. 5 by straightforward modifications of the arguments of [14].

Proof of Thm. 5 (a). Observe that $\partial_B^2 \phi_n = n^{-1} \mathbb{E}_n [\langle S^2 \rangle_n^B - (\langle S \rangle_n^B)^2]$ where S is $\sum_{i \in V_n} \sigma_i$ for Ising and $\sum_{i \in V_n} \bar{\sigma}_i$ for hard-core, so the ϕ_n are convex and hence so is the limit ϕ . Convex functions are absolutely continuous, so it holds for a.e. B that ϕ_n, ϕ are differentiable in B with $\partial_B \phi_n \to \partial_B \phi = \partial_B \Phi$ (by Thm. 4). It follows from [4, Propn. 2.4] that $\partial_B \Phi \equiv a^{\text{vx}}(B, h^+) = \mathbf{E}[\langle \sigma_o \rangle_{\nu^+}]$. But for any subsequential local weak limit ν of the ν_n , we also have

$$\partial_B \phi_n \equiv a_n^{\text{vx}}(B) = \mathbb{E}_n[\langle \sigma_{I_n} \rangle_n^B] \to \mathbf{E}[\langle \sigma_o \rangle_{\nu}].$$

Therefore $\mathbf{E}[\langle \sigma_o \rangle_{\nu}] = \mathbf{E}[\langle \sigma_o \rangle_{\nu^+}]$, and it follows from Lem. 2.3 and Lem. 2.6 that ν is a convex combination of the ν^{\pm} . Since the G_n are symmetric, we must have $\nu = (\nu^+ + \nu^-)/2$.

We now analyze the conditional measures ν_n^{\pm} , beginning with an easy observation:

Lemma 3.1. For anti-ferromagnetic two-spin models on $G_n \to_{loc} \mathbf{T}$,

$$\lim_{n \to \infty} \mathbb{E}_n \left[\nu_n \left(\sum_{i \in V_n} \tau_i \sigma_i = 0 \right) \right] = 0.$$

Proof. For the Ising model see [14, Lem. 4.1]. For the hard-core model, let A_n denote the set of vertices $i \in V_n$ with $B_2(i)$ isomorphic to \mathbb{T}^2_+ , the depth-two subtree of \mathbb{T}_+ ; then A_n is necessarily an independent set of black vertices. The probability that $\sum_{i \in A_n} \tau_i \bar{\sigma}_i = \sum_{i \in A_n} \bar{\sigma}_i$ takes value j, conditioned on all the spins $(\bar{\sigma}_i)_{i \notin A_n}$, is $\mathbb{P}(X = j)$ where X is a binomial random variable on $N = |\{i \in A_n : \bar{\sigma}_{\partial i} \equiv 0\}|$ number of trials with success probability $\lambda/(1 + \lambda)$. If $N \geq \epsilon n$ then $\mathbb{P}(X = j) = O(1/\sqrt{\epsilon n})$ uniformly in j (e.g. by the Berry-Esséen theorem). If $N < \epsilon n$ then $\sum_{i \in \partial A_n} \bar{\sigma}_i \geq (|A_n| - \epsilon n)/d$, so

$$\frac{1}{n} \sum_{i \in V_n} \tau_i \sigma_i = \frac{2}{n} \sum_{i \in V_n} \tau_i \bar{\sigma}_i - \frac{1}{n} \sum_{i \in V_n} \tau_i < \epsilon - \frac{|A_n|/n - \epsilon}{d} + \frac{|V_n \setminus (A_n \cup \partial A_n)|}{n} - \frac{1}{n} \sum_{i \in V_n} \tau_i.$$

As $n \to \infty$ the right-hand side tends in probability to $[-1/2 + \epsilon(d+1)]/d$, which is negative for small ϵ . Combining the above observations concludes the proof for the hard-core model. \square

In view of Lem. 3.1 we may without loss restrict attention to the measures ν_n^+ . Define the local functions (cf. [14, eq. (3.9)])

$$F_i^t \equiv F_i^t(\delta, \underline{\sigma}) \equiv \mathbf{1} \Big\{ \sum_{j \in B_t(i)} \tau_j \sigma_j \le -\delta |B_t(i)| \Big\};$$

 F_i^t indicates the vertices of G_n which are locally not in the + phase.

