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Finite State Models of Constrained 2D Data

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Abstract — **We consider a class of discrete finite alphabet 2D fields that can be characterized using tools from finite state machines and Markov chains. These fields have several properties that greatly simplify the analysis of 2D coding methods.**

I. INTRODUCTION

We define a class of fields that allow a detailed analysis by application of finite state models. First the set of outcomes on finite rectangular arrays is defined using methods from regular sets. All the required properties of these sets can be decided by standard methods of finite state machines. Probability distributions on these sets are then introduced, and we present a construction which allows the distributions to be completely specified in terms of finite state Markov chains.

II. FINITE STATE FIELDS

We consider finite arrays of arbitrary size that are constrained by the condition that any N by N outcome is an element of an admissible subset Ω of $A^{N \otimes N}$ [1]. We study causal models where each variable is specified conditioned on all values above and values to the left in the same row. The arrays are described by a finite state machine, where the states consist of a single connected set of recurrent states, and there are no periodic subsets.

By removing the top or bottom variable in the N band we get two N-1 bands described by the same state machine. The description can always be converted to a minimal deterministic machine.

We define a finite state 2D set as a set of arrays that satisfies the following three conditions:

Condition 1: The upper and lower N-1 band machines define the same sets.

When N-1 rows of the field are given, the set of compatible next rows can be described by a trellis with states from the set of N-1 band states.

Condition 2: For some fixed finite number d, a possible extension to the right of row j can be chosen when the previous N-1 columns including row j and the next d columns of the previous N-1 rows are know.

Condition 3: There is a finite number c, such that any combination of N-1 rows can be reached from any other in c transitions.

An array in such a set can be generated by a causal model. Let the symbols in rows j-N to j-1 be known at least up to column i+d. Symbol (j, i) can now be generated by the following steps:

- Extend rows j-N to j-1 from column i+d to i+d+1 using the finite state machine derived above.
- Extend the trellis of possible values for row j to column i+d+1, and remove all branches that are not connected to states in this column.
- Choose on of the possible values for the symbol in column i+1 of row j, and reduce the trellis to branches originating from the corresponding state.

III. PROBABILITY DISTRIBUTIONS ON FINITE STATE MODELS

Previously analytical methods have largely been restricted to constraints between neighboring symbols, and the classical result on fields defined by Markov chains is Pickard's construction of binary fields [2]. Such fields are defined from probability distributions on two by two symbols where the diagonal elements must satisfy an independence condition. Recently it has been observed that some modifications of the construction are possible without loss of the Markov property for rows and columns [3, 4]. However, these extensions still apply only to first order processes.

The following construction generalizes the Pickard condition to more rows and larger memory row processes, but in order to simplify the presentation we shall give details only for N=3. A stationary probability distribution on two rows can be defined by an m'th order Markov chain, m≥2, with alphabet A^2, and transition probabilities that are symmetric in the two rows. A finite state description of the probability distribution for 3 rows is based on states of the form

$$S(j,i)=\begin{bmatrix} - & \dots & - & a(j-2,i-1) & a(j-2,i) & a(j-2,i+1) & \dots & a(j-2,i+d) \\ a(j-1,i-d-1) & \dots & a(j-1,i-2) & a(j-1,i-1) & a(j-1,i) & a(j-1,i+1) & \dots & a(j-1,i+d) \\ a(j,i-d-1) & \dots & a(j,i-2) & a(j,i-1) & a(j,i) & - & \dots & - \end{bmatrix}$$

For this state description, the generalized Pickard condition can be expressed as

$$P\left[\begin{pmatrix} a(j-2,i+d+1) \\ a(j-1,i+d+1) \\ - \end{pmatrix} \middle| S(j,i)\right] = P\left[\begin{pmatrix} a(j-2,i+d+1) \\ a(j-1,i+d+1) \\ - \end{pmatrix} \middle| \begin{pmatrix} a(j-2,i+d) & a(j-2,i-d-1) \\ a(j-1,i+d) & a(j-1,i-d-1) \\ - & - \end{pmatrix}\right]$$

Theorem: If a stationary probability distribution for a finite state field exists such that the N row distribution of the states is an interior point in the set of distributions, and the distribution satisfies a mixing condition, the distribution can be approximated by a Markov chain of finite order.

IV. EXAMPLES

Several examples illustrating the construction of a field with a given set of constraints will be provided. The entropies can be calculated explicitly.

REFERENCES

[1] S. Forchhammer & J. Justesen, "Entropy bounds for constrained two-dimensional fields," IEEE Trans. Inform. Theory, vol. IT-45, pp. 118-127, January 1999.
[2] D.K. Pickard, "A curious binary lattice," J. Appl.Prob., vol. 14, pp. 717-731, 1977.
[3] R. M. Roth, P.H. Siegel, and J.K. Wolf, "Efficient coding schemes for the hard-square model", IEEE Trans. Inform. Theory, vol. 47, pp. 1166-76, March 2001.
[4] J. Justesen, "Fields from Markov chains," submitted to IEEE Trans. Inform. Theory.

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