Canonical decompositions of 3-connected graphs

Johannes Carmesin[♣] and Jan Kurkofka[♣]

Abstract. We offer a new structural basis for the theory of 3-connected graphs, providing a unique decomposition of every such graph into parts that are either quasi 4-connected, wheels, or thickened $K_{3,m}$'s. Our construction is explicit, canonical, and has the following applications: we obtain a new theorem characterising all Cayley graphs as either essentially 4-connected, cycles, or complete graphs on at most four vertices, and we provide an automatic proof of Tutte's wheel theorem.

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[♣]University of Birmingham, Birmingham, UK, funded by EPSRC, grant number EP/T016221/1

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Introduction

A tried and tested approach to a fair share of problems in structural and topological graph theory – such as the two-paths problem [55, 57, 58] or Kuratowski's theorem [59] – is to first solve the problem for 4-connected[1] graphs. Then, in an intermediate step, the solutions for the 4-connected graphs are extended to the 3-connected graphs, by drawing from a theory of 3-connected graphs that has been established to this end. Finally, the solutions for the 3-connected graphs are extended to all graphs, in a systematic way by employing decompositions of general graphs along their cutvertices and 2-separators.

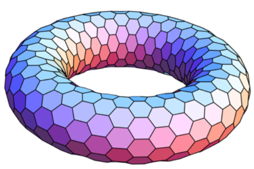
The intermediate step of this strategy seems curious: why should the step from 4-connected to 3-connected require an entirely different treatment than the systematic step from 3-connected to the general case? Indeed, the intermediate step carries the implicit believe that it is not possible to decompose 3-connected graphs along 3-separators in a way that is on a par with the renowned decompositions along separators of size at most two. Our main result offers a solution to this long-standing hindrance. To explain this, we start by giving a brief overview of the renowned decompositions along low-order separators.

Graphs trivially decompose into their components, which either are 1-connected or consist of isolated vertices. The 1-connected graphs are easily decomposed further, along their cutvertices, into subgraphs that either are 2-connected or K_2's which stem from bridges.

When decomposing 2-connected graphs further, however, things begin to get more complicated. Indeed, a 2-separator – a set of two vertices such that deleting the two vertices disconnects the graph – may separate the vertices of another 2-separator. Then if we choose one of them to decompose the graph by cutting at the 2-separator, we loose the other. In particular, it is not possible to decompose a 2-connected graph simply by cutting at all its 2-separators. An illustrative example for this are the 2-separators of a cycle.

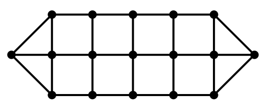
There is an elegant way to resolve this problem. If two 2-separators are compatible with each other, in the sense that they do not cut through each other, then we say that these 2-separators are *nested* with each other. Let us call a 2-separator *totally-nested* if it is nested with every 2-separator of the graph. The solution is that every 2-connected graph decomposes into 3-connected graphs, cycles and K_2's, by cutting precisely at its totally-nested 2-separators. Tutte [63] found this decomposition first, but with a different description. The description via total nestedness was discovered later by Cunningham and Edmonds [20].

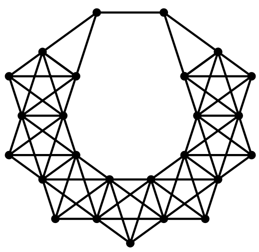
The obvious guess how the solution might extend to 3-separators of 3-connected graphs is this: every 3-connected graph decomposes into 4-connected graphs, wheels and K_3's, by cutting precisely at its totally-nested 3-separators. This guess turns out to be wrong, as the following three examples demonstrate.



Let G be a toroidal hex-grid as depicted on the left [50], and note that G is 3-connected. The neighbourhoods of the vertices of G are precisely the 3-separators of G, so no 3-separator of G is totally-nested. However, G is neither 4-connected nor a wheel. But we will see that G is 'quasi 4-connected', as no 3-separator cuts off more from G than just one vertex.

$3 \times k$ grids as depicted on the right, slightly extended to make them 3-connected, have no totally-nested 3-separators; yet they are neither 4-connected nor wheels.





Let G be the graph on the left. Every 3-separator of G consists of one of the two top vertices of degree three, and two vertices in the intersection of two neighbouring K_5's; or it is the neighbourhood of either degree-three vertex. Hence G has no totally-nested 3-separators. This remains true if we replace the K_5's in G with arbitrary 3-connected graphs. Thus, G represents a class of counterexamples that is as complex as the class of 3-connected graphs.

We resolve these problems with a twofold approach:

(1) We relax the notion of 4-connectivity to that of *quasi 4-connectivity*. We learned about this idea from Grohe's work [38].

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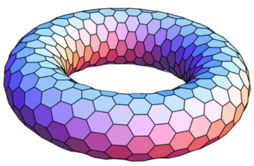
[1] A graph G is *k-connected*, for a $k \in \mathbb{N}$, if G has more than k vertices and deleting fewer than k vertices from G does not disconnect G.

(2) We introduce the new notion of a *tri-separation*, which we use instead of 3-separators. The key difference is that tri-separations may use edges in addition to vertices to separate the graph.

A *mixed-separation* of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ are nonempty. We refer to A and B as the *sides* of the mixed-separation. The *separator* of (A, B) is the disjoint union of the vertex set $A \cap B$ and the edge set $E(A \setminus B, B \setminus A)$. If the separator of (A, B) has size three, we call (A, B) a *mixed 3-separation*.

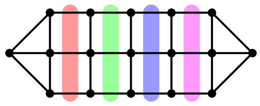
Definition (Tri-separation). A *tri-separation* of a graph G is a mixed 3-separation (A, B) of G such that every vertex in $A \cap B$ has at least two neighbours in both $G[A]$ and $G[B]$.

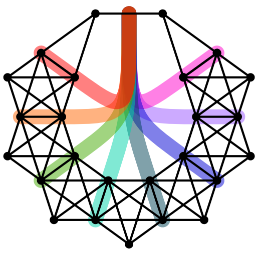
Two mixed-separations (A, B) and (C, D) of G are *nested* if, after possibly switching the name A with B or the name C with D, we have $A \subseteq C$ and $B \supseteq D$. A tri-separation of G is *totally-nested* if it is nested with every tri-separation of G. A tri-separation (A, B) of a 3-connected graph G is *trivial* if A and B are the sides of a 3-edge-cut with a side of size one. The tri-separations that we will use to decompose G are the totally-nested nontrivial tri-separations of G.



Every vertex of the toroidal hex-grid G forms the singleton side of a trivial tri-separation. Since there are no other tri-separations, all these tri-separations are totally-nested – but they are trivial. While G is not 4-connected, it is *quasi 4-connected*: G is 3-connected, has more than four vertices, and every 3-separation of G (a mixed 3-separation whose separator consists of vertices only) has a side of size at most four.

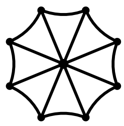
The coloured 3-edge-cuts determine nontrivial tri-separations of the slightly extended $3 \times k$ grid, and these are precisely the totally-nested nontrivial tri-separations.





Every nontrivial tri-separation of the graph on the left has a separator that consists of the top edge together with two vertices in a coloured set. As these tri-separations are pairwise nested, they are precisely the totally-nested nontrivial tri-separations.

Wheels have no totally-nested tri-separations.



Given a 3-connected graph G and a set N of pairwise nested tri-separations, we can say which parts we obtain by decomposing G along N. Roughly speaking, these are maximal subgraphs of G that lie on the same side of every tri-separation in N, with some edges added to represent external connectivity in G. We call the resulting minors of G the *torsos* of N, as they generalise the well known torsos of tree-decompositions from the theory of graph minors. See Section 2.2 for details.

According to the 2-separator theorem, some of the building blocks for 2-connected graphs are K_2's. The analogue of these building blocks for 3-connected graphs turn out to be *thickened* $K_{3,m}$*'s* with $m \geqslant 0$: these are obtained from $K_{3,m}$ by adding edges to its left class of size three to turn it into a triangle.

Theorem 1. *Let G be a 3-connected graph and let N denote its set of totally-nested nontrivial tri-separations. Each torso τ of N is a minor of G and satisfies one of the following:*

(1) *τ is quasi 4-connected;*
(2) *τ is a wheel;*
(3) *τ is a thickened $K_{3,m}$ or $G = K_{3,m}$ with $m \geqslant 0$.*

We emphasise that the sets $N = N(G)$ obviously are canonical, meaning that they commute with graph-isomorphisms: $N(\varphi(G)) = \varphi(N(G))$ for all $\varphi\colon G \to G'$. Our proof of Theorem 1 offers additional structural insights which can be used to refine Theorem 1; see Theorem 2.2.8. All graphs in this paper are finite or infinite unless stated otherwise; in particular, Theorem 1 includes infinite 3-connected graphs. It is not clear to us how the sets $N(G)$ could determine tree-decompositions, but we have a natural explanation for this: there exists a notion more general than tree-decomposition which can be used to express the sets $N(G)$; see Section 3.1.

Applications. We provide the following applications of our work. It is well known that all Cayley graphs of finite groups are either 3-connected, cycles, or complete graphs on at most two vertices [27]. By heavily exploiting the fact that our decomposition of 3-connected graphs is canonical, we can refine this fact:

Corollary 2. *Every vertex-transitive finite connected graph G either is essentially 4-connected, a cycle, or a complete graph on at most four vertices.*

We give the precise definition of 'essentially 4-connected' in Section 1.4; the main difference to 'quasi 4-connected' is that we allow 3-edge-cuts that have a side which is equal to a triangle. Corollary 2 strengthens [37, Theorem 3.4.2], a classical tool in geometric group theory from the textbook of Godsil and Royle, in a special case.

Another application comes in the form of an automatic proof of Tutte's wheel theorem [62]. In the upcoming work [17], Theorem 1 will be used to construct an FPT algorithm for connectivity augmentation from 0 to 4, and the property of total nestedness is crucial for that; see Chapter 3.

When canonicity and an explicit description matter. Recall that the tri-separation decomposition of Theorem 1 is canonical and is explicitly described so that it is uniquely determined for every 3-connected graph. These two of its aspects are *absolutely crucial* for a number of its applications:

(1) For vertex-transitive graphs, such as Cayley graphs, exploiting the combination of canonicity and total-nestedness makes up the entire proof of Corollary 2. Just recently, this combination has also been exploited when using the Tutte-decomposition in the proof of a low-order Stallings-type theorem for finite nilpotent groups [13]. An obvious next step in this direction would be to exploit this combination with the tri-separation decomposition.
(2) For Connectivity Augmentation, canonicity and access to an explicit description are key [17].
(3) Total-nestedness is incredibly desirable in Parallel Computing, the foundation of every supercomputer. Splitting the workload of finding the decomposition is a lot easier when all the partial solutions, which would come in the form of sets of already found totally-nested tri-separations, can always be combined without conflict.
(4) Recently, coverings as known from Topology have been employed to systematically construct *graph-decompositions*, tree-decompositions where the tree may be any arbitrary graph, by applying classical theorems about tree-decompostions to the covering of a graph, and then folding the tree-decomposition to a graph-decomposition [26]. For the Tutte-decomposition, it is known how to achieve this directly without employing coverings [7]. For the covering approach, canoncity is paramount: it makes the entire construction work. For the direct approach, the description of the Tutte-decomposition via total-nestedness is key. The construction from [26] can be generalised so that it can be applied to the tri-separation decomposition; we would be excited to see this happen.
(5) Finally, as the Tutte-decomposition is canonical and explicit, we believe that any decomposition result that claims to generalise Tutte should be both canonical and explicit.

Grohe showed in pioneering work that every 3-connected graph has a tree-decomposition of adhesion 3 into torsos that are quasi 4-connected, K_4 or K_3 [38]. Grohe's decomposition is exciting and has indeed quite a few applications, however, they do not include (1)–(5). Since our decomposition in particular satisfies (5), we regard it as an analogue of Tutte's decomposition.

More related work. Our work complements existing work on decomposing graphs along separators of arbitrary size and the corresponding theory of tangles [2, 6, 9, 10, 11, 12, 16, 22, 23, 24, 25, 28, 29, 30, 39, 40, 42, 56]. It would be most exciting to try to extend our work to separators of larger size, see

Chapter 3 for an open question in this direction. A fair share of the work on 3-connected graphs studies which substructures are '(in-)essential' to 3-connectivity (in the sense that their contraction or deletion preserves 3-connectivity); this includes the work of Ando, Enomoto and Saito [3] and of Kriesell [43, 44, 45, 46, 47, 48]. The structure of 3-separations in matroids is a well-studied topic; a fundamental result in this area is the decomposition result of Oxley, Semple and Whittle [52]. It would be most natural to extend our ideas to matroids and this problem is discussed in Chapter 3. Hopes for a generalisation to directed graphs are fuelled by recent work of Bowler, Gut, Hatzel, Kawarabayashi, Muzi and Reich [4]. Our work is related to recent work of Esperet, Giocanti and Legrand-Duchesne [31], who employ Grohe's techniques for 3-connected graphs to prove a general decomposition result of infinite graphs without an infinite clique-minor, and our result might provide an alternative perspective.

Organisation of the paper. An overview of the proof of Theorem 1 is given in Section 0.1. The remainder of the paper consists of three chapters and an appendix. Each chapter will feature its own comprehensive overview. In the first chapter, we introduce and prove the Angry Tri-Separation Theorem (1.1.5); this classifies the 3-connected graphs which have no totally-nested nontrivial tri-separations, and it will be the key ingredient of the proof of Theorem 1 as it deals with the special case $N(G) = \emptyset$. Corollary 2 can already be derived from the Angry Tri-Separation Theorem, so the first chapter also includes the proof of Corollary 2. In the second chapter, we prove Theorem 1. The third chapter provides an outlook, which includes a discussion of the relation to tree-decompositions. Finally, the appendix offers a proof of the 2-separator theorem, but phrased in the language of this paper and with a structural strengthening added, for the sake of convenience and completeness.

0.1. Overview of the proof

Let G be a 3-connected graph, and let N denote the set of totally-nested nontrivial tri-separations of G. Let τ be an arbitrary torso of N. It is routine to verify that τ is 3-connected or a K_3, and that τ is a minor of G. So it remains to show that τ either is quasi 4-connected, a wheel, a thickened $K_{3,m}$ or $G = K_{3,m}$ with $m \geqslant 0$.

Our approach is to link these three outcomes to the structure of the tri-separations of G that 'interlace' the torso τ, as follows. Let (A, B) be a nontrivial tri-separation of G. Roughly speaking, we say that (A, B) *interlaces* τ if τ has vertices in $A \setminus B$ and in $B \setminus A$, so (A, B) 'cuts through' τ. If additionally $G[A \setminus B]$ or $G[B \setminus A]$ is connected, then we say that (A, B) interlaces τ *heavily*. Else if both $G[A \setminus B]$ and $G[B \setminus A]$ have at least two components, then we say that (A, B) interlaces τ *lightly*. This allows for the following structural strengthening of Theorem 1:

(1) if τ is not interlaced, then τ is quasi 4-connected or a K_4 or K_3;
(2) if τ is heavily interlaced, then τ is a wheel;
(3) if τ is lightly interlaced, then τ is a thickened $K_{3,m}$ or $G = K_{3,m}$.

For the proof of (1), suppose that τ is not interlaced. Let us assume for a contradiction that τ is neither quasi 4-connected, nor a K_4 nor K_3. Then τ has a 3-separation (A, B) with two sides of size at least five. In a 3-page technical argument, we 'lift' (A, B) from τ to a tri-separation of G that interlaces τ – a contradiction.

The proof of (2) is a bit more tricky and will be explained below.

For the proof of (3), suppose that τ is lightly interlaced by a tri-separation (A, B). Then we first note that the separator S of (A, B) consists of three vertices, and that $G \setminus S$ has at least four components. The four components ensure that S is '4-connected' in G, as every two vertices in S can be linked by four internally vertex-disjoint paths in G through the four components. The 4-connectivity of S can then be used to show that every component C of $G \setminus S$ determines a totally-nested tri-separation of G, with C on one side, whose separator consists of vertices of S or edges that join C to S. The nontrivial ones amongst the totally-nested tri-separations determined by the components of $G \setminus S$ are precisely the ones that bound the torso τ. If $G[S]$ is edgeless and all components of $G \setminus S$ have size one, then $G = K_{3,m}$ where m is equal to the number of components of $G \setminus S$. Otherwise, a brief analysis shows that τ is a thickened $K_{3,m}$, where m is equal to the number of components C of $G \setminus S$ such that $|C| = 1$ and $G[S]$ is edgeless.

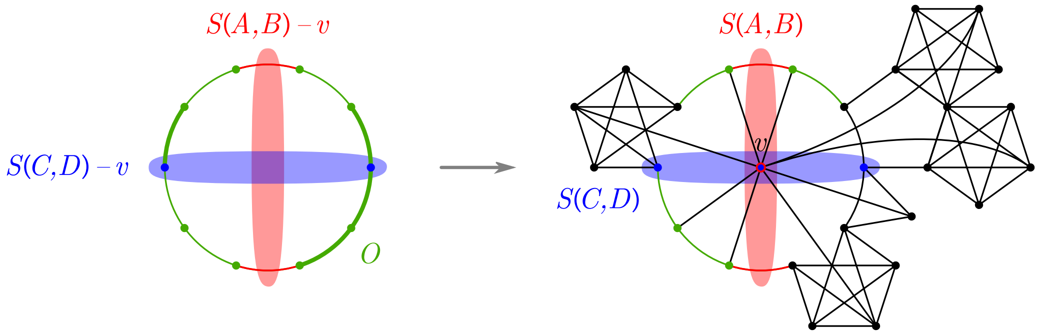


FIGURE 1. The graph G (on the right) can be constructed from a cycle O (on the left) by replacing its bold edges by 2-connected graphs and adding the vertex v together with its incident edges. Here $S(A, B)$ and $S(C, D)$ denote the separators of (A, B) and (C, D), respectively. They intersect in the vertex v. The cycle O is highlighted in green.

Proof of (2). Assume we are given a tri-separation (A, B) of G that interlaces τ heavily. Since (A, B) is not in N, it is *crossed* by a nontrivial tri-separation (C, D) of G, meaning that (A, B) and (C, D) are not nested with each other. The first step is to extend the standard theory of crossing separations to our context of tri-separations. From this we learn that the separators of (A, B) and (C, D) intersect in exactly one vertex; call it v. The next step is to apply the 2-separator theorem to the graph $G - v$. This tells us that the graph $G - v$ can be obtained from a cycle O by replacing some of its edges by 2-connected subgraphs of G. We refer to these replaced edges of O as *bold*. The separators of (A, B) and (C, D) without v alternate on the cycle O, see Figure 1. Intuitively, one might hope that the vertex v together with the two endvertices of a bold edge of O forms the separator of a totally-nested nontrivial tri-separation. This is almost true, but some of the vertices in the separator might fail to have two neighbours on some side. We resolve this issue by replacing these vertices with one of their incident edges. The resulting tri-separation is referred to as the *pseudo-reduction* at the bold edge.

A key challenge is to show that the pseudo-reductions at the bold edges of O are totally-nested. We approach this challenge in a systematic way by showing that a connectivity property of the separator of a tri-separation implies total nestedness (we very much believe that this can be turned into a characterisation of total nestedness, but given the length of the paper, we do not attempt this here). For this approach to succeed, we need to verify that the separators of the pseudo-reductions satisfy this connectivity property. The connectivity property itself is a little technical, as we require different amounts of connectivity depending on whether edges or vertices are in the separator. For simplicity, let us assume that the separator consists of three vertices. Then our task is to find three internally vertex-disjoint paths between every pair of non-adjacent vertices in the separator avoiding the third vertex of the separator. Between the two endvertices of a bold edge e of O we find two paths in the 2-connected subgraph that is associated with the bold edge e, and a third path follows the course of the path $O - e$. So it remains to construct three internally vertex-disjoint paths between an endvertex of the bold edge e and v. If O is short, we need to consider a few cases, and if O is long (length five suffices), then we study how the bold edges are distributed on O. We identify five possible patterns that cover all cases and verify that three internally vertex-disjoint paths exist for each of the five patterns, see Figure 2. Hence the pseudo-reductions at bold edges are totally-nested.

Having shown that the pseudo-reductions at the bold edges of O are totally-nested, we know that they bound a torso τ'; this is not necessarily a torso of N, but of the set of pseudo-reductions. Using our knowledge of the structure of $G - v$ provided by O, and 3-connectedness of G, it is straightforward to show that τ' is a wheel. So all that remains to show is that τ' is equal to τ. For this, we have a uniqueness-lemma, by which it suffices to show that no totally-nested nontrivial tri-separation of G interlaces τ'. So we assume for a contradiction that some totally-nested nontrivial tri-separation (U, W) of G interlaces τ'. Roughly speaking,

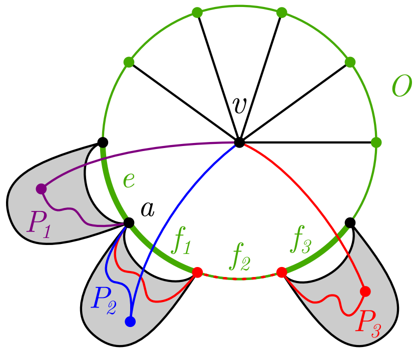


FIGURE 2. The graph G together with the bold edges of O. For each bold edge of O we indicate in grey its corresponding 2-connected replacement graph as given in the construction of G from O. In this figure we refer to them as *bags*. The vertex a is an endvertex of the bold edge e. Here the cycle O has the *pattern* `btx` with regard to e and a: the first letter `b` indicates that the edge f_1 on O incident with a aside from e is bold, the second letter `t` indicates that the edge f_2 after that on O is not bold (`timid`), and the letter `x` indicates that the edge f_3 after that can be arbitrary, bold or not. In our example, it is bold. For this pattern, there are three internally vertex-disjoint paths from a to v: the first path P_1 connects to v from the bag at e, the second path P_2 connects to v from the bag at f_1, and the final path P_3 uses a path disjoint from $P_2 - a$ through the bag at f_1, then traverses f_2 and eventually connects to v from the bag at f_3.

(U, W) induces a mixed 2-separation of O. This induced mixed 2-separation cuts O into two intervals. We carefully select a vertex or non-bold edge from each interval, and then add v to obtain the separator of a mixed 3-separation (E, F) of G. Then (E, F) crosses (U, W) by standard arguments, and with a bit of extra work we turn (E, F) into a tri-separation that still crosses (U, W), yielding a contradiction. Hence τ' is equal to τ. As we have already shown that τ' is a wheel, the overview of the proof of (2) is complete.

CHAPTER 1

An angry theorem for tri-separations

1.1. Overview of this chapter

This chapter provides the key ingredient for the proof of Theorem 1, which we call the Angry Tri-Separation Theorem. Essentially, it states that Theorem 1 holds in the special case where $N(G)$, the set of all totally-nested non-trivial tri-separations of G, is empty; that is: if G is not itself quasi 4-connected, and if every nontrivial tri-separation of G is crossed, then G must be either a wheel or a $K_{3,m}$ for some $m \geqslant 3$.

1.1.1. Statement of the Angry Tri-Separation Theorem. Here we give all the necessary definitions to then state the Angry Tri-Separation Theorem (1.1.5). An (oriented) *mixed-separation* of a graph G is an ordered pair (A, B) such that $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ are non-empty. We call A and B the *sides* of (A, B). The *separator* of (A, B) is the disjoint union of the vertex set $A \cap B$ and the edge set $E(A \setminus B, B \setminus A)$. We denote the separator of (A, B) by $S(A, B)$. The *order* of (A, B) is the size $|S(A, B)|$ of its separator. A mixed-separation of order k for $k \in \mathbb{N}$ is called a *mixed k-separation* for short. The separator of a mixed k-separation is a *mixed k-separator*. A mixed-separation (A, B) of G with no edges in its separator is called a *separation* of G. Separations of order k are called *k-separations* and their separators are called *k-separators*.

A mixed 3-separation (A, B) of G is *nontrivial* if both $G[A]$ and $G[B]$ include a cycle.

Definition 1.1.1 (Tri-separation). A *tri-separation* of a graph G is a mixed-separation (A, B) of G of order three such that every vertex in $A \cap B$ has at least two neighbours in both $G[A]$ and $G[B]$. The separator of a tri-separation of G is a *tri-separator* of G. A tri-separation is *strong* if every vertex in its separator has degree at least four.

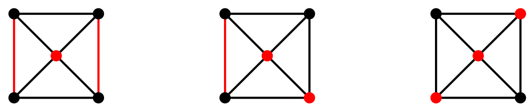


Figure 1. Separators of nontrivial tri-separations of the 4-wheel.

Example 1.1.2. Let G be a wheel with rim O of length at least four. Let v denote the centre of G. Adding v to any mixed 2-separator of O yields the separator of a nontrivial tri-separation of G. In fact, every nontrivial tri-separation of G can be obtained in this way.

If G is a 3-wheel, aka K_4, then every separator of a nontrivial tri-separation of G consists of two vertices and one edge.

Similarly as is common for separations [21], we define a partial ordering on the mixed-separations of any graph by letting $(A, B) \leqslant (C, D)$ if and only if $A \subseteq C$ and $B \supseteq D$. Two mixed-separations (A, B) and (C, D) are *nested* if, after possibly switching the name A with B or the name C with D, we have $A \subseteq C$ and $B \supseteq D$. If two mixed-separations are not nested, they *cross*. A set of mixed-separations is *nested* if its elements are pairwise nested.

Definition 1.1.3 (Totally nested). A tri-separation of G is *totally nested* if it is nested with every tri-separation of G.