Proof of Thm. 5 (b). We outline the steps of the proof of (1.5) following [14], describing modifications where needed.

- Let ν^* denote any subsequential local weak limit of the ν_n^+ . Then $\nu^* \in \mathcal{G}_{\mathcal{T}}$ (see [14, Lem. 3.4]). By Lem. 3.1, ν_n^+ has free energy density converging to ϕ , so the proof of Thm. 5 (a) implies that $\nu^* = (1-q)\nu^+ + q\nu^-$ for some $q \in [0,1]$.
- By local weak convergence, $\lim_{n\to\infty} \mathbb{E}_n[\langle F_{I_n}^t \rangle_n] = \mathbf{E}[\langle F_o^t \rangle_{\nu^*}]$; further, if J_n denotes a uniformly random neighbor of I_n , then

$$\lim_{n \to \infty} \mathbb{E}_n[\langle \mathbf{1}\{F_{I_n}^t \neq F_{J_n}^t\} \rangle_n] = \mathbf{E}[\langle \mathbf{1}\{F_o^t \neq F_j^t\} \rangle_{\nu^*}], \quad j \in \partial o$$

(cf. [14, Lem. 3.7]).

• For the hard-core or anti-ferromagnetic Ising model in non-uniqueness regimes, there exists $\delta > 0$ such that

$$\begin{split} &\lim_{t\to\infty} \langle F_o^t \rangle_{\nu^+} = 0 = 1 - \lim_{t\to\infty} \langle F_o^t \rangle_{\nu^-}, \\ &\lim_{t\to\infty} \langle F_o^t \neq F_j^t \rangle_{\nu^+} = 0 = \lim_{t\to\infty} \langle F_o^t \neq F_j^t \rangle_{\nu^-} \end{split}$$

(cf. [14, Lem. 3.8]). It follows that for sufficiently large t

$$\lim_{n \to \infty} \mathbb{E}_n[\langle F_{I_n}^t \rangle_n] \ge q - \epsilon, \quad \lim_{n \to \infty} \mathbb{E}_n[\langle \mathbf{1} \{ F_o^t \ne F_j^t \} \rangle_n] \le \epsilon.$$

The argument of [14, Propn. 3.9] (using the edge-expansion hypothesis) now gives a contradiction unless q=0 establishing (1.5). The proof of (1.6) then follows from applying the proof of [14, Thm. 2.5] to the bipartite case.

4. Computational Hardness

In this section we construct the bipartite expander gadgets to be used in the reduction to MAX-CUT (Lem. 4.1) and refine Thm. 5 to an approximate conditional independence statement for the gadgets (Propn. 4.2). We conclude with the proof of our main results Thms. 1 and 2.

For any fixed positive integer k, G_{2n}^k will be a bipartite graph on 2n vertices with n even, defined as follows:

- Let H_n be a graph on n vertices of maximum degree d, generated by the configuration model as follows: take a uniformly random matching \mathfrak{m} of [dn], and put an edge (i,j) in H_n for every edge $(i',j') \in \mathfrak{m}$ with $i' \in i+n\mathbb{Z}, j' \in j+n\mathbb{Z}$ (self-loops and multi-edges allowed).
- Take G_{2n} to be the bipartite double cover of H_n : the two parts of G_{2n} are $(i_+)_{i=1}^n$ and $(i_-)_{i=1}^n$, and we put two edges (i_+, j_-) and (j_+, i_-) in G_{2n} for every edge $(i, j) \in H_n$ (multi-edges allowed).
- Choose k vertices $(i^{\ell})_{\ell=1}^k$ uniformly at random from H_n , and for each ℓ choose $j^{\ell} \in \partial i^{\ell}$ uniformly at random. G_{2n}^k is the simple bipartite graph formed by deleting the edges $(i_{\pm}^{\ell}, j_{\mp}^{\ell})$ from G_{2n} and merging any remaining multi-edges in the graph into single edges. Write $W^{\pm} \equiv \{i_{\pm}^{\ell}, j_{\pm}^{\ell}\}_{\ell=1}^k$ and $W \equiv W^+ \cup W^-$.

