CORTICAL SURFACE FLATTENING USING LEAST SQUARE CONFORMAL MAPPING WITH MINIMAL METRIC DISTORTION

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ABSTRACT

Although flattening a cortical surface necessarily introduces metric distortion due to the non-constant Gaussian curvature of the surface, the Riemann Mapping Theorem states that continuously differentiable surfaces can be mapped without angular distortion. We apply the so-called least-square conformal mapping approach to flatten a patch of the cortical surface onto planar regions and to produce spherical conformal maps of the entire cortex while minimizing metric distortion within the class of conformal maps. Our method, which preserves angular information and controls metric distortion, only involves the solution of a linear system and a nonlinear minimization problem with three parameters and is a very fast approach.

I. INTRODUCTION

The human Cortex is a highly convoluted surface, i.e., functional foci are often buried with cortical sulci and appear in a number of discrete slices or widely seperated foci on oppisite walls of a sulcus may appear to be close together, that makes it difficult to view functional brain activity in a meaningful way. Additionally, It also makes it difficult to compare the locations and patterns of functional activity in humans across subjects because of individual differences in cortical folding. The surface-based approach is a useful tool to address these problems.

It is well-known that flattening a cortical surface necessarily introduces metric and areal distortion due to the non-constant Gaussian curvature of the surface. A number of techniques have been proposed. An approach that purports to substantially minimize the metric distortion (FreeSurfer) was suggested by Fischl *et al.* [4] and another one that attempts to reduce the areal distortion (CARET) by Drury and Van Essen *et al.* [3]. Both methods have been successful in comparative and functional investigation studies. On the other hand, prospects for a ngular distortion are better since the Riemann Mapping Theorem [1] states that continuously differentiable surfaces can be mapped onto each other without angular distortion. Hurdal *et al.* [6] proposed a method using circle packing for quasi-conformal flattening of cortical surfaces that can handle both topological disc and sphere cases and yields an upper bound for distortion, however, it runs slowly. Angenent *et al.* [2] proposed another PDE-based method (Laplace-Beltrami operator) that is much faster than the former one but cannot handle the topological disc case in general expcept mapping onto a rectangle. Gu and Yau [5] recently suggested a new method for computing conformal structures by minimizing the harmonic energy iteratively but still computationally expensive.

In this paper, we first apply the so-called *least square conformal mapping* (LSCM) method introduced in [7] to flatten a patch of cortical surface, then generalize it to produce spherical conformal maps of the entire cortex while minimizing metric distortion within the class of conformal maps. We also make a comparison with FreeSurfer [4] and CARET [3] about performance and distortion.

II. DISCRETE CONFORMAL MAPPING

A conformal mapping of a Riemannian surface to another one is a continuous one-to-one function that preserves all angle measures locally, i.e., *locally isotropic*.

II-A. Planar conformal map using LSCM

The least square conformal mapping is a planar quasi-conformal parameterization method based on a least-square approximation of the Cauchy-Reimann equation. Here we give a brief description, see [7] for details.

Let $\mathcal{K}$ represent a simply-connected triangulated surface of topological disc

$$\left\{\{\mathbf{v}_i\}_{i=1}^n, \mathcal{T} = \{(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \mathbf{v}_{i_3})\}_{i=1}^m\right\}$$

where $\{\mathbf{v}_i\}_{i=1}^n$ is a set of n vertices with $n \geq 3$ and $\mathcal{T}$ is a set of m triangles consisting of triples of vertices. We assume that $\mathcal{K}$ is consistently oriented, then each triangle of $\mathcal{T}$ has a uniquely defined normal; Furthermore, each

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triangle can be imposed a local orthonormal basis (x, y) with the normal along the z-axis.

