

On improving security of GPT cryptosystems

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Abstract—The public key cryptosystem based on rank error correcting codes (the GPT cryptosystem) was proposed in 1991. Use of rank codes in cryptographic applications is advantageous since it is practically impossible to utilize combinatoric decoding. This enabled using public keys of a smaller size. Several attacks against this system were published, including Gibson’s attacks and more recently Overbeck’s attacks. A few modifications were proposed withstanding Gibson’s attack but at least one of them was broken by the stronger attacks by Overbeck. A tool to prevent Overbeck’s attack is presented in [12]. In this paper, we apply this approach to other variants of the GPT cryptosystem.

I. INTRODUCTION

The first code-based public-key cryptosystem is introduced and investigated in [1]. The system is based on Goppa codes in the Hamming metric. It is a strong cryptosystem but the size of a public key is too large for practical implementations to be efficient.

The public key cryptosystem based on *rank* error correcting codes was proposed in [2], [3] and is now called the GPT cryptosystem.

Rank codes are well structured. It makes easier creation of attacks. Subsequently in a series of works, Gibson [4], [5] developed attacks that break the GPT system for public keys of about 5 Kbits which are efficient for practical values of parameters $n \leq 30$, where n is length of rank codes with the field \mathbb{F}_{2^n} as an alphabet.

Several variants of the GPT PKC were introduced to withstand Gibson’s attacks [6], [7]. One proposal is use of a rectangular row scramble matrix instead of a square matrix. This allows to work with subcodes of rank codes having much more complicated structure. Another proposal exploits a modification of Maximum Rank Distance (MRD) codes where the concept of a *column* scramble ma-

trix was also introduced. A new class of rank codes, so called, reducible codes, are also implemented to modify the GPT cryptosystem [8], [9]. All these variants withstand Gibson’s attack.

Recently, R. Overbeck [10], [11] proposed a new attack which is more effective than any of Gibson’s attacks. His method is based on the fact that a column scrambler is defined over the *base field*. A generalization and development of one Gibson’s idea allows him to break many instances of the GPT cryptosystem. It was found in [12] that a cryptographer can define a proper column scrambler over the *extension field without violation* of the standard mode of the PKC. It turns out that Overbeck’s attack fails in this case.

In this paper, we implement an idea of a proper choice of column scramblers over the extension field to other variants of the GPT cryptosystem. This choice withstands Overbeck’s attacks as well as Gibson’s attacks.

II. THE GPT CRYPTOSYSTEM

A. Rank codes

Let \mathbb{F}_q be a finite field of q elements and let \mathbb{F}_{q^N} be an extension field of degree N .

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be a vector with coordinates in \mathbb{F}_{q^N} .

The *Rank* norm of \mathbf{x} is denoted $\text{Rk}(\mathbf{x} | \mathbb{F}_q)$ and is defined as the *maximal* number of x_i , which are linearly independent over the *base field* \mathbb{F}_q .

Similarly, for a matrix \mathbf{M} with entries in \mathbb{F}_{q^N} the *column rank* is defined as the *maximal* number of columns, which are linearly independent over the base field \mathbb{F}_q , and is denoted $\text{Rk}(\mathbf{M} | \mathbb{F}_q)$.

The *Rank* distance between \mathbf{x} and \mathbf{y} is defined as the rank norm of the difference $\mathbf{x} - \mathbf{y}$, i.e. $d(\mathbf{x}, \mathbf{y}) = \text{Rk}(\mathbf{x} - \mathbf{y} | \mathbb{F}_q)$.

The theory of optimal MRD (Maximal Rank Distance) codes is given in [13]. A *generator* matrix \mathbf{G}_k of a MRD code is defined by

$$\mathbf{G}_k = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \\ g_1^{[1]} & g_2^{[1]} & \cdots & g_n^{[1]} \\ g_1^{[2]} & g_2^{[2]} & \cdots & g_n^{[2]} \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{[k-1]} & g_2^{[k-1]} & \cdots & g_n^{[k-1]} \end{bmatrix}, \quad (1)$$

where g_1, g_2, \dots, g_n are any set of elements of the extension field \mathbb{F}_{q^n} which are linearly independent over the base field \mathbb{F}_q .

The notation $g^{[i]} := g^{q^{i \bmod n}}$ means the i th Frobenius power of g .