The graphs G_{2n} are d-regular with probability bounded away from zero as $n \to \infty$ (see e.g. [10, Ch. 9]).

Lemma 4.1. Let k be fixed. For all $\delta > 0$ there exists $\lambda_{\delta} > 0$ such that the G_{2n}^k are $(\delta, 1/2, \lambda_{\delta})$ -edge expanders with high probability as $n \to \infty$.

Proof. By stochastic domination we may assume d=3. For $S \subset H_n$ with |S|=m, the probability that there are exactly j edges in H_n between S and its complement is

$$P_{j,m} = I_{j,m} \frac{\binom{3m}{j} \binom{3(n-m)}{j} j! M_{3(m-j)} M_{3(n-m-j)}}{M_{3n}},$$

where $I_{j,m}$ is the indicator that m-j is even, and $M_{\ell}=(\ell-1)!!=\pi^{-1/2}\Gamma[(\ell+1)/2]2^{\ell/2}$ is the number of matchings on $[\ell]$ for ℓ even. By Stirling's approximation, if $\delta \leq m/n \leq 1-\delta$ and $j=\gamma n$, then

$$P_{j,m} = I_{j,m} \exp\left\{-n\left[\frac{3}{2}H(m/n) - \gamma\log\gamma + O_{\delta}(\gamma)\right] + o_{\delta}(n)\right\}$$

(where H(p) denotes the binary entropy function $-p \log p - (1-p) \log(1-p)$). There are $\leq e^{nH(m/n)}$ subsets of H_n of size m so there exists $\gamma_{\delta} > 0$ such that with probability at least $ne^{-nH(\delta)/4}$, all subsets of H_n of size between δn and $(1-\delta)n$ have expansion at least γ_{δ} .

We now show expansion for G_{2n}^k : since k does not change with n and the number of edges leaving any set of vertices decreases by at most a factor of 3 when multi-edges are merged into single edges, it suffices to show expansion for G_{2n} . Let S_{\pm} be subsets of the \pm sides of

 G_{2n} such that $S \equiv S_+ \cup S_-$ has size $\leq n$. If the projection πS of S in H_n has size $\leq (1-\delta)n$, then S has expansion at least $\gamma_{\delta}/2$. Suppose $|\pi S| \geq (1-\delta)n$: without loss $|S_+| \geq |S_-|$, so $|\pi S_+ \setminus \pi S_-| \geq (1/2-\delta)n$. If there are fewer than $\gamma |S|$ edges leaving S, then there must be at least $3(1/2-\delta)n-\gamma n$ edges between $\pi S_+ \setminus \pi S_-$ and its complement in H_n . A similar analysis as above shows that for sufficiently small δ there exists $\gamma_{\delta} > 0$ such that the probability G_{2n} has such a set S is $\leq e^{-n(\log 2)/4}$, and this concludes the proof.

Recall that we use W^{\pm} to denote the endpoints on the \pm sides of the 2k edges deleted from G_{2n} in the formation of G_{2n}^k . Recall also the definitions of $\mu^{\pm} \in \mathscr{G}_{\mathbb{T}}$, and write $h^{\pm} \equiv h_{o \to j}^{\mu^{\pm}} \in \Delta$. For $h, h' \in \Delta$ define $h \otimes_{\psi} h' \in \Delta_{\mathscr{X}^2}$ by

$$(h \otimes_{\psi} h')(\sigma, \sigma') = \frac{h(\sigma)\psi(\sigma, \sigma')h(\sigma')}{z(h \otimes_{\psi} h')}, \tag{4.1}$$

for $z(h \otimes_{\psi} h')$ the normalizing constant.