Now we consider a smooth mapping $\mathcal{U}$ from $\mathcal{K}$ to $\mathbb{R}^2$. When restricting $\mathcal{U}$ on one of the triangles of $\mathcal{T}$, say T, according to the above assumptions, we could write $\mathcal{U}$ in the following form:

$$\mathcal{U}|_T : (x, y) \to (u, v),$$

i.e., $U(x + iy) = u + iv$. The Cauchy-Riemann equation says that $\mathcal{U}$ is conformal on T if and only if the following equality

$$\frac{\partial \mathcal{U}}{\partial x} + i\frac{\partial \mathcal{U}}{\partial y} = 0 \tag{1}$$

holds true on the whole T. Clearly, this conformal condition generally cannot be strictly satisfied on the whole triangulated surface $\mathcal{K}$, so the minimization of the violation of this condition was suggested in [7] to construct the quasi-conformal map in the least square sense:

$$\min_{\mathcal{U}} C(\mathcal{K}) = \sum_{T\in\mathcal{T}} \int_T \left|\frac{\partial \mathcal{U}}{\partial x} + i\frac{\partial \mathcal{U}}{\partial y}\right|^2 dA \tag{2}$$

If we suppose the mapping $\mathcal{U}$ is linear on each triangle T, then we know that $\frac{\partial \mathcal{U}}{\partial x} + i\frac{\partial \mathcal{U}}{\partial y}$ is a constant complex number on T, consequently,

$$C(\mathcal{K}) = \sum_{T\in\mathcal{T}} \left|\frac{\partial \mathcal{U}}{\partial x} + i\frac{\partial \mathcal{U}}{\partial y}\right|^2 A(T) \tag{3}$$

where $A(T)$ denotes the area of the triangle T. Furthermore, suppose that $\mathbf{u}_i = u_i + iv_i = \mathcal{U}(\mathbf{v}_i)$ for $i = 1, \cdots, n$, then $C(\mathcal{K})$ can be written in the quadratic form such as

$$C(\mathcal{K}) = \mathbf{u}^* M^* M \mathbf{u} \tag{4}$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n)$ and M is a sparse $m \times n$ complex matrix.

To make the minimization problem (2) has a unique and non-trivial solution, some of the $\mathbf{u}_i's$ must be pre-decided. Let us re-arrange the vector $\mathbf{u}$ such that $\mathbf{u} = (\mathbf{u}_f, \mathbf{u}_p)$ where $\mathbf{u}_f$ consists of $n-q$ free coordinates and $\mathbf{u}_p$ consists of q pinned coordinates. Then the equation (4) can be rewritten as

$$C(\mathcal{K}) = ||M_f \mathbf{u}_f + M_p \mathbf{u}_p||^2 \tag{5}$$

where $M = (M_f, M_p)$ such that M_f is a $m \times (n-q)$ matrix and M_f is a $m \times q$ matrix. The minimization problem of lesat square type (5) can be efficiently solved using *Conjugate Gradient Method*.

It has been shown that the minimization problem has a unique solution when $q \geq 2$ and the solution is invariant by a similarity transformation in the mapping space. In order to obtain the best conformality of the planar map, q should be set to 2. In our numerical experiments, the two vertices maximizing the length of the shorted path between them were pinned following the suggestion in [7].

II-B. Some measures of distortion

Since our final goal is to analyze the brain imaging data using cortical surface falttening as a tool, we have to take care of the quality of the resulting flat maps. Consequently, we need a uniform way to measure the distortion between the original cortical surface and the corresponding flat map. It is very important that these measures should be invariant under the similarity transformations.

The *angular distortion* is defined in the following:

$$\begin{aligned}\mathcal{F}_{ang}(\mathcal{U}) \;\; = \;\; & \frac{1}{3m} \sum_{Faces(\mathcal{K})}^{(\mathbf{v}_i,\mathbf{v}_j,\mathbf{v}_k)} \Big(|\theta^{\mathcal{U}}_{ijk} - \theta_{ijk}| + |\theta^{\mathcal{U}}_{jki} - \theta_{jki}| \\ & + |\theta^{\mathcal{U}}_{kij} - \theta_{kij}|\Big)\end{aligned} \tag{6}$$

where θ_{ijk} and $\theta^{\mathcal{U}}_{ijk}$ denote the angles $\angle \mathbf{v}_i\mathbf{v}_j\mathbf{v}_k$ and $\angle \mathcal{U}(\mathbf{v}_i)\mathcal{U}(\mathbf{v}_j)\mathcal{U}(\mathbf{v}_k)$ respectively. Although a conformal mapping from our piecewise flat surface to a planar region in fact preserves the "market share" of angles at vertices [6], our definition for angular distortion is still valid since the cortical surface is almost flat very locally.