Example 1.1.4. Every nontrivial tri-separation in the 4-wheel, as found in Example 1.1.2, is crossed by another nontrivial tri-separation; see Figure 1. By contrast, every 3-cut with both sides of size at least two in a 3-connected graph determines a totally-nested nontrivial tri-separation (we will see this in Corollary 1.3.14).

A graph G is *internally 4-connected* if it is 3-connected, every 3-separation of G has a side that induces a claw, and $G \notin \{K_4, K_{3,3}\}$. Internally 4-connected graphs are quasi 4-connected.

Theorem 1.1.5 (Angry Tri-Separation Theorem). *For every 3-connected graph G, exactly one of the following is true:*

(1) *G has a totally-nested nontrivial tri-separation;*
(2) *G is a wheel or a $K_{3,m}$ for some $m \geqslant 3$;*
(3) *G is internally 4-connected.*

1.1.2. Organisation of this chapter. In Section 1.2, we collect useful properties of tri-separations. In Section 1.3, we introduce tools that allow us to systematically study how tri-separations can cross. In Section 1.4, we deduce Corollary 2 from the Angry Tri-Separation Theorem. In Section 1.5, we employ the tools from the previous section to find necessary conditions for when a tri-separation is totally nested. In Section 1.6, we recall the 2-separation-version of Theorem 1, as we will need it in the proof of the Angry Tri-Separation Theorem. In Section 1.7, we will see why the 2-separation-version of Theorem 1 is helpful for finding totally-nested nontrivial tri-separations. In Section 1.8 and Section 1.9, we put together the tools developed in the previous sections and we complete the proof of the Angry Tri-Separation Theorem, where the former deals with special cases and the latter solves the general case.

1.2. Properties of tri-separations

A cut of a graph G is *atomic* if it is of the form $E(v, V(G) \setminus \{v\})$ for some vertex $v \in V(G)$.

Lemma 1.2.1. *The following are equivalent for every tri-separation (A, B) of a 3-connected graph G:*

(1) *(A, B) is trivial;*
(2) *A and B are the two sides of an atomic cut.*

Proof. The implication (2)→(1) is clear. For (1)→(2) suppose that $G[A]$ contains no cycle, say. We first show that the side B cannot have exactly two vertices. Indeed, if B has size two, then $G[B]$ has maximum degree at most one, and since (A, B) is a tri-separation it follows that $A \cap B$ must be empty. Thus, A and B are the two sides of a cut of size three. Hence some vertex in B has degree at most two in G, which contradicts 3-connectivity. So B does not have size two. As we are done otherwise, we from now on assume that the side B contains at least three vertices.

If two of the edges in $E(A \setminus B, B \setminus A)$ had the same endvertex in B, this vertex would be in a 2-separator of G. So as the graph G is 3-connected, no two edges in $E(A \setminus B, B \setminus A)$ can have the same endvertex in B. Let T be the graph obtained from $G[A]$ by adding all the edges from the separator of (A, B). By the above, T is a tree. The tree T has three leaves in B. Since G is 3-connected, T has no other leaves and also no vertices of degree two. Hence T is a $K_{1,3}$ and (2) follows. □

Corollary 1.2.2. *The trivial tri-separations of a 3-connected graph G are nested with all strong tri-separations of G.*

Proof. Let any trivial tri-separation be given. By Lemma 1.2.1, it is of the form $(\{v\}, V \setminus \{v\})$, say. Then the vertex v has degree three in G. Let (C, D) be any strong tri-separation. Since (C, D) is strong, the vertex v is not in $C \cap D$. So it is in precisely one of $C \setminus D$ and $D \setminus C$, say in $C \setminus D$. Then $D \subseteq V \setminus \{v\}$, which gives $(\{v\}, V \setminus \{v\}) \leqslant (C, D)$. □

Lemma 1.2.3. *Let G be a 3-connected graph, and let (A, B) be a nontrivial mixed 3-separation of G. Then the edges in $S(A, B)$ form a matching between $A \setminus B$ and $B \setminus A$.*

Proof. Let us show first that no two edges in $S(A, B)$ share an end. For this, let us suppose for a contradiction that two edges $e, f \in S(A, B)$ share an endvertex $v \in A \setminus B$, say. Let x be the remaining element of $S(A, B)$ besides e, f if it is a vertex, and otherwise let x denote the endvertex in $A \setminus B$ of the edge in $S(A, B)$ besides e, f. Let O be a cycle in $G[A]$. Since O has at least three vertices, one of them is distinct from v and x, and so is not in $B \cup \{v, x\}$. Hence the pair $(A, B \cup \{v, x\})$ is a 2-separation of G with separator equal to $\{v, x\}$. This contradicts the fact that G is 3-connected. □

Lemma 1.2.4. *Let (X_1, X_2) be a mixed 3-separation of a 3-connected graph G. Then for every vertex v in the separator of (X_1, X_2), exactly one of the following holds:*

(1) *The vertex v has two neighbours in both X_1 and X_2.*
(2) *There exists a unique index $i \in \{1, 2\}$ such that v has precisely one neighbour in X_i but two neighbours in X_{3-i}. The neighbour of v in X_i lies in $X_i \setminus X_{3-i}$, so v has two neighbours in X_{3-i} that lie in $X_{3-i} \setminus X_i$.*

Proof. Since G is 3-connected, v has neighbours in $X_1 \setminus X_2$ and in $X_2 \setminus X_1$, and v has degree three. So if (1) fails, we have (2). □

Definition 1.2.5 (Reduction). Let (X_1, X_2) be a mixed 3-separation of a 3-connected graph. We obtain a tri-separation from (X_1, X_2) by deleting vertices from X_1 or X_2, as follows. For every vertex $v \in X_1 \cap X_2$ that has fewer than two neighbours in some side X_i, the index $i =: i(v)$ is unique, v has a unique neighbour $x(v)$ in X_i that lies in $X_i \setminus X_{3-i}$, and v has two neighbours in $X_{3-i} \setminus X_i$ by Lemma 1.2.4. For both $j \in \{1, 2\}$ we obtain X'_j from X_j by deleting all vertices v with $i(v) = j$. Then (X'_1, X'_2) is a tri-separation of G. Every vertex $v \in X_1 \cap X_2$ that was removed from some side is not in the separator of (X'_1, X'_2), but instead the edge $\{v, x(v)\}$ is in $S(X'_1, X'_2)$. In this context, we say that v was *reduced* to the edge $\{v, x(v)\}$. We call (X'_1, X'_2) the *reduction* of (X_1, X_2). Note that the reduction (X'_1, X'_2) is nontrivial if (X_1, X_2) is nontrivial.

Let (A, B) be a mixed 3-separation of a graph G. A *strengthening* of (A, B) is a mixed 3-separation (A', B') that it is obtained from (A, B) by deleting all vertices of $A \cap B$ that have degree three in G and have a neighbour in $A \cap B$ from one of the sides, then taking a reduction.

Observation 1.2.6. *If (A', B') is a strengthening of (A, B), then $A \setminus B \subseteq A' \subseteq A$ and $B \setminus A \subseteq B' \subseteq B$. All strengthenings are strong tri-separations.* □

Lemma 1.2.7. *Every mixed 3-separation (A, B) of a 3-connected graph G has a strengthening (A', B'). Moreover, if there is an edge uv in G with both ends u, v in the separator of (A, B), then we may choose (A', B') so that $u, v \in B'$.*

Proof. Given (A, B), we obtain A'' from A by deleting all vertices that lie in $A \cap B$ and have degree three in G, and we put $B'' := B$. Then we let (A', B') be the reduction of (A'', B''). Suppose now that uv is an edge with ends $u, v \in A \cap B$. Then u and v have two neighbours in $B = B''$, so $u, v \in B'$. □

Proposition 1.2.8. *For every 3-connected graph G, the following assertions are equivalent:*

(1) *G is internally 4-connected or $G \in \{K_4, K_{3,3}\}$;*
(2) *every 3-separation of G is trivial or $G = K_4$;*
(3) *all tri-separations of G are trivial or $G = K_4$;*
(4) *all strong tri-separations of G are trivial.*

Proof. $\neg$(2)$\to$ $\neg$(1). As all 3-separations of $K_{3,3}$ are trivial and K_4 is excluded, G is none of these graphs. Let (A, B) be a nontrivial 3-separation of G. If a cycle in the side A included only one vertex of $A \setminus B$, then two vertices of the separator $A \cap B$ are adjacent. Making the same argument with the roles of 'A' and 'B' interchanged, we deduce that the separator $A \cap B$ contains two adjacent vertices or $A \setminus B$ and $B \setminus A$ both have size at least two. Thus G is not internally 4-connected.

$\neg$(3)$\to$ $\neg$(2). Let (A, B) be a nontrivial tri-separation of G. For each edge in $S(A, B)$ we pick one of its endvertices and add it to both sides. We pick these endvertices so that we preserve that $A \setminus B$ and $B \setminus A$ are nonempty. This is possible as G has at least five vertices. As this preserves nontriviality, we end up with a nontrivial 3-separation of G.

Clearly (3)$\to$(4).

$\neg$(1)$\to$ $\neg$(4). As G is not internally 4-connected and $G \notin \{K_4, K_{3,3}\}$, we may let (A, B) be a 3-separation of G none of whose sides induces a claw. Let $X := A \cap B$. If G had at most four vertices, then G would be a K_4, contradicting our assumption, so we have $|G| \geqslant 5$.

Case 1: the induced subgraph $G[X]$ has no edges. Then $|A \setminus B| \geqslant 2$ and $|B \setminus A| \geqslant 2$. By Lemma 1.2.7, there is a strong tri-separation (A', B') of (A, B) with $A \setminus B \subseteq A'$ and $B \setminus A \subseteq B'$. Since $A \setminus B$ and $B \setminus A$ have size at least two, so have A' and B'. Thus (A', B') is nontrivial by Lemma 1.2.1.

Case 2: the induced subgraph $G[X]$ contains an edge x_1x_2. Recall that $|G| \geqslant 5$. The only 3-connected graphs on exactly five vertices that have a 3-separator are the 4-wheel and K_5^- (K_5 minus one edge). The unique 3-separator of K_5^- is a triangle and thus it is the separator of a nontrivial tri-separation. The 4-wheel has two nontrivial strong tri-separations. So it remains to consider the case that G has at least six vertices. By symmetry, we may assume that $|A \setminus B| \geqslant 2$. Since x_1x_2 is an edge of G with both ends in $X = A \cap B$, we find a strong tri-separation (A', B') of G with $A \setminus B \subseteq A'$ and $B \setminus A \subseteq B'$ such that $x_1, x_2 \in B'$. Hence, the side B' misses at most one vertex of B. As $|B| \geqslant 4$, this gives $|B'| \geqslant 3$. We also have $|A'| \geqslant |A \setminus B|$, and $|A \setminus B| \geqslant 2$ by assumption. Hence the strong tri-separation (A', B') is nontrivial by Lemma 1.2.1. $\square$

1.3. Nested or crossed: analysing corner diagrams

What can we say about two mixed-separations if they cross? In this section we address this question by introducing corner diagrams for mixed-separations.

Let (A, B) and (C, D) be mixed-separations of a graph G. The following definitions all depend on the context that (A, B) and (C, D) are given. They are supported by Figure 2, which is commonly referred to as a 'corner diagram'.

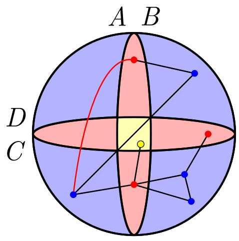


FIGURE 2. (A, B) and (C, D) cross. Corners are blue, links are red, the centre is yellow.

The *corner* for the pair $\{A, C\}$ is the vertex set $(A \setminus B) \cap (C \setminus D)$. For each pair of sides, one from (A, B) and one from (C, D), we define its corner in the analogous way. Thus there are four corners in total. Two corners are *adjacent* if their pairs share a side, otherwise they are *opposite*. Note that there is a unique corner opposite of each corner and that each corner is the opposite of its opposite corner. Each corner has exactly two adjacent corners.

EXAMPLE 1.3.1. The corner for $\{A, C\}$ is opposite to the corner for $\{B, D\}$, and it is adjacent to the corner for $\{A, D\}$ and to the corner for $\{B, C\}$.

An edge of G is *diagonal* if its endvertices lie in opposite corners. Note that an edge is diagonal if and only if it is contained in the separators of both separations (A, B) and (C, D). The *centre* consists of the diagonal edges together with the vertex set $A \cap B \cap C \cap D$.

An edge e in the separator of (A, B) is *in the edge-link for C* if it is not diagonal and has an endvertex in one of the corners for C; that is, in one of the corners for $\{A, C\}$ or $\{B, C\}$. The *link for C* is the union of the edge-link for C and the vertex set $(A \cap B) \setminus D$. In a slight abuse of notation we will sometimes say things like 'a vertex of the link $(A \cap B) \setminus D$' instead of the formally precise 'a vertex of the link for C'. We define 'the link for D' as 'the link for C' with 'D' in place of 'C'. Note that every edge in the separator of (A, B) that is not diagonal lies in at most one of the edge-links for C and D. We define the links for the sides A and B of (A, B) analogously with the separations '(A, B)' and '(C, D)' interchanged. The link for a side X is *adjacent* to the two corners for the pairs that contain the side X. Two links are *adjacent* if there is a corner they are both adjacent to. Every link is adjacent to all but one link; we refer to that link as its *opposite* link.

EXAMPLE 1.3.2. The link for C is adjacent to the two corners for the pairs $\{A, C\}$ and $\{B, C\}$. It is adjacent to the links for A and B. It is opposite to the link for D.

LEMMA 1.3.3. *Two mixed-separations (A, B) and (C, D) of a graph are nested if and only if they admit a corner such that it and the two adjacent links are empty. Thus, (A, B) and (C, D) cross as soon as two opposite links are nonempty.*

PROOF. Indeed, we have $A \subseteq C$ and $B \supseteq D$ if and only if the corner for $\{A, D\}$ and its two adjacent links are empty. $\square$

Lemma 1.3.3 offers an alternative definition of nestedness. We may and will use the two definitions interchangeably.

Suppose that we are given sides $X \in \{A, B\}$ and $Y \in \{C, D\}$. The *corner-separator* $L(X, Y)$ at the corner for $\{X, Y\}$ is the union of the two links adjacent to the corner for $\{X, Y\}$ together with the centre but without those diagonal edges that do not have an endvertex in the corner for $\{X, Y\}$; see Figure 3 for a picture. Two corner-separators are *opposite* or *adjacent* if their respective corners are.

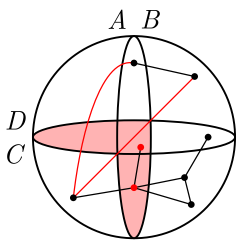


FIGURE 3. (A, B) and (C, D) cross. The corner separator $L(A, C)$ contains all red vertices and edges.

An edge joining vertices in two opposite links is called a *jumping edge*; see Figure 4. It is straightforward to check that jumping edges are the only edges in the separators $S(A, B)$ and $S(C, D)$ that are not in any links or the centre and thus not in any corner separators.

As for separations, we get the following submodularity property:

LEMMA 1.3.4. *For two mixed-separations (A, B) and (C, D) of a graph, we have*

$$|L(A, C)| + |L(B, D)| \leqslant |S(A, B)| + |S(C, D)|.$$

Moreover, if we have equality, there are no jumping edges, and every diagonal edge has its endvertices in the corners for $\{A, C\}$ and $\{B, D\}$.

PROOF. This is a standard argument. We just check for each vertex or edge counted in $|L(A, C)| + |L(B, D)|$ that it is counted in $|S(A, B)| + |S(C, D)|$ with the same or greater multiplicity. $\square$

LEMMA 1.3.5. *Let G be a 3-connected graph. Let (A, B) and (C, D) be two mixed 3-separations of G that cross so that two opposite corner-separators have size three. Then either*

(1) *all links have the same size ℓ, for some $\ell \in \{0, 1\}$; or*
(2) *two adjacent links have size i and the other two links have size $3 - i$, for some $i \in \{1, 2\}$.*

PROOF. Let a, b, c, d denote the sizes of the links for A, B, C, D, respectively. Let x denote the size of the centre. Without loss of generality, the separators at the corners for $\{A, C\}$ and $\{B, D\}$ have size three. By Lemma 1.3.4, every diagonal edge has its endvertices in the corners for $\{A, C\}$ and $\{B, D\}$. Hence $a + c + x = 3$ and $b + d + x = 3$. Since (A, B) and (C, D) have order three, we further have $c + d + x = 3$ and $a + b + x = 3$. Considering the two equations that contain a, we find that $b = c$. Considering the two equations that contain c, we find that $a = d$. Without loss of generality, $a = d \leqslant b = c$.

Suppose first that $a, d = 0$. Then, since (A, B) and (C, D) cross, the corner for $\{A, D\}$ must be nonempty. As G is 3-connected, it follows that the centre has size $x = 3$. Hence we can read from the equations that $b, c = 0$, giving outcome (1).

Otherwise $a, d = 1$, since $a, d \geqslant 2$ would imply $b, c \leqslant 1 < a, d$. Hence $b, c \leqslant 3 - a = 2$. But also $1 = a, d \leqslant b, c$. So b, c take the same value in $\{1, 2\}$, giving outcome (1) or (2). $\square$

Corollary 1.3.6. *If two tri-separations of a 3-connected graph cross so that two opposite corner-separators have size three, then all links have the same size ℓ, for some $\ell \in \{0,1\}$.*

Proof. Suppose for a contradiction that this fails. Then two adjacent links have size one, and the other two links have size two, by Lemma 1.3.5. Hence the centre is empty. Let X denote the corner whose adjacent links have size one. As the corner-separator for X has size two, but G is 3-connected, the corner X is empty. Since X is empty, not both adjacent links can consist of edges only, so one link contains a vertex v. Since the two links adjacent to X have size one and the corner X is empty, it follows that the vertex v has at most one neighbour in one of the sides of the two crossing tri-separations. This contradicts the definition of tri-separation. □

Lemma 1.3.7. *If two tri-separations (A,B) and (C,D) of a 3-connected graph G cross so that two opposite corner-separators have size three and all links have the same size $\ell \in \{0,1\}$, then there are no diagonal edges.*

Proof. Without loss of generality, the separators at the corners for $\{A,C\}$ and $\{B,D\}$ have size three. Suppose for a contradiction that there is a diagonal edge uv. By Lemma 1.3.4, the ends u and v lie in the corners for $\{A,C\}$ and $\{B,D\}$, respectively say.

Claim 1.3.7.1. *All links and the centre have size one.*

Proof of Claim. By Lemma 1.3.4, there are no jumping edges. Hence it suffices to show that all links have size $\ell = 1$. Suppose for a contradiction that all links are empty, i.e. that $\ell = 0$. Then the centre has size three. But since the centre contains the diagonal edge uv, the separators at the corners for $\{A,D\}$ and $\{B,C\}$ have size two. Then, since G is 3-connected, the corners for $\{A,D\}$ and $\{B,C\}$ are empty. Since all links are empty as well, it follows that (A,B) and (C,D) are nested, contradicting our assumption that they cross. ♢

By Claim 1.3.7.1, all links and the centre have size one. So the centre only consists of the diagonal edge uv. Hence the separators at the two corners for $\{A,D\}$ and $\{B,C\}$ have size two. As G is 3-connected, the two corners for $\{A,D\}$ and $\{B,C\}$ are empty. It follows that all four links contain no vertices, since any vertex in a link would fail to have two neighbours in some side of (A,B) or (C,D), contradicting that (A,B) and (C,D) are tri-separations. Hence all four links contain edges, and only edges. But then each of these edges must have an end in the corner for $\{A,D\}$ or $\{B,C\}$, contradicting that these corners are empty. □

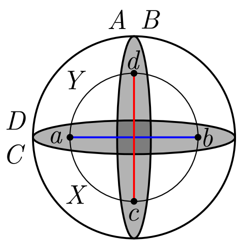


Figure 4. Two tri-separations of a K_4 cross with two jumping edges (red and blue)

Lemma 1.3.8. *If two nontrivial tri-separations (A,B) and (C,D) of a 3-connected graph G cross so that no two opposite corner-separators have size three, then $G = K_4$.*

Proof. This proof is supported by Figure 4. Since no two opposite corner-separators have size three, we find two adjacent corners whose separators have size at most two by Lemma 1.3.4. Say these are the corners X and Y for $\{A,C\}$ and $\{A,D\}$, respectively. As G is 3-connected, both corners X and Y must be empty. Since (A,B) is a tri-separation, $A \setminus B$ contains some vertex a. Since X and Y are empty, the vertex a lies in the link for A. As G is 3-connected, a has degree at least three. Since X and Y are empty, and since the corner-separators at X and Y have size at most two, some edge incident to a is a jumping edge. Let b denote the other endvertex of this jumping edge, so b lies in the link for B.

Since (A,B) is nontrivial, there is a cycle O included in $G[A]$. As X and Y are empty, the vertices of this cycle lie in the two corner-separators at X and Y. The two corner-separators share the vertex a, and have size at most two, hence O must be a triangle which contains a and whose other two vertices c, d lie in the links for C and D, respectively. Thus cd is another jumping edge. Therefore, the separators at the two corners besides X and Y have size at most two. By symmetry, we find that the two corners besides X and Y are empty, and that bcd is a triangle. Hence $G = K_4$. □

A mixed-separation (A, B) of a graph G is *half-connected* if $G[A \setminus B]$ or $G[B \setminus A]$ are connected.

LEMMA 1.3.9. *Let (A, B) and (C, D) be crossing mixed 3-separations of a graph G. If (A, B) is half-connected, then the centre cannot have size three.*

PROOF. Without loss of generality, $G[A \setminus B]$ is connected. Assume for a contradiction that the centre has size three. Then all links are empty. As $G[A \setminus B]$ is non-empty, we know that at least one of the two corners included in $A \setminus B$ is non-empty. But since $G[A \setminus B]$ is connected, the other of the two corners must be empty, contradicting that (A, B) and (C, D) are crossing. □

LEMMA 1.3.10 (Crossing Lemma). *Let G be a 3-connected graph other than K_4. Let (A, B) and (C, D) be two nontrivial tri-separations of G that cross. Then exactly one of the following holds:*

(1) *all links have size one and the centre consists of a single vertex;*
(2) *all links are empty and the centre consists of three vertices.*

In particular, there are no jumping edges. Moreover, if (A, B) or (C, D) is half-connected, then (1) holds.

PROOF. Since $G \neq K_4$, it follows from Lemma 1.3.8 that two opposite corner-separators have size three. Then all links have the same size $\ell \in \{0, 1\}$ by Corollary 1.3.6. By Lemma 1.3.4, there are no jumping edges. By Lemma 1.3.7, there are no diagonal edges either. Hence the centre contains no edges, and the size of the centre is determined by ℓ. If $\ell = 0$, then the centre has size three; if $\ell = 1$, then the centre has size one.

The 'Moreover' part follows from Lemma 1.3.9. □

LEMMA 1.3.11. *Let G be a 3-connected graph. If a strong nontrivial tri-separation (A, B) of G is crossed by a tri-separation of G, then (A, B) is also crossed by a tri-separation of G that is strong.*

PROOF. Suppose that (C, D) is a tri-separation of G that crosses (A, B). If (C, D) is strong, we are done, so we may assume that some vertex u of G of degree three lies in the separator of (C, D). Since (A, B) is a strong tri-separation, the vertex u cannot lie in the centre, so u lies in a link, say it lies in the link for A. As a K_4 has no strong nontrivial tri-separation, G is not a K_4. By Corollary 1.2.2, (C, D) is nontrivial. Hence we may apply the Crossing Lemma (1.3.10) to find that the existence of u implies that all links have size one while the centre consists of a single vertex, and that there are no jumping edges. Since (C, D) is a tri-separation and u has degree three, u has a neighbour v in $C \cap D$. As there are no jumping edges, the neighbour v of u can only lie in the centre. Since (A, B) is a tri-separation, v has a neighbour w in A besides u. By symmetry $w \in C$.

CLAIM 1.3.11.1. *The vertex v has three neighbours in C.*

Proof of Claim. If the corner $\{B, C\}$ is nonempty, then as G is 3-connected the corner contains a neighbour of v, and together with u and w we have found three neighbours of v in C. Thus assume that the corner $\{B, C\}$ is empty. Since not both adjacent links can consist of an edge, at least one adjacent link contains a vertex y. Since the corner for $\{B, C\}$ is empty but y has two neighbours either in B or in C (depending on which tri-separator $S(A, B)$ or $S(C, D)$ contains y), it follows that y is adjacent to v. If y is in the link for B, then u, y and w are three distinct neighbours of v in C, and we were done. So assume that y is in the separator of the strong tri-separation (A, B). Thus y has degree at least four. And since the corner $\{B, C\}$ is empty, the vertex y must have one of its neighbours outside the mixed 3-separator $S(C, D)$ in the corner $\{A, C\}$, and this nonempty corner contains a neighbour of v by 3-connectivity, which is different from u and y. ◇

Let c denote the unique neighbour of u in $C \setminus D$.