A code with the generator matrix (1) is referred to as a (n, k, d) code, where n is the code length, k is the number of information symbols, d is the code distance. For MRD codes, $d = n - k + 1$.

Let $\mathbf{m} = (m_1, m_2, \dots, m_k)$ be an information vector of dimension k . The corresponding code vector is the n -vector

$$\mathbf{g}(\mathbf{m}) = \mathbf{m}\mathbf{G}_k.$$

If $\mathbf{y} = \mathbf{g}(\mathbf{m}) + \mathbf{e}$ and $\text{Rk}(\mathbf{e}) = s \leq t = \frac{d-1}{2}$, then the information vector \mathbf{m} can be recovered uniquely from \mathbf{y} by some decoding algorithm.

There exist *fast* decoding algorithms for MRD codes [13], [14]. A decoding procedure requires elements of the $(n - k) \times n$ parity check matrix \mathbf{H} such that $\mathbf{G}_k \mathbf{H}^\top = \mathbf{0}$. For decoding, the matrix \mathbf{H} should be of the form

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_1^{[1]} & h_2^{[1]} & \cdots & h_n^{[1]} \\ h_1^{[2]} & h_2^{[2]} & \cdots & h_n^{[2]} \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{[d-2]} & h_2^{[d-2]} & \cdots & h_n^{[d-2]} \end{bmatrix}, \quad (2)$$

where elements h_1, h_2, \dots, h_n are in the extension field \mathbb{F}_{q^n} and are linearly independent over the base field \mathbb{F}_q .

B. Description of the standard GPT cryptosystem

The GPT cryptosystem is described as follows.

1) *Possible generator matrices using as public keys:* Denote by \mathbf{G}_{pub} the public key, which is a generator matrix of a code.

1)

$$\mathbf{G}_{\text{pub}} = \mathbf{S}\mathbf{G}_k\mathbf{P}. \quad (3)$$

The main matrix \mathbf{G}_k is given by Eq. (1). It is used to correct rank errors. Errors of rank not greater than $t = \lfloor \frac{n-k}{2} \rfloor$ can be corrected.

A square $k \times k$ matrix \mathbf{S} over the extension field \mathbb{F}_{q^n} is called the row scrambling matrix. It is used to destroy any visible structure of the matrix \mathbf{G}_k by mixing its rows.

A matrix $\mathbf{P} = [p_{ij}]$ is called the column scrambler. This matrix is a non singular square matrix of order n . It is used to mix columns of \mathbf{G}_k .

If \mathbf{P} is a matrix over the *base field* \mathbb{F}_q , then a matrix $\mathbf{G}_k\mathbf{P}$ has just the same structure as the matrix \mathbf{G}_k with a different first row. Hence, from the point of view of breaking, matrices $\mathbf{G}_{\text{pub}} = \mathbf{S}\mathbf{G}_k\mathbf{P}$ and $\mathbf{G}_{\text{pub}} = \mathbf{S}\mathbf{G}_k$ are equivalent. A cryptographer may not use a matrix \mathbf{P} at all.

On the other hand, if entries p_{ij} are in the extension field \mathbb{F}_{q^n} , then a matrix \mathbf{P} makes breaking much harder. We shall analyze this case.

- 2) Another generator matrix is obtained by an extension of matrix \mathbf{G}_k :

$$\mathbf{G}_{\text{pub}} = \mathbf{S} [\mathbf{X} \ \mathbf{G}_k] \mathbf{P}. \quad (4)$$

A matrix \mathbf{X} of size $k \times t_1$ is called a distortion source matrix. This matrix is a part of the concatenation $[\mathbf{X} \ \mathbf{G}_k]$. The column rank of \mathbf{X} is $\text{Rk}(\mathbf{X} | \mathbb{F}_q) = t_1$. The number t_1 is a *design* parameter. Another *design* parameter is the ordinary rank which can take values from 1 to t_1 . The rank distance of a code generated by the matrix \mathbf{G}_{pub} is not less than the rank distance of a code generated by the matrix $\mathbf{S} [\mathbf{O} \ \mathbf{G}_k] \mathbf{P}$.

A matrix \mathbf{P} is called the column scrambler. This matrix is a non-singular square matrix of order $n + t_1$. It is used to mix and to corrupt columns of \mathbf{G}_k by means of the distortion source matrix \mathbf{X} .