Proposition 4.2. The conditional measure $\nu_{G_{2n}^{\pm}}^{\pm}(\underline{\sigma}_{W}=\cdot)$ converges to the product measure

$$Q_W^{\pm}(\underline{\sigma}) \equiv \prod_{w \in W^+} h^{\pm}(\sigma_w) \prod_{w \in W^-} h^{\mp}(\sigma_w).$$

Proof. Let B_t denote the union of the balls $B_t(w) \subseteq G_{2n}$ over $w \in \{i_{\pm}^{\ell}\}_{\ell=1}^k$; assume that B_t is a disjoint union of graphs isomorphic to \mathbb{T}^t with internal boundary $S_t \equiv B_t \setminus B_{t-1}$, which is the case with high probability. For $\eta \in \mathscr{X}^{S_t}$ let

$$\xi_{t,\ell,\underline{\eta}}^{\pm}(\cdot) \equiv \nu_{G_{2n}^{k}}(\sigma_{i_{\pm}^{\ell}} = \cdot \mid \underline{\sigma}_{S_{t}} = \underline{\eta}),$$

$$\zeta_{t,\ell,\underline{\eta}}^{\pm}(\cdot) \equiv \nu_{G_{2n}^{k}}(\sigma_{j_{\pm}^{\ell}} = \cdot \mid \underline{\sigma}_{S_{t}} = \underline{\eta}),$$

so that

$$\nu_{G_{2n}}[(\sigma_{i_+^{\ell}}, \sigma_{j_-^{\ell}}) = \cdot \mid \underline{\sigma}_{S_t} = \underline{\eta}] = \xi_{t,\ell,\eta}^+ \otimes_{\psi} \zeta_{t,\ell,\eta}^-.$$

By Thm. 5, the conditional measures $\nu_{G_{2n}}^+(\underline{\sigma}_{B_t(i_+^\ell)} = \cdot)$ converge to μ^+ . But by (1.6), $Y(\underline{\sigma})$ agrees with $Y_t(\underline{\sigma}) \equiv \operatorname{sgn} \sum_{i \in V \setminus B_t} \tau_i \sigma_i$ with high probability, so that convergence also holds if we replace $\nu_{G_{2n}}^+$ by $\nu_{G_{2n}}^{\pm t}(\cdot) \equiv \nu_{G_{2n}}(\cdot \mid Y_t(\underline{\sigma}) = \pm)$. In particular,

$$0 = \lim_{t \to \infty} \lim_{n \to \infty} \mathbb{E}_{2n} \left[\left\| \sum_{\underline{\eta}} \nu_{G_{2n}}^{+t} (\underline{\sigma}_{S_t} = \underline{\eta}) \xi_{t,\ell,\underline{\eta}}^{+} \otimes_{\psi} \zeta_{t,\ell,\underline{\eta}}^{-} - h^{+} \otimes_{\psi} h^{-} \right\|_{\text{TV}} \right]$$
$$= \lim_{t \to \infty} \lim_{n \to \infty} \mathbb{E}_{2n} \left[\left\| \langle \xi_{t,\ell,\underline{\sigma}_{S_t}}^{+} \otimes_{\psi} \zeta_{t,\ell,\underline{\sigma}_{S_t}}^{-} \rangle_{\nu_{G_{2n}}^{+t}} - h^{+} \otimes_{\psi} h^{-} \right\|_{\text{TV}} \right]$$