CLAIM 1.3.11.2. *The edge uc does not lie in the link for C.*

Proof of Claim. Suppose for a contradiction that uc lies in the link for C. Then w must lie in the corner for $\{A, C\}$. In particular, the corner for $\{A, C\}$ is nonempty. So $\{u, v\}$ is a 2-separator, a contradiction to 3-connectivity. ♢

Let $C' := C - u$ and $D' := D$. Then the separator of (C', D') arises from the separator of (C, D) by replacing the vertex u with the edge uc. The only vertex in the separator of (C, D) that might loose a neighbour when moving to (C', D') is the vertex v, which looses its neighbour u in C'. However, Claim 1.3.11.1 ensures that v has two neighbours in C'. Hence (C', D') is a tri-separation, and it has less vertices of degree three in its separator than (C, D). Moreover, (C', D') crosses (A, B) with the same links and centre as for (C, D), with just one exception: the link for A, which consisted of u for (C, D), consists of the edge uc for (C', D'). By iterating at most two times, we obtain a strong tri-separation that crosses (A, B). □

LEMMA 1.3.12. *If a mixed 3-separation of a 3-connected graph is not strong, then it is crossed by a trivial tri-separation.*

PROOF. Let (A, B) be a mixed 3-separation of a 3-connected graph G, and let $v \in A \cap B$ be a vertex of degree three. Since $A \setminus B$ and $B \setminus A$ are nonempty, and since G is 3-connected, the vertex v must have neighbours in $A \setminus B$ and in $B \setminus A$. Hence the trivial tri-separation with $\{v\}$ as one side crosses (A, B). □

PROPOSITION 1.3.13. *Let G be a 3-connected graph, and let (A, B) be a nontrivial tri-separation of G. Then the following assertions are equivalent:*

(1) *(A, B) is totally nested;*
(2) *(A, B) is strong and nested with every strong nontrivial tri-separation of G.*

PROOF. (1)→(2). We only have to show that (A, B) is strong. This follows from Lemma 1.3.12.

(2)→(1). Suppose for a contradiction that (A, B) is crossed by a tri-separation (C, D) of G. By Lemma 1.3.11, we may assume that (C, D) is strong. Since (A, B) is strong, (C, D) is nontrivial by Corollary 1.2.2. This contradicts (1). □

COROLLARY 1.3.14. *Let G be a 3-connected graph. Let A and B be the sides of a non-atomic 3-cut of G. Then (A, B) is a totally-nested nontrivial tri-separation of G.*

PROOF. By Lemma 1.2.1, (A, B) is nontrivial. Since $S(A, B)$ consists of edges, (A, B) is strong. By Proposition 1.3.13, it suffices to show that (A, B) is nested with every strong nontrivial tri-separation of G. And indeed, since $S(A, B)$ contains no vertices and since K_4 has no non-atomic 3-cut, (A, B) is nested with every strong nontrivial tri-separation of G by the Crossing Lemma (1.3.10). □

1.4. Proof of Corollary 2

Before we prove the Angry Tri-Separation Theorem, let us see how it implies Corollary 2. A graph G is *essentially 4-connected* if it is 3-connected, every nontrivial strong tri-separation has three edges in its separator such that the subgraph induced by one side is equal to a triangle, and $G \neq K_4$. A graph G is *vertex-transitive* if the automorphism group of G acts transitively on its vertex set $V(G)$.

PROOF OF COROLLARY 2. Let G be a vertex-transitive finite connected graph. We have to show that G either is essentially 4-connected, a cycle, or a complete graph on at most four vertices. By Corollary 3.3.5, G is a cycle, K_2, K_1 or 3-connected. Since we are done otherwise, let us assume that G is 3-connected. By the Angry Tri-Separation Theorem (1.1.5), G is internally 4-connected, a $K_{3,m}$ with $m \geqslant 3$, a wheel, or G has a totally-nested nontrivial tri-separation. If G is internally 4-connected, then $G \notin \{K_4, K_{3,3}\}$ by definition, and all strong tri-separations of G are trivial by Proposition 1.2.8; in particular, G is essentially 4-connected. If G is a $K_{3,m}$ for some $m \geqslant 3$, then $m = 3$ since G is vertex-transitive, and $G = K_{3,3}$ is essentially 4-connected since all its strong tri-separations are trivial. If G is a wheel, then G can only be a K_4 by vertex-transitivity, and K_4 is a possible outcome.

As we are done otherwise, we may assume that G has a totally-nested nontrivial tri-separation (A, B). Every automorphism φ of G takes (A, B) to $(\varphi(A), \varphi(B))$. Let O denote the union of the orbits of (A, B)

and (B, A) under the automorphism group of G. As G is finite, we may let (U, W) be $\leqslant$-minimal in O; so $(U, W) \leqslant (C, D)$ or $(U, W) \leqslant (D, C)$ for all $(C, D) \in O$ as (U, W) is totally-nested.

Claim 1.4.0.1. *The separator of (U, W) consists of three edges.*

Proof of Claim. Suppose for a contradiction that there is a vertex $v \in U \cap W$. Since (U, W) is a mixed-separation, there is a vertex $u \in U \setminus W$. Let $\varphi \in \mathrm{Aut}(G)$ send v to u. Then $(\varphi(U), \varphi(W)) \leqslant (U, W)$ or $(\varphi(W), \varphi(U)) \leqslant (U, W)$ since (U, W) is totally-nested, and in either case the inequality is strict since u does not lie in $S(U, W)$ but does so after the application of φ. This contradicts the choice of (U, W). ♢

Claim 1.4.0.2. $G[U] = K_3$.

Proof of Claim. Since G is 3-connected, since (U, W) is nontrivial and since $S(U, W)$ consists of three edges by Claim 1.4.0.1, it suffices to show that every vertex in U is incident with an edge in $S(U, W)$. The proof is analogue to the proof of Claim 1.4.0.1. ♢

By Claim 1.4.0.1 and Claim 1.4.0.2, every tri-separation in O has three edges in its separator and a side that induces a triangle. As (A, B) was chosen arbitrarily, every totally-nested nontrivial tri-separation has three edges in its separator and a side that induces a triangle. Hence to show that every nontrivial strong tri-separation also has three edges in its separator and a side that induces a triangle, it suffices to show that

Claim 1.4.0.3. *Every nontrivial strong tri-separation of G is totally-nested.*

Proof of Claim. Since $S(U, W)$ consists of three edges (which share no ends by Lemma 1.2.3) and $G[U] = K_3$, all vertices of G have degree three. Hence all vertices of G have degree three. Let (C, D) be an arbitrary nontrivial strong tri-separation of G. As (C, D) is strong, the separator of (C, D) consists of three edges. Then (C, D) is totally-nested by Corollary 1.3.14. ♢

Combining Claim 1.4.0.1, Claim 1.4.0.2 and Claim 1.4.0.3 yields that G is essentially 4-connected. □

Open Problem 1.4.1. *Can Corollary 2 be used to simplify existing characterisations of classes of finite Cayley graphs (like characterisations of the finite Cayley graphs that embed in the torus or some other surface, as in or similar to* [**53, 60, 61**]*)?*

Another area where our ideas might turn out to be fruitful is in the study of infinite planar Cayley graphs, see [**33, 34, 35, 36**].

1.5. Understanding nestedness through connectivity

In this section, we provide sufficient conditions for when a tri-separation is totally nested. Let v be a vertex of a graph G. We say that a vertex w of G is *v-free* if it is not adjacent to v or if it has degree at most three; that is, a vertex is not v-free if it is adjacent to v and has degree at least four.

Given a mixed 3-separator $\{x_1, x_2, x_3\}$ of G, we say that $\{x_1, x_2, x_3\}$ is *externally tri-connected around* a vertex x_i with $i \in \mathbb{Z}_3$ if one of the following holds:

- (:) The pair $\{x_{i+1}, x_{i+2}\}$ consists of two vertices and these vertices are adjacent or joined by three internally disjoint paths in $G - x_i$.
- ($\dot{-}$) The pair $\{x_{i+1}, x_{i+2}\}$ consists of one vertex x (say) and one edge e (say) such that e has an x_i-free endvertex y for which there are two internally disjoint x–y paths in $G - x_i - e$.
- ($=$) The pair $\{x_{i+1}, x_{i+2}\}$ consists of two edges which have x_i-free endvertices y_{i+1} and y_{i+2}, respectively, such that there are two internally disjoint y_{i+1}–y_{i+2} paths in $G - x_1 - x_2 - x_3$.

We say that a mixed 3-separator $\{x_1, x_2, x_3\}$ is *externally tri-connected* if $\{x_1, x_2, x_3\}$ is externally tri-connected around each vertex $x_i \in \{x_1, x_2, x_3\}$. We say that a mixed 3-separation is *externally tri-connected* if its separator is externally tri-connected.

Example 1.5.1. A mixed-separator that consists of three edges or that induces a clique is externally tri-connected.

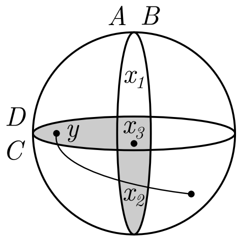


FIGURE 5. The situation excluded by Lemma 1.5.2

For a depiction of the situation excluded by our next lemma, see Figure 5.

LEMMA 1.5.2. *Let G be a 3-connected graph and (A, B) a half-connected tri-separation of G. Denote the separator of (A, B) by $\{x_1, x_2, x_3\}$. Assume that x_2 is an edge with an x_3-free endvertex y. If (A, B) is crossed by a strong tri-separation of G so that x_3 lies in the centre, then y cannot lie in a link.*

PROOF. Let (C, D) be a strong tri-separation of G that crosses (A, B) so that x_3 is in the centre. Since K_4 has no strong tri-separation, G cannot be a K_4. By the Crossing Lemma (1.3.10) and since (A, B) is half-connected, x_3 is a vertex and the only element of the centre, all links have size one, and there are no jumping edges. Without loss of generality, x_2 lies in the link for C, and $y \in A \setminus B$. If y lies in the corner for $\{A, C\}$, then we are done. Otherwise, y must lie in the link for A. The corner for $\{A, C\}$ must be empty, since otherwise $\{y, x_3\}$ would be a 2-separator of G, contradicting 3-connectivity. As $y \in S(C, D)$ must have two neighbours in $G[C]$, and since there are no jumping edges, it follows that yx_3 must be an edge in G. But $y \in S(C, D)$ also means that y cannot have degree three as (C, D) is strong, and since y is x_3-free this means that the edge yx_3 must not be present in G, a contradiction. □

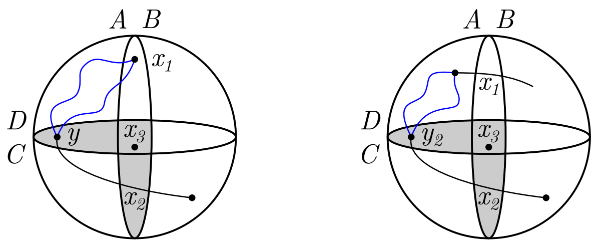


FIGURE 6. The situation in the second and third case of the proof of Lemma 1.5.3

LEMMA 1.5.3. *Let G be a 3-connected graph and (A, B) a half-connected tri-separation of G. If $S(A, B)$ is externally tri-connected around some vertex in $S(A, B)$, then no strong tri-separation of G can cross (A, B) so that this vertex is in the centre.*

PROOF. Let us denote the separator of (A, B) by $\{x_1, x_2, x_3\}$, and let us assume that x_3 is a vertex and that the separator is externally tri-connected around x_3. Let us assume for a contradiction that (A, B) is crossed by a strong tri-separation (C, D) so that x_3 lies in the centre. Since K_4 has no strong tri-separation, G is not a K_4. By the Crossing Lemma (1.3.10), all links have size one, x_3 is the only element of the centre, and there are no jumping edges. We distinguish three cases.

(:) In the first case, x_1 and x_2 are vertices. Since there are no jumping edges, x_1 and x_2 are not adjacent. So by external tri-connectivity, there are three internally disjoint paths in G from x_1 to x_2 avoiding x_3. Each of them has to meet the two links that contain neither x_1 nor x_2, which is not possible as there are three paths and the two links have size one, a contradiction.

($\dot{-}$) In the second case, x_1 is a vertex and x_2 is an edge, say. Without loss of generality, x_1 lies in the link for D while x_2 lies in the link for C. For a depiction of the situation, see Figure 6. By external tri-connectivity, there are two internally disjoint paths P, Q from x_1 to an endvertex y of x_2 that is x_3-free, and these paths avoid x_3 and x_2. Without loss of generality, y lies in $A \setminus B$. Then the two paths P, Q are contained in $G[A]$. Since the link for A has size one, not both of the two paths P, Q can meet it in internal vertices. Hence the vertex y must lie in the link for A. This contradicts Lemma 1.5.2.

($=$) In the third case, x_1 and x_2 are edges. By external tri-connectivity, these edges have x_3-free endvertices y_1 and y_2 and there are two internally disjoint paths P, Q from y_1 to y_2 avoiding x_1, x_2, x_3. By symmetry assume that the vertex y_1 lies in the side A. Then the two paths P, Q are contained in $G[A]$ and y_2 is in A as well. Since the link for A has size one, not both of the two paths P, Q can meet it in internal vertices. So y_1 or y_2 must lie in the link for A. This contradicts Lemma 1.5.2. □

Proposition 1.5.4. *Let G be a 3-connected graph and let (A, B) be a tri-separation of G. If (A, B) is externally tri-connected, half-connected, strong and nontrivial, then (A, B) is totally nested.*

Proof. Let (A, B) be a tri-separation of G that is externally tri-connected, half-connected, strong and nontrivial. Assume for a contradiction that (A, B) is crossed by a tri-separation (C, D) of G. By Proposition 1.3.13, we may assume that (C, D) is strong and nontrivial. As K_4 has no strong nontrivial tri-separation, G is not a K_4. Hence by the Crossing Lemma (1.3.10), (A, B) and (C, D) cross so that the centre contains a vertex. This contradicts Lemma 1.5.3. □

Example 1.5.5. In a 3-connected graph G, every strong and nontrivial tri-separation (A, B) with $G[A \setminus B]$ connected and $G[B \setminus A]$ disconnected is totally nested.

Proof of Example 1.5.5. Note first that $E(A \setminus B, B \setminus A)$ is empty since G is 3-connected. Hence it suffices to find three internally disjoint paths between any pair of vertices in the separator $A \cap B$ avoiding the third vertex, by criterion ($:$) and Proposition 1.5.4. By assumption, $G \setminus (A \cap B)$ has at least three components, and each component has neighbourhood equal to $A \cap B$ since G is 3-connected. Thus we find three internally disjoint paths for each pair of vertices in $A \cap B$ through these components. □

Lemma 1.5.6. *Let G be a 3-connected graph, and X a set of three vertices in G such that $G \setminus X$ has at least three components. Let K be a component of $G \setminus X$, let $A := V(K) \cup X$ and let $B := V(G \setminus K)$. We denote by (A', B') the reduction of the 3-separation (A, B). Then the following assertions hold:*

(1) *$B' = B$ and $(A', B') \leqslant (A, B)$;*
(2) *(A', B') is half-connected and strong.*
(3) *If (A', B') is nontrivial, then it is totally nested.*
(4) *(A', B') is nontrivial if and only if two vertices in X are adjacent or $|K| \geqslant 2$.*

Proof. (1). Since $G[B]$ has minimum degree two, we deduce that $B' = B$, and so $(A', B') \leqslant (A, B)$.

(2). Since the vertex set of the component K is equal to $A' \setminus B'$, the tri-separation (A', B') is half-connected. To see that (A', B') is strong, let v be a vertex in the separator $A' \cap B'$. As (A', B') is a tri-separation, v has two neighbours in A'. Furthermore, v has two neighbours in $B \setminus A$, one in each component by 3-connectivity. Note that $B \setminus A \subseteq B' \setminus A'$. So v has at least four neighbours.

(3). Suppose that (A', B') is nontrivial; we have to show that (A', B') is totally nested. For this, it suffices to show that (A', B') is externally tri-connected, by Proposition 1.5.4. In the case ($:$) we construct the three internally-disjoint paths so that they have their internal vertices in different components of $G \setminus X$. So assume that we are in the cases ($\dot{-}$) or ($=$). Every vertex $x \in X$ that is reduced to an edge in $S(A', B')$ is x'-free for every other vertex $x' \in X$, as x and x' are not adjacent in this case. Hence to show that (A', B') is externally tri-connected, it suffices to find two internally disjoint paths in $G[B']$ between every two vertices in X avoiding the third vertex in X; these are picked so that their internal vertices are in the two components of $G \setminus X$ aside from K.

(4). If (A', B') is nontrivial, then $G[A]$ contains a cycle, so two vertices in X are adjacent or $|K| \geqslant 2$. Conversely, suppose now that two vertices in X are adjacent or that $|K| \geqslant 2$. Since $B' = B$ and $|B| \geqslant 2$, it follows that $|B'| \geqslant 2$. Thus it suffices to show that A' contains at least two vertices, by Lemma 1.2.1. If two vertices in X are adjacent, then these two vertices are not reduced to edges in $S(A', B')$, so they lie in A' and we are done. So assume that $|V(K)| \geqslant 2$. Since $V(K) \subseteq A'$, we are done as well. □

Corollary 1.5.7. *Let G be a 3-connected graph with a tri-separation (C, D) that is not half-connected. If G has no totally-nested nontrivial tri-separation, then $G = K_{3,m}$ for some $m \geqslant 4$.*

Proof. As (C, D) is not half-connected, by 3-connectivity of G its separator X consists of three vertices; and $G \setminus X$ has at least four components. By Lemma 1.5.6, no two vertices in X are adjacent, and every component of $G \setminus X$ is trivial. As G is 3-connected, every component of $G \setminus X$ has neighbourhood equal to X. So G is a $K_{3,m}$ for some $m \geqslant 4$. □

1.6. Background on 2-separations

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$ indexed by the nodes t of T such that the following conditions are satisfied:

(T1) $G = \bigcup_{t \in T} G[V_t]$;

(T2) for every $v \in V(G)$, the vertex set $\{\, t \in T \mid v \in V_t \,\}$ is connected in T.

The vertex sets V_t and the subgraphs $G[V_t]$ they induce are the *bags* of this decomposition. The intersections $V_{t_1} \cap V_{t_2}$ for edges $t_1 t_2 \in E(T)$ are the *adhesion sets* of $(T, \mathcal{V})$. The *adhesion* of $(T, \mathcal{V})$ is the maximum size of an adhesion set of $(T, \mathcal{V})$. The *torso* of a bag is the graph obtained from $G[V_t]$ by adding for every neighbour t' of t in T every possible edge xy with both endvertices in the adhesion set $V_t \cap V_{t'}$. We point out that the edges xy are not required to be edges of G, so each adhesion set $V_t \cap V_{t'}$ induces a complete graph in the torso of $G[V_t]$, and in particular torsos need not be subgraphs of G. The edges of a torso that are not edges of the bag are called *torso edges*.

Every edge $t_1 t_2$ of T, when directed from t_1 to t_2 say, *induces* the separation (X_1, X_2) of G for $X_i := \bigcup_{t \in T_i} V_t$, where T_i is the component of $T - t_1 t_2$ that contains t_i, provided that both $X_1 \setminus X_2$ and $X_2 \setminus X_1$ are non-empty. We call these separations the *induced* separations of $(T, \mathcal{V})$. In this paper, all tree-decompositions have the property that all their edges induce separations. The separator of (X_1, X_2) is the adhesion set $V_{t_1} \cap V_{t_2}$, which is why we also refer to the adhesion sets of $(T, \mathcal{V})$ as the *separators* of $(T, \mathcal{V})$.

Let us call a set S of separations of G *symmetric* if $(A, B) \in S$ implies $(B, A) \in S$ for all $(A, B) \in S$. A set S of separations of G *induces* a tree-decomposition $(T, \mathcal{V})$ of G if the map $(t_1, t_2) \mapsto (X_1, X_2)$ is a bijection between the directed edges of T and the set S.

We shall use the 2-Separation Theorem of Tutte [63] with the total-nestedness description by Cunningham and Edmonds [20], which we recall below with the notation most suitable here. Let us say that a 2-separation of a graph is *totally nested* if it is nested with every 2-separation of the graph.

Theorem 1.6.1 (2-Separation Theorem). *For every 2-connected graph G, the totally-nested 2-separations of G induce a tree-decomposition $(T, \mathcal{V})$ of G all whose torsos are minors of G and are 3-connected, cycles, or K_2's. Moreover, $(T, \mathcal{V})$ is canonical and has the following two properties:*

(1) *If (A, B) and (C, D) are two mixed 2-separations of G that cross so that all four links have size one (and the centre is empty), then there exists a unique node $t \in T$ such that the associated torso is a cycle which alternates between $S(A, B)$ and $S(C, D)$.*

(2) *If the torso associated with a $t \in T$ is 3-connected or a cycle, then the adhesion sets induced by the edges $st \in E(T)$ are pairwise distinct.*

We refer to the tree-decompositions provided by the 2-Separation Theorem as *Tutte-decompositions* as customary. We provide a proof of the 2-Separation Theorem in Section 3.3. A far reaching extension of the 2-separation theorem (that also applies to infinite matroids and extends [27, 54]) was proved by Aigner-Horev, Diestel and Postle [1].

1.7. Apex-decompositions

Recall that a *star* is a rooted tree with at most two levels. The root of the star is commonly referred to as its *centre*. A *star-decomposition* means a tree-decomposition whose decomposition tree is a star.

Let G be a graph and $v \in V(G)$ a vertex. An *apex-decomposition* of G with *centre* v is a star-decomposition $\mathcal{A}$ of $G - v$ of adhesion two such that its central torso is a cycle O and all adhesion sets are pairwise distinct. We refer to O as the *central torso-cycle* of $\mathcal{A}$. The intersection of a leaf-bag B_ℓ of $\mathcal{A}$ with the centre-bag of $\mathcal{A}$ is the *adhesion set of* B_ℓ. We call the edges of O that are spanned by adhesion sets of leaf-bags *bold*, and all other edges of O are *timid*. Note that the timid edges of O exist in G, while possibly some but not necessarily all bold edges of O exist in G. An apex-decomposition is *2-connected* if all its leaf-bags are 2-connected.

LEMMA 1.7.1. *Let G be a 3-connected graph, let $\mathcal{A}$ be a 2-connected apex-decomposition of G with centre v, and let O denote the central torso-cycle of $\mathcal{A}$. In G, the vertex v has a neighbour in $B_\ell \setminus V(O)$ for every leaf-bag B_ℓ of $\mathcal{A}$.*

PROOF. Since $\mathcal{A}$ is 2-connected, B_ℓ has at least three vertices, so $B_\ell \setminus V(O)$ is non-empty. If v had no neighbour in $B_\ell \setminus V(O)$, then the adhesion set of B_ℓ would form a 2-separator of G, contradicting that G is 3-connected. □

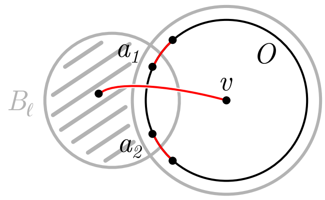


FIGURE 7. In the depicted situation, the separator of the pseudo-reduction induced by ℓ consists of the red edges

Let $\mathcal{A} = (S, \mathcal{B})$ be a 2-connected apex-decomposition of G with centre v. We call a vertex u in the adhesion set of a leaf-bag B_ℓ of $\mathcal{A}$ *edgy* if all but exactly one of the neighbours of u in G lie in B_ℓ.

OBSERVATION 1.7.2. *If a vertex u in the adhesion set of B_ℓ is edgy, then*

(1) *v is not a neighbour of u in G, and*
(2) *the two edges of O incident with u are bold and timid.* □

Each leaf ℓ of S *induces* the 2-separation $(B_\ell, \bigcup_{t \in S-\ell} B_t) =: (X_\ell, Y_\ell)$ of $G - v$. The *pseudo-reduction* of (X_ℓ, Y_ℓ) is the mixed 3-separation (X, Y) of G defined as follows (see Figure 7):

- X is obtained from X_ℓ by adding v unless v has at most one neighbour in X_ℓ, and
- Y is obtained from $Y_\ell + v$ by removing any vertex that lies in the adhesion set of B_ℓ and is edgy.

A set $\sigma = \{ (A_i, B_i) : i \in I \}$ of mixed-separations of G is a *star* with *leaves* A_i if $(A_i, B_i) \leqslant (B_j, A_j)$ for all distinct indices $i, j \in I$. In this context, we also refer to A_i as the *leaf-side* of (A_i, B_i).

EXAMPLE 1.7.3. Stars of genuine separations correspond to star-decompositions, as follows. On the one hand, if $(S, \mathcal{V})$ is a star-decomposition of G and c is the central node of S, then the separations induced by the edges of S incident with c and directed to c form a star σ_c of separations with leaves V_ℓ where the nodes ℓ are the leaves of S. On the other hand, if $\sigma = \{ (A_i, B_i) : i \in I \}$ is a star of separations with leaves A_i, then it defines a star-decomposition $(S, \mathcal{V})$ of G with leaf-bags A_i and which induces σ in the sense that $\sigma = \sigma_c$, where

- S is a star whose set of leaves is equal to I, and
- $V_i := A_i$ for $i \in I$ while $V_c := V(G) \setminus \bigcup \{A_i : i \in I\}$, with c denoting the central node of S.

The set of pseudo-reductions of the separations induced by $\mathcal{A}$ is a star of mixed 3-separations of G, which we call the *tri-star* of $\mathcal{A}$. We will show in Proposition 1.7.9 that the elements of the tri-star are tri-separations, provided that O essentially is not too short.

REMARK 1.7.4. The pseudo-reduction of (X_ℓ, Y_ℓ) need not be a reduction of the 3-separation $(X_\ell + v, Y_\ell + v)$ of G. Indeed, suppose that G is a 3-connected graph which has an apex-decomposition $\mathcal{A} = (S, \mathcal{B})$ with centre v. Suppose further that $\mathcal{A}$ has a leaf-bag B_ℓ with adhesion set $\{a_1, a_2\}$ such that no other leaf-bag

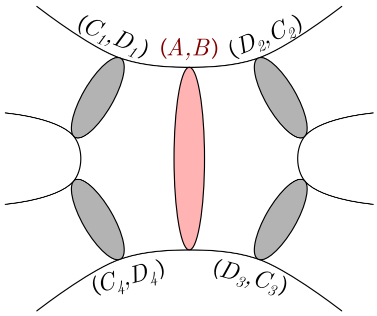


FIGURE 8. A mixed-separation (A, B) interlaces a star $\{\, (C_i, D_i) : i \in [4]\,\}$

contains any a_i, that a_1a_2 is an edge in G but neither a_1 nor a_2 is adjacent to v, and that v has at least two neighbours in B_ℓ and at least two neighbours on O. Then a_1 and a_2 are edgy, and so they are not in the separator of the pseudo-reduction induced by ℓ. However, the 3-separation $(X_\ell + v, Y_\ell + v)$ is a tri-separation of G. So in this case the pseudo-reduction is not a reduction of the 3-separation $(X_\ell + v, Y_\ell + v)$ of G.