Note that in previous works, the matrix \mathbf{P} has all its entries in the base field \mathbb{F}_q . Overbeck's attack against this PKC succeeded due to this fact. But the attack fails for the proper choice of \mathbf{P} over the extension field \mathbb{F}_{q^n} [12].

3)

$$\mathbf{G}_{\text{pub}} = \mathbf{S} [\mathbf{X} \ \mathbf{G}_k] \mathbf{P}. \quad (5)$$

Here a scrambling matrix \mathbf{S} is a rectangular $(k - p) \times k$ matrix.

4)

$$\mathbf{G}_{\text{pub}} = \mathbf{S} ([\mathbf{O} \ \mathbf{G}_k] + [\mathbf{X}_1 \ \mathbf{X}_2]) \mathbf{P}. \quad (6)$$

Here: the row scrambler \mathbf{S} is a square non-singular matrix of order k with entries in \mathbb{F}_{q^n} chosen at random; \mathbf{O} is the $k \times m$ matrix

of 0's; \mathbf{X}_1 is some $k \times m$ matrix — the first distortion matrix; \mathbf{X}_2 is a $k \times n$ matrix with $r(\mathbf{X}_2|\mathbb{F}_1) = t_1$ — the second distortion matrix; the column scrambler \mathbf{P} is a non-singular matrix of order $n + m$ with entries in \mathbb{F}_q .

2) *Plaintext*: For public keys (3), (4) and (6), a **plaintext** is any k -vector $\mathbf{m} = (m_1, m_2, \dots, m_k)$, $m_s \in \mathbb{F}_{q^n}$, $s = 1, 2, \dots, k$. For the public key (5), a plaintext is a $(k - p)$ -vector.

3) *Private keys*: The **Private keys** are matrices $\mathbf{S}, \mathbf{G}_k, \mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{P}$ separately and (explicitly) a fast decoding algorithm of an MRD code. Note also, that the matrices $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2$ are not used to decrypt a ciphertext and can be deleted after calculating the Public key.

4) *Encryption*: Let $\mathbf{m} = (m_1, m_2, \dots, m_k)$ be a plaintext. The corresponding ciphertext is given by

$$\mathbf{c} = \mathbf{m}\mathbf{G}_{pub} + \mathbf{e} = \mathbf{m}\mathbf{S}[\mathbf{X}|\mathbf{G}_k]\mathbf{P} + \mathbf{e}, \quad (7)$$

where \mathbf{e} is an artificial vector of errors of rank t_2 or less, randomly chosen and added by the sending party. The number t_2 is the third design parameter.

5) *Decryption*: The legitimate receiver upon receiving \mathbf{c} calculates

$$\mathbf{c}' = \mathbf{c}\mathbf{P}^{-1} = \mathbf{m}\mathbf{S}[\mathbf{X}|\mathbf{G}_k] + \mathbf{e}\mathbf{P}^{-1}.$$

Then he extracts from \mathbf{c}' the plaintext \mathbf{m} using decoding algorithms and properties of public keys.

III. THE OVERBECK ATTACK - AN IDEA

In [10], [11], a new attack is proposed on the GPT PKC described by means of Eq. (4).

It is claimed, that similar attacks can be proposed on all the variants of GPT PKC.

We can not describe the attack in detail but recall briefly an idea of this attack.

We need some notations.

For $x \in \mathbb{F}_{q^n}$, let $\sigma : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$, $\sigma(x) = x^q$ be the Frobenius automorphism.

For the matrix $\mathbf{T} = (t_{ij})$ over \mathbb{F}_{q^n} , let $\sigma(\mathbf{T}) = (\sigma(t_{ij})) = (t_{ij}^q)$.

For any integer s , let $\sigma^s(\mathbf{T}) = \sigma(\sigma^{s-1}(\mathbf{T}))$.

It is clear that $\sigma^n = \sigma$. Thus the inverse exists $\sigma^{-1} = \sigma^{n-1}$.

The following simple properties of σ are useful:

- $\sigma(a + b) = \sigma(a) + \sigma(b)$.
- $\sigma(ab) = \sigma(a)\sigma(b)$.
- In general, for matrices $\sigma(\mathbf{T}) \neq \mathbf{T}$.
- If \mathbf{P} is a matrix over the *base field* \mathbb{F}_q , then $\sigma(\mathbf{P}) = \mathbf{P}$.