Before defining the metric distortion, we must deal with the computation of geodesic distances on the triangulated cortical surface $\mathcal{K}$. For each vertex, we label each of its nearest neighbors as a 1-neighbor, then we label each neighbor of a 1-neighbor that is not already labeled as a 2-neighbor. Repeat this process, we could define k-neighbors for each vertex. Denote by $d(\mathbf{v}_i, \mathbf{v}_j)$ and $d_{\mathcal{U}}(\mathbf{v}_i, \mathbf{v}_j)$ the geodesic distances between the vertex $\mathbf{v}_i$ and $\mathbf{v}_j$ on $\mathcal{K}$ and its flat map $\mathcal{U}(\mathcal{K})$ respectively. There are many practical algorithms to compute $d(\mathbf{v}_i, \mathbf{v}_j)$, we take the one proposed in [4], which employs the *Dijkstra Algorithm* and requires dynamic programming due to the memory restriction.

Then, the *metric distortion* is defined in the following:

$$\mathcal{F}_{met}(\mathcal{U}) = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{N(i)}\sum_{j\in N(i)} |d^{\mathcal{U}}_{ij} - d_{ij}|/d_{ij}\Big) \tag{7}$$

where $N(i)$ denotes the set of vertices which are predefined neighbors of vertex i, d_{ij} and $d^{\mathcal{U}}_{ij}$ are normalized geodesic distances between $\mathbf{v}_i$ and $\mathbf{v}_j$ by the sum of length of all edges on $\mathcal{K}$ and its flat map $\mathcal{U}(\mathcal{K})$ respectively to avoid similarity transformations influence.

II-C. Spherical conformal map

In the former Section II-A, the LSCM approach for flattening a surface of topological disc on a planar region has been discussed, we are now left with another problem: how to produce its discrete spherical conformal if the triangulated surface $\mathcal{K}$ is a topological sphere?

Our spherical conformal mapping $\mathcal{U}$ from $\mathcal{K}$ to $\mathcal{S}^2$ proceeds via the map for discs using a trick. First, an arbitrary vertex $\mathbf{v}^*$ choosen from $\{\mathbf{v}_i\}_{i=1}^n$ and all edges containing it are removed from the input triangulated mesh $\mathcal{K}$, clearly, the pruned mesh $\mathcal{K}'$ becomes a topological disc. Then we generate the conformal planar map $\mathcal{U}'$ of the $\mathcal{K}'$ using the LSCM approach. Finally, the map $\mathcal{U}'$ is stereographically projected to the unit sphere $\mathcal{S}^2$ while $\mathbf{v}^*$ is mapped to the "north pole" of $\mathcal{S}^2$; i.e., $\forall\ \mathbf{v} \in \{\mathbf{v}_i\}_{i=1}^n$,

$$\mathcal{U}(\mathbf{v}) = \begin{cases} (0,0,1), & \mathbf{v} = \mathbf{v}^*, \\ \mathcal{P}(\mathcal{U}'(\mathbf{v})), & \text{otherwise.} \end{cases}$$

where $\mathcal{P}$ is the stereographic projection. While such a process does give us a spherical conformal map of $\mathcal{K}$, the spherical conformal maps of $\mathcal{K}$ are not unique since the sphere $\mathcal{S}^2$ has a rich group of *automorphisms*, i.e., one-to-one conformal maps from $\mathcal{S}^2$ to itself. The automorphism group of $\mathcal{S}^2$, $Aut(\mathcal{S}^2)$ is the group of all Mobius transformations of $\mathcal{S}^2$, i.e.,

$$Aut(\mathcal{S}^2) = \{\psi \mid \psi : z \to \frac{az+b}{cz+d}, a,b,c,d \in \mathbb{C}, ad-bc \neq 0\}.$$