On the other hand, it is easily seen that $(h \otimes_{\psi} h')(1,0)$ is maximized by taking h(1) and h'(0) as large as possible. But in the limit $t \to \infty$ the values $\xi_{t,\ell,\underline{\eta}}^{\pm}(1), \zeta_{t,\ell,\underline{\eta}}^{\pm}(1)$ (with $\underline{\eta}$ arbitrary) are sandwiched between $h^{\pm}(1)$, so it must be that

$$0 = \lim_{t \to \infty} \lim_{n \to \infty} \mathbb{E}_{2n} \left[\left\langle \| \xi_{t,\ell,\underline{\sigma}_{S_t}}^+ - h^+ \|_{\text{TV}} + \| \zeta_{t,\ell,\underline{\sigma}_{S_t}}^- - h^- \|_{\text{TV}} \right\rangle_{\nu_{G_{2n}}^{+t}} \right]. \tag{4.2}$$

We now claim that (4.2) continues to hold after removal of the edges $(i_+^{\ell}, j_{\pm}^{\ell})$. Indeed,

$$\frac{\nu_{G_{2n}}^{+t}(\underline{\sigma}_{S_t} = \underline{\eta})}{\nu_{G_{2n}^{+t}}^{+t}(\underline{\sigma}_{S_t} = \underline{\eta})} = \frac{Z_{\text{out}}^{+t}(\underline{\eta})Z_{\text{in}}(\underline{\eta})}{Z_{\text{out}}^{+t}(\underline{\eta})Z_{\text{in}}^{k}(\underline{\eta})} \cdot \frac{\sum_{\underline{\eta'}} Z_{\text{out}}^{+t}(\underline{\eta'})Z_{\text{in}}^{k}(\underline{\eta'})}{\sum_{\underline{\eta'}} Z_{\text{out}}^{+t}(\underline{\eta'})Z_{\text{in}}(\underline{\eta'})}$$
(4.3)

where

$$Z_{\text{out}}^{\pm t}(\underline{\eta}) \equiv Z_{G_{2n} \setminus B_{t-1}} [\{\underline{\sigma}_{G_{2n} \setminus B_{t-1}} : Y_t(\underline{\sigma}) = \pm \text{ and } \underline{\sigma}_{S_t} = \underline{\eta}\}],$$

$$Z_{\text{in}}(\underline{\eta}) \equiv Z_{B_t} [\{\underline{\sigma}_{B_t} : \underline{\sigma}_{S_t} = \underline{\eta}\}],$$

$$Z_{\text{in}}^k(\underline{\eta}) \equiv Z_{B_t \cap G_{2n}^k} [\{\underline{\sigma}_{B_t} : \underline{\sigma}_{S_t} = \underline{\eta}\}].$$

Now note that for k bounded and t large we have $Z_{\rm in}(\eta) \simeq Z_{\rm in}^k(\eta)$ uniformly over η : for Ising interactions at non-zero temperature this is obvious, while for the hard-core model

$$\frac{Z_{\rm in}(\underline{\eta})}{Z_{\rm in}^k(\underline{\eta})} = \prod_{\ell=1}^k \left\{ [1 - \xi_{t,\ell,\underline{\eta}}^+(1)\zeta_{t,\ell,\underline{\eta}}^-(1)][1 - \xi_{t,\ell,\underline{\eta}}^-(1)\zeta_{t,\ell,\underline{\eta}}^+(1)] \right\}$$

which for t large is ≈ 1 uniformly over $\underline{\eta}$. Since the $\xi_{t,\ell,\underline{\eta}}^{\pm}$ and $\zeta_{t,\ell,\underline{\eta}}^{\pm}$ are $\underline{\eta}$ -measurable, it follows from (4.3) that (4.2) continues to hold with $\nu_{G_{2n}^k}^{+t}$ in place of $\nu_{G_{2n}}^{+t}$. Since the spins $(\sigma_w)_{w\in W}$ are independent under $\nu_{G_{2n}^k}^{\pm t}(\cdot \mid \underline{\sigma}_{S_t})$, this further implies

$$0 = \lim_{t \to \infty} \lim_{n \to \infty} \mathbb{E}_{2n} \left[\| \nu_{G_{2n}^k}^{+t} (\underline{\sigma}_W = \cdot) - Q_W^+ \|_{\text{TV}} \right]. \tag{4.4}$$