6) *An idea of Overbeck's attack*: To break a system, a cryptanalyst constructs from the public key $\mathbf{G}_{pub} = \mathbf{S}[\mathbf{X} \ \mathbf{G}_k]\mathbf{P}$ the *extended* public key as follows:

$$\mathbf{G}_{ext, pub} = \left\| \begin{array}{c} \mathbf{G}_{pub} \\ \sigma(\mathbf{G}_{pub}) \\ \sigma^2(\mathbf{G}_{pub}) \\ \vdots \\ \sigma^u(\mathbf{G}_{pub}) \end{array} \right\|. \quad (8)$$

The property that $\sigma(\mathbf{P}) = \mathbf{P}$, if \mathbf{P} is a matrix over the *base field* \mathbb{F}_q , is used in (8). Further transformations of Eq. (8) allows to obtain the first row of the check matrix \mathbf{H} of the rank code used. It is enough to break the cryptosystem.

If \mathbf{P} is a matrix over the *extension field* \mathbb{F}_{q^n} , then $\sigma(\mathbf{P}) \neq \mathbf{P}$.

We have to stress that Overbeck's attack fails in this case.

Moreover Gibson's attacks use also in implicit form the condition $\sigma(\mathbf{P}) = \mathbf{P}$ and can not be implemented without it.

Our intention is to show that there exist column scramblers \mathbf{P} in the *extension field* \mathbb{F}_{q^n} such that the GPT PKC works and is secure against all known attacks.

IV. OTHER ATTACKS ON THE GPT PKC

An important part of a decryption procedure is correcting rank errors using a fast decoding algorithm known to the legitimate party. An unauthorized party may want to correct rank errors by a general algorithm without any knowledge of the structure of a rank code. We consider algorithms described in [15] and in the recent paper [16].

The authors of [15] proposed two algorithms for decoding an arbitrary (n, k) linear rank distance code over \mathbb{F}_{q^N} . These algorithms correct errors of rank $t = \lfloor \frac{n-k}{2} \rfloor$ in $O((Nt)^3 q^{(t-1)(k+1)})$ and $O((k+t)^3 t^3 q^{(t-1)(N-t)})$ operations in \mathbb{F}_q respectively.

Consider as an example a case when we use a $(28, 14)$ rank code with $N = n = 28, k = 14, q = 2, d = 15, t = 7$. The size of the public key is equal to $Nnk = 10976$ bits. To correct 7-fold rank errors, Ourivski–Johansson's algorithms [15] require 2^{113} and 2^{147} operations in \mathbb{F}_2 . Thus these attacks are infeasible for practical implementations.

The algorithm of [16] requires $O(\log(q)N^{3(N-t)})$ operations. We have for the above example 2^{302} operations. Thus this attack is also infeasible for practical implementations.

V. THE SIMPLE GPT PKC

Consider the public key of Eq. (3). No distortion matrix \mathbf{X} is used. A ciphertext has the form

$$\mathbf{c} = \mathbf{m}\mathbf{S}\mathbf{G}_k\mathbf{P} + \mathbf{e}, \quad (9)$$

where the rank $\text{Rk}(\mathbf{e} \mid \mathbb{F}_q) = t_1$ of an artificial error \mathbf{e} is less or equal to $t = \lfloor \frac{n-k}{2} \rfloor$.

Brute-force attacks are based on the exhaustive search of possible artificial errors \mathbf{e} . It depends on the number of error vectors. If artificial errors are all possible n -vectors of rank t_1 , then the number of operations to search is $O(q^{nt_1})$.

Attacks on the public key contemplate to find unknown factors (to a cryptanalyst) $\tilde{\mathbf{S}}$, $\tilde{\mathbf{G}}_k$ and $\tilde{\mathbf{P}}$, or, to find matrices $\tilde{\mathbf{S}}$, $\tilde{\mathbf{G}}_k$ and $\tilde{\mathbf{P}}$ such that $\mathbf{S}\mathbf{G}_k\mathbf{P} = \tilde{\mathbf{S}}\tilde{\mathbf{G}}_k\tilde{\mathbf{P}}$ from the known public key matrix $\mathbf{S}\mathbf{G}_k\mathbf{P}$.