Let σ be a star of mixed-separations of a graph G. We say that a mixed-separation (A, B) of G *interlaces* σ if for every $(C, D) \in \sigma$ either $(C, D) < (A, B)$ or $(C, D) < (B, A)$; see Figure 8.

LEMMA 1.7.5. *Let G be a 3-connected graph with two tri-separations (A, B) and (C, D) that cross so that their separators intersect only in a vertex v and all links have size one. Then G has a 2-connected apex-decomposition $\mathcal{A}$ with centre v such that (A, B) and (C, D) interlace the tri-star of $\mathcal{A}$ and the central torso-cycle of $\mathcal{A}$ alternates between $S(A, B) - v$ and $S(C, D) - v$.*

PROOF. Let us consider the 2-connected graph $G' := G - v$. Let $\mathcal{T} = (T, \mathcal{V})$ be the Tutte-decomposition of G provided by the 2-Separation Theorem (1.6.1). Since (A, B) and (C, D) cross in G so that the centre consists of v and all links have size one, their induced mixed 2-separations of G' cross with empty centre and all links of size one. Hence there is a bag $V_t \in \mathcal{V}$ whose torso is a cycle O that alternates between $S(A, B) - v$ and $S(C, D) - v$.

Let S be the star obtained from T by contracting all edges of T not incident with t. Put $B_t := V_t$. For each leaf ℓ of S, we let B_ℓ be the union of all bags $V_s \in \mathcal{V}$ with $s \in \ell$. Then $\mathcal{A} := (S, \mathcal{B})$ with $\mathcal{B} := (\, B_s : s \in V(S)\,)$ is an apex-decomposition of G'. Note that O is equal to the torso of B_t.

Next, we show that the leaf-bags of $\mathcal{A}$ are 2-connected subgraphs of G'. Let B_ℓ be any leaf-bag of $\mathcal{A}$ and let $\{a_1, a_2\}$ denote its adhesion set. By construction, the torso of the bag B_ℓ is 2-connected, so B_ℓ has at least three vertices. Furthermore, B_ℓ is a side of totally-nested 2-separation (B_ℓ, X) of G' with separator $\{a_1, a_2\}$. So at least one of $G'[B_\ell]$ and $G'[X]$ is 2-connected. If a_1 and a_2 are adjacent in G, both are 2-connected and we are done. Otherwise every vertex of O other than a_1, a_2 witnesses that $G'[X]$ is not 2-connected, so $G'[B_\ell]$ is 2-connected as desired.

It remains to show that both (A, B) and (C, D) interlace the tri-star of $\mathcal{A}$. By symmetry, it suffices to show this for (A, B). Consider any pseudo-reduction (X, Y) induced by a leaf ℓ of S. Without loss of generality, the leaf-bag B_ℓ is included in A. We claim that $(X, Y) \leqslant (A, B)$. The side X is obtained from B_ℓ by possibly adding the vertex v. Since v lies in the separator of (A, B) by assumption, this gives $X \subseteq A$. The side Y is obtained from $(V(G) \setminus B_\ell) \cup \{a_1, a_2\}$ by possibly removing some of the vertices in the adhesion set $\{a_1, a_2\}$ of B_ℓ. Since B is included in $(V(G) \setminus B_\ell) \cup \{a_1, a_2\}$, it suffices to show that $a_i \notin Y$ implies $a_i \notin B$ for both $i = 1, 2$. If a_i is not contained in Y, then this is because a_i is edgy, i.e. a_i has just one neighbour outside B_ℓ in G. If B contains a_i, then a_i has two neighbours in B since (A, B) is a tri-separation. The neighbour of a_i in $B \cap B_\ell$ can only be a_{3-i} since $B \cap B_\ell \subseteq \{a_1, a_2\}$. But then O cannot alternate between the separators of (A, B) and (C, D), as $\{a_1, a_2\} \subseteq S(A, B)$ but a_1a_2 is a bold edge of O and therefore cannot lie in $S(C, D)$. □

We label the edges of the central torso-cycle O of an apex-decomposition $\mathcal{A}$ with the letters `b` or `t`, depending on whether they are bold or timid, respectively. The cyclic sequence of these letters is the *type* of O. When O has type `btbt`, we say that O has the type $\texttt{btbt}^-$ if O additionally has a timid edge both of whose endvertices are not adjacent to v; otherwise we say that O has the type $\texttt{btbt}^+$.

Observation 1.7.6. *A central torso-cycle of type* $\texttt{btbt}^+$ *has two non-adjacent vertices that are adjacent to* v *or else the two endvertices of one bold edge are neighbours of* v *while no endvertex of the other bold edge is adjacent to* v. □

Setting 1.7.7. *Let* $\mathcal{A} = (S, \mathcal{B})$ *be a 2-connected apex-decomposition of a 3-connected* G *with centre* v. *Denote the central torso-cycle of* $\mathcal{A}$ *by* O.

When we say that we assume Setting 1.7.7 *with crossing tri-separations*, this means that we also assume that the tri-star of $\mathcal{A}$ is interlaced by two crossing tri-separations of G such that their separators intersect exactly in the vertex v and such that O alternates between the two separators (minus v).

Lemma 1.7.8. *In Setting 1.7.7 with crossing tri-separations,* O *does not have type* `bbt`*,* `bbb` *or* $\texttt{btbt}^-$.

Proof. Let (A, B) and (C, D) denote the two crossing tri-separations of G from the assumption. Let us suppose for a contradiction that O has one of the types we claim it hasn't.

Case `bbt`. Since O alternates between $S(A, B) - v$ and $S(C, D) - v$ by assumption, but neither $S(A, B)$ nor $S(C, D)$ can contain a bold edge of O, it follows that one of $S(A, B)$ and $S(C, D)$ contains both endvertices of the timid edge of O, say $S(A, B)$ contains them. But then one of $A \setminus B$ or $B \setminus A$ is empty, contradicting that (A, B) is a tri-separation.

Case `bbb`. Here we find that $S(A, B)$ and $S(C, D)$ must share a vertex on O, a contradiction.

Case $\texttt{btbt}^-$. Let $e = xy$ be a timid edge of O such that neither x nor y is adjacent to v in G. Let us write $O =: wxyz$. The edges wx and yz lie in neither $S(A, B)$ nor $S(C, D)$ since they are bold.

We claim that the vertices x and y lie in neither $S(A, B)$ nor $S(C, D)$ as well. Assume for a contradiction that $x \in S(A, B)$, say. Then $w \notin S(A, B)$, since O alternates between $S(A, B) - v$ and $S(C, D) - v$, and since wx is bold. Hence the leaf-bag B_ℓ of $\mathcal{A}$ with adhesion set $\{w, x\}$ meets $S(A, B)$ only in x. As $G[B_\ell]$ is 2-connected, $G[B_\ell] - x$ is connected, so B_ℓ is included in $A \setminus B$ or in $B \setminus A$, say in $A \setminus B$. But since v is not a neighbour of x in G, all neighbours of x in G besides y lie in B_ℓ. Thus x has at most one neighbour in B, contradicting that (A, B) is a tri-separation.

So neither vertex x, y and neither edge wx, yz lies in $S(A, B)$ or $S(C, D)$. Since O alternates between $S(A, B)$ and $S(C, D)$ and only the vertices w, z and the edges e, wz can lie in $S(A, B)$ or $S(C, D)$, we find that $S(A, B)$ or $S(C, D)$ must contain two elements of $\{w, wz, z\}$. Say $S(A, B)$ contains two elements. These two elements can only be w and z, since separators of mixed-separations do not contain both a vertex and an edge incident to that vertex. But then $A \setminus B$ or $B \setminus A$ is empty, a contradiction. □

Proposition 1.7.9. *Assume Setting 1.7.7 with crossing tri-separations. Then the tri-star of* $\mathcal{A}$ *consists of totally-nested strong nontrivial tri-separations.*

We prove Proposition 1.7.9 across the next two sections.

1.8. Proof of Proposition 1.7.9: Special cases

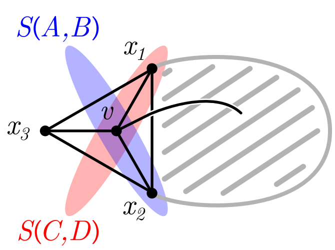


Figure 9. The situation in the proof of Lemma 1.8.1

LEMMA 1.8.1. *Assume Setting 1.7.7 with crossing tri-separations. If O has length three, then the tri-star of $\mathcal{A}$ consists of totally-nested strong nontrivial tri-separations, O has type* `ttt` *or* `btt`*, and v is adjacent to all vertices of O.*

PROOF. Let (A, B) and (C, D) denote the two crossing tri-separations from the assumptions. If O has type `ttt`, then the tri-star of $\mathcal{A}$ is empty, and we are done. So O has at least one bold edge. By Lemma 1.7.8, O does not have type `bbt` or `bbb`, so O must have type `btt`. Let $x_1x_2x_3 := O$ so that x_1x_2 is the bold edge of O; see Figure 9.

Since O alternates between $S(A, B) - v$ and $S(C, D) - v$, since x_1x_2 cannot lie in $S(A, B)$ or $S(C, D)$, and since neither separator can contain both endvertices of x_2x_3 or of x_3x_1, we find that x_3 cannot lie in either separator. Thus only four elements of $V(O) \sqcup E(O)$ can lie in $S(A, B)$ or $S(C, D)$. Hence the separators of (A, B) and (C, D) are determined up to symmetry, say $S(A, B) - v = \{x_2, x_1x_3\}$ and $S(C, D) - v = \{x_1, x_2x_3\}$; see Figure 9. Using that (A, B) and (C, D) are tri-separations, we infer that v is adjacent to all three vertices x_1, x_2, x_3 of O. We also recall that v has a neighbour in the leaf-bag of $\mathcal{A}$ other than x_1 and x_2 by Lemma 1.7.1. Hence $(E, F) := (V(G) - x_3, \{x_1, x_2, x_3, v\})$ with $S(E, F) = \{v, x_1, x_2\}$ is a strong tri-separation of G and the unique element of the tri-star of $\mathcal{A}$. The tri-separation (E, F) is nontrivial by Lemma 1.2.1. Since $G[F \setminus E]$ is a K_1, the tri-separation (E, F) is half-connected.

We claim that (E, F) is totally nested. By Proposition 1.5.4, it suffices to show that (E, F) is externally tri-connected. As v is joined to x_1 and x_2 by edges, it remains to show that $S(E, F)$ is externally tri-connected around v. For this, note that x_1 and x_2 are joined by three internally disjoint paths avoiding v: we find two paths in the 2-connected leaf-bag, and the third path is $x_1x_3x_2$. □

LEMMA 1.8.2. *Assume Setting 1.7.7. If O has type* `btbt`$^+$*, then the tri-star of $\mathcal{A}$ consists of totally-nested strong nontrivial tri-separations.*

PROOF. Since O has type `btbt` and $\mathcal{A}$ is 2-connected, $G - v$ is obtained from the disjoint union of two 2-connected graphs X and Y by adding a matching of size two between X and Y. Call the two matching edges e_1 and e_2. We denote the endvertices of e_i in X and in Y by x_i and y_i, respectively, for both $i = 1, 2$. Since the torso-cycle O has type `btbt`$^+$, we can use Observation 1.7.6 to find that O has two opposite vertices that are adjacent to v or else the two ends of some bold edge of O are neighbours of v while no endvertex of the other bold edge is adjacent to v. We consider the two cases separately.

Case 1: O has two opposite vertices that are adjacent to v in G. Without loss of generality, x_1 and y_2 are adjacent to v in G. By symmetry, it suffices to show that the pseudo-reduction induced by X (viewing X as a leaf-bag of $\mathcal{A}$) is a totally-nested nontrivial tri-separation of G. Since x_1v is an edge in G, and since v has a neighbour in $X \setminus \{x_1, x_2\}$ by Lemma 1.7.1, the pseudo-reduction induced by X is either $(X + v, Y \cup \{v, x_1, x_2\})$ with separator $\{x_1, v, x_2\}$ or $(X + v, Y \cup \{v, x_1\})$ with separator $\{x_1, v, e_2\}$, depending on whether x_2v is an edge in G or not, respectively. In either case, since v also has a neighbour in $Y \setminus \{y_1, y_2\}$ by Lemma 1.7.1, we have a half-connected nontrivial tri-separation that is strong. So by Proposition 1.5.4, it remains to show external tri-connectivity.

Subcase 1a: x_2v is an edge in G. Then the separator is $\{x_1, v, x_2\}$. External tri-connectivity around x_i is witnessed by the edge $x_{3-i}v$ for both $i = 1, 2$. External tri-connectivity around v is witnessed by two internally disjoint x_1–x_2 paths through X and a third x_1–x_2 path which passes through Y via the edges e_1 and e_2.

Subcase 1b: x_2v is not an edge in G. Then the separator is $\{x_1, v, e_2\}$. The endvertex x_2 of e_2 is v-free as x_2v is not an edge. For external tri-connectivity around v, we find two internally disjoint x_1–x_2 paths in X. The endvertex y_2 of e_2 is x_1-free as x_1y_2 is not an edge in G. For external tri-connectivity around x_1, we find two internally disjoint y_2–v paths in $G - x_1 - e_2$, one through Y and via a neighbour of v in $Y \setminus \{y_1, y_2\}$ (which exists by Lemma 1.7.1), and the second one is y_2v.

Case 2: x_1 and x_2 are adjacent to v while y_1 and y_2 are not, say. First, we consider the pseudo-reduction induced by X. This is $(X + v, Y \cup \{v, x_1, x_2\})$ with separator $\{v, x_1, x_2\}$. We verify external tri-connectivity as in Subcase 1a. The pseudo-reduction induced by Y is $(X + v, Y + v)$ or $(X + v, Y)$, depending on whether

v has more than one or just one neighbour in $Y \setminus \{y_1, y_2\}$, respectively. In either case, we have a strong half-connected nontrivial tri-separation, and by Proposition 1.5.4 it remains to verify external tri-connectivity. For $(X + v, Y + v)$ we notice that y_1 and y_2 are v-free and find two internally disjoint y_1–y_2 paths in Y, which suffices as v is the only vertex in the separator $\{v, e_1, e_2\}$. For $(X + v, Y)$ there is nothing to show as its separator consists of three edges. □

1.9. Proof of Proposition 1.7.9: General case

In the previous section, we have seen that the conclusion of Proposition 1.7.9 holds if O has length three (Lemma 1.8.1) or if O has type `btbt` (Lemma 1.7.8 with Lemma 1.8.2). In this section, we show that the conclusion of Proposition 1.7.9 also holds if O has length at least four and O does not have type `btbt`, thereby completing the proof of Proposition 1.7.9.

LEMMA 1.9.1. *Assume Setting 1.7.7. If O has length at least four and O does not have type* `btbt`*, then the tri-star of $\mathcal{A}$ consists of totally-nested strong nontrivial tri-separations.*

To prove Lemma 1.9.1 systematically, we need some machinery. Lemma 1.9.2 below will help us to find paths for verifying external tri-connectivity of mixed 3-separators. To make Lemma 1.9.2 applicable in various settings, we introduce the following definitions which will allow us to deal with cases systematically in Lemma 1.9.2 and its applications. Assume Setting 1.7.7. A *pattern* is any of the following five words:

`bb`, `btx`, `tbb`, `tbtx`, `ttx`.

We say that a finite sequence of consecutive edges in a given cyclic orientation of O *has pattern* p if p is a pattern and the labels of the edges in the sequence start with the pattern p after possibly replacing an occurrence of `x` in p with either `b` or `t`. Note that an edge-sequence of length at least four has a unique pattern; we refer to this pattern as *its* pattern. If a sequence $e_0, \dots, e_n$ has pattern p, and e_i is the last edge in the sequence which contributes to p, then the endvertex of e_i that is not incident with e_{i-1} is called the *capstone* of the sequence $e_0, \dots, e_n$. If a sequence $e_0, \dots, e_n$ has pattern p, then the *pre-reservoir* of this sequence is the union of all leaf-bags of $\mathcal{A}$ (viewed as induced subgraphs of $G - v$) whose adhesion set span edges e_i which contribute to p, plus all the timid edges e_i which contribute to p. The *reservoir* of $e_0, \dots, e_n$ is obtained from the pre-reservoir of $e_0, \dots, e_n$ by adding the vertex v plus all the edges in G from v to the pre-reservoir and then deleting the capstone of $e_0, \dots, e_n$. Note that the reservoir is a subgraph of G.

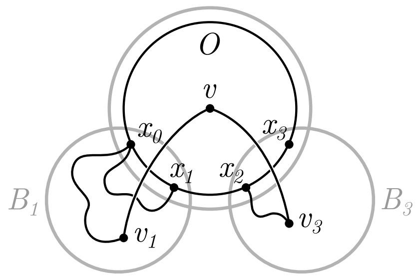


FIGURE 10. The situation of Lemma 1.9.2 for $p =$ `btx` with `x` $:=$ `b`

LEMMA 1.9.2 (Linking Lemma). *Assume Setting 1.7.7. Suppose that O has length at least four. Let e_1, e_2, e_3, e_4 be consecutive edges in a cyclic orientation of O with pattern p. Denote the endvertices of e_1 by x_0 and x_1 so that x_1 is incident with e_2.*

(1) *If the first letter of p is* `b`*, then there are two internally disjoint paths from x_0 to v included in the reservoir of $e_1, \dots, e_3$.*
(2) *Otherwise, there are two internally disjoint paths from x_1 to v included in the reservoir of $e_2, \dots, e_4$ avoiding x_0.*

Recall that a *2-fan* from u to x and y is the union of a u–x path with a u–y path where the two paths meet precisely in u.

FACT 1.9.3. *In a 2-connected graph, there exists a 2-fan from u to x and y for every three vertices u, x, y in the graph.* □

PROOF OF THE LINKING LEMMA (1.9.2). We consider each of the five possible patterns in turn. Let $G' := G - v$. Whenever an edge e_i is bold, we let $B_i := G'[B_\ell]$ for the unique leaf-bag B_ℓ of $\mathcal{A}$ whose adhesion set consists of the endvertices of e_i (tacitly assuming that i is not a node of S). By Lemma 1.7.1, the vertex v has a neighbour in $B_i \setminus O$ whenever B_i exists, and we choose such a neighbour v_i for each eligible i. We denote the vertex of O that is incident with both e_i and e_{i+1} by x_i. This is consistent with the naming of x_1 in the statement of the lemma.

(`bb`) By Fact 1.9.3, we find a 2-fan in B_1 from x_0 to v_1 and x_1. Since B_2 is 2-connected, we find an x_1–v_2 path in B_2 which avoids x_2. The subgraph of G obtained from the union of the 2-fan with the x_1–v_2 path by adding v and the edges vv_1 and vv_2 contains two desired paths.

(`btx`) By Fact 1.9.3, we find a 2-fan in B_1 from x_0 to v_1 and x_1. If `x` is equal to `b`, then B_3 exists, and since B_3 is 2-connected, there is an x_2–v_3 path in B_3 which avoids x_3. Then the subgraph of G obtained from the union of the 2-fan with the path by adding the timid edge e_2 as well as v and the two edges vv_1 and vv_3 contains two desired paths. Otherwise `x` is equal to `t`. Then the two edges e_2 and e_3 are timid, hence 3-connectivity implies that x_2 is adjacent to v. Then the subgraph of G obtained from the 2-fan by adding the edges vv_1, e_2 and vx_2 contains two desired paths.

(`tbb`) Since `bb` is a suffix of `tbb` and since the sought paths are allowed to start in x_1 instead of x_0, we may follow the argumentation of (`bb`).

(`tbtx`) Since `btx` is a suffix of `tbtx` and since the sought paths are allowed to start in x_1 instead of x_0, we may follow the argumentation of (`btx`).

(`ttx`) By 3-connectivity, the edge x_1v must exist in G. Suppose first that `x` is equal to `b`. Then B_3 exists. Since $\mathcal{A}$ is 2-connected, we find an x_2–v_3 path in B_3 avoiding x_3. Then x_1v is one path and adding both edges e_2 and vv_3 to the x_2–v_3 path yields the second path. Otherwise `x` is equal to `t`. By 3-connectivity, the edge x_2v must exist in G. Hence x_1v and x_1x_2v are two desired paths. □

LEMMA 1.9.4. *Assume Setting 1.7.7. Suppose that O has length at least four. Let $\{a_1, a_2\}$ be the adhesion set of a leaf-bag of $\mathcal{A}$ and let $i \in \{1, 2\}$. If a_i is not edgy, then either there are three internally disjoint paths from a_i to v avoiding a_{3-i}, or va_i is an edge in G.*

PROOF. Without loss of generality we have $i = 2$. If va_2 is an edge in G we are done, so let us suppose that v is not adjacent to a_2. Let e_1, e_2, e_3 and e_4 be the four edges of O that come after a_1a_2 on O in the cyclic orientation of O in which a_1 precedes a_2. Since a_2 is not edgy and v is not a neighbour of a_2, the edge e_1 is bold. So the sequence e_1, e_2, e_3, e_4 has pattern `bb` or `btx`, both of which have length at most three. Now we apply the Linking Lemma (1.9.2) to the sequence e_1, e_2, e_3, e_4. This gives us two internally disjoint paths from a_2 to v included in the reservoir. By assumption, O has length at least four, so the vertex a_1 is distinct from the endvertices of the edges e_1 and e_2. Hence, the two internally disjoint paths avoid $B_\ell - a_2$, where B_ℓ is the leaf-bag with adhesion set $\{a_1, a_2\}$. By Lemma 1.7.1, we find a third path from a_2 to v included in $G[B_\ell + v]$, completing the proof. □

LEMMA 1.9.5. *Assume Setting 1.7.7. Suppose that O has length at least four and that O does not have type* `btbt`*. Let $\{a_1, a_2\}$ be the adhesion set of a leaf-bag of $\mathcal{A}$ and let $i \in \{1, 2\}$. Denote the unique neighbour of a_i on O other than a_{3-i} by a_i'. If a_i is edgy, then a_ia_i' is an edge in G while a_iv is not, and there are two internally disjoint paths from a_i' to v in G that avoid B_ℓ.*

PROOF. Without loss of generality, we have $i = 2$. Let e_1, e_2, e_3 and e_4 be the four consecutive edges which come after a_1a_2 in the cyclic orientation of O in which a_1 precedes a_2. Since a_2 is edgy and $\mathcal{A}$ is 2-connected, e_1 is timid. We apply the Linking Lemma (1.9.2) to the sequence e_1, e_2, e_3, e_4. Since the pattern of this sequence starts with `t` and since O has length at least four, we obtain two internally disjoint paths from a_2' to v included in the reservoir and avoiding a_2.

If O has length at least five, the vertex a_1 is distinct from the endvertices of the edges e_1, e_2 and e_3. In particular, the two internally disjoint paths avoid the unique leaf-bag B_ℓ of $\mathcal{A}$ with adhesion set $\{a_1, a_2\}$.

So it remains to consider the case that O has length four. The existence of the bag B_ℓ implies that the edge e_4 is bold. In combination with our assumption that O does not have the type `btbt`, it follows that the pattern `tbtx` is not possible (indeed, here $\mathtt{x} = \mathtt{b}$ since e_4 is bold). Hence the pattern has length at most three. Thus the fact that the vertex a_1 is distinct from the endvertices of the edges e_1 and e_2 suffices to deduce that the two internally disjoint paths avoid B_ℓ. This completes the proof. □

Lemma 1.9.6. *Assume Setting 1.7.7. If O has length at least four and does not have type* `btbt`*, then v has two neighbours in $G \setminus B_\ell$ for every leaf-bag B_ℓ of $\mathcal{A}$.*

Proof. Let B_ℓ be a leaf-bag of $\mathcal{A}$, and let $\{a_1, a_2\}$ denote its adhesion set. Let e_1, e_2, e_3 and e_4 be the four edges on O which come after a_1a_2 in the cyclic orientation of O in which a_1 precedes a_2. Let p be the pattern of this sequence. For each bold e_i, let B_i denote the leaf-bag of $\mathcal{A}$ witnessing that e_i is bold.

Case $p =$ `bb`. By Lemma 1.7.1, the vertex v has two neighbours, one in $B_1 \setminus O$ and one in $B_2 \setminus O$.

Case $p =$ `btx`. By Lemma 1.7.1, the vertex v has one neighbour in $B_1 \setminus O$. If the third edge is bold, we find a second neighbour in $B_3 \setminus O$. Otherwise, we consider the endvertex that is shared by e_2 and e_3. By 3-connectivity, this endvertex must be adjacent to v. So v has two neighbours outside of B_ℓ.

Case $p =$ `tbb`. Since `bb` is a suffix of `tbb`, we may argue as in the case $p =$ `bb`.

Case $p =$ `tbtx`. We consider two subcases. If O has length at least five, then we may argue as in the case $p =$ `btx`, since `btx` is a suffix of `tbtx`. Otherwise O has length four. Then the existence of the leaf-bag B_ℓ entails that the edge e_4 is bold, and so this case is excluded as O does not have type `btbt`.