Finally, by a similar argument as before $\lim_{n\to\infty} \nu_{G_{2n}^k}(Y(\underline{\sigma}) = Y_t(\underline{\sigma})) = 1$, so (4.4) holds with $\nu_{G_{2}^{+}}^{+}$ in place of $\nu_{G_{2}^{+}}^{+t}$ which gives the result.

We now demonstrate how to use Propn. 4.2 to establish a randomized reduction from approximating the partition function to the problem of approximate MAX-CUT on 3-regular graphs, which is NP-hard [1].

Let H be a 3-regular graph on m vertices and construct the bipartite graph $G = G_{2n}^{3k}$ by the procedure described above. By Lem. 3.1 and Propn. 4.2, for any $\epsilon > 0$ there exists $n(\epsilon)$ large enough such that the following hold with positive probability:

- (I) G_{2n}^{3k} was formed by removing 3k distinct edges from a d-regular graph G_{2n} ;
- (II) $\nu_{G_{2n}^{3k}}(Y(\underline{\sigma}) = +) \leq (1 + \epsilon)/2$; and (III) $\nu_{G_{2n}^{3k}}^{\pm}(\underline{\sigma}_W)/Q_W^{\pm}(\underline{\sigma}_W) \in [1 \epsilon, 1 + \epsilon]$ for all $\underline{\sigma}_W$.

Consequently, for given ϵ we may find G_{2n}^{3k} satisfying properties (I)-(III) within finite time by deterministic search. We then construct from H and G a new graph H^G as follows:

- For each vertex $x \in H$ let G_x be a copy of G, and denote by W_x^{\pm} the vertices of G_x corresponding to W^{\pm} in G. Let \widehat{H}^{G} be the disjoint union of the G_x , $x \in H$.
- For every edge $(x,y) \in H$, add 2k edges between W_x^+ and W_y^+ and similarly 2k edges between W_x^- and W_y^- . This can be done deterministically in such a way that the resulting graph, which we denote H^G , is d-regular.

We write a spin configuration on \widehat{H}^G or H^G as $\underline{\sigma} \equiv (\underline{\sigma}_x)_{x \in H}$ where $\underline{\sigma}_x$ is the restriction of $\underline{\sigma}$ to G_x . We write $Y_x \equiv Y(\underline{\sigma}_x)$ for the phase of each $\underline{\sigma}_x$, and $\mathcal{Y}(\underline{\sigma}) \equiv (Y(\underline{\sigma}_x))_{x \in H} \in \{0, 1\}^H$. Write $Z_{H^G}(\mathcal{Y})$ for the partition function for the two-spin model on H^G restricted to configurations of phase \mathcal{Y} , and define likewise $Z_{\widehat{H}^G}(\mathcal{Y})$.

Recalling (4.1), let

$$\Gamma \equiv z(h^+ \otimes_{\psi} h^+) z(h^- \otimes_{\psi} h^-), \quad \Theta \equiv z(h^+ \otimes_{\psi} h^-)^2,$$

and note that for anti-ferromagnetic two-spin models in non-uniqueness regimes, $\Theta > \Gamma$.