Assume first that the column scrambler \mathbf{P} is a matrix over the base field \mathbb{F}_q . The legitimate user knows the secret key \mathbf{P} and \mathbf{P}^{-1} . His algorithm is as follows.

- 1) Get a ciphertext $\mathbf{c} = \mathbf{m}\mathbf{S}\mathbf{G}_k\mathbf{P} + \mathbf{e}$.
- 2) Multiply to the right by \mathbf{P}^{-1} . Get an intermediate ciphertext

$$\mathbf{c}' = \mathbf{c}\mathbf{P}^{-1} = \mathbf{m}\mathbf{S}\mathbf{G}_k + \mathbf{e}\mathbf{P}^{-1}. \quad (10)$$

Note that $\text{Rk}(\mathbf{e}\mathbf{P}^{-1} \mid \mathbb{F}_q) = \text{Rk}(\mathbf{e} \mid \mathbb{F}_q) = t_1 \leq t = \lfloor \frac{n-k}{2} \rfloor$ since \mathbf{P}^{-1} is in the base field \mathbb{F}_q .

- 3) Decode \mathbf{c}' using a fast decoding algorithm and get $\mathbf{m}\mathbf{S}$.
- 4) Get a plaintext \mathbf{m} as $(\mathbf{m}\mathbf{S})\mathbf{S}^{-1}$.

On the other hand, the cryptanalyst can get a successful representation $\mathbf{G}_{\text{pub}} = \tilde{\mathbf{S}}\tilde{\mathbf{G}}_k$ for the equivalent rank code with the generator matrix $\tilde{\mathbf{G}}_k$ from the public key $\mathbf{S}\mathbf{G}_k\mathbf{P}$. It can be done by means of Gibson–Overbeck’s attacks and therefore break the system.

The situation is quite different if \mathbf{P} is a matrix over the extension field \mathbb{F}_{q^N} . For the general matrix \mathbf{P} , it is unknown how to solve the following problems: to find the public key factors $\tilde{\mathbf{S}}$, $\tilde{\mathbf{G}}_k$ and $\tilde{\mathbf{P}}$, or, to find matrices $\tilde{\mathbf{S}}$, $\tilde{\mathbf{G}}_k$ and $\tilde{\mathbf{P}}$ such that $\mathbf{S}\mathbf{G}_k\mathbf{P} = \tilde{\mathbf{S}}\tilde{\mathbf{G}}_k\tilde{\mathbf{P}}$ from the known public key matrix $\mathbf{S}\mathbf{G}_k\mathbf{P}$. Gibson–Overbeck’s attacks are not applicable if a matrix \mathbf{P} is chosen in the extension field \mathbb{F}_{q^N} .

We can assume from now on that Gibson’s and Overbeck’s attacks can not be implemented. But the cryptographer should select a *secret* column scrambler \mathbf{P} in the *extension field* \mathbb{F}_{q^N} and a *public* set \mathcal{E} of artificial errors \mathbf{e} such that

$$\text{Rk}(\mathbf{e}\mathbf{P}^{-1} \mid \mathbb{F}_q) \leq t = \left\lfloor \frac{n-k}{2} \right\rfloor, \quad (11)$$

where $\mathbf{e}\mathbf{P}^{-1}$ is an error in the intermediate ciphertext (10).

a) *Choice of \mathcal{E}* : The public set of artificial errors is chosen as the set consisting of all n -vectors in $\mathbb{F}_{q^N}^n$ with rank $t_1 < t$:

$$\mathcal{E} = \left\{ \mathbf{e} : \mathbf{e} \in \mathbb{F}_{q^N}^n, \text{Rk}(\mathbf{e} \mid \mathbb{F}_q) = t_1 \right\}.$$

b) *Choice of \mathbf{P}* : The cryptographer chooses an inverse matrix \mathbf{P}^{-1} in the form $\mathbf{P}^{-1} = [\mathbf{Q}_1 \ \mathbf{Q}_2]$, where \mathbf{Q}_1 is a submatrix of size $n \times (t - t_1)$ with entries in the *extension field* \mathbb{F}_{q^N} while \mathbf{Q}_2 is a submatrix of size $n \times (n - t + t_1)$ with entries in the *base field* \mathbb{F}_q .

Lemma 1: Let \mathbf{e} be any n -vector of rank t_1 . Then the condition Eq. (11) is hold.