Case $p =$ `ttx`. Here we may argue similarly as in the case $p =$ `btx`. □

Lemma 1.9.7. *Assume Setting 1.7.7. If O has length at least four and does not have type* `btbt`*, then the tri-star of $\mathcal{A}$ consists of strong nontrivial tri-separations.*

Proof. Let (X, Y) be a pseudo-reduction in the tri-star of $\mathcal{A}$, induced by a leaf ℓ of S. Let us denote the adhesion set of the leaf-bag B_ℓ by $\{a_1, a_2\}$. We claim that very vertex $u \in S(X, Y)$ has degree at least four in G and has at least two neighbours in both X and Y.

Case 1: $u = a_i$ for some $i \in \{1, 2\}$. Since $G'[B_\ell]$ is 2-connected, a_i has at least two neighbours in $B_\ell \subseteq X$. As a_i lies in $S(X, Y)$, it is not edgy, so it has at least two neighbours in $V(G) \setminus B_\ell \subseteq Y$. As these neighbours are distinct, a_i has degree at least four in G.

Case 2: $u = v$. Since v lies in $S(X, Y)$, it has at least two neighbours in $B_\ell \subseteq X$. By Lemma 1.9.6, v has two neighbours in $V(G) \setminus B_\ell \subseteq Y$.

Therefore, (X, Y) is a strong tri-separation. It remains to show that (X, Y) is nontrivial. Since $G'[B_\ell]$ is 2-connected, it contains a cycle, which is included in $G[X]$. To see that $G[Y]$ contains a cycle, by Lemma 1.2.1 it suffices to show that $|Y \setminus X| \geqslant 2$, which follows from O having length at least four. □

Proof of Lemma 1.9.1. Assume Setting 1.7.7. Further suppose that O has length at least four and that O does not have type `btbt`. We have to show that the tri-star of $\mathcal{A}$ consists of totally-nested strong nontrivial tri-separations. By Lemma 1.9.7, the tri-star of $\mathcal{A}$ consists of strong nontrivial tri-separations. So it remains to show that these are totally nested.

Let (X, Y) be a pseudo-reduction in the tri-star of $\mathcal{A}$. Let ℓ be the leaf of S which induces (X, Y), and let B_ℓ denote the leaf-bag of $\mathcal{A}$ assigned to ℓ. Let $\{a_1, a_2\}$ denote the adhesion set of B_ℓ. By definition, $X \cap Y$ is a subset of $\{v, a_1, a_2\}$.

Claim 1.9.7.1. *If the separator of (X, Y) contains v, then it is externally tri-connected around v.*

Proof of Claim. We assume $v \in S(X, Y)$. The vertices a_1, a_2 either lie in the separator of (X, Y) or are v-free. If a vertex a_i is not in $S(X, Y)$, then $S(X, Y)$ contains the edge on O that joins a_i to its neighbour on O other than a_{3-i}. So if at least one of a_1 and a_2 is not in $S(X, Y)$, then the two internally disjoint a_1–a_2 paths through $G'[B_\ell]$ provided by 2-connectedness witness that $S(X, Y)$ is externally tri-connected around v, according to criterion $(=)$ or $(\dot{-})$. It remains to consider the case where $S(X, Y)$ contains both a_1 and a_2. Then, to satisfy criterion $(:)$, we accompany the two a_1–a_2 paths through $G'[B_\ell]$ with a third path, internally disjoint from the former two, which we obtain from the a_1–a_2 path $O - a_1a_2$ by replacing torso-edges with detours through their corresponding leaf-bags if necessary. $\diamondsuit$

CLAIM 1.9.7.2. *If the separator of (X, Y) contains an a_i, then it is externally tri-connected around a_i.*

Proof of Claim. Suppose that $a_2 \in S(X, Y)$, say. We distinguish two cases.

Case 1: the vertex a_1 lies in the separator of (X, Y) as well. Then a_1 is not edgy and Lemma 1.9.4 either yields three internally disjoint a_1–v paths avoiding a_2 or that a_1v is an edge in G. So if $S(X, Y)$ contains v, it is externally tri-connected around a_2 by criterion (:). Otherwise, v is not in $S(X, Y)$, and the unique neighbour u of v in B_ℓ is distinct from a_1 and a_2 by Lemma 1.7.1, so v is a_2-free. If there exist three internally disjoint a_1–v paths avoiding a_2, at least two paths also avoid the edge $uv \in S(X, Y)$; or a_1v is an edge; so $S(X, Y)$ is externally tri-connected around a_2 by criterion ($\dot{-}$).

Case 2: not Case 1. Then instead of the vertex a_1, the edge a_1a_1' lies in $S(X, Y)$, where a_1' denotes the unique neighbour of a_1 in G outside B_ℓ. Note that $a_1' \in O$. By assumption, O has length at least four and does not have type `btbt`. So by Lemma 1.9.5, there are two internally disjoint paths from a_1' to v in G that avoid B_ℓ. If $S(X, Y)$ contains v, the two paths witness that $S(X, Y)$ is externally tri-connected around a_2 by criterion ($\dot{-}$). So we may assume that v is not in $S(X, Y)$, so $S(X, Y)$ instead contains the edge uv where $u \in B_\ell \setminus \{a_1, a_2\}$ is the unique neighbour of v in B_ℓ. Hence v is a_2-free. Since the vertex u lies in B_ℓ, it is avoided by both paths. As v is a_2-free, the two paths witness that $S(X, Y)$ is externally tri-connected around a_2 by criterion (=). $\diamondsuit$

By Claim 1.9.7.1 and Claim 1.9.7.2, $S(X, Y)$ is externally tri-connected. Since (X, Y) also is half-connected, Proposition 1.5.4 gives that (X, Y) is totally nested. $\square$

PROOF OF PROPOSITION 1.7.9. We combine Lemma 1.7.8, Lemma 1.8.1, Lemma 1.8.2 and Lemma 1.9.1. $\square$

PROOF OF THE ANGRY TRI-SEPARATION THEOREM (1.1.5). Let us assume for a contradiction that there exists a 3-connected graph G that fails all three outcomes of Theorem 1.1.5, that is: all nontrivial tri-separations of G are crossed; G is neither a wheel nor a $K_{3,n}$ for any $n \geqslant 3$; and G is not internally 4-connected. Then G has a nontrivial strong tri-separation (A, B) by Proposition 1.2.8, which is half-connected by Corollary 1.5.7. By assumption, the tri-separation (A, B) is crossed by another tri-separation (C, D). The tri-separation (C, D) is nontrivial by Corollary 1.2.2. By Lemma 1.3.10 and using that G is not a wheel such as K_4, all four links have size one, and the centre consists of a single vertex v. By Lemma 1.7.5, G has a 2-connected apex-decomposition $\mathcal{A}$ with centre v, such that (A, B) and (C, D) interlace the tri-star of $\mathcal{A}$, and such that the central torso-cycle of $\mathcal{A}$ alternates between $S(A, B) - v$ and $S(C, D) - v$. As G is not a wheel, $\mathcal{A}$ has at least one leaf-bag, and so the tri-star of $\mathcal{A}$ is non-empty. By Proposition 1.7.9, the tri-star of $\mathcal{A}$ consists of totally-nested nontrivial tri-separations, contradicting our assumption that all nontrivial tri-separations of G are crossed. $\square$

CHAPTER 2

Decomposing 3-connected graphs

2.1. Overview of this chapter

In this chapter, we prove the main result of the paper, Theorem 1. The proof of Theorem 1 offers additional structural insights, which lead us to a refinement of Theorem 1 that comes in the form of Theorem 2.2.8.

This chapter is organised as follows. In the next section we introduce the notation we need to then state Theorem 2.2.8. Like Theorem 1, this theorem will have three possible outcomes for the torsos, and we devote a section to the analysis of each possible outcome.

2.2. Basics

2.2.1. Generalised wheels. The following definitions are supported by Figure 1. A *Y-graph* is a 3-star $K_{1,3}$ and the set of its 3 leaves is referred to as its *attachment set*. A *concrete generalised wheel* is a triple (W, O, v) where W is a graph obtained from a cycle O and a vertex v not on O by doing the following, subject only to the condition that the resulting graph has minimum degree three:

(1) for every vertex on O, we may (but need not) join it to v, and
(2) for every edge xy on O, we may (but need not) disjointly add a Y-graph and identify its attachment set with $\{x, y, v\}$.

We refer to O as the *rim* of this concreted generalised wheel, and we refer to v as its *centre*. For convenience, we write W instead of (W, O, v), and refer to W as a concrete generalised wheel by a slight abuse of notation. Since concrete generalised wheels have minimum degree three, it is straightforward to show that they are 3-connected. The *length* of a concrete generalised wheel means the length of its rim.

A *generalised wheel* is a triple $(W, \mathcal{A}, v)$ where W is a 3-connected graph, v is a vertex of W, and $\mathcal{A}$ is an apex-decomposition of W with centre v such that all leaf-bags are triangles. The *rim* of a generalised wheel is the cycle that is given by the torso of the central bag of the apex-decomposition. The *length* of a generalised wheel means the length of its rim. The vertex v is its *centre*. For convenience, we write W instead of $(W, \mathcal{A}, v)$ and refer to W as a generalised wheel by a slight abuse of notation.

Lemma 2.2.1. *A graph G is a generalised wheel with centre v and rim O if and only if it is a concrete generalised wheel with centre v and rim O.*

Proof. Clearly, every concrete generalised wheel is a generalised wheel with the same rim and centre. As generalised wheels W are 3-connected, the centre is adjacent to every vertex that has degree two in $W - v$, which implies that W has the structure of a concrete generalised wheel with the same rim and centre. □

2.2.2. Splitting stars. Recall that a set $\sigma = \{ (A_i, B_i) : i \in I \}$ of (oriented) mixed-separations of G is a *star* with *leaves* A_i if $(A_i, B_i) < (B_j, A_j)$ for all distinct indices $i, j \in I$. We have seen in Example 1.7.3 that these stars naturally correspond to star-decompositions if they consist of separations only.

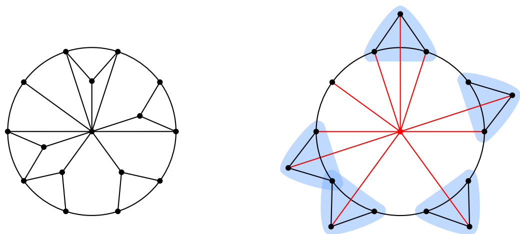


Figure 1. A concrete generalised wheel (left) and its apex-decomposition (right), where leaf-bags are indicated in blue and the centre plus its incident edges are red

Let S be a set mixed-separations of G. A star $\{ (A_i, B_i) : i \in I \} \subseteq S$ with leaves A_i is *splitting* if for every $(C, D) \in S$ there is $i \in I$ with either $(C, D) \leqslant (A_i, B_i)$ or $(D, C) \leqslant (A_i, B_i)$.

EXAMPLE 2.2.2. Let $(T, \mathcal{V})$ be a tree-decomposition of G, and let S denote the set of induced separations of $(T, \mathcal{V})$. For every node $t \in T$, let σ_t denote star of separations induced by the edges of T incident with t and directed to t. The splitting stars of S are precisely the stars σ_t with $t \in T$.

LEMMA 2.2.3. *Let N be a nested set of mixed-separations of a graph G, and let $\sigma \subseteq N$ be a star. Then the following assertions are equivalent:*

(1) *σ is a splitting star of N;*
(2) *no element of N interlaces σ.*

PROOF. (1)→(2). Let σ be a splitting star of N, and assume for a contradiction that $(A, B) \in N$ interlaces σ. Then there is $(C, D) \in \sigma$ such that $(A, B) \leqslant (C, D)$ or $(B, A) \leqslant (C, D)$. But we also have $(C, D) < (A, B)$ or $(C, D) < (B, A)$ since (A, B) interlaces σ. In two cases we obtain immediate contradictions, and in the other two cases we obtain $A \subseteq B$ or $B \subseteq A$ which contradicts the definition of a separation.

(2)→(1). Assume that no element of N interlaces σ; we show that σ is a splitting star of N. Let $(A, B) \in N$, and assume for a contradiction that there is no $(C, D) \in \sigma$ such that $(A, B) \leqslant (C, D)$ or $(B, A) \leqslant (C, D)$. Then, since N is nested, for every $(C, D) \in \sigma$ we have $(C, D) < (A, B)$ or $(C, D) < (B, A)$, so (A, B) interlaces σ. □

LEMMA 2.2.4. *Let M be a nested set of mixed-separations of a graph G, and let σ and τ be two distinct splitting stars of M. Then there are $(A, B) \in \sigma$ and $(C, D) \in \tau$ such that $(B, A) \leqslant (C, D)$.*

PROOF. Let $(X, Y) \in \sigma$ be arbitrary. Since τ is a splitting star, there is $(C, D) \in \tau$ such that $(X, Y) \leqslant (C, D)$ or $(Y, X) \leqslant (C, D)$. In the latter case, we put $(A, B) := (X, Y)$ and are done. In the former case, we use that σ is a splitting star to find $(A, B) \in \sigma$ such that $(C, D) \leqslant (A, B)$ or $(D, C) \leqslant (A, B)$. It suffices to derive a contradiction from $(C, D) \leqslant (A, B)$. Indeed, then $(X, Y) \leqslant (C, D) \leqslant (A, B) \leqslant (Y, X)$ gives $X \subseteq Y$, contradicting that (X, Y) is a mixed-separation. □

LEMMA 2.2.5. *Given a nested set of mixed-separations M of a graph G, a mixed-separation of G interlaces at most one splitting star of M.*

PROOF. Assume for a contradiction that some mixed-separation (A, B) of G interlaces two distinct splitting stars σ_1 and σ_2 of M. By Lemma 2.2.4, there exist $(C_1, D_1) \in \sigma_1$ and $(C_2, D_2) \in \sigma_2$ such that $(D_1, C_1) \leqslant (C_2, D_2)$. Since (A, B) interlaces σ_1 and σ_2, and since $(A, B) \not< (A, B)$ nor $(B, A) \not< (B, A)$, we either have

$$(A, B) < (D_1, C_1) \leqslant (C_2, D_2) < (B, A) \quad \text{or} \quad (B, A) < (D_1, C_1) \leqslant (C_2, D_2) < (A, B).$$

Then $A \subseteq B$ or $B \subseteq A$, contradicting that (A, B) is a mixed-separation. □

2.2.3. Torsos. A *mixed-separation$^+$* of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and no two edges in $E(A \setminus B, B \setminus A)$ share endvertices. We stress that we allow $A \setminus B$ and $B \setminus A$ to be empty. All the usual concepts for mixed-separations extend to mixed-separations$^+$ in the obvious way.

EXAMPLE 2.2.6. All nontrivial mixed 3-separations of a 3-connected graph are mixed 3-separations$^+$ by Lemma 1.2.3. Pairs $(A, V(G))$ for $A \subseteq V(G)$ are separations$^+$ but not separations.

Let $\sigma = \{ (A_i, B_i) : i \in I \}$ be a star of mixed-separations$^+$ of a graph G, with leaf-sides A_i. The *bag* of σ is the intersection $\bigcap_{i \in I} B_i$ of all non-leaf sides B_i. We follow the convention that the bag of the empty star is equal to the vertex-set of G.

If all (A_i, B_i) are separations$^+$, then the *torso* of σ is the graph obtained from $G[\beta]$, where β is the bag of σ, by making every separator $A_i \cap B_i$ into a clique (by adding all possible edges inside $A_i \cap B_i$ for all $i \in I$). In general, however, there are (at least) two ways how the notion of a torso can be generalised to stars of mixed-separations$^+$. Here we present two ways, supported by Figure 2.

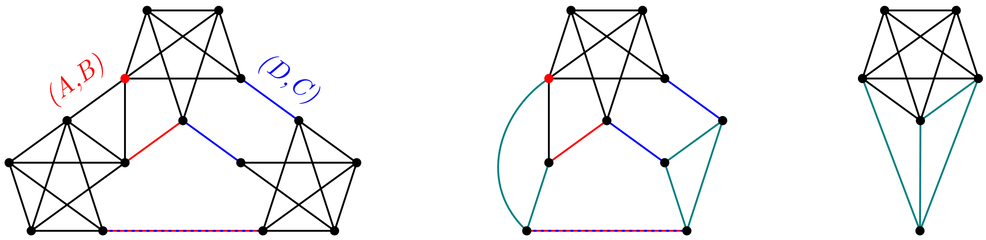


FIGURE 2. Left: the star $\sigma = \{\,(A,B),(C,D)\,\}$. Middle: the expanded torso of σ. Right: the compressed torso of σ.

The *compressed torso* of σ is the graph that is obtained from G by contracting all edges in separators of elements of σ, reducing parallel edges to simple ones, and then taking the torso as defined above. The torsos in Theorem 1 that were mentioned in the introduction are the compressed torsos.

The *expanded torso* of σ is the graph that is obtained from G as follows. We obtain (A_i', B_i') from $(A_i, B_i) \in \sigma$ by letting $A_i' := A_i$ and we obtain B_i' from B_i by adding all endvertices of edges in the separator of (A_i, B_i). Then (A_i', B_i') is a separation$^+$ with the same order as (A_i, B_i). We take the torso of the star $\{\,(A_i', B_i') : i \in I\,\}$ of separations$^+$ as the expanded torso of σ.

Note that the compressed torso can be obtained from the expanded torso by contracting all edges in the separators $S(A_i, B_i)$. If all (A_i, B_i) are separations$^+$, then the compressed torso and the expanded torso coincide. If N is a nested set of mixed-separations of G, then the (compressed/expanded) torsos of N are the (compressed/expanded) torsos of the splitting stars of N. We remark that if N is the set of induced separations of a tree-decomposition $(T, \mathcal{V})$ of G, then the torsos of N are precisely the torsos of $(T, \mathcal{V})$.

LEMMA 2.2.7. *Let N be a nonempty nested set of nontrivial mixed 3-separations of a 3-connected graph G. Then all compressed torsos and expanded torsos of N are minors of G.*

PROOF. By nontriviality, the leaves of the splitting stars induce subgraphs of G that include cycles, and using Menger's theorem we can contract these cycles onto the respective triangles in the torsos. □

2.2.4. Statement of the main theorem. Let σ be a splitting star of a nested set N of tri-separations of G. We say that a strong nontrivial tri-separation (A, B) of G interlaces σ *lightly* if both $G[A \setminus B]$ and $G[B \setminus A]$ have at least two components. Otherwise (A, B) interlaces σ *heavily*. We stress that tri-separations that fail to be strong or nontrivial interlace σ neither lightly nor heavily by definition.

A *thickened* $K_{3,m}$ is obtained from the bipartite graph $K_{3,m}$ by making a bipartition class of size three complete; that is, we add the three edges of a triangle to that set. We allow the degenerated case of a triangle as a thickened $K_{3,0}$.

THEOREM 2.2.8. *Let G be a 3-connected graph and let N denote its set of totally-nested nontrivial tri-separations. Each splitting star σ of N has the following structure:*

(i) *if σ is interlaced lightly, then its compressed torso is a thickened $K_{3,m}$ or $G = K_{3,m}$ for some $m \geqslant 0$;*
(ii) *if σ is interlaced heavily, then its compressed torso is a wheel, and its expanded torso is a generalised wheel;*
(iii) *if σ is not interlaced by a strong nontrivial tri-separation, then its compressed torso is quasi 4-connected or a K_4 or K_3.*

Moreover, all expanded torsos and compressed torsos of N are minors of G.

REMARK 2.2.9. For every integer $m \geqslant 0$, there exist G and σ as in the statement of Theorem 2.2.8 such that (i) holds and the compressed torso of σ is a thickened $K_{3,m}$. Indeed, let m be given. Let X and Y be disjoint vertex sets of size m and three, respectively. We let G be the graph obtained from the complete bipartite graph on (X, Y) by disjointly adding four triangles $\Delta_1, \ldots, \Delta_4$ and joining each triangle Δ_i to the three vertices in Y by a matching of size three. Then $\sigma := \{\,(\Delta_i, V(G \setminus \Delta_i)) : i \in [4]\,\}$ is a splitting star of N. The splitting star σ is lightly interlaced by the tri-separation $(\Delta_1 \cup \Delta_2 \cup Y, V(G \setminus (\Delta_1 \cup \Delta_2))\,)$. The compressed

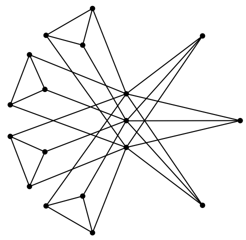


FIGURE 3. A graph with a thickened $K_{3,3}$ as torso; see Remark 2.2.9

torso of σ is obtained from G by contracting all edges in the matchings between Y and the triangles Δ_i, so it is a thickened $K_{3,m}$.

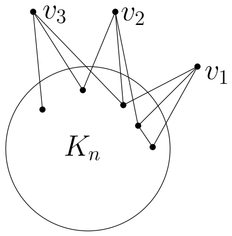


FIGURE 4. This graph is obtained from a K_n with $n = 100$ by first attaching the three vertices v_1, v_2 and v_3 of degree three as illustrated, and then deleting all edges with both ends in the neighbourhood of a v_i, except one edge which lies as in the figure.

REMARK 2.2.10. In (iii), we cannot replace 'quasi 4-connected' by 'internally 4-connected'. Indeed, let G be the graph depicted in Figure 4. The graph G has precisely one strong nontrivial tri-separation (up to flipping sides), which has the form $(A, B) = (v_1 + x + y, V(G) - v_1)$ for xy the unique edge with both ends in the neighbourhood of v_1. Hence $\{ (A, B) \}$ is a splitting star of N. The compressed torso of σ is obtained from G by making the neighbourhood of the vertex v_1 complete and then deleting v_1. Thus this compressed torso has a nontrivial tri-separation, similar to the tri-separation (A, B) with 'v_2' taking the role of 'v_1'. Hence the compressed torso is quasi 4-connected but not internally 4-connected, compare Proposition 1.2.8.

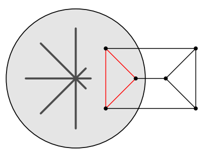


FIGURE 5. An expanded torso that is not quasi 4-connected; red edges are missing

REMARK 2.2.11. In (iii), we cannot replace 'compressed torso' by 'expanded torso'. Indeed, let G be the graph obtained from K_{10} by picking a triangle Δ within and attaching a new triangle Δ' to Δ via a matching, and then deleting the edges of Δ; see Figure 5. The matching is a 3-edge cut of G and determines a splitting star $\{ (\Delta', G \setminus \Delta') \}$ of N. Hence G is an expanded torso of N, but the splitting star is not interlaced by a tri-separation and G is not quasi 4-connected.

2.3. Proof of (i)

Our strategy to prove (i) is to construct for every tri-separation (A, B) that is not half-connected a splitting star σ interlaced by (A, B). The construction is explicit and allows us to deduce that this splitting star has the structure for (i). Then we apply Lemma 2.2.5 to deduce that every splitting star interlaced by (A, B) must be equal to σ, completing the proof. The details are as follows.

Let G be a 3-connected graph. Let U be a set of three vertices of G. The *star of 3-separations induced by U* is the star

$$\{\,(A_K, B_K) : K \text{ is a component of } G \setminus U\,\} \quad \text{with leaves} \quad A_K := V(K) \cup U, \text{ where } B_K := V(G \setminus K).$$

Suppose now that $G \setminus U$ has at least four components. By Lemma 1.5.6, for every 3-separation (A_K, B_K) in this star the reduction (A'_K, B'_K), which in particular is a tri-separation, satisfies $(A'_K, B'_K) \leqslant (A_K, B_K)$. The *star of tri-separations induced by U* is the star that consists of all the reductions of the 3-separations in the star of 3-separations induced by U.

Lemma 2.3.1. *Let (A, B) be a mixed 3-separation of a 3-connected graph G, and let (A', B') be the reduction of (A, B). Then no tri-separation (C, D) of G satisfies $(A', B') < (C, D) \leqslant (A, B)$.*

Proof. Suppose for a contradiction that (C, D) is a tri-separation of G with $(A', B') < (C, D) \leqslant (A, B)$. Since (A', B') is the reduction of (A, B), we have $B' \subseteq B$. From $(A', B') \leqslant (C, D) \leqslant (A, B)$ we obtain $B' \supseteq D \supseteq B$, so $B' = D = B$. Since $(A', B') < (C, D)$ and $B' = D$, the inclusion $A' \subseteq C$ must be proper. As $C \subseteq A$, this means that some vertex $v \in A \cap B$ lies in C but has been removed from A to obtain A'. So v has only one neighbour in A. But then v has only one neighbour in $C \subseteq A$, contradicting that (C, D) is a tri-separation with $v \in C \cap D$. □

Lemma 2.3.2. *Let G be a 3-connected graph and $U \subseteq V(G)$ of size three such that $G \setminus U$ has at least three components. Then $U \subseteq A$ or $U \subseteq B$ for every nontrivial tri-separation of G.*

Proof. Let $U =: \{u, v, w\}$. Suppose for a contradiction that there is a nontrivial tri-separation (A, B) of G with $u \in A \setminus B$ and $v \in B \setminus A$. Since $G \setminus U$ has at least three components, and since every component K has neighbourhood equal to U by 3-connectivity, there exist three independent u–v paths P_K in G with all internal vertices included in K. Hence $S(A, B)$ must contain an edge or an internal vertex from each path P_K, and therefore w cannot be contained in $S(A, B)$. So $w \in B \setminus A$, say. Hence u is a cutvertex of $G[A]$. As (A, B) is nontrivial, there is a cycle in $G[A]$. Let K^* denote the component of $G \setminus U$ that contains a vertex of this cycle. The cycle has a vertex x, say, that is distinct from u and not in $S(A, B)$ (since otherwise $S(A, B)$ would intersect K^* in two vertices, a contradiction). But then the element of $S(A, B)$ on P_{K^*} together with u separate x from the other components of $G \setminus U$, contradicting that G is 3-connected. □

Lemma 2.3.3. *Let G be a 3-connected graph and $U \subseteq V(G)$ of size three such that $G \setminus U$ has at least three components. Then for every half-connected nontrivial tri-separation (C, D) of G there is a component K of $G \setminus U$ such that $(C, D) \leqslant (A_K, B_K)$ or $(D, C) \leqslant (A_K, B_K)$.*

Proof. By Lemma 2.3.2 we may assume that U avoids $C \setminus D$, say. Hence $U \subseteq D$. If $S(C, D) = U$, then (C, D) being half-connected implies that (C, D) or (D, C) is equal to a 3-separation (A_K, B_K) for some component K of $G \setminus U$, and we are done. So assume that $S(C, D) \neq U$. Let K' be an arbitrary component of $G[C \setminus D]$. Since U is included in D, the component K' avoids U. So K' is included in a unique component K of $G \setminus U$. We claim that $(C, D) \leqslant (A_K, B_K)$.