Lemma 4.3. For G satisfying properties (I)-(III),

$$[(1-\epsilon)/2]^m \le \frac{Z_{H^G}/Z_{\widehat{H}^G}}{\Gamma^{2k|E(H)|}(\Theta/\Gamma)^{2k_{\text{MAX-CUT}}(H)}} \le (1+\epsilon)^m.$$

Proof. By (II),

$$(1 - \epsilon)^m \le 2^m \frac{Z_{\widehat{H}^G(\mathcal{Y})}}{Z_{\widehat{H}^G}} \le (1 + \epsilon)^m \tag{4.5}$$

for all $\mathcal{Y} \in \{0,1\}^H$. By (III), the ratio

$$\frac{Z_{H^G}(\mathcal{Y})}{Z_{\widehat{H}^G}(\mathcal{Y})} = \sum_{x \in H} \sum_{\underline{\sigma}_{W_x}} \nu_{G_x}^{Y_x}(\underline{\sigma}_{W_x}) \prod_{(i,j) \in E(H^G) \setminus E(\widehat{H}^G)} \psi(\sigma_i, \sigma_j)$$

is within a $(1 \pm \epsilon)^m$ factor of

$$\sum_{x \in H} \sum_{\underline{\sigma}_{W_x}} Q^{Y_x}(\underline{\sigma}_{W_x^+}) \prod_{(i,j) \in E(H^G) \setminus E(\widehat{H}^G)} \psi(\sigma_i, \sigma_j),$$

which by direct calculation equals

$$\Gamma^{2k|E(H)|}(\Theta/\Gamma)^{2k\operatorname{cut}(\mathcal{Y})}$$

where $\operatorname{cut}(\mathcal{Y}) \equiv |\{(x,y) \in E(H) : Y_x \neq Y_y\}|$, the number of edges crossing the cut of H induced by \mathcal{Y} . Combining with (4.5) gives

$$Z_{H^G} = \sum_{\mathcal{Y}} \frac{Z_{H^G}(\mathcal{Y})}{Z_{\widehat{H}^G}(\mathcal{Y})} Z_{\widehat{H}^G}(\mathcal{Y}) \le (1 + \epsilon)^{2m} \Gamma^{2k|E(H)|} (\Theta/\Gamma)^{2k \text{ MAX-CUT}(H)} Z_{\widehat{H}^G}$$

and similarly

$$Z_{H^G} \ge 2^{-m} (1 - \epsilon)^{2m} \Gamma^{2k|E(H)|} (\Theta/\Gamma)^{2k \text{ max-cut}(H)} Z_{\widehat{H}^G}.$$

Rearranging gives the stated result.

Using this lemma we now complete the reduction to approximate MAX-CUT:

Proof of Thms. 1 and 2. Let H be a 3-regular graph on m vertices, and note that the maximum cut of H is at least 3m/4, the expected value of a random cut. Construct \widehat{H}^G , H^G as above. Since \widehat{H}^G is a disjoint collection of constant-size graphs, its partition function can

be computed in polynomial time. Suppose Z_{H^G} could be approximated within a factor of $e^{c|H^G|}$ in polynomial time for any c > 0: rearranging the result of Lem. 4.3 gives

$$\frac{\log\left(\frac{Z_{H^G}/Z_{\widehat{H}^G}}{\Gamma^{2k|E(H)|}(1+\epsilon)^m}\right)}{2k\log(\Theta/\Gamma)} \le \text{MAX-CUT}(H) \le \frac{\log\left(\frac{Z_{H^G}/Z_{\widehat{H}^G}}{\Gamma^{2k|E(H)|}[(1-\epsilon)/2]^m}\right)}{2k\log(\Theta/\Gamma)}, \quad (4.6)$$

so within polynomial time one obtains upper and lower bounds for MAX-CUT(H) which differ by O[(c|G|+1)m/k]. Taking k large and c small then allows to compute MAX-CUT(H) up to an arbitrarily small multiplicative error: that is, we have completed the reduction to a PRAS for MAX-CUT on 3-regular graphs, in contradiction of the result of [1].

Acknowledgements. We thank Andreas Galanis, Daniel Štefankovič, and Eric Vigoda for describing to us their methods and for sending us a draft of their paper. We thank Amir Dembo, David Gamarnik, Andrea Montanari, Alistair Sinclair, Piyush Srivastava, and David Wilson for helpful conversations.