Proof: We have $\mathbf{e}\mathbf{P}^{-1} = \mathbf{e}[\mathbf{Q}_1 \ \mathbf{Q}_2] = [\mathbf{e}\mathbf{Q}_1 \ \mathbf{e}\mathbf{Q}_2]$. A vector \mathbf{e} can be represented as $\mathbf{e} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \dots \ \mathbf{w}_{t_1}] \mathbf{A}$, where \mathbf{w}_j ’s are linearly independent over \mathbb{F}_q and \mathbf{A} is the $t_1 \times n$ matrix over \mathbb{F}_q of rank t_1 . Then $\mathbf{e}\mathbf{Q}_1 = [\mathbf{w}_1 \ \mathbf{w}_2 \ \dots \ \mathbf{w}_{t_1}] \mathbf{B}_1$, where $\mathbf{B}_1 = \mathbf{A}\mathbf{Q}_1$ is the $t_1 \times (t - t_1)$ matrix over the extension field \mathbb{F}_{q^N} . It is clear that $\text{Rk}(\mathbf{e}\mathbf{Q}_1 \mid \mathbb{F}_q) \leq t - t_1$. Similarly, $\mathbf{e}\mathbf{Q}_2 = [\mathbf{w}_1 \ \mathbf{w}_2 \ \dots \ \mathbf{w}_{t_1}] \mathbf{B}_2$, where $\mathbf{B}_2 = \mathbf{A}\mathbf{Q}_2$ is the $t_1 \times (n - t + t_1)$ matrix over the *base field* \mathbb{F}_q . It follows that $\text{Rk}(\mathbf{e}\mathbf{Q}_2 \mid \mathbb{F}_q) = \min(t_1, n - t + t_1) \leq t_1$. Hence

$$\text{Rk}(\mathbf{e}\mathbf{P}^{-1} \mid \mathbb{F}_q) \leq \text{Rk}(\mathbf{e}\mathbf{Q}_1 \mid \mathbb{F}_q) + \text{Rk}(\mathbf{e}\mathbf{Q}_2 \mid \mathbb{F}_q) \leq (t - t_1) + t_1 = t = \left\lfloor \frac{n-k}{2} \right\rfloor. \quad \blacksquare$$

Remark 1: The matrix \mathbf{P}^{-1} can be replaced by a matrix $\tilde{\mathbf{P}}^{-1} = \mathbf{P}^{-1}\mathbf{Q}$, where \mathbf{Q} is any $n \times n$ non singular matrix over the base field \mathbb{F}_q .

Example 1: Consider again the case when we use a (28, 14) rank code with $N = n = 28, k = 14, q = 2, d = 15, t = 7$. Possible systems are listed below.

$t_1 = 0$, \mathbf{P} in the extension field, attacks on PK – Information sets attacks, brute-force attacks – not needed, status – *insecure*.

$t_1 = 1$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{24} , status – *insecure*.

$t_1 = 2$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{48} , status – *insecure*.

$t_1 = 3$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{72} , status – *secure*.

$t_1 = 4$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{96} , status – *secure*.

$t_1 = 5$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{120} , status – *secure*.

$t_1 = 6$, \mathbf{P} in the extension field, attacks on PK – unknown, brute-force attacks – 2^{144} , status – secure.

$t_1 = 7$, \mathbf{P} in the *base* field, attacks on PK – Gibson–Overbeck, brute-force attacks – 2^{168} , status – *insecure*.

For $t_1 = 0 \dots 2$, the system is insecure due to brute-force attacks. For $t_1 = 7$, the system is insecure because of Gibson–Overbeck’s attacks since in this case the matrix \mathbf{P} is in the base field \mathbb{F}_q . But for $t_1 = 3 \dots 6$, the system is secure against all known attacks. We recommend the value $t_1 = 3$, or the value $t_1 = 4$.

VI. OTHER VARIANTS OF THE GPT PKC

We can repeat word for word all previous considerations for variants (4)- (6) and choose for each case a proper column scrambler \mathbf{P} over the *extension field* \mathbb{F}_{q^N} . This prevents Overbeck’s and Gibson’s attacks.

VII. CONCLUSION

An approach is presented to withstand attacks on the GPT Public key cryptosystem based on rank codes.