First, we show $C \subseteq A_K$. It suffices to show $G[C \setminus D] \subseteq K$ since this implies

$$C \subseteq (C \setminus D) \cup N(C \setminus D) \subseteq V(K) \cup N(K) = A_K.$$

If $S(C, D)$ contains an edge, then by 3-connectivity every component of $G[C \setminus D]$ must contain the end of this edge in C, and so $G[C \setminus D] = K' \subseteq K$ as desired. Hence we may assume that $S(C, D)$ consists of three vertices, and since $S(C, D) \neq U$ there is a vertex $v \in S(C, D) \setminus U$. By 3-connectivity, every component of $G[C \setminus D]$ has neighbourhood equal to $C \cap D$, and so $K' \subseteq K$ with $A_K \cap B_K = U$ implies $v \in A_K \setminus B_K = V(K)$. As all components of $G[C \setminus D]$ avoid U but have v in their neighbourhoods, they must all be included in the component K of $G \setminus U$ that contains v, so $G[C \setminus D] \subseteq K$ as desired.

For $D \supseteq B_K$, we use $C \subseteq A_K$ to get $B_K \setminus A_K \subseteq D \setminus C$, and recall that $B_K \cap A_K = U \subseteq D$. □

Lemma 2.3.4. *Every totally-nested tri-separation of a 3-connected graph is half-connected.*

Proof. We show the contrapositive. Let (A,B) be non-half-connected tri-separation of a 3-connected graph. Then the separator of (A,B) consists of vertices only. Pick arbitrary components α and β of $G[A\setminus B]$ and $G[B\setminus A]$, respectively. Let

$$C := (A\cap B)\cup V(\alpha)\cup V(\beta) \quad \text{and} \quad D := V(G\setminus(\alpha\cup\beta))$$

Since G is 3-connected, every component of $G-(A\cap B)$ has neighbourhood equal to $A\cap B$. Hence (C,D) is a tri-separation of G. It is straightforward to check that (C,D) crosses (A,B). $\square$

Lemma 2.3.5. *Let G be a 3-connected graph and $U\subseteq V(G)$ of size three such that $G\setminus U$ has at least three components. Let σ' denote the star of tri-separations induced by U, and let $\sigma\subseteq\sigma'$ consist of its nontrivial elements. Then σ is a splitting star of the set of all totally-nested nontrivial tri-separations of G.*

Proof. The elements of σ are totally-nested nontrivial tri-separations of G by Lemma 1.5.6.

Suppose for a contradiction that σ is not splitting. Then some totally-nested nontrivial tri-separation (C,D) of G interlaces σ by Lemma 2.2.3. Since (C,D) is totally nested, it is half-connected by Lemma 2.3.4. By Lemma 2.3.3 there is a component K of $G\setminus U$ such that $(C,D)\leqslant(A_K,B_K)$, say. Let $(A'_K,B'_K)\in\sigma'$ denote the reduction of (A_K,B_K).

If (A'_K,B'_K) is nontrivial, it lies in σ, so $(A'_K,B'_K)<(C,D)\leqslant(A_K,B_K)$ as (C,D) interlaces σ. This contradicts Lemma 2.3.1. Hence (A'_K,B'_K) is trivial; so A'_K and B'_K are the sides of an atomic 3-cut by Lemma 1.2.1. The only possibility here is that $|A'_K|=1$, since B'_K includes U. But then $G[A_K]$ is a $K_{1,3}$, so $G[C]$ contains no cycle and (C,D) is trivial, a contradiction. $\square$

Proof of Theorem 2.2.8 (i). Let σ be a splitting star of N that is interlaced lightly by a tri-separation (A,B) of G. Then $U=A\cap B$ is a 3-separator of G such that $G\setminus U$ has at least four components. Let $\bar\sigma$ be the star of tri-separations induced by U, and note that (A,B) interlaces $\bar\sigma$ as well. Let σ' consist of the nontrivial tri-separations in $\bar\sigma$. By Lemma 2.3.5, σ' is a splitting star of N. As (A,B) interlaces the splitting stars σ and σ' of N, these splitting stars need to be equal by Lemma 2.2.5.

If $G=K_{3,m}$ for some $m\geqslant 4$, then we are done. So we may assume that the graph $G[U]$ has an edge or $G\setminus U$ has a component of size at least two. Thus σ' is non-empty by Lemma 1.5.6. The compressed torso of σ' can be obtained from G by first removing every component K of $G\setminus U$ with $|K|\geqslant 2$, then removing every component K of $G\setminus U$ with $|K|=1$ if $G[U]$ has an edge, and finally making U into a clique (for the latter we need that σ' is nonempty). Hence the compressed torso of $\sigma=\sigma'$ is a thickened $K_{3,m}$ with $m\geqslant 0$. $\square$

2.4. Tools to prove (ii) and (iii)

In this short section we prove a few lemmas that we will use in the proofs of (ii) and (iii).

Definition 2.4.1 (Almost interlacing). We say that a mixed-separation$^+$ (C,D) of a graph G *almost interlaces* a star σ of mixed-separations$^+$ of G if $(A,B)\leqslant(C,D)$ or $(A,B)\leqslant(D,C)$ for all $(A,B)\in\sigma$.

The notion of 'almost interlaces' is more general than the notion of 'interlaces' in two ways: on the one hand, we consider mixed-separations$^+$, and on the other hand we no longer require that (A,B) and its inverse are not in σ.

Lemma 2.4.2. *Assume Setting 1.7.7. If a tri-separation (C,D) of G almost interlaces the tri-star σ of $\mathcal{A}$, then $S(C,D)$ contains v or an edge incident with v.*

Proof. Since (C,D) almost interlaces σ, the elements of the separator $S(C,D)$ are vertices or edges of $O+v$ or edges incident with v. Suppose for a contradiction that $S(C,D)$ contains neither v nor an edge incident to v. Then $S(C,D)\subseteq O$. Every component of $G\setminus(O+v)$ contains a neighbour of v, since G is 3-connected. Every vertex on O that is not a neighbour of a component of $G\setminus(O+v)$ is incident with two timid edges on O, and hence is adjacent to v by 3-connectivity. Hence $G\setminus S(C,D)$ is connected, a contradiction. $\square$

Lemma 2.4.3. *Let G be a 3-connected graph. Let (A, B) be a mixed 3-separation of G, and let (A', B') be a strengthening of (A, B). Then for every strong tri-separation (C, D) of G with $(C, D) \leqslant (A, B)$ we also have $(C, D) \leqslant (A', B')$.*

Proof. Clearly $B' \subseteq B \subseteq D$. Let v be a vertex in $A \cap B$. Assume that v is in C. It remains to show that v is in A'. Since $B \subseteq D$, we have that $v \in C \cap D$. As (C, D) is a strong tri-separation, v has degree four in G and two neighbours in C. Hence the set C witnesses that in the construction of the strengthening (A', B') no vertex of C can be deleted from A. So $v \in A'$ as desired. □

Corollary 2.4.4. *Let σ be a star of strong tri-separations of a 3-connected graph G. If a mixed 3-separation (A, B) of G almost interlaces σ, then every strengthening of (A, B) almost interlaces σ.* □

Similar to Lemma 2.4.3, we prove the following (differences are underlined).

Lemma 2.4.5. *Let G be a 3-connected graph. Let (A, B) be a mixed 3-separation of G, and let (A', B') be the <u>reduction</u> of (A, B). Then for every <u>tri-separation</u> (C, D) of G with $(C, D) \leqslant (A, B)$ we also have $(C, D) \leqslant (A', B')$.* □

Corollary 2.4.6. *Let σ be a star of tri-separations of a 3-connected graph G. If a mixed 3-separation (A, B) of G almost interlaces σ, then the reduction of (A, B) almost interlaces σ as well.* □

2.5. Proof of (ii)

A key step in the proof of (ii) is to show that the tri-star of the apex-decomposition from Setting 1.7.7 is splitting, see Lemma 2.5.10 below. Then we finish the proof similarly to the proof of (i). We start preparing to prove Lemma 2.5.10.

Lemma 2.5.1. *Assume Setting 1.7.7 with crossing tri-separations. If O has type* `btt`*, then the tri-star of $\mathcal{A}$ is a splitting star of the set of all totally-nested nontrivial tri-separations of G.*

Proof. By Lemma 1.8.1, the tri-star of $\mathcal{A}$ consists of totally-nested nontrivial tri-separations, and v is adjacent to all three vertices of O. It remains to show that the tri-star of $\mathcal{A}$ is splitting. Let x_1, x_2, x_3 denote the vertices of O so that the edge x_2x_3 is bold. Let B_ℓ denote the unique leaf-bag of $\mathcal{A}$; so $\{x_2, x_3\}$ is the adhesion set of B_ℓ. Since all x_i are neighbours of v, the pseudo-reduction (C, D) induced by ℓ is given by $C := B_\ell + v$ and $D := \{x_1, x_2, x_3, v\}$.

Let (U, W) be a nontrivial tri-separation of G with $(C, D) < (U, W)$. We have to show that (U, W) is crossed by a tri-separation.

Claim 2.5.1.1. *$(U, W) = (C, D - y)$ for some $y \in \{x_2, x_3, v\}$.*

Proof of Claim. Since $D \setminus C = \{x_1\}$ and $W \setminus U$ is nonempty, we deduce from $(C, D) \leqslant (U, W)$ that $x_1 \in W \setminus U$ and $C = U$. Thus since $(C, D) < (U, W)$, the side W is a proper subset of $D = \{x_1, x_2, x_3, v\}$. As $G[W]$ contains a cycle, it has exactly three vertices. ♢

It is straightforward to check that each $(C, D - y)$ with $y \in \{x_2, x_3, v\}$ is a tri-separation, and that these cross for different values of y. □

Definition 2.5.2 (Red). Assume Setting 1.7.7. A vertex of O is *red* if it is adjacent to v or incident with two bold edges of O.

Example 2.5.3. Vertices incident with two timid edges of O are red: since G is 3-connected, they need to have a third neighbour in G, and this can only be v.

A *mixed 2-separatör*[1] of O is a mixed 2-separator of O or else it consists of the two endvertices of a bold edge of O. It is *red* if all edges in it are timid and all vertices in it are red.

[1] It is a technical variant of a mixed 2-separator, hence the similar name.

Remark 2.5.4. (Motivation) In what follows, we offer a way to understand the tri-separations interlacing the tri-star of $\mathcal{A}$ in Setting 1.7.7 via red mixed 2-separatörs. They have smaller order and hence are easier to analyse.

Given a mixed 2-separatör X, we denote by O_X the topological space obtained from the geometric realisation of O (which is homeomorphic to $\mathbb{S}^1$) by removing all vertices of X and all interior points of edges in X. We refer to the two connected components of O_X as the *intervals* of O_X.

In the following, when (C, D) is a tri-separation, we write $S(C, D) \cap O$ as an abbreviation of $S(C, D) \cap (V(O) \cup E(O))$.

Lemma 2.5.5. *Assume Setting 1.7.7. If a strong tri-separation (C, D) of G almost interlaces the tri-star σ of $\mathcal{A}$, then $S(C, D) \cap O$ is a red mixed 2-separatör.*

Proof. Since (C, D) almost interlaces σ, the elements of the separator $S(C, D)$ are vertices or edges of $O + v$ or edges incident with v. By Lemma 2.4.2, $S(C, D)$ contains v or an edge incident with v. By Lemma 1.2.3, no two edges in $S(C, D)$ share ends. Thus, $X := S(C, D) \cap O$ has size two.

Claim 2.5.5.1. *Each interval of O_X contains (the interior points of) a bold edge or a vertex.*

Proof of Claim. Suppose not for a contradiction. Then one of the intervals of O_X is equal to the set of interior points of a timid edge $e = xy$. So one of the sides of (C, D), say D, contains all vertices of O. So C intersects $V(O)$ precisely in the vertices x and y. As the leaf-bags of $\mathcal{A}$ are 2-connected and e is timid, the graph $G - v - x - y$ is connected; hence C contains no other vertex of $G - v$. Since $C \setminus D$ is nonempty, it must contain a vertex and the only possibility is that $C \setminus D = \{v\}$. By Lemma 1.2.3 the vertex v has at most one neighbour in $D \setminus C = D - x - y$. So by Lemma 1.7.1, at most one edge of O can be bold. Thus one of the vertices x and y, say x, is incident with two timid edges. So x has degree at most three. As $x \in C \cap D$, this contradicts the assumption that (C, D) is strong. ♢

By Claim 2.5.5.1, X is a mixed 2-separatör. It remains to show that X is red. Every edge in X is timid, so let x be a vertex in X. Since we are done otherwise, assume that x is not adjacent to v. If x is not adjacent to a vertex y in X, then the fact that it has at least two neighbours in the sides C and D implies that both its incident edges on O must be bold. So assume that x has a neighbour y in X; that is, $X = \{x, y\}$. The only way this is possible is that $xy =: e$ is an edge of O. By Claim 2.5.5.1, the edge e must be bold. Let f be the edge of O incident with x aside from e.

Suppose for a contradiction that f is timid. Then as x is not adjacent to v, the edge f is in the separator of the pseudo-reduction corresponding to e. Denote this pseudo-reduction by (E, F) with leaf-side E. We have shown that the vertex x is in $C \cap D$ and in $E \setminus F$. As (C, D) almost interlaces, we have that $C \cap D \subseteq F$, a contradiction. So both edges incident with x are bold. Hence x is red. □

Lemma 2.5.6. *Assume Setting 1.7.7. For every red mixed 2-separatör X of G, there is a tri-separation (C, D) of G that almost interlaces the tri-star σ of $\mathcal{A}$ and satisfies $S(C, D) \cap O = X$.*

Proof. Denote the intervals of O_X by C_1 and D_1. We obtain C_2 from C_1 by replacing every bold edge e of O with interior in C_1 by the leaf-side A_i of the tri-separation $(A_i, B_i) \in \sigma$ that corresponds to e, and adding v. We define D_2 analogously. Since X is red, it contains no bold edges and $C_2 \setminus D_2$ and $D_2 \setminus C_2$ are nonempty. Thus (C_2, D_2) is a mixed-separation of G that almost interlaces σ. Its separator is $X + v$, so it is a mixed 3-separation. Vertices of X are red, so have two neighbours in C_2 and D_2. By 3-connectivity, v has a neighbour in $C_2 \setminus D_2$ and in $D_2 \setminus C_2$. So if v has a neighbour in X, the mixed 3-separation (C_2, D_2) is the desired tri-separation. Otherwise the reduction (C, D) of (C_2, D_2) satisfies $S(C, D) = X$ and almost interlaces σ by Corollary 2.4.6, so it is the desired tri-separation. □

The *boundary* of an edge e of O is the 2-element set that, for each endvertex u of e, contains u if u is red, and otherwise contains the unique edge of O other than e that is incident with u.

Example 2.5.7. If e is bold, then its boundary is a red mixed 2-separatör.

LEMMA 2.5.8. *Assume Setting 1.7.7 with crossing tri-separations and that O does not have the type* `btt`. *If a tri-separation (C,D) of G almost interlaces the tri-star σ of $\mathcal{A}$ and $S(C,D)\cap O$ is equal to the boundary of a bold edge of O, then (C,D) or (D,C) is in the tri-star of $\mathcal{A}$.*

PROOF. Let e be a bold edge of O such that $S(C,D)\cap O$ is equal to the boundary of e; if there are two choices for e, we denote the other choice by f. Let (A,B) be the tri-separation in σ that corresponds to the edge e. As (C,D) almost interlaces σ, we have $(A,B)\leqslant(C,D)$ or $(A,B)\leqslant(D,C)$, say the former. Abbreviate $X:=S(C,D)\cap O$. Let P denote the interval of O_X that does not include the interior of e.

CLAIM 2.5.8.1. *Possibly after exchanging the roles of 'e' and 'f' and adjusting P, the sum of neighbours of v on P plus bold edges of P is at least two.*

Proof of Claim. Suppose first for a contradiction that P contains no bold edges and at most one neighbour of v. Then all edges of O except for e are timid. So all vertices not incident with e are neighbours of v by Example 2.5.3 and they are in P. So O has length three and the type `btt`. By assumption this type is excluded, so we reach a contradiction.

It remains to suppose for a contradiction that P contains no neighbour of v and exactly one bold edge. If a vertex of O is incident with two timid edges, this vertex is a neighbour of v by Example 2.5.3 and is in P, which is excluded. So every vertex of O is incident with a bold edge. So O is a cycle with exactly two bold edges such that all its vertices are incident with a bold edge. So O contains at most four vertices. By Lemma 1.7.8 and since O does not have the type `btt` by assumption, O has the type $\texttt{btbt}^+$. As P contains no neighbour of v, by Observation 1.7.6 the two endvertices of e or of f are adjacent to v. Hence, after possibly exchanging the roles of 'e' and 'f' and adjusting P, we find that P contains two neighbours of v. $\diamondsuit$

By Claim 2.5.8.1 we may assume that the sum of neighbours of v on P plus bold edges of P is at least two. So v has two neighbours in D by Lemma 1.7.1. By Lemma 1.2.3, two edges incident with v cannot both be in $S(C,D)$, so $v\in D$.

We shall show that $(A,B)=(C,D)$. When restricting to $G-v$, this equality is immediate. By definition of σ, the vertex v is in B. So $B=D$. Since $(A,B)\leqslant(C,D)$, it remains to show that if $v\in C$, then $v\in A$. So assume $v\in C$. Since (C,D) is a tri-separation, v has two neighbours in C. As $C-v=A-v$, the vertex v has two neighbours in $A-v$. So by the definition of (A,B), the vertex v is in A. This completes the proof. $\square$

We say that a mixed 2-separatör X is crossed by a mixed 2-separatör Y if the two intervals of O_X contain elements of Y; note that crossing is a symmetric relation for mixed 2-separatörs.

LEMMA 2.5.9. *Assume Setting 1.7.7. A mixed 2-separatör X of O is crossed by a red mixed 2-separatör of O if and only if X is not equal to the boundary of a bold edge.*

PROOF. If X is equal to the boundary of a bold edge e, then one of the intervals of O_X consists only of the interior of e plus possibly some non-red endvertices of e; thus X is not crossed by a red mixed 2-separatör. Conversely, if X is not equal to the boundary of a bold edge, then both intervals of O_X either contain a red vertex, a timid edge or at least two edges. Since we are done otherwise immediately, assume that we have the third outcome: two edges in an interval, and as we do not have the second outcome assume all the edges in the interval are bold. Then the interval has an internal vertex (a vertex that is not in the boundary of the interval), which is incident with two bold edges and thus is red. Hence X is crossed by a red mixed 2-separatör. $\square$

LEMMA 2.5.10. *Assume Setting 1.7.7 with crossing tri-separations. The tri-star of $\mathcal{A}$ is a splitting star of the set of totally-nested nontrivial tri-separations.*

PROOF. Let (C,D) be a nontrivial totally nested tri-separation of G that almost interlaces the tri-star of $\mathcal{A}$. Since every tri-separation with a degree-3-vertex x in its separator is crossed (by the atomic cut at x), the totally nested tri-separation (C,D) is strong. By Lemma 2.5.5, $X:=S(C,D)\cap O$ is a red mixed 2-separatör. By Lemma 2.5.9, either X is crossed by a red mixed 2-separator Y or X is equal to the boundary of a bold edge of O. In the second case, by Lemma 2.5.8 and since we are otherwise done by Lemma 2.5.1, the tri-separation (C, D) or (D, C) is in the tri-star of $\mathcal{A}$. In the first case, by Lemma 2.5.6 there is a tri-separation (E, F) almost interlacing the tri-star of $\mathcal{A}$ such that $S(E, F) \cap O = Y$. Since X and Y cross on O, the sets X and Y together ensure that all four links of the corner diagram for the tri-separations (C, D) and (E, F) are nonempty; thus the tri-separations (C, D) and (E, F) cross. We have shown that any nontrivial tri-separation (C, D) that interlaces the tri-star of $\mathcal{A}$ is crossed by a tri-separation. To summarise, the tri-star of $\mathcal{A}$, which consists of totally-nested nontrivial tri-separations by Proposition 1.7.9, is splitting within the set of all totally-nested nontrivial tri-separations by Lemma 2.2.3. □

Lemma 2.5.11. *Let $\mathcal{A}$ be a 2-connected apex-decomposition with central torso-cycle O of a 3-connected graph, and let σ denote the tri-star of $\mathcal{A}$. Then the expanded torso of σ is a generalised wheel with rim O, and the compressed torso of σ is a wheel with rim O.*

Proof. An edge xy of O is *good* if there is some $(A, B) \in \sigma$ such that $\{x, y\} \subseteq A \cap B$ and v is not in the leaf-side A. Let X denote the expanded torso of σ. The graph $X - v$ is isomorphic to the graph obtained from O by adding for every good edge xy of O a new vertex and joining it to x and y (so that the three vertices form a triangle). Hence X is a concrete generalised wheel with rim O. By Lemma 2.2.1, X is also a generalised wheel, with the same rim. The compressed torso of σ is obtained from X by contracting all edges that join v to newly added vertices. Since X is a concrete generalised wheel with rim O, it follows that the compressed torso is a wheel with rim O. □

Proof of Theorem 2.2.8 (ii). Let G be a 3-connected graph, N its set of totally-nested nontrivial tri-separations, and σ a splitting star of N. Suppose that σ is heavily interlaced by a tri-separation (A, B) of G. So (A, B) is half-connected, strong and nontrivial. As (A, B) is not in N, it is crossed by a tri-separation (C, D) of G. By Proposition 1.3.13, we may assume that (C, D) is nontrivial and strong. By the Crossing Lemma (1.3.10), (A, B) and (C, D) cross so that the centre consists of a single vertex v and all links have size one.

By Lemma 1.7.5, G has a 2-connected apex-decomposition $\mathcal{A}$ with centre v whose tri-star σ' is interlaced by (A, B) and (C, D), and whose central torso-cycle O alternates between $S(A, B) - v$ and $S(C, D) - v$; that is to say that we may assume Setting 1.7.7 with crossing tri-separations. By Lemma 2.5.10, the tri-star σ' is a splitting star of N. By Lemma 2.2.5, the fact that (A, B) interlaces the two splitting stars σ and σ' implies $\sigma' = \sigma$. Finally, by Lemma 2.5.11, the compressed and expanded torsos of $\sigma' = \sigma$ are a wheel and generalised wheel, respectively. □

2.6. Proof of (iii)

A key step in this proof will be to understand how separations from the compressed torso for a splitting star σ can be lifted to mixed-separations of G that interlace σ; this is then used to show that the compressed torso of σ can only have very specific 3-separations when σ is not interlaced at all, which roughly speaking is the essence of (iii). Next we prepare to lift.

Lemma 2.6.1. *Let G be a graph, and let σ be a star of mixed-separations$^+$ of G. Then every edge of G lies in the separators of at most two elements of σ.*

Proof. This follows immediately from the observation that the vertex sets $A \setminus B$ are disjoint for distinct elements $(A, B) \in \sigma$. □

Definition 2.6.2 ($\dot{G}$, $(\dot{A}, \dot{B})$ and $\dot{\sigma}$). Suppose now that G is a graph and σ is a star of mixed-separations$^+$ of G. In this context, we define the graph $\dot{G}$ to be the graph obtained from G by subdividing every edge that lies in the separators of exactly two elements of σ. Let $(A, B) \in \sigma$ be arbitrary. We define $\dot{A}$ to be the vertex set obtained from A by adding for every edge $e \in S(A, B)$ the subdividing vertex of e if existent and the endvertex of e in B otherwise. We define $\dot{B}$ to be the vertex set obtained from B by adding for every edge $e \in S(A, B)$ its subdividing vertex if existent (the endvertex of e in A is not added). Then $(\dot{A}, \dot{B})$ is a separation$^+$ of $\dot{G}$, which has the same order as (A, B). We write $\dot{\sigma} := \{ (\dot{A}, \dot{B}) : (A, B) \in \sigma \}$.