References

- [1] P. Alimonti and V. Kann. Hardness of approximating problems on cubic graphs. In *Algorithms and complexity (Rome, 1997)*, volume 1203 of *Lecture Notes in Comput. Sci.*, pages 288–298. Springer, Berlin, 1997.
- [2] A. Dembo and A. Montanari. Gibbs measures and phase transitions on sparse random graphs. *Braz. J. Probab. Stat.*, 24(2):137–211, 2010.
- [3] A. Dembo and A. Montanari. Ising models on locally tree-like graphs. Ann. Appl. Probab., 20(2):565–592, 2010.
- [4] A. Dembo, A. Montanari, and N. Sun. Factor models on locally tree-like graphs. Preprint, arXiv:1110.4821v1, 2011.
- [5] S. Dommers, C. Giardinà, and R. van der Hofstad. Ising models on power-law random graphs. J. Stat. Phys., 141(4):638–660, 2010.
- [6] M. Dyer, A. Frieze, and M. Jerrum. On counting independent sets in sparse graphs. In 40th Annual Symposium on Foundations of Computer Science (New York, 1999), pages 210–217. IEEE Computer Soc., Los Alamitos, CA, 1999.
- [7] M. Dyer, A. Frieze, and M. Jerrum. On counting independent sets in sparse graphs. SIAM J. Comput., 31(5):1527-1541 (electronic), 2002.
- [8] A. Galanis, Q. Ge, D. Štefankovič, E. Vigoda, and L. Yang. Improved inapproximability results for counting independent sets in the hard-core model. In L. Goldberg, K. Jansen, R. Ravi, and J. Rolim, editors, Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, volume 6845 of Lecture Notes in Computer Science, pages 567–578. Springer Berlin / Heidelberg, 2011. 10.1007/978-3-642-22935-0-48.
- [9] A. Galanis, D. Štefankovič, and E. Vigoda. Inapproximability of the partition function for the antiferromagnetic Ising and hard-core models. Preprint, 2012.
- [10] S. Janson, T. Łuczak, and A. Rucinski. Random graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
- [11] M. Jerrum and A. Sinclair. Polynomial-time approximation algorithms for the Ising model. SIAM J. Comput., 22(5):1087–1116, 1993.
- [12] L. Li, P. Lu, and Y. Yin. Correlation decay up to uniqueness in spin systems. Preprint, arXiv:1111.7064v1, 2011.
- [13] M. Luby and E. Vigoda. Fast convergence of the Glauber dynamics for sampling independent sets. *Random Structures Algorithms*, 15(3-4):229–241, 1999. Statistical physics methods in discrete probability, combinatorics, and theoretical computer science (Princeton, NJ, 1997).

- [14] A. Montanari, E. Mossel, and A. Sly. The weak limit of Ising models on locally tree-like graphs. *Probability Theory and Related Fields*, 152:31–51, 2012. 10.1007/s00440-010-0315-6.
- [15] E. Mossel, D. Weitz, and N. Wormald. On the hardness of sampling independent sets beyond the tree threshold. *Probab. Theory Related Fields*, 143(3-4):401–439, 2009.
- [16] A. Sinclair, P. Srivastava, and M. Thurley. Approximation algorithms for two-state anti-ferromagnetic spin systems on bounded degree graphs. Preprint, arXiv:1107.2368v3, 2011.
- [17] A. Sly. Computational transition at the uniqueness threshold. Foundations of Computer Science, Annual IEEE Symposium on, 0:287–296, 2010.
- [18] D. Weitz. Counting independent sets up to the tree threshold. In STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, pages 140–149. ACM, New York, 2006.
- [19] S. Zachary. Countable state space Markov random fields and Markov chains on trees. *Ann. Probab.*, 11(4):894–903, 1983.

DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY EVANS HALL, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF STATISTICS, STANFORD UNIVERSITY SEQUOIA HALL, 390 SERRA MALL, STANFORD, CALIFORNIA 94305