Since the above process also may produce a highly distorted spherical map, our idea here is to normalize the map $\mathcal{U}$ by an automorphism $\psi \in Aut(\mathcal{S}^2)$ to minimize the metric distortion defined in (7), i.e., we need to solve the following minimization problem:

$$\min_{\psi \in Aut(\mathcal{S})} \mathcal{F}_{met}(\mathcal{U}^*(\mathcal{K})) = \mathcal{F}_{met}(\mathcal{P}\psi\mathcal{P}^{-1}\mathcal{U}(\mathcal{K})) \qquad (8)$$

This a very small-scale nonlinear optimization problem, and we also want to mention several points:

- If we fix $\mathcal{U}^*(\mathbf{v}^*) = (0,0,1)$ and get rid of the rotation influence, then ψ could be simplified to: $\psi(z) = az+b$ where $a \in \mathbb{R}$, $b \in \mathbb{C}$. Then it is only a three-parameter minimization problem.
- No derivative information available since it is very complicated to compute. *Powell Method* can be used to solve this minimization problem. Some global searching methods may produce better results.
- Since our conformal maps preserve angles locally, it is very reasonable to use only 1-neighbor to define $N(i)$ in $\mathcal{F}_{met}$ that will great reduce the computation time and not affect the resulting map much.

III. CORTICAL FLAT MAPS

We illustrate our method by flattening a human cerebellar cortex and a cerebral cortex and comparing with the popular flattening softwares FreeSurfer and CARET.

III-A. Cerebellar cortex

We first chose to flatten a cerebellum which was extracted from a high-resolution T1-weighted MRI volume in a consistent manner across subjects. Cortical regions defined by different lobes and fissures were colored for identification purposes. The parcellated surface consistsis of 56,676 triangles and 28,340 vertices and is equivalent to a topological sphere, see Fig. 1-**A**. As discussed in Section II-C, for each vertex $\mathbf{v}_i$, $N(i)$ was set to be all its 1-neighbors during the minization process (8). The spherical conformal map of the entire cerebellar cortex were then created using our method and shown in Fig. 1-**B**.

To be flattened in a planar region, the cortical surface must be a topological disc. The lobe IV and V patch (Fig.1-**C**) was cut off from the cerebellar cortex and then its planar conformal map was produced with only two vertices prepinned (Fig.1-**D**). This patch has 7,482 triangles and 3,903 vertices with 322 vertices on the boundary. It is easy to find that the shape of the patch seems to be well-preserved. We also would like to point out that the patch size in general should not be too large in order to obtain planar map with better metric distortion; otherwise, artificial cuts should be made for the patch.

In addition, we pre-decided a planar region, in particular, we chose an elipse such as $\mathcal{E} = \{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2/0.4^2 = 1\}$ and fixed a homeomorphism $g : \partial\mathcal{K} \to \partial\mathcal{E}$ using the following procedure: The boundary vertices of $\mathcal{U}(\mathcal{K})$, were positioned on the boundary of $\mathcal{E}$ such that the sides subtend angles basically propositional to the length of the boundary edges of $\mathcal{K}$ joining the corresponding corners. Then we flattened the lobe IV and V patch in the region $\mathcal{E}$ (Fig.1-**E**). Of course, some conformality will be lost, but we get the control of the shape. The CPU time and measures of distortion for the above flat maps are reported in Table I. Note that the metric distortion is computed by setting $N(i)$ to be all neighbors up to the 5th level.

III-B. Cerebral cortex

The parcellated surface of left cerebral hemisphere has totally 383,444 triangles and 191,724 vertices, see Fig. 2 (left) for the colored lobes map. The corresponding spherical conformal map is shown in the Fig. 2 (right). The occipital lobe patch (Fig.3-**A**) was cut off from the cerebral cortex which is a topological disc, having 11,670 triangles and 5,918 vertices with 290 vertices on the boundary. Like before, we first flattened the occipital lobe patch with only two vertices pinned on a plane (Fig.3-**B**), and then did it on a pre-defined planar region (Fig.3-**C**), in particular, the unit circle $\mathcal{D} = \{(u,v) \mid u^2 + v^2 = 1\}$. The CPU time and measures of distortion are reported in Table I.

III-C. Comparisons with other approaches

FreeSurfer and CARET are two commonly used softwares for flattening cortical surfaces at present. In order to evaluate the quality and performance of our quasi-conformal flat maps, we also flattened the above cerebellar and cerebral cortices using FreeSurfer and CARET. The computational results were reported in Table II.