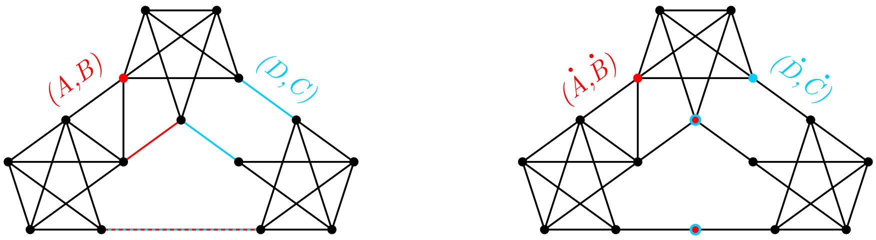


FIGURE 6. σ and $\dot\sigma$

Let G be a 3-connected graph, and let σ be a star of nontrivial tri-separations of G. Let X denote the compressed torso of σ. We define a map $\iota\colon V(X)\to V(\dot G)$ as follows. Let v be any vertex of X. If v is not a contraction vertex, then v also is a vertex of G and we let $\iota(v):=v$. Otherwise v is a contraction vertex with branch set U. If U is spanned in G by a single edge e that lies in the separators of two elements of σ, then we let ι take v to the subdividing vertex of e in $\dot G$. Else U intersects the bag of σ in a unique vertex, by Lemma 1.2.3, and we let ι take v to this unique vertex. Let i be the restriction of ι onto its image.

LEMMA 2.6.3. *Let G be a 3-connected graph, and let σ be a star of nontrivial tri-separations of G. Then $\dot\sigma$ is a star of 3-separations$^+$ of $\dot G$, and i is a graph-isomorphism between the compressed torso of σ in G and the torso of $\dot\sigma$ in $\dot G$.* □

LEMMA 2.6.4 (Lifting Lemma). *Let σ be a star of separations$^+$ of a graph G and let X denote the torso of σ. For every separation (A,B) of X there exists a separation $(\hat A,\hat B)$ of G such that $\hat A\cap V(X)=A$ and $\hat B\cap V(X)=B$ and $\hat A\cap\hat B=A\cap B$. Moreover, $(\hat A,\hat B)$ almost interlaces σ.*

PROOF. Let (A,B) be given. For every $(C,D)\in\sigma$, the separator $C\cap D\subseteq V(X)$ is complete in X, so $C\cap D$ is included in A or in B (possibly in both). We obtain $\hat A$ from A by adding all vertices in $C\setminus D$ from elements $(C,D)\in\sigma$ with $C\cap D\subseteq A$, and we obtain $\hat B$ from B by adding all vertices in $C\setminus D$ from elements $(C,D)\in\sigma$ with $C\cap D\nsubseteq A$. Then $\hat A\cup\hat B=V(G)$. Let us assume for a contradiction that G contains an edge ab with $a\in\hat A\setminus\hat B$ and $b\in\hat B\setminus\hat A$. Since (A,B) is a separation of X, not both a and b can lie in $V(X)$. So $a\in C\setminus D$ for some $(C,D)\in\sigma$ with $C\cap D\subseteq A$, say. Since (C,D) is a separation$^+$, it follows that b must lie in C, contradicting that $C\subseteq\hat A$.

The equalities $\hat A\cap V(X)=A$ and $\hat B\cap V(X)=B$ are immediate from the fact that $C\setminus D$ avoids $V(X)$ for all $(C,D)\in\sigma$. The equality $\hat A\cap\hat B=A\cap B$ follows from the fact that $C\setminus D$ is disjoint from $C'\setminus D'$ for every distinct two $(C,D),(C',D')\in\sigma$. □

In the context of Lemma 2.6.4, we say that (A,B) *lifts* to $(\hat A,\hat B)$, and call $(\hat A,\hat B)$ a *lift* of (A,B).

COROLLARY 2.6.5. *If G is a subdivision of a 3-connected graph, and σ is a star of 3-separations$^+$ of G, and the bag of σ does not include a degree-two vertex plus both its neighbours, then the torso of σ is 3-connected or a K_3.* □

DEFINITION 2.6.6 (Hyper-lift). Let G be a 3-connected graph, and let σ be a star of nontrivial tri-separations of G. Let (C,D) be a separation of the compressed torso X of σ. A *hyper-lift* of (C,D) to G is a mixed-separation $(\hat C,\hat D)$ of G that is obtained from (C,D) in the following way. First, we view (C,D) as a separation of the torso of $\dot\sigma$, using Lemma 2.6.3. Next, we lift (C,D) from the torso of $\dot\sigma$ to a separation (C',D') of $\dot G$, using the Lifting Lemma (2.6.4). Finally, we let $\hat C:=C'\cap V(G)$ and $\hat D:=D'\cap V(G)$.

LEMMA 2.6.7. *Let G be a 3-connected graph, and let σ be a star of nontrivial tri-separations of G. Let (C,D) be a separation of the compressed torso X of σ, and let $(\hat C,\hat D)$ be a hyper-lift of (C,D) to G. Then:*

(1) *the order of $(\hat C,\hat D)$ is at most the order of (C,D);*
(2) *$(\hat C,\hat D)$ almost interlaces σ;*
(3) *$|\hat C\setminus\hat D|\geqslant|C\setminus D|$;*
(4) *for every $(A,B)\in\sigma$ we have $|(\hat C\setminus\hat D)\cap B|\geqslant|C\setminus D|$.*

Proof. Let (C', D') denote the separation of $\dot{G}$ that was used to obtain the hyper-lift $(\hat{C}, \hat{D})$.

(1). First, we shall define an injection from $S(\hat{C}, \hat{D})$ to $S(C', D')$. To this end, let x be an edge in $S(\hat{C}, \hat{D})$. Since (C', D') has no edges in its separator, the edge x must be an edge of G that is not an edge of $\dot{G}$. Denote by y the unique subdivision vertex of x in $\dot{G}$. Since the two neighbours of y are in each of $C' \setminus D'$ and $D' \setminus C'$, and $S(C', D')$ contains no edges, the vertex y is in $S(C', D')$. Let $\varphi(x) := y$. We extend φ to a map from $S(\hat{C}, \hat{D})$ to $S(C', D')$ by taking the identity on vertices. This map is injective since subdivision vertices $y \in V(\dot{G})$ of distinct edges of G are distinct. Hence the order of $(\hat{C}, \hat{D})$ is at most the order of (C', D'). By Lemma 2.6.4, the order of (C', D') is equal to the order of (C, D).

(2). We prove the stronger statement that (C', D') almost interlaces $\dot{\sigma}$ (which gives the desired result as $V(G) \subseteq V(\dot{G})$ and we just need to restrict sides). This follows from Lemma 2.6.4.

(3). If σ is empty, we are done and otherwise (3) follows from (4), so it remains to prove (4).

(4). Let β denote the bag of $\dot{\sigma}$ in $\dot{G}$, and let $(A, B) \in \sigma$. By Lemma 2.6.4, we have that $|(C' \setminus D') \cap \beta| = |C \setminus D|$. We define an injection from $(C' \setminus D') \cap \beta$ to $(\hat{C} \setminus \hat{D}) \cap B$. Assume that $v \in C' \setminus D'$ is a subdivision vertex of an edge e of G (so in particular $v \in \beta$). Then there is some $(E, F) \in \sigma$ that is different from (A, B) that has the edge e in its separator. Let x be the endvertex of e in $E \setminus F$.

Claim 2.6.7.1. $x \in (\hat{C} \setminus \hat{D}) \cap B$.

Proof of Claim. In the proof of (2) we proved that (C', D') almost interlaces $\dot{\sigma}$. As $v \in C' \setminus D'$, we have that $(\dot{E}, \dot{F}) \leqslant (C', D')$. So $\dot{E} \setminus \dot{F} \subseteq C' \setminus D'$. So $x \in C' \setminus D'$. Since $x \in V(G)$, we deduce that $x \in \hat{C} \setminus \hat{D}$. As $(E, F) \leqslant (B, A)$, we have that $x \in B$. $\diamondsuit$

Let φ denote the map from $(C' \setminus D') \cap \beta$ to $(\hat{C} \setminus \hat{D}) \cap B$ that is the identity on vertices of G and maps subdivision vertices v to vertices x as defined above. Since x is not in β, the sets $E' \setminus F'$ for $(E', F') \in \sigma$ are disjoint, and x is incident with at most one edge of $S(E, F)$ by Lemma 1.2.3, the map φ is injective. $\square$

Lemma 2.6.8. *Let G be a 3-connected graph. Let (A, B) be a strong tri-separation of G, and let (C, D) be a mixed 3-separation of G such that $(A, B) \leqslant (C, D)$. If $|(C \setminus D) \cap B| \geqslant 1$, then every strengthening (C', D') of (C, D) satisfies $(A, B) < (C', D')$.*

Proof. By Lemma 2.4.3, we have $(A, B) \leqslant (C', D')$. By assumption, there is a vertex $v \in C \setminus D$ that lies in B. Then v also lies in $(C' \setminus D') \cap B$. Hence the inclusion $B \supseteq D'$ is proper. $\square$

Lemma 2.6.9. *Let G be a 3-connected graph. Let σ be a star of strong tri-separations of G. Suppose that the compressed torso of σ has a 3-separation (C, D) such that both sides have size at least five. Then σ is interlaced by a strong nontrivial tri-separation of G.*

Proof. Let $(\hat{C}, \hat{D})$ be a hyper-lift of (C, D) to G. By Lemma 2.6.7 (applied to (C, D) and (D, C)), $(\hat{C}, \hat{D})$ is a mixed 3-separation of G that almost interlaces σ, and it satisfies $|\hat{C} \setminus \hat{D}| \geqslant 2$ and $|\hat{D} \setminus \hat{C}| \geqslant 2$ by (3). And by (4), for every $(A, B) \in \sigma$ we have $|(\hat{C} \setminus \hat{D}) \cap B| \geqslant 2$ and $|(\hat{D} \setminus \hat{C}) \cap B| \geqslant 2$. Let $(\bar{C}, \bar{D})$ be a strengthening of $(\hat{C}, \hat{D})$. Since $\bar{C} \setminus \bar{D} = \hat{C} \setminus \hat{D}$ and $\bar{D} \setminus \bar{C} = \hat{D} \setminus \hat{C}$, it follows with Lemma 1.2.1 that $(\bar{C}, \bar{D})$ is nontrivial. By Corollary 2.4.4, $(\bar{C}, \bar{D})$ almost interlaces σ. By Lemma 2.6.8, neither $(\bar{C}, \bar{D})$ nor $(\bar{D}, \bar{C})$ lies in σ. Hence $(\bar{C}, \bar{D})$ interlaces σ. $\square$

Proof of Theorem 2.2.8 (iii). Let G be a 3-connected graph, let N denote its set of totally-nested nontrivial tri-separations, and let σ be a splitting star of N. Suppose that σ is not interlaced by a strong nontrivial tri-separation of G. We denote by X the compressed torso of σ. By Lemma 2.6.3 and Corollary 2.6.5, X is 3-connected or a K_3, and we are done in the latter case. The tri-separations in σ are strong by Lemma 1.3.12. Hence by the contrapositive of Lemma 2.6.9, every 3-separation of X has a side with at most four vertices. Hence X is quasi 4-connected or a K_4. $\square$

Proof of Theorem 2.2.8. We have proved (i), (ii) and (iii) in the respective sections above. The 'Moreover' part holds by Lemma 2.2.7. $\square$

Proof of Theorem 1. Theorem 2.2.8 implies Theorem 1. $\square$

2.7. Tutte's Wheel Theorem

If G is a graph and e is an edge of G, we denote by G/e the (multi-)graph that arises from G by contracting e. Recall that a 3-connected (multi-)graph does not have parallel edges. A graph G is *minimally 3-connected* if it is 3-connected and for every edge e of G neither $G - e$ nor G/e is 3-connected.

THEOREM 2.7.1 (Tutte's Wheel Theorem [62]). *Every minimally 3-connected finite graph G is a wheel.*

In this section we give an automatic proof of Tutte's wheel theorem; the proof strategy is as follows. Take a minimally 3-connected graph G. First, we show that all totally-nested tri-separators of G consist of three vertices that do not span any edge. Now consider a 'leaf-torso'[2] of the set of totally-nested nontrivial tri-separations. By Theorem 2.2.8 there are three options how this torso might look like and one easily checks that none of them is possible. Hence G has no totally-nested nontrivial tri-separation, and again by Theorem 2.2.8 we have three options how G might look like; two are excluded for the same reasons and thus the only possibility is that G is a wheel. The details are as follows.

OBSERVATION 2.7.2. *Let e be an arbitrary edge of a 3-connected graph G. Then:*

(c) *if the ends of e do not lie in the same 3-separator of G and e does not lie in a triangle of G, then G/e is 3-connected;*

(d) *if e does not lie in a mixed 3-separator of G, then $G - e$ is 3-connected.* □

An edge e of G is of *type c or d* if it satisfies the premises of conditions (c) or (d) in Observation 2.7.2, respectively.

OBSERVATION 2.7.3. *Minimally 3-connected graphs do not have any edges of type c or d.* □

LEMMA 2.7.4. *Let G be a minimally 3-connected graph. The separator of every totally-nested tri-separation (C, D) of G consists of three vertices that do not span any edge.*

PROOF. Let U denote the set of vertices of $S(C, D)$ together with the endvertices of edges in $S(C, D)$. We claim that every edge $e = vw \notin S(C, D)$ between two vertices of U is of type d. Indeed, the reduction (E, F) of any mixed 3-separation with e in the separator crosses (C, D) with v and w in opposite links, which is not possible by total-nestedness of (C, D). As G is minimally 3-connected, the claim follows by Observation 2.7.3.

We claim that every edge $e = vw$ in $S(C, D)$ is of type c. Indeed, by the above e does not lie in a triangle. Moreover, the reduction (E, F) of any mixed 3-separation with v and w in the separator also contains v and w in its separator (since vw is an edge), and hence (E, F) crosses (C, D) with v and w in opposite links, which is not possible by total-nestedness of (C, D). As G is minimally 3-connected, $S(C, D)$ does not contain any edge by Observation 2.7.3. This completes the proof. □

LEMMA 2.7.5. *Let G be a minimally 3-connected graph and let X be a nonempty set of vertices of G such that the neighbourhood $N(X)$ does not span any edge. Then there is a nontrivial tri-separation (U, W) of G whose separator contains a vertex of X or an edge that is incident with a vertex of X.*

PROOF. For this, let e be an arbitrary edge of G with an endvertex in X. Since e is not of type d by Observation 2.7.3, it lies in the separator of a mixed 3-separation (A, B) of G. Since we are done otherwise, we may assume that the reduction of (A, B) is trivial. So an endvertex v of e has degree three. The lemma is trivial for $G = K_4$. So assume that there is a mixed 3-separation (C, D) of G with separator equal to $N(v)$.

CLAIM 2.7.5.1. *If $N(v)$ spans an edge $f = ab$, then there is a nontrivial tri-separation of G whose separator contains a vertex of X or an edge that is incident with a vertex of X.*

Proof of Claim. The mixed 3-separation $(\{a, b, v\}, V(G) - v)$ is a nontrivial tri-separation of G. Every endvertex of the edge e is incident with the unique edge of its separator or is in its separator. Thus the endvertex of e in X witnesses that this tri-separation has the desired property. ◇

[2] A *leaf-torso* means a compressed torso of a splitting star $\{ (A, B) \}$ where (A, B) is a $\leqslant$-maximal totally-nested nontrivial tri-separation.

Since we are done otherwise by Claim 2.7.5.1, assume that $N(v)$ does not span an edge; that is, e is not in a triangle. Since $e = vw$ is not of type c by Observation 2.7.3, there is a 3-separation (E, F) with v and w in its separator; its reduction has the neighbours v and w in its separator and hence is nontrivial by Lemma 1.2.1. As one of v and w is in X, this gives the desired result. □

Observation 2.7.6. *The nontrivial tri-separation (U, W) in Lemma 2.7.5 can be chosen strong.*

Proof. Via Lemma 1.2.7, take a strengthening of the tri-separation given by Lemma 2.7.5. □

Lemma 2.7.7. *A minimally 3-connected finite graph G has no totally-nested nontrivial tri-separation.*

Proof. Suppose for a contradiction that G has a totally-nested nontrivial tri-separation (A, B). Pick such an (A, B) that is maximal with regard to the partial order $\leqslant$ on mixed-separations. Then $\sigma := \{ (A, B) \}$ is a splitting star of the set of totally-nested nontrivial tri-separations of G. Let $U := B \setminus A$. By Lemma 2.7.4, the separator of (A, B) consists of three vertices that do not span an edge. In particular, $N(U) = A \cap B$. Applying Lemma 2.7.5 together with Observation 2.7.6 to U yields that there is a strong nontrivial tri-separation (C, D) of G such that $S(C, D)$ contains a vertex of U or an edge incident with a vertex of U. Since (A, B) is totally-nested, this implies that $(A, B) < (C, D)$ or $(A, B) < (D, C)$. So (C, D) interlaces σ. By Theorem 2.2.8, the compressed torso X of σ either is a wheel or a thickened $K_{3,m}$ or $G = K_{3,m}$ with $m \geqslant 0$. As $K_{3,m}$ has no totally-nested nontrivial tri-separation, G is not a $K_{3,m}$. Note that as $S(A, B)$ consists only of vertices, X is a genuine torso. Recall that $S(C, D)$ contains a vertex of U or an edge.

We claim that X is a wheel, and suppose that X is a thickened $K_{3,m}$. Then $G[B]$ is a $K_{3,m}$ with $A \cap B$ equal to the left class of size three. As (C, D) is strong, its separator contains no vertex of U, so $S(C, D)$ contains an edge. Then both $G[C \setminus D]$ and $G[D \setminus C]$ are connected, and so (C, D) interlaces σ heavily and X is a wheel W.

The set $A \cap B$ spans a triangle in W. Since this set spans no edge in G, the graph $G[B]$ is obtained from W by deleting the edges of a triangle. Since all but at most one vertex of W have degree three, this leaves a vertex of $A \cap B$ with degree one in $G[B]$, a contradiction to the assumption that (A, B) is a tri-separation. □

Automatic proof of Theorem 2.7.1. Every edge of the graph $K_{3,m}$ with $m \geqslant 3$ is of type c. So by Observation 2.7.3, G is not a $K_{3,m}$ with $m \geqslant 3$. Applying Lemma 2.7.5 with $X = V(G)$ yields that G has a nontrivial tri-separation, so G is not internally 4-connected by Proposition 1.2.8. By Lemma 2.7.7, G has no totally-nested nontrivial tri-separation. Hence by the Angry Tri-Separation Theorem (1.1.5), G is a wheel. □

A natural problem in this area is to understand which edges of 3-connected graph are *essential* in that they cannot be contracted or deleted without destroying 3-connectivity; see for example [3], and [47] for further extensions. Theorem 2.2.8 and our automatic proof of Tutte's wheel theorem provide a new perspective on essential edges, and it is not unreasonable to conjecture that these ideas can be used to resolve this problem.

CHAPTER 3

Concluding remarks

3.1. Tree-like decomposition

It is well-known that there is a natural correspondence between nested sets of separations and tree-decompositions from the Theory of Graph Minors [56]. This correspondence does not extend to *mixed*-separations. Hence it is not clear to us how the sets $N(G)$ in Theorem 1 could determine tree-decompositions. However, we believe that this is not surprising, and there is a natural solution. To explain this, we need a bit more background first.

Wollan introduced tree-cut decompositions to study the immersion-relation, an alternative to the graph-minor relation [65]. His tree-cut decompositions naturally correspond to nested sets of edge-cuts. Since tree-decompositions correspond to separations (with vertex-separators), and tree-cut decompositions correspond to edge-cuts, the two notions of decomposition are not more general than each other, yet they are closely related. As mixed-separations generalise both separations and edge-cuts, they should correspond to a notion of tree-like decomposition that generalises both tree-decompositions and tree-cut decompositions. Indeed, such a notion exists.

Let G be a graph. Let us call a pair $(T,(V_t)_{t\in T})$ of a tree T and a family of vertex sets $V_t \subseteq V(G)$ indexed by the nodes $t \in T$ a *mixed-tree-decomposition* of G if it satisfies the following two conditions:

(M1) $V(G) = \bigcup_{t\in T} V_t$;

(M2) the subgraph of T induced by $\{\, t \in T : v \in V_t \,\}$ is connected for every vertex $v \in G$.

We refer to the vertex sets V_t as *bags*.

The difference to tree-decompositions is that edges are not required to have both ends in some bag V_t. The differences to tree-cut decompositions are that, on the one hand, we allow bags associated to distinct nodes to intersect, and on the other hand, we additionally require (M2). It is straightforward to check that all tree-decompositions and all tree-cut decompositions are mixed-tree-decompositions.

Let $\mathcal{T} := (T,(V_t)_{t\in T})$ be a mixed-tree-decomposition of a graph G. Write $\vec{E}(T) := \{\, (x,y) : xy \in E(T) \,\}$ for the set of all possible directions of edges in T. We can define a map $\alpha_{\mathcal{T}}$ with domain $\vec{E}(T)$ that assigns to each $(t_1,t_2) \in \vec{E}(T)$ the pair (U_1,U_2), where U_i is the union of all bags V_t with t contained in the component of $T - t_1t_2$ that includes t_i (for $i = 1,2$). A set M of mixed-separations is *symmetric* if for every $(A,B) \in M$ it also contains (B,A). From the abstract theory of [23], it follows that every nested symmetric set M of mixed-separations of G uniquely determines (up to isomorphism) a mixed-tree-decomposition $\mathcal{T}$ of G such that $\alpha_{\mathcal{T}}$ is bijective with image equal to M. In particular, the sets $N(G)$ in Theorem 1 can be expressed through mixed-tree-decompositions.

3.2. Outlook

We start by reviewing directions to continue this research. Similarly as for graphs, decompositions along 3-separations are a key tool to study matroids, for example in the context of matroids representable over finite fields [32] and for splitter theorems (and strengthenings thereof) [5, 18, 19].

Open Problem 3.2.1. *Extend Theorem 1.1.5 (and then Theorem 2.2.8) to 3-connected matroids.*

To this end, a natural way to define tri-separations of matroids is the following. Given a 3-connected matroid M, a *nontrivial mixed 3-separation* of G is a triple (A,S,B) such that A,S,B partition the ground set of M and for every bipartition $S = A' \sqcup B'$ we have that $(A\cup A', B\cup B')$ is a 3-separation of M. If S is inclusionwise maximal, meaning that there is no $S' \supsetneq S$ such that $(A\setminus S', S', B\setminus S')$ is a nontrivial mixed 3-separation of M, then (A,S,B) is a *nontrivial tri-separation* of M. Note that (A,S,B) is a nontrivial tri-separation of M if and only if it is a nontrivial tri-separation of the dual M^* of M.

Example 3.2.2. In $U_{3,m}$ for $m \geqslant 6$ every nontrivial tri-separation is crossed by a nontrivial tri-separation.

Another direction for future research is the following:

Open Problem 3.2.3. *Extend Theorem 1.1.5 (and then Theorem 2.2.8) to separators of size larger than 3.*

An instructive example concerning Open Problem 3.2.3 is the line-graph of the 3-dimensional cube. In this graph, there are three 4-separations that cross '3-dimensionally', as depicted in Figure 1 below.

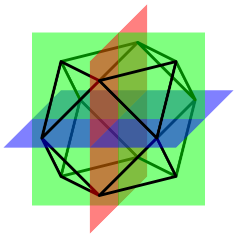


Figure 1. Three 4-separations crossing 3-dimensionally

(Mixed) k-separations of order $k < 4$ do not cross '3-dimensionally': this is trivial for $k = 1$; for $k = 2$ we can read it from the 2-Separation Theorem (1.6.1); and for $k = 3$ this follows by combining the Crossing Lemma (1.3.10) with Lemma 1.7.5. Some hope towards a solution of Open Problem 3.2.3 stems from the results of [8] (building on earlier work of [64]), where it is proved that if a k-connected but not $(k+1)$-connected graph has minimum degree larger than $\frac{3k}{2} - 1$, then it has a totally-nested k-separation (and in fact every k-separation (A, B) with A minimal is totally-nested).

Remark 3.2.4. We expect that many results about 3-connected graphs in the literature can be derived in a fairly straightforward way from Theorem 2.2.8, for example those in [21] or [51].

Our main result Theorem 2.2.8 has quite a few applications in addition to the ones presented here. Whilst for some of these applications, our papers are at an early stage, the following applications will appear on the arXiv shortly (or already have appeared):

(1) Consider the following *connectivity augmentation problem from 0 to 4*. Suppose that we are given a graph G, a set $F \subseteq [V(G)]^2$ of edges not in $E(G)$, and an integer $k \geqslant 0$. Decide whether there is a k-element subset X of F such that $G + X$ is 4-connected. In upcoming work, the first author and Sridharan present an algorithm that solves this problem and that is an FPT-algorithm: its running time is upper-bounded by some function in k times a polynomial in $|V(G)|$. The property of total-nestedness is crucial for this algorithm [17].
(2) We characterised 4-tangles through a connectivity property [15].
(3) A wheel-minor W of a 3-connected graph G is *stellar* if G admits a star-decomposition of adhesion three such that W is equal to the central torso and all leaf-bags include a cycle. We shall show that every stellar wheel-minor of G where the rim is sufficiently large is a minor of an expanded torso of the set of totally-nested nontrivial tri-separations of G [14].

In the following, we compare the decomposition of this paper with Grohe's [38] and with the findings of the upcoming work [15, 14]. Details that we skip here will be addressed in [15, 14]. The results of this paper and related works give rise to three types of decompositions of 3-connected graphs, labelled below by (D1) to (D3). The decomposition (D1) is obtained by taking an inclusion-wise maximal set of pairwise nested nontrivial 3-separations; this is essentially the decomposition constructed by Grohe [38] and we refer to the upcoming work [15] for a refined analysis of this decomposition. The decomposition (D3) is that of Theorem 2.2.8. The decomposition (D2) is obtained from (D3) by applying to each quasi 4-connected compressed torso the decomposition (D1); so (D2) refines (D3).

We made a list of desirable properties for such decompositions, (A1)–(C3) below, and compare the decompositions on the basis of these properties in the following chart.