It is shown that there exist column scramblers \mathbf{P} over the extension field \mathbb{F}_{q^N} which allow decryption for the authorized party while an unauthorized party can not break the system by means of known attacks.

REFERENCES

- [1] R.J. McEliece, “A Public Key Cryptosystem Based on Algebraic Coding Theory,” *JPL DSN Progress Report 42–44*, Pasadena, CA, pp. 114–116, 1978.
- [2] E.M. Gabidulin, A.V. Paramonov, O.V. Tretjakov, “Ideals over a Non-commutative Ring and Their Application in Cryptology”, in: *Advances in Cryptology – Eurocrypt ’91*, Editor: D.W. Davies, Lecture Notes in Computer Science, No. 547, pp. 482–489, Berlin and Heidelberg: Springer-Verlag, 1991.
- [3] E.M. Gabidulin, “Public-Key Cryptosystems Based on Linear Codes over Large Alphabets: Efficiency and Weakness,” in: *Codes and Ciphers*, Editor: P.G. Farrell, pp. 17–32, Essex: Formara Limited, 1995.
- [4] J. K. Gibson, “Severely denting the Gabidulin version of the McEliece public key cryptosystem,” // *Designs, Codes and Cryptography*, 6(1), 1995, pp. 37–45.
- [5] J. K. Gibson, “The security of the Gabidulin public-key cryptosystem,” in: U. M. Maurer, ed. // *Advances in Cryptology – EUROCRYPT’96, LNCS 1070*, 1996, pp. 212–223.
- [6] E.M. Gabidulin, A.V. Ourivski, “Improved GPT Public Key Cryptosystems.” // In: P. Farrell, M. Darnell, B. Honary (Ed’s), “*Coding, Communications, and Broadcasting*”, Research Studies Press, 2000, pp. 73-102.
- [7] A. V. Ourivski, E. M. Gabidulin, “Column Scrambler for the GPT Cryptosystem.” // *Discrete Applied Mathematics*. 128(1): 207-221 (2003).
- [8] E. M. Gabidulin, A. V. Ourivski, B. Honary, B. Ammar, “Reducible Rank Codes and Their Applications to Cryptography.” // *IEEE Transactions on Information Theory*. 49(12): 3289-3293 (2003).
- [9] A. S. Kshevetskiy, E. M. Gabidulin, “High-weight errors in public-key cryptosystems based on reducible rank codes.” // In: *Proc. of ISCTA*, 2005.
- [10] R. Overbeck, “A new brute-force attack for GPT and variants.” In: *Proc. of Mycrypt’2005*, vol. 3517 of LNCS, pp. 5-63, Springer-Verlag, 2005.
- [11] Overbeck R., “Brute-force attacks Public Key Cryptosystem Based on Gabidulin codes.” *J. Cryptology*, 21(2): 280-301 (2008).
- [12] E. M. Gabidulin, “Attacks and counter-attacks on the GPT public key cryptosystem,” *Designs, Codes and Cryptography*. V. 48, No. 2/ August 2008. Pp. 171-177, Springer Netherlands, DOI 10.1007/s10623-007-9160-8.
- [13] E.M. Gabidulin, “Theory of Codes with Maximum Rank Distance,” *Probl. Inform. Transm.*, vol. 21, No. 1, pp. 1–12, July, 1985.
- [14] E. M. Gabidulin, “A Fast Matrix Decoding Algorithm For Rank-Error-Correcting Codes.” In: (Eds G. Cohen, S. Litsyn, A. Lobstein, G. Zemor), *Algebraic coding*, pp. 126-132, Lecture Notes in Computer Science No. 573, Springer-Verlag, Berlin, 1992.
- [15] T. Johansson, A.V. Ourivski, “New technique for decoding codes in the rank metric and its cryptography applications,” *Problems Inform. Transm.* 38(3), 237246 (2002).
- [16] F. Levy-dit-Vehell, J.-Ch. Jean-Charles Faug’ere, and L. Perret, “Cryptanalysis of MinRank.” *Advances in Cryptology - CRYPTO 2008, 28th Annual International Cryptology Conference*, Santa Barbara, CA, USA, August 17-21, 2008, Proceedings. Series: Lecture Notes in Computer Science. Subseries: Security and Cryptology, Vol. 5157. Wagner, David (Ed.). 2008. Pp. 280-296.