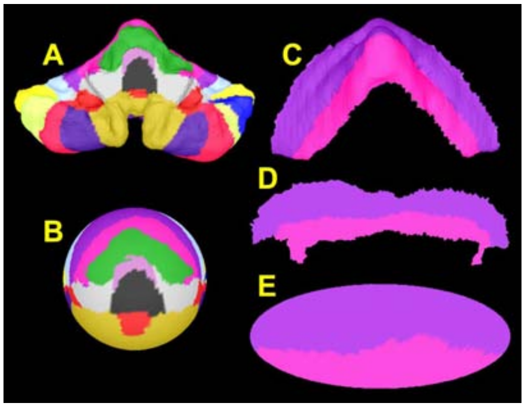

Fig. 1. **A**. Parcellated surface of the human cerebellum; **B**. Spherical conformal map of **A**; **C**. Cortical patch of cerebellar Lobes IV and V; **D**. Planar conformal map of **C** when two vertices are fixed; **E**. Planar conformal map of **C** when all boundary vertices are pre-defined on ellipse $\mathcal{E}$.

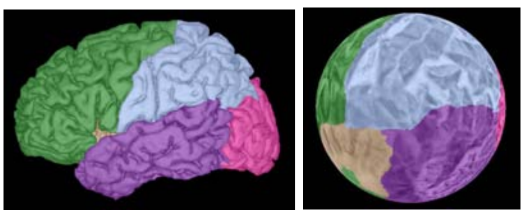

Fig. 2. **Left**. Parcellated surface of the left cerebral hemisphere; **Right**. Spherical conformal map of the left hemispheral cortex.

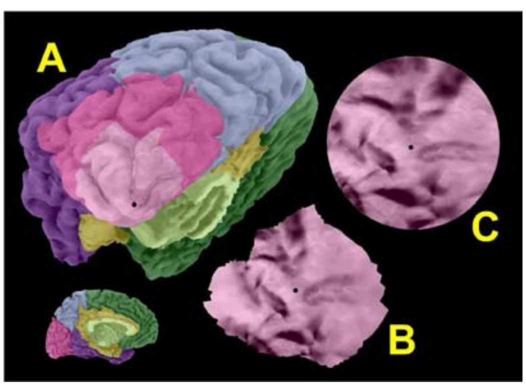

Fig. 3. **A**. Occipital lobe patch (pink), the occipital pole is indicated by a black dot; **B**. Planar conformal map of **A** when two vertices are fixed; **C**. Planar conformal map of **A** when all boundary vertices are pre-defined on disc $\mathcal{D}$.

IV. DISCUSSION AND CONCLUSION

Our discrete conformal method can be applied to *either* a user-defined patches *or* to the entire cortical surface and it shows that angular information is truely preserved while metric distortion is minimized. Metric distortion is similar to that produced by FreeSurfer and CARET, but computation time is greatly reduced relative to both.

	Cerebellar Cortex		
Flat Maps	**A**	**B**	**C**
CPU Time (sec.)	56.9	7.9	2.7
Angular Distortion	1.89°	1.01°	22.20°
Metric Distortion	42.89%	45.56%	33.64%
	Cerebral Cortex		
Flat Maps	**A**	**B**	**C**
CPU Time (sec.)	779.6	20.7	16.3
Angular Distortion	4.46°	2.02°	10.71°
Metric Distortion	40.09%	38.76%	27.50%

Table I. CPU time and angular and metric distortion produced by the LSCM method. **A**. spherical map of the cerebellar cortex or the cortex of the left cerebral hemisphere; **B**. planar map of a cortical patch when two vertices are fixed; **C**. planar map of the same patch when all boundary vertices are pre-defined.

	Cerebellar Cortex		Cerebral Cortex	
Methods	FreeSurfer	CARET	FreeSurfer	CARET
CPU Time (hr.)	0.5	2	10	-
Angular Distortion	23.34°	21.22°	18.70°	-
Metric Distortion	32.47%	31.06%	33.50%	-

Table II. CPU time and angular and metric distortion of spherical maps produced by CARET and FreeSurfer.

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