(D1)	(D2)	(D3)		Property
×	×	×	(A1)	4-tangles appear[1] as torsos
✓	✓	×	(A2)	non-cubic[2] 4-tangles appear as torsos
✓	✓	✓	(A3)	non-cubic 4-tangles live in different quasi 4-connected torsos
✓	✓	✓	(B1)	every non-cubic internally 4-connected minor of G is a minor of some torso
×	✓	✓	(B2)	every stellar m-wheel minor of G with $m \geqslant 5$ is a minor of some torso
✓	✓	×	(C1)	all torsos are internally 4-connected, thickened $K_{3,m}$'s or generalised wheels
✓	✓	✓	(C2)	all torsos are quasi 4-connected, thickened $K_{3,m}$'s or generalised wheels
×	×	✓	(C3)	canonical

To see that (A1) fails, construct a graph G as follows. Start with a set X of four vertices and glue a clique K_{10} at each 3-element subset of X. Then G has a cubic 4-tangle θ that lives on X. In every decomposition (Di) the set X determines a K_4 torso such that θ can only possible live in that torso, but K_4 has no 4-tangle. The results from the upcoming work [15] show that the properties (A3) and (C2) hold for all three decompositions, and that (A2) and (C1) hold for (D1) and (D2). Remark 2.2.10 shows that (A2) and (C1) fail for (D3). (B1) follows from (A3) via our characterisation of 4-tangles from [15]. In the upcoming work [14] we show that (B2) holds for (D2) and (D3) and that $m \geqslant 5$ is necessary, and we also show that no tree-decomposition can possess this property, so in particular not (D1). Clearly, (C3) holds for (D3). The necklace of K_5's from the introduction shows that (C3) fails for (D1). Remark 2.2.10 shows that (C3) fails for (D2) as well.

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[1] The 4-tangles of G *appear as torsos* of (Di) if there is a natural injection ι from the 4-tangles to the torsos of (Di) such that $\iota(\theta)$ is internally 4-connected and its unique 4-tangle lifts to θ; see [15] for details.

[2] A set $X = \{v_1, v_2, v_3, v_4\}$ of four vertices of a graph G is *cubic* if there are components C_1, C_2, C_3, C_4 of $G \setminus X$ such that $N(C_i) = X - v_i$ for $i = 1, 2, 3, 4$, and no component of $G \setminus X$ has the whole of X in its neighbourhood. If X is cubic, then G has the cube Q_3 as a minor where one bipartition class is X and the other is $\{C_1, C_2, C_3, C_4\}$. A 4-tangle is *cubic* if it lives on a cubic vertex set X; that is, every big side in the tangle contains X.

3.3. Appendix: Reviewing the 2-Separation Theorem

3.3.1. Overview of this chapter. A basic fact about graphs states that every connected graph can be cut along its cutvertices in a tree-like way into maximal 2-connected subgraphs and bridges. 2-connected graphs can be decomposed further in the same vein, which is useful to study planar embeddings of graphs, but it is no longer obvious where to best cut these graphs. MacLane found for every 2-connected graph G a tree-decomposition of adhesion two all whose torsos are 3-connected, cycles or K_2's [49]. Tutte [63] later found a canonical such tree-decomposition, for which Cunningham and Edmonds discovered an elegant one-step construction [20]. Here we review and prove a structural version of Tutte's result with the description by Cunningham and Edmonds, using the terminology of this paper, and then derive the 2-Separation Theorem (1.6.1) from it.

3.3.2. Characterising nestedness through connectivity.

FACT 3.3.1. *If G is 2-connected and (A, B) is a 2-separation of G, then $G[A]$ and $G[B]$ are connected, and neither vertex in $A \cap B$ is a cutvertex of $G[A]$ or $G[B]$.*

PROOF. Every component of $G - (A \cap B)$ has neighbourhood equal to $A \cap B$. □

The two situations in (X1) and (X2) of the following lemma are depicted in Figure 2. Recall that a separation (A, B) *separates* two vertices u, v if $u \in A \setminus B$ and $v \in B \setminus A$ or vice versa.

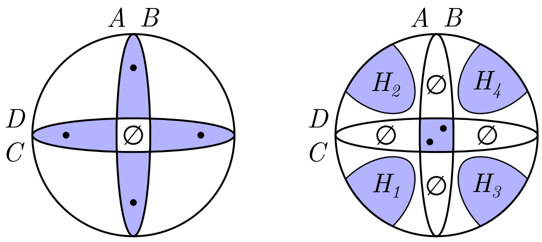

FIGURE 2. The two ways in which 2-separations can cross

LEMMA 3.3.2. *Two 2-separations (A, B) and (C, D) of a 2-connected graph G cross if and only if one of the following assertions holds:*

(X1) *(A, B) separates the two vertices in $C \cap D$ while (C, D) separates the two vertices in $A \cap B$; or*

(X2) *$A \cap B = C \cap D$ and there are four components $H_1, \dots, H_4$ of $G - (A \cap B)$ such that $H_1, H_2 \subseteq G[A]$ and $H_3, H_4 \subseteq G[B]$, while $H_1, H_3 \subseteq G[C]$ and $H_2, H_4 \subseteq G[D]$.*

If (A, B) and (C, D) cross as in (X1), we say that they cross *like in a cycle*. If (A, B) and (C, D) cross as in (X2), we say that they cross *with a four-flip*.

PROOF OF LEMMA 3.3.2. The backward implication is straightforward. For the forward implication, suppose that (A, B) and (C, D) cross. Let us write $X := A \cap B$ and $Y := C \cap D$. We consider three cases.

Case $X \cap Y = \emptyset$. We have to show that (A, B) and (C, D) cross like in a cycle. Let us suppose for a contradiction that they don't. Then the two vertices in X, say, are not separated by (C, D). With $X \cap Y = \emptyset$, it follows that $X \subseteq C \setminus D$, say. As $G[D]$ is connected by Fact 3.3.1 and avoids X, it follows that $G[D]$ is included in a unique component I of $G - X$. Without loss of generality, $I \subseteq G[B]$, so $B \supseteq D$. From $Y \subseteq I \subseteq G[B]$ and $X \cap Y = \emptyset$ we deduce that $G[A]$, which is connected by Fact 3.3.1, is a connected subgraph of $G - Y$, and lies in the component J of $G - Y$ that contains the subset $X \subseteq A$. Thus $G[A] \subseteq J \subseteq G[C]$, that is, $A \subseteq C$. Hence $(A, B) \leqslant (C, D)$, a contradiction.

Case $|X \cap Y| = 1$. We will show that this case is impossible. Let us denote the vertex in the intersection $X \cap Y$ by z, and let us denote the vertices in $X \setminus Y$ and $Y \setminus X$ by x and y, respectively; so $X = \{x, z\}$ and $Y = \{y, z\}$. Let $K(X, y)$ denote the component of $G - X$ that contains y, and let $K(Y, x)$ denote the component of $G - Y$ that contains x. Without loss of generality, $K(X, y) \subseteq G[B]$ and $K(Y, x) \subseteq G[C]$. Since $G[A] - z$ is connected by Fact 3.3.1 and a subgraph of $G - Y$ that contains x, it must be included in the component $K(Y, x)$ of $G - Y$ that contains x. Hence $G[A] \subseteq K(Y, x) \cup N(K(Y, x)) \subseteq G[C]$. A symmetric argument shows $G[D] \subseteq G[B]$. So $(A, B) \leqslant (C, D)$, a contradiction.

Case $X = Y$. Then it is straightforward to deduce that (A, B) and (C, D) cross with a four-flip. □

A 2-separation (A, B) of a graph G is *externally 2-connected* if

- at least one of $G[A]$ and $G[B]$ is 2-connected, and
- at least one of $G[A \setminus B]$ and $G[B \setminus A]$ is connected.

Lemma 3.3.2 implies the following characterisation of total nestedness through external connectivity:

COROLLARY 3.3.3. *A 2-separation of a 2-connected graph is totally nested if and only if it is externally 2-connected.* □

3.3.3. When all 2-separations are crossed.

THEOREM 3.3.4 (Angry 2-Separation Theorem). *If a 2-connected graph G has a 2-separation and every 2-separation of G is crossed by another 2-separation, then G is a cycle of length $\geqslant 4$.*

PROOF OF THEOREM 3.3.4. Let (A,B) be a 2-separation of G. We may choose (A,B) so that $G[A\setminus B]$ is connected. Let (C,D) be a 2-separation of G that crosses (A,B). Since $G[A\setminus B]$ is connected, (C,D) must cross (A,B) like in a cycle. Let T_1 and T_2 be the block graphs of $G[A]$ and of $G[B]$, respectively, where we use the definition of block graphs as in [**21**, §3.1].

The two vertices in $A\cap B$ are separated in $G[A]$ by the vertex of $C\cap D$ that lies in A, so they lie in distinct blocks of $G[A]$. Since G is 2-connected, the vertices in $A\cap B$ are not cutvertices of $G[A]$, so the blocks containing them are unique. Let P_1 be the unique path in T_1 that links these two blocks. Then $T_1=P_1$, because otherwise some edge of T_1 leaving P_1 would induce a 1-separation of $G[A]$ with $A\cap B$ contained in one side, which in turn would extend to a 1-separation of G, contradicting that G is 2-connected. Similarly, we find that T_2 is a path linking the unique blocks of $G[B]$ containing the two vertices in $A\cap B$. So it remains to show that all blocks of $G[A]$ and of $G[B]$ are K_2's.

Let us assume for a contradiction that $G[A]$, say, has a 2-connected block X. Let Y denote the union of all blocks of $G[A]$ and of $G[B]$ except X. Then $\{V(X),V(Y)\}$ is a 2-separation of G. By Corollary 3.3.3, it it totally nested as X is 2-connected and $Y\setminus X$ is connected, a contradiction. □

COROLLARY 3.3.5. [**27**, Theorem 3] *Every vertex-transitive finite connected graph either is 3-connected, a cycle, a K_2 or a K_1.*

PROOF. Let G be a finite connected vertex-transitive graph. If $|G|\leqslant 3$, then G is a complete graph on $\leqslant 3$ vertices, so we may assume that $|G|\geqslant 4$.

We claim that G is 2-connected. Otherwise G has a cutvertex. Then every vertex of G is a cutvertex. Let T be the block graph of G, and let t be a leaf of T. Then t is a block, but contains at most one cutvertex. So some vertex in t is not a cutvertex of G, a contradiction.

Let us suppose now that G is not 3-connected, so G has a 2-separation. If every 2-separation of G is crossed by another one, then G is a cycle of length $\geqslant 4$ by the Angry 2-Separation Theorem (3.3.4). Otherwise G has a totally-nested 2-separation. Let O denote its orbit under the action of the automorphism group of G, and pick $(A,B)\in O$ such that A is minimal. Pick any vertex $v\in A\setminus B$. By vertex-transitivity, there is $(C,D)\in O$ such that $v\in C\cap D$. Since O is nested, and since v obstructs both of $(A,B)\leqslant(C,D)$ and $(A,B)\leqslant(D,C)$, we have $(C,D)\leqslant(A,B)$ or $(D,C)\leqslant(A,B)$. Hence $C\subseteq A$ or $D\subseteq A$. As v lies in $C\cap D$ but not in B, the inclusion $C\subseteq A$ or $D\subseteq A$ must be proper, contradicting the choice of (A,B). □

3.3.4. A structural 2-Separation Theorem. We say that σ is U*-principal* for a vertex set $U\subseteq V(G)$ if $G\setminus U$ has at least three components and

$$\sigma=\{\,s_K\colon K\text{ is a component of }G\setminus U\,\}$$

where

$$s_K:=\big(\,V(K)\cup U,\,V(G)\setminus V(K)\,\big).$$

The bag of a U-principal star σ is equal to $G[U]$, and the separators of the elements of σ are equal to U.

THEOREM 3.3.6 (Structural 2-Separation Theorem). *Let G be a 2-connected graph, and let σ be any splitting star with torso X of the set N of all totally-nested 2-separations of G. If $|X|\leqslant 2$, then X is a K_2 and σ is $V(X)$-principal. Otherwise $|X|\geqslant 3$, and exactly one of the following is true:*

(1) *σ is interlaced by a 2-separation of G that is crossed like in a cycle, and X is a cycle of length $\geqslant 4$;*
(2) *σ is not interlaced by a 2-separation of G, and X is 3-connected or a triangle.*

We remark that the set N in Theorem 3.3.6 is canonical.

LEMMA 3.3.7. *Let G be a 2-connected graph and $U\subseteq V(G)$ a set of two vertices such that $G\setminus U$ has at least three components. Then the U-principal star σ of separations is a splitting star of the set of all totally-nested 2-separations of G.*

Proof. Clearly, σ is a star. Its elements are totally nested by Corollary 3.3.3. Let (C, D) be any 2-separation of G that interlaces σ. Then (C, D) defines a bipartition $(\mathscr{C}, \mathscr{D})$ of the set of components of $G \setminus U$, where $\mathscr{C}$ and $\mathscr{D}$ consist of the components contained in $G[C]$ and in $G[D]$, respectively; also $C \cap D \subseteq U$, and hence $C \cap D = U$. Since (C, D) is not in σ, both $\mathscr{C}$ and $\mathscr{D}$ contain at least two components. Hence (C, D) is not totally-nested by Corollary 3.3.3. $\square$

Lemma 3.3.8. *Let G be a 2-connected graph. Let N be a nested set of half-connected 2-separations of G. Then there is no $(\omega+1)$-chain in N, and for every ω-chain $(A_0, B_0) < (A_1, B_1) < \ldots$ in N we have $\bigcap_{n\in\mathbb{N}} (B_n \setminus A_n) = \emptyset$.*

Proof. Suppose for a contradiction that $(A_0, B_0) < (A_1, B_1) < \ldots < (A_\omega, B_\omega)$ is an $(\omega+1)$-chain in N. Since the elements of N are half-connected, the separations in any 3-chain in N do not all have the same separators. Therefore, we may assume without loss of generality that the (A_i, B_i) have pairwise distinct separators. Let $x \in A_0 \setminus B_0$ and $y \in B_\omega \setminus A_\omega$. Then x and y are separated by infinitely many pairwise distinct separators of size two. Since G is 2-connected, these separators are inclusionwise minimal x–y separators in G. This contradicts a lemma of Halin [41, 2.4], which states that any two vertices u, v in a graph are separated by only finitely many inclusionwise minimal u–v separators of size at most an arbitrarily prescribed $k \in \mathbb{N}$. The same argument also shows that $\bigcap_{n\in\mathbb{N}} (B_n \setminus A_n) = \emptyset$ for every ω-chain $(A_0, B_0) < (A_1, B_1) < \ldots$ in N. $\square$

Corollary 3.3.9. *Let G be a connected graph, and let N be a nested set of half-connected 2-separations of G. Then every separation (A, B) of G with $(A, B) \notin N$ that is nested with every separation in N interlaces a unique splitting star of N.*

Proof. The maximal elements of

$$\{ (C, D) \in N : (C, D) < (A, B) \text{ or } (C, D) < (B, A) \}$$

form a star $\sigma \subseteq N$ that is interlaced by (A, B). By Lemma 3.3.8, the star σ is a splitting star of N. By Lemma 2.2.5, (A, B) interlaces no other splitting star of N. $\square$

Proof of Theorem 3.3.6. Let σ be a splitting star of N with torso X. If X has at most two vertices, then $X = K_2$, so we may assume that X has at least three vertices.

We claim that X is 2-connected, and assume for a contradiction that it is not. Then X has a separation (A, B) of order at most one. By the Lifting Lemma (2.6.4), (A, B) lifts to a separation $(\hat{A}, \hat{B})$ of G of order at most one, contradicting that G is 2-connected. So X is 2-connected.

If X has precisely three vertices, then $X = K_3$ as X is 2-connected, so we may assume that X has at least four vertices.

(i). Suppose that σ is not interlaced by a 2-separation of G. If X is not 3-connected, then X has a 2-separation (A, B) (since X has at least four vertices). By the Lifting Lemma (2.6.4), (A, B) lifts to a 2-separation $(\hat{A}, \hat{B})$ of G, where it interlaces σ, a contradiction.

(ii). Suppose that σ is interlaced by a 2-separation of G.

Claim 3.3.9.1. *Every 2-separation (A, B) of G that interlaces σ induces a 2-separation $(A \cap V(X), B \cap V(X))$ of X that is crossed by a 2-separation of X.*

Proof of Claim. Since (A, B) interlaces σ, it is not in N. Hence (A, B) is crossed by a 2-separation (C, D) of G. We note that (C, D) interlaces σ as well, since otherwise (C, D) would be nested with (A, B). So the separators $A \cap B$ and $C \cap D$ are included in X. If (A, B) and (C, D) cross like in a cycle, then $(A \cap V(X), B \cap V(X))$ and $(C \cap V(X), D \cap V(X))$ are two crossing 2-separations of X as desired. So it remains to show that (A, B) and (C, D) cannot cross with a four-flip. Indeed, otherwise $A \cap B = C \cap D$, and $G \setminus (A \cap B)$ has at least four components which define a splitting star of N as in Lemma 3.3.7. As this splitting star is interlaced by (A, B), it must be equal to σ by Lemma 2.2.5. But then $V(X) = A \cap B$ contradicts our assumption that X has at least four vertices. $\diamondsuit$

We recall that X is 2-connected. By Claim 3.3.9.1 and our assumption, X has a 2-separation. Every 2-separation (A, B) of X lifts to a 2-separation of G by the Lifting Lemma (2.6.4), which interlaces σ and through Claim 3.3.9.1 yields a 2-separation of X that crosses (A, B). Hence X is a cycle of length $\geqslant 4$ by the Angry 2-Separation Theorem (3.3.4). □

3.3.5. Proof of the 2-Separation Theorem (1.6.1). The *bag* of a star $\sigma = \{ (A_i, B_i) : i \in I \}$ of separations of a graph G is the graph obtained from G by deleting $A_i \setminus B_i$ for all $i \in I$. For example, if $(T, \mathcal{V})$ is a tree-decomposition of G and t is a node of T, then the separations induced by the edges of T incident with t and directed to t form a star σ_t of separations. The bag of σ_t is equal to the bag $G[V_t]$ associated with t, where $V_t \in \mathcal{V}$. The *torso* of a star σ of separations of G is obtained from the bag of σ by making $A \cap B$ complete for every $(A, B) \in \sigma$. The torsos of the stars σ_t coincide with the torsos of the bags of $(T, \mathcal{V})$.

Let N be a nested set of separations of G. We define a candidate $\mathcal{T}(N) = (T, \mathcal{V})$ for a tree-decomposition of G, as follows. The vertices of T are the splitting stars of N. We make two nodes $t_1 \neq t_2$ of T adjacent if $(A, B) \in t_1$ and $(B, A) \in t_2$ for some separation (A, B) of G. For each splitting star $t \in T$ we let V_t be the vertex set of the bag of the star t, and put $\mathcal{V} = (V_t)_{t \in T}$.

Lemma 3.3.10. *Let G be a connected graph and N a symmetric nested set of separations of G. Then the following two assertions are equivalent:*

(1) *$\mathcal{T}(N)$ is a tree-decomposition of G whose set of induced separations is equal to N;*
(2) *there is no $(\omega + 1)$-chain in N, and for every ω-chain $(A_0, B_0) < (A_1, B_1) < \ldots$ in N we have $\bigcap_{n \in \mathbb{N}} (B_n \setminus A_n) = \emptyset$.*

Proof. The proof of (i)→(ii) is straightforward. The proof of [29, Lemma 2.7] shows (ii)→(i), even though the statement of [29, Lemma 2.7] says otherwise. □

Lemma 3.3.11. *Let G be a 2-connected graph, and let N denote the set of totally-nested 2-separations of G. Let σ be a splitting star of N such that the torso of σ is a cycle O. Then, for every $(A, B) \in \sigma$, the side $G[A]$ is 2-connected.*

Proof. If $G[A]$ is not 2-connected, then $G[A]$ has a cutvertex u. Since G is 2-connected, u must be contained in $A \setminus B$. Let v be any vertex on O that is not in A. Then $\{u, v\}$ is a 2-separator of G, and every 2-separation of G with separator $\{u, v\}$ crosses (A, B) like in a cycle, contradicting $(A, B) \in N$. □

Lemma 3.3.12. *Let G be a 2-connected graph, and let N denote the set of totally-nested 2-separations of G. Let σ be a splitting star of N such that the torso of σ is 3-connected. Then, for every $(A, B) \in \sigma$, the side $G[B]$ is 2-connected.*

Proof. If $G[B]$ has a cutvertex v, then v separates the two vertices in $A \cap B$ as G is 2-connected. Hence v together with the edge in the torso joining the two vertices in $A \cap B$ forms a mixed 2-separator of the torso, contradicting that the torso is 3-connected. □

Fact 3.3.13. *Let G be a 2-connected graph with a vertex v of degree two such that v lies in a 2-separator of G. Then, for every totally-nested 2-separation (A, B) of G, we have $v \notin S(A, B)$ and $S(A, B)$ contains at most one neighbour of v.*

Proof. If $v \in S(A, B)$, then $(N(v) + v, V(G) - v)$ is a 2-separation of G that crosses (A, B). If $S(A, B) = N(v)$, then (A, B) is crossed by a 2-separation of G which contains v in its separator. □

Proof of Theorem 1.6.1. Let G be a 2-connected graph, and let N denote the set of totally-nested 2-separations of G. By Lemma 3.3.8 and Lemma 3.3.10, N induces a tree-decomposition $\mathcal{T}(N) =: (T, \mathcal{V})$ of G. Since G is 2-connected and $(T, \mathcal{V})$ has adhesion two, it follows with Menger's theorem that all torsos of $(T, \mathcal{V})$ are minors of G. By Theorem 3.3.6, the torsos of $(T, \mathcal{V})$ (which coincide with the torsos of N) are 3-connected, cycles or K_2's.

(1). Let (A, B) and (C, D) be two mixed 2-separations of G that cross so that all four links have size one (and the centre is empty). Let F denote the set of all edges in $S(A, B)$ or $S(C, D)$. Let G' be the graph obtained from G by subdividing all the edges in F. For each edge $e \in F$, we denote the subdividing vertex by v_e.

If (U, W) is a mixed 2-separation of G, then every edge $e \in F$ has an end that is not in $S(U, W)$. We obtain U' from U by adding all vertices v_e for which e has an end in $U \setminus W$. Similarly, we obtain W' from W by adding all vertices v_e for which e has an end in $W \setminus U$. Then (U', W') is a 2-separation of G: the separator $S(U', W')$ is obtained from $S(U, W)$ by replacing every edge e in it that is in F with v_e.

Let N' denote the set of all totally-nested 2-separations of G'. As (A', B') and (C', D') cross like in a cycle, they are not members of N'. By Corollary 3.3.9, (A', B') interlaces a unique splitting star σ' of N', and (C', D') interlaces σ' as well (since otherwise (C', D') would be nested with (A', B')). By the Structural 2-Separation Theorem (3.3.6), the torso of σ' is a cycle; let us denote this cycle by O'. Since (A', B') and (C', D') cross like in a cycle, O' alternates between the separators $S(A', B')$ and $S(C', D')$.

Claim 3.3.13.1. *The map $\varphi\colon (U, W) \mapsto (U', W')$ is a bijection between N and N'.*

Proof of Claim. Let $(U, W) \in N$. By Corollary 3.3.3, (U, W) is externally 2-connected. This is preserved by subdivision, so (U', W') is externally 2-connected, and totally-nested by Corollary 3.3.3. Hence $(U', W') \in N'$.

Clearly, the map φ is injective. It remains to show that it is surjective, so let $(X, Y) \in N'$ be given. By Fact 3.3.13, the separator of (X, Y) contains no subdividing vertices, so (U, W) where $U := X \cap V(G)$ and $W := Y \cap V(G)$ is a 2-separation of G which φ sends to (X, Y). As above, applying Corollary 3.3.3 twice gives $(U, W) \in N$. $\diamondsuit$

By Claim 3.3.13.1, $\sigma := \varphi^{-1}(\sigma')$ is a splitting star of N. Since the separators of the elements of σ' contain no subdividing vertices, the torso O of σ is obtained from O' be replacing every subpath xv_ey where $e = xy \in F \cap E(O')$ with the edge e. Thus, O is a cycle. As the cycle O' alternates between $S(A', B')$ and $S(C', D')$, the cycle O alternates between $S(A, B)$ and $S(C, D)$.

(2). Assume that the torso X associated with $t \in T$ is 3-connected or a cycle. Let xy be an edge of X, and let $\{ s_i t : i \in I \} =: F$ be the set of all edges of T incident with t that induce the adhesion set $\{x, y\}$. If $|I| \leqslant 1$ we are done, so let us suppose for a contradiction that $|I| \geqslant 2$. For each $i \in I$, let T_i denote the component of $T - s_i t$ that contains s_i. Let T_t denote the component of $T - F$ that contains t. Putting

$$A := \bigcup_{i \in I} \bigcup_{r \in T_i} V_r \quad \text{and} \quad B := \bigcup_{r \in T_t} V_r$$

defines a separation (A, B) of G with separator $\{x, y\}$.

We claim that $(A, B) \in N$. If X is 3-connected, then $G[B]$ is 2-connected by Lemma 3.3.12. If X is a cycle, then $G[A]$ is 2-connected by Lemma 3.3.11. Hence at least one of $G[A]$ or $G[B]$ is 2-connected. Since G is obtained from X by replacing some edges with connected graphs containing their endvertices, and since the graph $X \setminus A$ is connected, also the graph $G[B \setminus A]$ is connected. Hence $(A, B) \in N$ by Corollary 3.3.3.

But then (A, B) is an element of N that interlaces t (viewed as a splitting star of N), which contradicts Lemma 2.2.3. $\square$

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